DYNAMIC COORDINATION IN MANUFACTURING AND HEALTHCARE SYSTEMS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Zhongjie Ma

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

December 2018

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Pengyi Shi, Chair
School of Management
Dr. Qi Annabelle Feng
School of Management
Dr. Susan Feng Lu
School of Management
Dr. J.George Shanthikumar
School of Management

Approved by:

Dr. Yanjun Li
Head of the School Graduate Program

Dedicated to my family.

ACKNOWLEDGMENTS

First and foremost, I would like to express my sincere gratitude to my advisors Professor Pengyi Shi and Qi (Annabelle) Feng for the continuous support of my Ph.D. study and related research. I appreciate Professor Shi's contributions of time, efforts, and guidance to make my Ph.D. experience productive and stimulating. I am grateful for Professor Feng's patience, advice, and immense knowledge. Her guidance helped me in all the time of the study, research and writing of this dissertation.

I would like to thank Professor George Shanthikumar for teaching me Stochastic Model and Stochastic Dynamic Programming and his guidance on my research. It has been my great pleasure working with him. He is one of the most intelligent researchers I have ever met, and his brilliant ideas in the class or in our research discussions always enlighten me. His great sense of humor has brought me much joy in this journey and taught me to stay positive.

I would also like to express my great appreciation to Professor Jonathan Helm and Sebastian Heese for guiding me in my research, for spending hours discussing with me, and for raising challenging questions to encourage me to think further. Their insightful comments and questions have incented me to widen my research from various perspectives.

I want to thank Professor Susan Lu for serving on my dissertation committee and providing generous advice in my career choice. I would like to thank my fellow Ph.D. students and friends in Krannert School of Management for the stimulating discussions and for all the fun we have had in the last five years.

Finally, I would like to thank my parents for supporting me throughout my Ph.D. journey. I appreciate the unconditional love and care from them. I would like to thank in particular my girlfriend, Caiying Zhu, for her support and encouragement.

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Ma, Zhongjie Ph.D., Purdue University, December 2018. Dynamic Coordination in Manufacturing and Healthcare Systems. Major Professor: Pengyi Shi.

As the manufacturing and healthcare systems becomes more complex, efficiently managing these systems requires cooperation and coordination between different parties. This dissertation examines the coordination issues in a supply chain problem and diagnostic decision making in the healthcare system. Below, we provide a brief description of the problem and results achieved.

With supply chain becoming increasingly extended, the uncertainty in the upstream production process can greatly affect the material flow that aims toward meeting the uncertain demand at the downstream. In Chapter 2, we analyze a two-location system in which the upstream production facility experiences random capacities and the downstream store faces random demands. Instead of decomposing the profit function widely used to treat multi-echelon systems, our approach builds on the notions of stochastic functions, in particular, the stochastic linearity in midpoint and the directional concavity in midpoint, which establishes the concavity and submodularity of the profit functions. In general, it is optimal to follow a two-level state-dependent threshold policy such that an order is issued at a location if and only if the inventory position of that location is below the corresponding threshold. When the salvage values of the ending inventories are linear, the profit function becomes decomposable in the inventory positions at different locations and the optimal threshold policy reduces to the echelon base-stock policy. The effect of production and demand uncertainty on inventory levels depends critically on whether the production capacity is limited or ample in relation to the demand. Only when the capacity is about the demand, the upstream facility holds positive inventory; otherwise, all units produced are im-

mediately shipped to the downstream. We further extend our analysis to situations with general stochastic production functions and with multiple locations.

In Chapter 3, we examine the two-stage supply chain problem (described in Chapter 2) under the decentralized control. We consider two scenarios. In the first scenario, the retail store does not have any supply information including the inventory level at the manufacturing facility. We show that the upstream and downstream can be dynamically coordinated with proper transfer payment defined on local inventories and their own value function in the dynamic recursion. In the second scenario, the demand distribution is unknown to the manufacturing facility as well as the retail store does not know the supply information. We characterize the optimal transfer contracts under which coordination can be achieved, and propose an iterative algorithm to compute the optimal transfer contracts in the decentralized setting. The total profit of the decentralized system under our algorithm is guaranteed to converge to the centralized optimal channel profit for any demand and capacity distribution functions.

In Chapter 4, we provide a case study for the framework developed in [1]. The authors study the evaluation and integration of new medical research considering the operational impacts. As a case study, we first describe their two-station queueing control model using the MDP framework. We then present the structural properties of the MDP model. Since multiple classes of patients are considered in the MDP model, it becomes challenging to solve when the the number of patient classes increases. We describe an efficient heuristic algorithm developed by [1] to overcome the curse of dimensionality. We also test the numerical performance of their heuristic algorithm, and find that the largest optimality gap is less than 1.50% among all the experiments.

1. INTRODUCTION

In the first part of this study, we address the issue of dynamic coordination in supply chain with limited information visibility. Specifically, we consider a downstream retail store and an upstream manufacturing facility, and the upstream faces a random supply as well as the downstream demand is random. In the second part, we provide a case study for the framework developed in [1]. The authors address the issue of evaluating the operational impacts of new diagnostic tests. We briefly introduce this dissertation in the below sections.

1.1 Inventory Control in Multi-Stage Supply Chain with Supply Uncertainty

With the supply chain becoming increasingly global and extended, there have been considerable attentions in practice and research paid to managing risks deep in the production processes. The failure of machines and equipment deployed for production, the variation of worker skill and operating conditions, the fluctuation of raw material flows, and the lack of visibility of management process at the manufacturing facilities can all result in uncertainties in the supply process at the far upstream of the supply chain. Though there is a vast literature on multi-stage systems, little is done to understand the impact of upstream supply uncertainty on the dynamics of the material flows in meeting the uncertain demands at the downstream. This is the objective of our study.

In particular, we consider a two-location system with an upstream manufacturing facility producing the product and a downstream store selling the product. The production capacity at the upstream and the customer demand at the downstream are both random. At the beginning of each period, the on-hand inventory levels at both locations are reviewed, and a production order at the upstream and a replenishment order at the downstream are initiated. The production output, which is the minimum of production order and the random production capacity, is generated at the end of the period. Upon completion of the production, a shipment is dispatched from the upstream to reach the downstream in the next period. The realized demand at the downstream is fulfilled using the available inventory and the unmet demand is fully backordered.

The consideration of random production capacity makes the problem challenging even in the single-location setting (see, e.g., [2, 3]). For our problem involving serial system, we deployed the notion of stochastic functions, in particular, the stochastic linearity in midpoint and the directionally concave order, to transform the objective into a joint concave function provided that the terminal value obtained at the end of the planning horizon is concave and submodular in the ending inventory positions. This approach is an alternative to the conventional decomposition approach developed by [4] that has been widely adopted to derive structural properties of multi-echelon systems. To apply the latter approach, one needs to establish the separability of the profit function in the inventory positions for all locations, whereas our approach does not require such a separability.

With the joint concavity established, the optimal management policy can be characterized by two state-dependent thresholds, one for each location. A positive order is initiated at a location if and only if the inventory position at that location is below its threshold. We also show that the threshold at one location is decreasing in the prior order inventory position at the other location. This monotone property suggests an substitutable relationship between the inventory positions, as both inventory positions are targeted toward meeting the eventual customer demand at the downstream.

When the terminal value is separable in the inventory positions, the threshold policy reduces to the echelon base-stock policy. The base-stock level for the upstream inventory position represents the target level of the total inventory, which is independent of prior-ordering inventory levels. The base-stock level for the downstream inventory position is a function cannot exceed the post-order inventory position of the entire system and is thus dependent on the prior-order inventory position at the upstream.

Under the optimal policy, the target inventory positions increase when either the production capacity or the customer demand increases stochastically. The distribution of the inventory in the system, however, changes disproportionately with the capacity and the demand. In particular, we demonstrate that as the capacity becomes ample relative to the demand, the average on-hand inventory at the downstream and the average in-transit order to the downstream are both increasing. However, the upstream facility only hold inventory when the average production capacity is around the average consumer demand. Otherwise, production output is always immediately shipped to the downstream. When the production capacity is limited [ample] relative to the consumer demand, an increase in either capacity or demand uncertainty leads to an decreased [increased] average system inventory. We also observe that, while an increase in the holding costs or a decrease in the backorder cost induces reduced target inventory positions, the local inventory levels may respond oppositely. Specifically, the upstream may retain a larger amount of inventory for a lower backorder cost when production capacity is limited. An increased amount of inventory is held at the upstream [downstream] when the holding cost increases at the downstream [upstream].

We further show that our analysis can be extended when the upstream production follow a general stochastic function, which satisfies the single-crossing property for stochastic linearity in midpoint. Moreover, we show that our analysis can be carried out for serial systems with multiple locations.

1.2 Dynamic Supply Chain Coordination with Supply Uncertainty and Limited Information

When the upstream and downstream make the ordering decisions based on self interests, the natural question is: Whether the supply chain can achieve coordination in such a dynamic environment. To answer this question, we consider two scenarios. In the first scenario, the downstream does not know the distribution of the upstream random supply and the upstream's inventory level; in the second, the upstream does not know the demand distribution while the downstream does not have the supply information.

By recognizing some separability property of the optimal value function in the centralized problem, we show that the upstream and downstream can be dynamically coordinated with proper transfer payment defined on local inventories and their own value function in the dynamic recursion in the first scenario.

In the second scenario, when no party has full information, the coordination becomes much more challenging. The traditional principal-agent framework, though has been applied to dynamic contracting [5], it mostly works when one party knows the exact value of a specific parameter and the other does not. In our model, in an essential contrast, private knowledge is a function (i.e., the distribution). Applying the principal-agent approach would require the downstream to specify the set of possible distributions and a distribution over this set. Such a knowledge structure is very complex and difficult to specify in practice. Instead, we take a different angle. Using an iterative process that one party provides a transfer payment and then the other makes its ordering decision, we show that coordination can be eventually achieved despite the fact that the downstream has no knowledge of the supply distribution while the upstream has no knowledge of the demand distribution. This iterative process mimics a negotiation process, which can be implemented easily.

1.3 Case Study: New Diagnostic Test Evaluation

In medical research, new diagnostic tests are developed and evaluated solely on their efficacy in detecting an illness [6], [7], [8]. In the working paper [1], which motivates Chapter 4, the authors show that both the ED and patients could be worse off after introducing the new diagnostic test without considering the operational impacts. They further propose a framework for evaluating the operational impact of the new diagnostic tests and integrating the new tests into the existing clinical workflow.

Chapter 4 serves as a case study for the framework developed in [1]. Specifically, we first describe the two-station queueing network model developed by [1]. The authors propose an infinite-horizon MDP framework to model the test routing problem. We then present useful structural properties about the dominance among different classes: if a negative test result is more accurate, it is preferable to use the test on low-risk patients to rule out the disease; on the other hand, it is preferred to using the test on high-risk patients to confirm the disease if a positive test result is more accurate. Since the number of each class patients needs to be kept track of in the MDP model, the size of the state space becomes large as the number of patient classes increases. To overcome this curse of dimensionality, an efficient heuristic algorithm is developed by [1] based on the decomposition idea. We conduct numerical experiments to validate the performance of their heuristic algorithm, and find that the largest optimality gap is less than 1.50% among all the experiments.

1.4 Organization of the Dissertation

The remainder of this dissertation is organized as follows. In Chapter 2, we formulate a two-stage periodic-review inventory model. Using the notion of directional concave order, we analyze the structural properties of the optimal value function and characterize the optimal policy. We further analyze the responses of the optimal decisions to the changes of cost structure, production capacity, demand and planning horizon. This chapter is based on [9].

In Chapter 3, we study the two-stage supply chain problem under decentralized control. We consider two scenarios of information visibility. In the first scenario, the downstream does not have any supply information including the upstream's inventory level; in the second, the upstream does not know demand information while the downstream does not have supply information. In the first scenario, we show that the upstream and downstream can be dynamically coordinated with a proper transfer payment defined on local inventories and their own value function in the dynamic recursion. In the second scenario, we first characterize the form of the transfer contracts to achieve coordination. We further propose an iterative algorithm to compute the transfer contracts in the decentralized setting, and we prove the decentralized total profit from our algorithm converges to the centralized optimal channel profit.

In Chapter 4, we provide a case study for the developed framework in [1]. As a case study, we first describe the queueing control model in [1]. We present the structural properties of the optimal routing policies. We describe an efficient heuristic algorithm developed in [1] to overcome the curse of dimensionality in the MDP model. We validate the heuristic performance through extensive numerical experiments.

In Chapter 5, we conclude the dissertation and provide suggestions for future research.

2. MULTI-STAGE SUPPLY CHAIN WITH SUPPLY UNCERTAINTY

2.1 Synopsis

With the supply chain becoming increasingly global and extended, there have been considerable attentions in practice and research paid to managing risks deep in the production processes. Though there is a vast literature on multi-stage systems, little is done to understand the impact of upstream supply uncertainty on the dynamics of the material flows in meeting the uncertain demands at the downstream. This is the objective of our study.

In particular, we consider a two-location system with an upstream manufacturing facility producing the product and a downstream store selling the product. The production capacity at the upstream and the customer demand at the downstream are both random. The consideration of random production capacity makes the problem challenging even in the single-location setting (see, e.g., [2, 3]). For our problem involving serial system, we deployed the notion of stochastic functions, in particular, the stochastic linearity in midpoint and the directionally concave order, to transform the objective into a joint concave function provided that the terminal value obtained at the end of the planning horizon is concave and submodular in the ending inventory positions.

With the joint concavity established, the optimal management policy can be characterized by two state-dependent thresholds, one for each location. A positive order is initiated at a location if and only if the inventory position at that location is below its threshold. When the terminal value is separable in the inventory positions, the threshold policy reduces to the echelon base-stock policy.

We further show that our analysis can be extended when the upstream production follow a general stochastic function, which satisfies the single-crossing property for stochastic linearity in midpoint. Moreover, we show that our analysis can be carried out for serial systems with multiple locations.

The remainder of this chapter is organized as follows. In the next section, we discuss the related literature and our contribution. We lay out the model in Section 2.3 and develop a transformation applying the notations of the stochastic functions in Section 2.4. The characterization of the optimal policy is presented in Section 2.5. We present two extensions of our model in Section 2.6 and conclude our study in Section 2.7.

2.2 Literature Review

Our work lies at the intersection of studies concerning procurement management with uncertain supply and multi-echelon inventory management.

Most of the studies on managing uncertain supplies focus on single-location inventory systems. There is significant amount of work in this area focuses on single-period models with random demands (e.g., [10–14]) and long-term planning with constant demand rate (see the review by [15]). For dynamic problems involving demand uncertainty, the proportional random yield model is mostly studied. Under this supply function, the optimality of a threshold ordering policy or a reorder-point policy is established in various contexts (see, e.g., [16–18]. Several authors explored the model of random supply capacity and find that it is optimal to follow a base-stock policy when the demand is not price sensitive [2] or the capacity is observed before ordering [19]. When the demand is price dependent, however, the optimal policy is (almost) a threshold policy [3, 20]. Different from this line of work, we consider a serial system in which inventory can be held at different locations, as well as in transit. As a result, the state in our model is multi-dimensional and the interaction among the state variables plays a crucial role in determining an efficient management policy.

While our main model focuses on the case of random supply capacity, we show that our analysis can be generalized to a large class of supply functions by extending the recent development by [21].

There is a rich literature on multi-echelon inventory management. [4] introduce the notion of echelon inventories and prove that the optimal value function can be decomposed into separable functions of individual echelon inventories when the holding costs and the salvage values are linear. With this decomposition, an echelon base-stock policy is optimal. [22] and [23] extend this analysis for problems with infinite planning horizons. [24] derive upper and lower bounds for the echelon base-stock levels, and demonstrate that the simple-to-compute heuristic performs close to optimal policy. There are many extensions of multi-echelon inventory models by introducing two delivery modes [25], random yields [15, 26], and cash retention decisions [27]. There are only limited work that incorporates supply capacity. [28] are the first to consider a two-stage inventory system with deterministic capacities. They prove the optimality of a modified echelon base-stock policy provided that the downstream has a smaller capacity than the upstream does. Their assumptions of a two-stage system and zero lead time for upstream production are essential for their results to hold. [29] extends the analysis for multi-location systems and one-period production lead times. In general, the structure of the optimal decision for capacitated multi-echelon systems is known to be difficult to derive. Therefore, many studies restrict the analysis within the class of base-stock policies (e.g., [30–32]). The recent work by [33] and [34] make the same assumption of operating under base-stock policy and propose algorithms to find echelon base-stock levels by mapping the multi-stage serial system to a single-stage system. [35] propose a variation of the echelon base-stock policy and derive explicit forms for the base-stock levels. [36] consider a system with uncertain capacity, while allowing only a one-time production and immediate salvaging of any leftover inventory.

Our contribution to this literature is twofold. First, we model general stochastic supply functions at the upstream. This consideration destroys the concavity of the profit function. We develop a transformation by appropriately exploring the stochastic properties of the future inventory positions. Second, our approach is very different from the conventional decomposition discovered by [4]. With this approach, one does not require the terminal value function to be separable in the ending inventory positions.

2.3 Problem Description

We consider a two-stage supply chain planning problem over T-period as depicted in Figure 2.1. The upstream division is a manufacturing facility and the downstream division is a retail store. At the beginning of period t, the upstream facility reviews its current inventory level $x_{M,t}$, receives an order $q_{R,t}$ from the downstream and issues a production order of $q_{M,t}$. The production takes one period and the output is constrained by a random capacity K_t. At the end of the production, a shipment up to the downstream order $q_{R,t}$, i.e.,

$$W_t = q_{R,t} \wedge (x_{M,t} + q_{M,t} \wedge K_t) \tag{2.1}$$

is issued. Thus, the inventory dynamics at the upstream facility is

$$X_{M,t+1} \quad = \quad (x_{M,t} + q_{M,t} \wedge K_t - q_{R,t})^+,$$

where $a^+ = \max\{a, 0\}$. A shipment from the facility to the store takes one period. Upon receiving w_{t-1}, the downstream fulfills the random demand D_t occurred in period t. Any leftover inventory is carried over to the next period and any unmet demand is backordered. Thus, the inventory dynamics at the downstream store is

$$X_{R,t+1} \quad = \quad x_{R,t} + w_{t-1} - D_t$$

We assume that $\{K_t\}_{t=1}^T$ and $\{D_t\}_{t=1}^T$ are independent and identically distributed stochastic processes, though our analysis can be easily extended to non-stationary, Markov modulated processes.

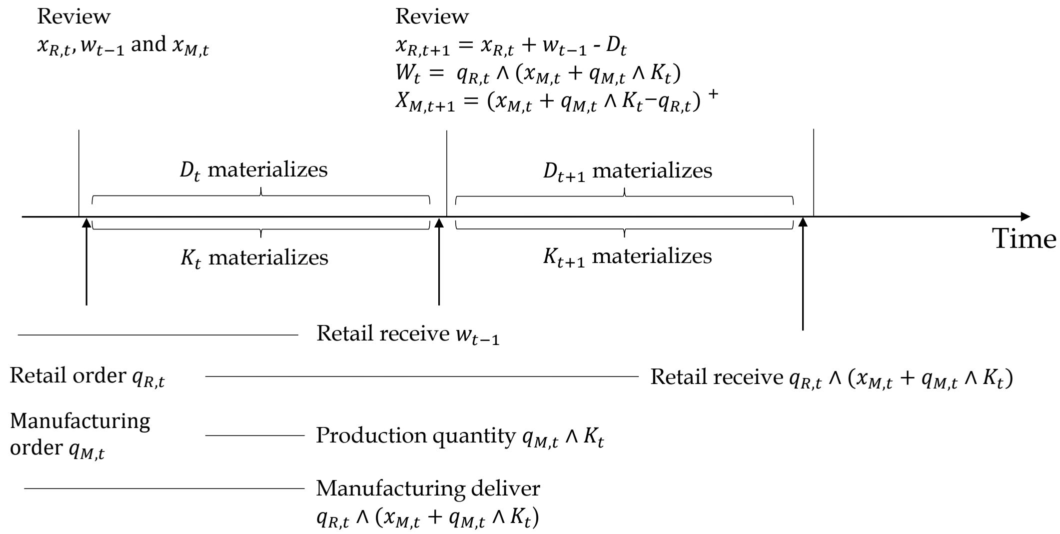


Fig. 2.1.: The sequence of events

As is in the case of classical multi-echelon systems, it is convenient to work the inventory positions (i.e., the amount of total stock from the current location to the downstream store). Let Y_S and Y_R denote the inventory position for the upstream and the downstream, respectively. Then,

$$Y_{S,t+1} = (y_{S,t} + q_{M,t}) \wedge (y_{S,t} + K_t) - D_t, \tag{2.2}$$

$$Y_{R,t+1} = y_{R,t} + W_t - D_t. \tag{2.3}$$

Note that $y_{S,t} = x_{M,t} + w_{t-1} + x_{R,t}$ is the total amount of inventory in the system and $y_{R,t} = x_{R,t} + w_{t-1}$ is the total of on-hand and in-transit stocks for the downstream.

It is easy to see that if $q_{M,t} < q_{R,t} - x_{M,t}$, then for any given $(q_{M,t}, q_{R,t})$ one can construct another pair of order quantities with $q'_{M,t} = q_{M,t}$ and $q'_{R,t} = q_{M,t} + x_{M,t}$, and the resulting expected profit does not change. Therefore, it is without loss of generality to assume $q_{M,t} \geq (q_{R,t} - x_{M,t})^+$. This implies that the firm should not intentionally under produce for the downstream order. With this observation and (2.2), we can rewrite (2.1) and (2.3), respectively, as

$$W_t = (y_{R,t} + q_{R,t}) \wedge (y_{S,t} + K_t) - y_{R,t}, \tag{2.4}$$

$$Y_{R,t+1} = (y_{R,t} + q_{R,t}) \wedge (y_{S,t} + K_t) - D_t. \tag{2.5}$$

The retail price for the product is p dollars per unit and the manufacturing cost is c dollars per unit. The per unit inventory holding cost at the upstream is h_M and that in transit is h_W. The holding/backlogging cost at the downstream is $H_R(\cdot) : \mathbb{R} \to \mathbb{R}^+$. We assume that $H_R(\cdot)$ is convex and $\lim_{x\to\pm\infty} H_R(x) = \infty$.

Let $\hat{J}_t(y_{R,t}, y_{S,t}, q_{R,t}, q_{M,t})$ denote the expected profit in period t when the current inventory positions are $(y_{R,t}, y_{S,t})$, and a production order of $q_{M,t}$ and a store order of $q_{R,t}$ are issued. Then,

$$\begin{aligned}
&\hat{J}_t(y_{R,t}, y_{S,t}, q_{R,t}, q_{M,t}) \qquad (2.6)\\
= \;& p\mathbb{E}[D_t] - c\mathbb{E}[q_{M,t} \wedge K_t] - \mathbb{E}[H_R(y_{R,t} - D_t - D_{t+1})]\\
& -\mathbb{E}[h_W((y_{R,t} + q_{R,t}) \wedge (y_{S,t} + K_t) - y_{R,t})]\\
& -\mathbb{E}[h_M((y_{S,t} + q_{M,t}) \wedge (y_{S,t} + K_t) - (y_{R,t} + q_{R,t}) \wedge (y_{S,t} + K_t))]\\
& +\mathbb{E}[V_{t+1}((y_{R,t} + q_{R,t}) \wedge (y_{S,t} + K_t) - D_t, (y_{S,t} + q_{M,t}) \wedge (y_{S,t} + K_t) - D_t)],
\end{aligned}$$

where

$$V_t(y_{R,t}, y_{S,t}) = \max_{\substack{q_{R,t}\geq 0,\\ q_{M,t}\geq (y_{R,t}+q_{R,t}-y_{S,t})^+}} \left\{\hat{J}_t(y_{R,t}, y_{S,t}, q_{R,t}, q_{M,t})\right\} \qquad (2.7)$$

is the optimal profit function in period t when the inventory positions are $(y_{R,t}, y_{S,t})$. At the end of the planning horizon T, the firm collects a terminal value $V_{T+1}(y_{R,T+1}, y_{S,T+1})$. The existing studies often use a piecewise linear and separable form for the terminal value function (i.e., $V_{T+1}(y_{R,T+1}, y_{S,T+1}) = s_M(y_{S,T+1} - y_{R,T+1}) + s_R y^+_{R,T+1} - by^-_{R,T+1})$, where s_M is the salvage value at the upstream, s_R is the salvage value at the downstream, and b is the penalty of unmet demand). We assume a general concave and submodular $V_{T+1}(\cdot,\cdot)$. When $y_{R,T+1} > 0$, $V_{T+1}(y_{R,T+1}, y_{S,T+1})$ represents the salvage value of the final stocks. When $y_{R,T+1} < 0$, $V_{T+1}(y_{R,T+1}, y_{S,T+1})$ may account the penalty for eventually unmet demands.

We shall note that we have assumed stationary system parameters for ease of exposition, though all formal results derived below except Proposition 3 extends easily to nonstationary systems. In our analysis below, we use a superscript $*$ to denote the quantities under an optimal solution. For ease of exposition, we may drop the time index from the subscript when it does not cause confusions.

2.4 Preliminaries and Model Transformation

In this section, we introduce some properties of stochastic functions that are useful for our analysis. These properties allow us to transform the problem in to a concave optimization problem and facilitate the analysis of the optimal policy in the next section.

2.4.1 Preliminaries

It is well known that the inventory control problem under random supply capacity is nonconcave even for single-location settings (see, e.g., [2,3]). Thus, the multidimensional problem defined in (2.6)-(2.7) is nonconcave in general. Before we can analyze the optimal policy, we must make appropriate transformations of the problem and establish structural properties of the profit function. In this subsection, we provide the needed background knowledge for our analysis.

[21] develop the notion of stochastic linearity in midpoint to describe general material flows in inventory systems, which also applies to our model with random production capacities. The notion of stochastic linearity is defined using the concave ordering [37]. Specifically, a random variable X_1 is said to be smaller than a random variable X_2 in the concave order, written as $X_1 \leq_{cv} X_2$, if $\mathbb{E}[\phi(X_1)] \leq \mathbb{E}[\phi(X_2)]$ for any concave function ϕ.

Definition 1 [21] A function $\{A(\mathbf{u}) : \mathbb{R}^n \to \mathbb{R}\}$ is **stochastically linear in midpoint**, written $\{A(\mathbf{u}), \mathbf{u} \in \mathbb{R}^n\} \in SL(mp)$, if for any $\mathbf{u}_a, \mathbf{u}_b \in \mathbb{R}^n$, there exist $\hat{A}(\mathbf{u}_a)$ and $\hat{A}(\mathbf{u}_b)$ defined on a common probability space such that

(i) $\hat{A}(\mathbf{u}_i) =^d A(\mathbf{u}_i),\ i = a, b$ (where $=^d$ stands for being equal in distribution), and

(ii) $\frac{\hat{A}(\mathbf{u}_a)+\hat{A}(\mathbf{u}_b)}{2} \leq_{cv} A(\frac{\mathbf{u}_a+\mathbf{u}_b}{2})$.

It is easy to check that any almost surely linear function (i.e., of the form $A(\mathbf{u}) = \sum_j \alpha_j u_j$ for real numbers α_j) satisfies Definition 1, and therefore stochastic linearity generalizes deterministic linearity. A sufficient condition for a function to be stochastically linear in midpoint is the following single-crossing property [21].

Definition 2 Suppose $A(\mathbf{u}) = \phi(\mathbf{u}, Z), \mathbf{u} \in \mathcal{U} \subset \mathbb{R}^n$, where Z is a random variable and $\phi(\mathbf{u}, z)$ is an increasing function in z, and $\mathbb{E}[A((\mathbf{u}_a + \mathbf{u}_b)/2)] = \mathbb{E}[(A(\mathbf{u}_a) + A(\mathbf{u}_b))/2]$, $\mathbf{u}_a, \mathbf{u}_b \in \mathcal{U}$. Then $\phi(\mathbf{u}, z)$ satisfies the **single-crossing property** if for any $\mathbf{u}_a, \mathbf{u}_b \in \mathcal{U}$, as z increases, the sign of

$$\phi(\frac{\mathbf{u}_a + \mathbf{u}_b}{2}, z) - \frac{\phi(\mathbf{u}_a, z) + \phi(\mathbf{u}_b, z)}{2}$$

can change at most once and the change is from positive to negative.

The next lemma states that stochastic linearity in midpoint is preserved under monotone transformation, a useful property for analyzing our model.

Lemma 1 *Suppose that $\{A(u) : u \in \mathcal{U} \subset \mathbb{R}\}$ has a transformation that is stochastically linear in midpoint and $g(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is monotone in the first argument. Let $v_0(\mu, \mathbf{v}) = \inf\{w : \mathbb{E}[A(g(w, \mathbf{v}))] \geq \mu\}, \mathbf{v} \in \mathcal{V} \subset \mathbb{R}^n$. Then $\{A(g(v_0(\mu, \mathbf{v}), \mathbf{v})) : g(v_0(\mu, \mathbf{v}), \mathbf{v}) \in \mathcal{U}\} \in SL(mp)$. Moreover, if $A(u) = \phi(u, z)$ satisfies the single-crossing property, so does $A(g(v_0(\mu, \mathbf{v}), \mathbf{v}))$.*

Proof of Lemma 1. Take μ_1, μ_2, $\mathbf{v}_1$ and $\mathbf{v}_2$. Define $u_1 = g(v_0(\mu_1, \mathbf{v}_1), \mathbf{v_1})$, $u_2 = g(v_0(\mu_2, \mathbf{v}_2), \mathbf{v_2})$, $\bar{\mu} = (\mu_1 + \mu_2)/2$, and $\bar{\mathbf{v}} = (\mathbf{v}_1 + \mathbf{v}_2)/2$. By definition, we must have $\mathbb{E}[A(u_1)] = \mu_1$, $\mathbb{E}[A(u_2)] = \mu_2$ and $\mathbb{E}[A(g(v_0(\bar{\mu}, \bar{\mathbf{v}}), \bar{\mathbf{v}}))] = \bar{\mu}$. Because $A(u)$ has a transformation that is stochastically linear in midpoint, there exist $A_1 =^d A(u_1)$, $A_2 =^d A(u_2)$ such that

$$\frac{A_1 + A_2}{2} \leq_{cv} A(g(v_0(\bar{\mu}, \bar{\mathbf{v}}), \bar{\mathbf{v}})).$$

Hence, $\{A(g(v_0(\mu, \mathbf{v}), \mathbf{v})) : g(v_0(\mu, \mathbf{v}), \mathbf{v}) \in \mathcal{U}\} \in SL(mp)$.

If the transformation of $A(u) = \phi(u, z)$ satisfies the single-crossing property, using same argument that $\mathbb{E}[A(u_1)] = \mu_1$, $\mathbb{E}[A(u_2)] = \mu_2$ and $\mathbb{E}[A(g(v_0(\bar{\mu}, \bar{\mathbf{v}}), \bar{\mathbf{v}}))] = \bar{\mu}$, thus

$$\phi(g(v_0(\bar{\mu}, \bar{\mathbf{v}}), \bar{\mathbf{v}}), z) - \frac{\phi(u_1, z) + \phi(u_2, z)}{2}$$

can change at most once and the change is from positive to negative.

Hence, $A(g(v_0(\mu, \mathbf{v}), \mathbf{v}))$ also satisfies the single-crossing property. □

As our analysis unfolds, it will become clear that the property of stochastic linearity in midpoint allows us to establish concavity of the profit function in the realized inventory levels provided that the future value function is concave. Given that our problem involves two locations, we also need additional properties describing the relationship between the inventory levels at the two locations. This requires the notions of directional concavity and directional concave order introduced by [37] for multidimesional stochastic functions.

Definition 3 [37] Let $\leq$ denote the coordinatewise ordering in $\mathbb{R}^n$. For $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$, use the notation $[\mathbf{u}, \mathbf{v}] \leq (\geq)\mathbf{w}$ as a shorthand for $\mathbf{u} \leq (\geq)\mathbf{w}$ and $\mathbf{v} \leq \mathbf{w}$. A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is said to be **directionally concave** if for any $\mathbf{u}_i \in \mathbb{R}^n, i = a, b, c, d$, such that $\mathbf{u}_a \leq [\mathbf{u}_b, \mathbf{u}_c] \leq \mathbf{u}_d$ and $\mathbf{u}_a + \mathbf{u}_d = \mathbf{u}_b + \mathbf{u}_c$, one has $\psi(\mathbf{u}_b) + \psi(\mathbf{u}_c) \geq \psi(\mathbf{u}_a) + \psi(\mathbf{u}_d)$

Definition 4 [37] Let $\mathbf{A} = (A_1, A_2, ..., A_n)$ and $\mathbf{B} = (B_1, B_2, ..., B_n)$ be two random vectors with $\mathbb{E}[\psi(\mathbf{A})] \leq \mathbb{E}[\psi(\mathbf{B})]$, for all (increasing) functions $\psi : \mathbb{R}^n \to \mathbb{R}$ that are directionally concave, provided that the expectations exist. Then $\mathbf{A}$ is said to be smaller than $\mathbf{B}$ in the **(increasing) directionally concave order**, written $\mathbf{A}(\leq_{idir-cv}) \leq_{dir-cv} \mathbf{B}$.

[37] show that a function $\phi(\cdot)$ is directionally concave if and only if $\phi(\cdot)$ is submodular and coordinatewise concave. The next lemma establishes the connection between the single-crossing property and the directionally concave order. This result is crucial for the analysis of our model.

Lemma 2 *Suppose $\psi_i(\mathbf{u}, z), \mathbf{u} \in \mathcal{U} \subset \mathbb{R}^n, i \in \{1, 2, \ldots, k\}$, are increasing in $z \in \mathcal{Z} \subset \mathbb{R}$ and satisfy the single-crossing property. For any $\mathbf{u}_a, \mathbf{u}_b \in \mathcal{U}$ and $\bar{\mathbf{u}} = (\mathbf{u}_a + \mathbf{u}_b)/2$, $\mathbb{E}[(\psi_i(\mathbf{u}_a, Z) + \psi_i(\mathbf{u}_b, Z))/2]$ and $\mathbb{E}[\psi_i(\bar{\mathbf{u}}, Z)]$ are finite for some random variable Z with support $\mathcal{Z}$. Then*

$$\left\{\frac{\psi_i(\mathbf{u}_a, Z) + \psi_i(\mathbf{u}_b, Z)}{2} + C\right\}_{i\in\{1,2,\ldots,k\}} \leq_{dir-cv} \left\{\psi_i(\bar{\mathbf{u}}, Z) + C\right\}_{i\in\{1,2,\ldots,k\}}, \tag{2.8}$$

where C is any random variable independent of Z with a finite mean.

Proof of Lemma 2. Take a function $f : \mathbb{R}^k \to \mathbb{R}$ that is increasing directional concave. By [37], f is also componentwise concave and submodular. We only consider the case where f is differentiable. Note that because f is componentwise concave, it can have at most a countable number of nondifferentiable points. At the nondifferentiable points, we can replace the first-order derivative by the left or right derivative and the following argument continue to hold. Let $\bar{\varphi}_i(z) = (\psi_i(\mathbf{u}_a, z) + \psi_i(\mathbf{u}_b, z))/2$ and $\varphi_i(z) = \psi_i((\mathbf{u}_a + \mathbf{u}_b)/2, z)$. Clearly $\bar{\varphi}_i$ and φ_i are increasing functions. We have, by componentwise concavity of f,

$$\begin{aligned}
&f(\varphi_1(z), \ldots, \varphi_k(z)) - f(\bar{\varphi}_1(z), \varphi_2(z), \ldots, \varphi_k(z)) \\
&\qquad \geq f_1(\varphi_1(z), \ldots, \varphi_k(z)) \cdot (\varphi_1(z) - \bar{\varphi}_1(z)), \\
&f(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_{j-1}(z), \varphi_j(z), \ldots, \varphi_k(z)) - f(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_j(z), \varphi_{j+1}(z), \ldots, \varphi_k(z)) \\
&\qquad \geq f_j(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_{j-1}(z), \varphi_j(z), \ldots, \varphi_k(z)) \cdot (\varphi_j(z) - \bar{\varphi}_j(z)), j = 2, \ldots, k-1, \\
&f(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_{k-1}(z), \varphi_k(z)) - f(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_k(z)) \\
&\qquad \geq f_k(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_{k-1}(z), \varphi_k(z)) \cdot (\varphi_k(z) - \bar{\varphi}_k(z)),
\end{aligned}$$

where f_j is the partial derivative of the jth argument. Adding all of the above together, we obtain

$$\begin{aligned}
f(\varphi(z)) - f(\bar{\varphi}(z)) &\geq \sum_{j=1}^{k} f_j(\bar{\varphi}_1(z), \ldots, \bar{\varphi}_{j-1}(z), \varphi_j(z), \ldots, \varphi_k(z)) \cdot (\varphi_j(z) - \bar{\varphi}_j(z)), \\
&\geq \sum_{j=1}^{k} f_j(\bar{\varphi}_1(z_j), \ldots, \bar{\varphi}_{j-1}(z_j), \varphi_j(z_j), \ldots, \varphi_k(z_j)) \cdot (\varphi_j(z) - \bar{\varphi}_j(z)).
\end{aligned}$$

where z_j is the unique crossing point of φ_j and $\bar{\varphi}_j$, i.e., $\varphi_j(z) - \bar{\varphi}_j(z) \geq (<)0$ for $z \leq (>)z_j$. The last inequality follows from two observations. First, because f is increasing directional concave, f_j is nonnegative and decreasing in each argument and thus f_j is decreasing in z. Second, for any decreasing function ϕ_1 and a ϕ_2 with $\phi_2(z) \geq (<)0$ for $z \leq (>)z^0$, $\phi_1(z)\phi_2(z) \geq \phi_1(z^0)\phi_2(z)$.

Taking the expectation on both sides in the above inequality and applying the reltaiton $\mathbb{E}[\varphi_j(Z)] = \mathbb{E}[\bar{\varphi}_j(Z)]$, we obtain

$$\mathbb{E}[f(\bar{\boldsymbol{\varphi}}(Z)) - f(\boldsymbol{\varphi}(Z))] \leq 0.$$

Now take a function $g : \mathbb{R}^k \to \mathbb{R}$ that is directionally concave. Then there exists an increasing directionally concave function f such that $g(\mathbf{x}) = f(\mathbf{x}) + \sum_{j=1}^{k} c_j x_j$ for some constants $c_1, c_2, \ldots, c_k$. Then, for any constant c,

$$\begin{aligned}
\mathbb{E}[g(\bar{\boldsymbol{\varphi}}(Z) + \mathbf{c})] &= \mathbb{E}[f(\bar{\boldsymbol{\varphi}}(Z) + \mathbf{c})] + \sum_{j=1}^{k} c_j \mathbb{E}[\bar{\varphi}_j(Z) + c] \\
&\leq \mathbb{E}[f(\boldsymbol{\varphi}(Z) + \mathbf{c})] + \sum_{j=1}^{k} c_j \mathbb{E}[\varphi_j(Z) + c] \\
&= \mathbb{E}[g(\boldsymbol{\varphi}(Z) + \mathbf{c})].
\end{aligned}$$

It is clear that the above inequality continue to hold when we replace c by a random variable C and take expectation over C. Hence, we conclude the proof. $\square$

Lemma 2 states that if multiple stochastic functions depend on the same random variable and all satisfy the single-crossing property, they are increasing directionally concave ordered in midpoint. Lemma 2 provides a way to tackle our multi-dimensional problem involving stochastically linear functions.

2.4.2 Problem Transformation and Properties of the Profit Function

To apply the notion of stochastic linearity in midpoint, we note that two random variables are concavely ordered only when they have the same mean. It is therefore natural to transform the stochastic material flows into their respective means. Define

$$\mu_M(q_{M,t}) = \mathbb{E}[q_{M,t} \wedge K_t] \quad \text{and} \quad \mu_R(y_{R,t}, y_{S,t}, q_{R,t}) = \mathbb{E}[(y_{R,t} - y_{S,t} + q_{R,t}) \wedge K_t].$$

It is easy to see that $\mu_M(\cdot)$ and $\mu_R(y_{R,t}, y_{S,t}, \cdot)$ are increasing. We can derive the inverse of these two functions, with a slight abuse of notation, as

$$q_{M,t}(\mu) = \inf\{q : \mu_M(q) \geq \mu\} \quad \text{and} \quad q_{R,t}(y_{R,t}, y_{S,t}, \mu) = \inf\{q : \mu_R(y_{R,t}, y_{S,t}, q) \geq \mu\} \tag{2.9}$$

To understand the above transformation, we note that production are initiated because of the need to increase the inventory at the upstream facility, at the downstream store or both. The *upstream-triggered production output* is

$$P_t(\mu_M) = q_{M,t}(\mu_M) \wedge K_t. \tag{2.10}$$

This output, however, may or may not be fully transferred to the retailer. The *downstream-triggered production output*, i.e., the difference between the downstream order and the upstream inventory, is

$$S_t(y_{R,t}, y_{S,t}, \mu_R) = (y_{R,t} - y_{S,t} + q_{R,t}(y_{R,t}, y_{S,t}, \mu_R)) \wedge K_t. \tag{2.11}$$

[21] have shown that functions with the form of $P_t(\mu)$ are stochastically linear in midpoint and satisfy the single-crossing property. With Lemma 1, it is immediate that the function $S_t(y_{R,t}, y_{S,t}, \mu_R)$ is also stochastically linear in midpoint and satisfies the single-crossing property. With (2.10) and (2.11), we can rewrite (2.5) and (2.2), respectively as,

$$\begin{aligned} Y_{R,t+1} &= (y_{R,t} + q_{R,t}(y_{R,t}, y_{S,t}, \mu_R)) \wedge (y_{S,t} + K_t) - D_t \\ &= y_{S,t} + S_t(y_{R,t}, y_{S,t}, \mu_R) - D_t, & (2.12) \\ Y_{S,t+1} &= (y_{S,t} + q_{M,t}(\mu_M)) \wedge (y_{S,t} + K_t) - D_t = y_{S,t} + P_t(\mu_M) - D_t. & (2.13) \end{aligned}$$

With these transformations, we can rewrite the dynamic programming equation as

$$V_t(y_{R,t}, y_{S,t}) = \max\{J_t(y_{R,t}, y_{S,t}, \mu_R, \mu_M) : \mu_R \geq (y_{R,t} - y_{S,t}),\ \mu_M \geq \mu_R^+\}, \tag{2.14}$$

where

$$J_t(y_{R,t}, y_{S,t}, \mu_R, \mu_M) = \hat{J}_t(y_{R,t}, y_{S,t}, q_{R,t}(y_{R,t}, y_{S,t}, \mu_R), q_{M,t}(\mu_M)). \tag{2.15}$$

In the above formulation, we have replaced the condition $q_{R,t} \geq 0$ by $\mu_R \geq (y_{R,t} - y_{S,t})$ and $q_{M,t} \geq (q_{R,t} - y_{S,t} + y_{R,t})^+$ by $\mu_M \geq \mu_R^+$.

With the development in Lemma 2, we can establish the properties of the profit function for our problem as described in the next lemma.

Lemma 3 *The function $\varphi(y_R, y_S) = \max\{\mathbb{E}[\psi(v_R \wedge (y_S + K), v_S \wedge (y_S + K))] : y_R \leq v_R \leq v_S, v_S \geq y_S\}$, where K is a nonnegative random variable, is submodular in (y_R, y_S) if ψ is concave and submodular.*

Proof of Lemma 3. Let $\phi(y_S, v_R, v_S) = \psi(v_R \wedge (y_R + K), v_S \wedge (y_S + K))]$. Take $y_R^a < y_R^b$ and $y_S^a < y_S^b$. Let (v_R^{ij}, v_S^{ij}) be the maximizer of $\{\mathbb{E}[\psi(v_R \wedge (y_S^j + K), v_S \wedge (y_S^j + K))] : y_R^i \leq v_R \leq v_S, v_S \geq y_S^j\}$. We first observe that if $v_R^{ij} > y_R^i$ for $i = a, b$ and $j = a, b$, we have

$$\begin{aligned} \varphi(y_R^b, y_S^b) - \varphi(y_R^a, y_S^a) &= \phi(y_S^b, v_R^{bb}, v_S^{bb}) - \phi(y_S^b, v_R^{ab}, v_S^{ab}) \\ &= 0 \\ &= \phi(y_S^b, v_R^{ba}, v_S^{ba}) - \phi(y_S^a, v_R^{aa}, v_S^{aa}) \\ &= \varphi(y_R^b, y_S^a) - \varphi(y_R^a, y_S^b) \end{aligned} \tag{2.16}$$

because y_R^i does not affect the optimal solutions. This observation together with Lemma 4 suggest the following three cases.

Case 1: $v_R^{ij} = y_R^i$ for $i = a, b$ and $j = a, b$. Because $y_R^i \leq y_S^i$, we have

$$\phi(y_S^i, v_R^{ij}, v_S^{ij}) = \mathbb{E}[\psi(y_R^i, v_S^{ij} \wedge (y_S^i + K_t))].$$

Note that $v_S^{ij} = \bar{v}_S(y_R^i) \vee y_S^i$, where $\bar{v}_S(\cdot)$ maximizes $\psi(y_R^i, v_{S,t})$ and is a decreasing function independent of y_S^i. If $\bar{v}_S(y_R^a) \leq y_S^a$ or $\bar{v}_S(y_R^b) \geq y_S^b$, we can verify that $\bar{v}_S(y_R^b) \vee y_S^b \wedge (y_S^b + K_t) \geq \bar{v}_S(y_R^a) \vee y_S^a \wedge (y_S^a + K_t)$ for each realization of K_t. Then,

$$\begin{aligned}
& \phi(y_S^b, v_R^{bb}, v_S^{bb}) - \phi(y_S^a, v_R^{ba}, v_S^{ba}) \\
= \; & \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^b) \vee y_S^b \wedge (y_S^b + K_t))] - \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^b) \vee y_S^a \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^b) \vee y_S^b \wedge (y_S^b + K_t))] - \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^a) \vee y_S^a \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^b) \vee y_S^b \wedge (y_S^b + K_t))] - \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \vee y_S^a \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \vee y_S^b \wedge (y_S^b + K_t))] - \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \vee y_S^a \wedge (y_S^a + K_t))], \\
= \; & \phi(y_S^b, v_R^{ab}, v_S^{ab}) - \phi(y_S^a, v_R^{aa}, v_S^{aa}).
\end{aligned}$$

The first inequality follows from the suboptimality of $\bar{v}_S(y_R^a)$ for the state (y_R^b, y_S^a), the second inequality follows from the submodularity of ψ, and the last inequality follows from the optimality of $\bar{v}_S(y_R^a)$ for the state (y_R^a, y_S^b).

If, however, $\bar{v}_S(y_R^a) > y_S^a$ and $\bar{v}_S(y_R^b) < y_S^b$, then,

$$\begin{aligned}
& \phi(y_S^b, v_R^{bb}, v_S^{bb}) - \phi(y_S^a, v_R^{ba}, v_S^{ba}) \\
= \; & \mathbb{E}[\psi(y_R^b, y_S^b + K_t)] - \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^b) \vee y_S^a \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^b, y_S^b + K_t)] - \mathbb{E}[\psi(y_R^b, \bar{v}_S(y_R^a) \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^a, y_S^b + K_t)] - \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \wedge (y_S^a + K_t))] \\
\leq \; & \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \vee y_S^b \wedge (y_S^b + K_t))] - \mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \wedge (y_S^a + K_t))] \\
= \; & \phi(y_S^b, v_R^{ab}, v_S^{ab}) - \phi(y_S^a, v_R^{aa}, v_S^{aa}).
\end{aligned}$$

The first inequality follows from the optimality of $\bar{v}_S(y_R^S)$ for the states (y_R^b, y_S^a), the second inequality follows from the submodularity of ψ, and the last inequality follows from the optimality of $\bar{v}_S(y_R^a) \vee y_S^b$.

Case 2: $v_R^{bb} = y_R^b$ and $v_R^{ij} > y_R^i$ for $(i, j) = (a, a), (a, b), (b, a)$. We must have $\varphi(y_R^b, y_S^b) \leq \psi(y_S^b, v_R^{ab}, v_S^{bb}) = \varphi(y_R^a, y_S^b)$. It follows that

$$\varphi(y_R^b, y_S^b) - \varphi(y_R^a, y_S^b) \leq 0 = \psi(y_S^a, v_R^{ba}, v_S^{ba}) - \phi(y_S^a, v_R^{aa}, v_S^{aa}) = \varphi(y_R^b, y_S^a) - \varphi(y_R^a, y_S^a).$$

The last equality holds because $v_R^{ia} > y_R^i$.

Case 3: $v_R^{bb} > y_R^b$ and $v_R^{ij} = y_R^i$ for $(i,j) = (a,a),(a,b),(b,a)$. Let y_R^m be such that $\mathbb{E}[\psi(y_R^m, \bar{v}_S(y_R^m) \wedge (y_S^a + K))] = \max_{v_R,v_S}\{\mathbb{E}[\psi(v_R \wedge (y_S^a + K), v_S \wedge (y_S^a + K))] : v_R \le v_S, v_S \ge y_S^a\}$. It is easy to see that $y_R^a < y_R^m \le y_R^b$. Also, let y_S^m be such that $\mathbb{E}[\psi(y_R^a, \bar{v}_S(y_R^a) \wedge (y_S^m + K))] = \max_{v_R,v_S}\{\mathbb{E}[\psi(v_R \wedge (y_S^m + K), v_S \wedge (y_S^m + K))] : v_R \le v_S, v_S \ge y_S^m\}$. Then, we must have $y_S^a < y_S^m \le y_S^b$. Let (v_R^{ij}, v_S^{ij}) be the maximizer with states (y_R^i, y_S^j) for $i \in \{a,m,b\}$ and $j \in \{a,m,b\}$. Applying the argument in (2.16) to $y_R^m, y_R^b, y_S^a, y_S^b$, we obtain

$$\varphi(y_R^b, y_S^b) - \varphi(y_R^b, y_S^a) \le \varphi(y_R^m, y_S^b) - \varphi(y_R^m, y_S^a). \tag{2.17}$$

Applying the argument for Case 1 to $y_R^a, y_R^m, y_S^m, y_S^b$, we obtain

$$\varphi(y_R^m, y_S^b) - \varphi(y_R^m, y_S^m) \le \varphi(y_R^a, y_S^b) - \varphi(y_R^a, y_S^m). \tag{2.18}$$

Applying the argument for Case 2 to $y_R^a, y_R^b, y_S^a, y_S^m$, we obtain

$$\varphi(y_R^m, y_S^m) - \varphi(y_R^m, y_S^a) \le \varphi(y_R^a, y_S^m) - \varphi(y_R^a, y_S^a). \tag{2.19}$$

The above three relations imply

$$\varphi(y_R^b, y_S^b) - \varphi(y_R^b, y_S^a) \le \varphi(y_R^b, y_S^a) - \varphi(y_R^a, y_S^a).$$

Combining the cases, we conclude the result. $\square$

Lemma 3 suggests that the optimal profit function is submodular provided that the objective function is concave and submodular. We are yet to complete the induction argument to establish the concavity for the value function defined in the dynamic program in (2.14).

Theorem 1 *The objective function J_t is jointly concave in (y_R, y_S, μ_R, μ_M) and the value function V_t is jointly concave and submodular in (y_R, y_S).*

Proof of Theorem 1. In this proof, we drop the subscript t of state and decision variables for ease of exposition. By assumption $V_{T+1}(y_R, y_S)$ is jointly concave and submodular. Assume that $V_{t+1}(y_R, y_S)$ is concave and submodular. Take

$(y_R^1, y_S^1, \mu_R^1, \mu_M^1)$ and $(y_R^2, y_S^2, \mu_R^2, \mu_M^2)$. Also define $\bar{y}_R = (y_R^1 + y_R^2)/2$, $\bar{y}_S = (y_S^1 + y_S^2)/2$, $\bar{\mu}_R = (\mu_R^1 + \mu_R^2)/2$ and $\bar{\mu}_M = (\mu_M^1 + \mu_M^2)/2$. By Lemma 2 and Definition 4, we have

$$\begin{aligned}
&\mathbb{E}[V_{t+1}(y_S^1 + S_t(y_R^1, y_S^1, \mu_R^1) - D_t, y_S^1 + P_t(\mu_M^1) - D_t) \\
&+V_{t+1}(y_S^2 + S_t(y_R^2, y_S^2, \mu_R^2) - D_t, y_S^2 + P_t(\mu_M^2) - D_t)] \\
&\leq \mathbb{E}[V_{t+1}(\bar{y}_S + S_t(\bar{y}_R, \bar{y}_S, \bar{\mu}_R) - D_t, \bar{y}_S + P_t(\bar{\mu}_M) - D_t)].
\end{aligned}$$

Thus $V_{t+1}(y_R, y_S, \mu_R, \mu_M)$ is jointly concave in (y_R, y_S, μ_R, μ_M). From (2.15) and (2.6), it is immediate that $J_{t+1}(y_R, y_S, \mu_R, \mu_M)$ is jointly concave. Because concavity is closed under maximization and the set $\{(\mu_R, \mu_M) : \mu_R \geq (y_R - y_S), \mu_M \geq \mu_R^+\}$ is convex, it follows that $V_t(y_R, y_S)$ is jointly concave.

To show the submodularity of $V_t(y_R, y_S)$, take $v_S = y_S + q_M$ and $v_R = y_R + q_R$. Define

$$\phi(y_S, v_R, v_S) = \hat{J}_t(y_R, y_S, q_R, q_M) - p\mathbb{E}[D_t] + cy_S + \mathbb{E}[H_R(y_R - D_t - D_{t+1})] - h_W y_R \tag{2.20}$$

From (2.6), we have

$$\phi(y_S, v_R, v_S) = \mathbb{E}[\psi(v_R \wedge (y_S + K_t), v_S \wedge (y_S + K_t))], \tag{2.21}$$

where

$$\psi(v_1, v_2) = -cv_2 - h_W v_1 - h_M(v_2 - v_1) + \mathbb{E}[V_{t+1}(v_1 - D_t, v_2 - D_t)].$$

Clearly, the optimal solution should correspond to $(v_R^*(y_R, y_S), v_S^*(y_R, y_S))$ that maximizes $\phi(y_S, v_R, v_S)$ subject to the constraints $v_R \geq y_R$ (replacing $q_R \geq 0$) and $v_S \geq y_S \vee v_R$ (replacing $q_M \geq (y_R - y_S + q_R)^+$). From our previous result, ψ is jointly concave and submodular. Applying Lemma 3, we conclude that $\phi(y_S, v_R^*(y_R, y_S), v_S^*(y_R, y_S))$ is submodular in (y_R, y_S) and thus $V_t(y_R, y_S)$ is submodular. Hence, we conclude the proof. $\square$

Theorem 1 establishes a substitutable relationship between the downstream inventory y_R and the system inventory y_S. This is expected because downstream inventory is directly used to meet the demand, while the system inventory is eventually flowing to the downstream to meet the demand. Moreover, the marginal values of the mean material flows and the inventory levels are subject to diminishing returns, implied by the concavity of the objective function. This result is an important technical development, which enables the characterization of the optimal policy.

2.5 Analysis of the Optimal Policy

In this section, we derive the optimal policy and analyze the dynamics of the system against changes in the system parameters.

2.5.1 Characterization of the Optimal Policy

With the properties established for the profit function, we can solve the problem as a concave optimization problem and analyze the structure of the optimal policy.

The proof of Theorem 2 uses the following lemma.

Lemma 4 *Let* $(v_R^*(y_R, y_S), v_S^*(y_R, y_S))$ *maximizes* $\{\mathbb{E}[\psi(v_R \wedge (y_S + K), v_S \wedge (y_S + K))] : y_R \leq v_R \leq v_S, v_S \geq y_S\}$, *where* ψ *is concave and submodular, and* K *is a nonnegative random variable. Then the following results hold.*

i) If $v_R^*(y_0, y_S) = y_0$*, then* $v_R^*(y_R, y_S) = y_R$ *for any* $y_R \geq y_0$.

ii) If $v_R^*(y_R, y_0) = y_R$*, then* $v_R^*(y_R, y_S) = y_R$ *for any* $y_S \geq y_0$.

iii) If $v_S^*(y_0, y_S) = y_S$*, then* $v_S^*(y_R, y_S) = y_S$ *for any* $y_R \geq y_0$.

iv) If $v_S^*(y_R, y_0) = y_0$*, then* $v_S^*(y_R, y_S) = y_S$ *for any* $y_S \geq y_0$.

Proof. To see i), we note that y_R affects the optimal value only through the constraint $v_R \geq y_R$. Let $\bar{v}_S(v_R, y_S) = \arg\max_{v_S}\{\mathbb{E}[\psi(v_R \wedge (y_S + K), v_S \wedge (y_S + K)) : v_S \geq v_R \vee y_S\}$. From the proof of Theorem 1, we deduce that $\mathbb{E}[\psi(v_R \wedge (y_S + K), \bar{v}_S(v_R, y_S) \wedge (y_S + K))]$ is unimodal in v_R. Let $\bar{v}_R(y_S)$ denote its maximizer. Then, $v^*(y_R, y_S) = \bar{v}_R(y_S) \vee y_R$. Thus, part i) follows.

In part ii), let $\bar{v}_S(v_R) = \arg\max_{v_S}\{\psi(v_R, v_S) : v_R \le v_S\}$. Because $v_R^*(y_R, y_0) = y_R$, we have $y_S^*(y_R, y_0) = \bar{v}_S(y_R) \vee y_0$. Because ψ is submodular, we have $\arg\max_{v_R}$ $\{\mathbb{E}[\psi(v_R, \bar{v}_S(y_R) \vee (y_0 + \delta) \wedge (y_0 + \delta + K))] : v_R \ge y_R\} = y_R$ for any $\delta > 0$. It follows that $\mathbb{E}[\psi(v_R, \bar{v}_S(v_R) \vee (y_0 + \delta) \wedge (y_0 + \delta + K))]$ for $v_R \ge y_R$ is maximized at $v_R = y_R$. Take $v_R^\epsilon = y_R + \epsilon$ for some $\epsilon \in (0, \delta)$. Then $v_R^\epsilon < y_0 + \delta$ as $y_R \le y_0$. Let $\varphi(v_R) = \max_{v_S \ge v_R \vee (y_0+\delta)} \mathbb{E}[\psi(v_R \wedge (y_0 + \delta + K), v_S \wedge (y_0 + \delta + K))]$. Then

$$\varphi(v_R^\epsilon) = \mathbb{E}[\psi(v_R^\epsilon, \bar{v}_S(v_R^\epsilon) \vee (y_0 + \delta) \wedge (y_0 + \delta + K))] \le \varphi(y_R).$$

From the proof of Theorem 1, φ is unimodal. Thus, the above relation suggests that $(v_R^*(y_R, y_0 + \delta), v_S^*(y_R, y_0 + \delta)) = (y_R, \bar{v}_S(y_R) \vee (y_0 + \delta))$.

To see part iii), we note that y_R only affects the optimality through the constraint $v_R \ge y_R$. If $v_R^*(y_0, y_S) > y_0$, then for a sufficiently small δ, the optimal solution would not change if y_R increases from y_0 to $y_0 + \delta$. Thus, we can focus on the case where $v_R^*(y_0, y_S) = y_0$. By part i), we have $v_R^*(y_0+\delta, y_S) = y_0+\delta$. Denote $y_S^\delta \equiv v_S^*(y_0+\delta, y_S)$. Suppose $v_S = v_S^\delta > y_S$ is optimal for $y_R = y_0 + \delta$, we have

$$\mathbb{E}[\psi(y_0 + \delta, v_S^\delta \wedge (y_S + K))] - \psi(y_0 + \delta, y_S) > 0.$$

Because $v_S = y_S$ is optimal for $y_R = y_0$, we have

$$\mathbb{E}[\psi(y_0, v_S^\delta \wedge (y_S + K))] - \psi(y_0, y_S) \le 0.$$

However, by the submodularity of ψ, we have for each realization of $K = k$

$$\psi(y_0 + \delta, v_S^\delta \wedge (y_S + k)) - \psi(y_0 + \delta, y_S) \le \psi(y_0, v_S^\delta \wedge (y_S + k)) - \psi(y_0, y_S),$$

which leads to a contradiction. Hence we obtain part iii).

Next we show part iv). Given that $v_S = y_0$, we must have $v_R^0 \equiv v_R^*(y_R, y_0) \le y_0$ maximizes $\psi(v_R, y_0)$ over $y_R \le v_R \le y_0$. We show that $v_S = y_0$ maximizes $\psi(v_R^0, v_S)$ over $v_S \ge y_0$. Suppose this is not true, then exists a sufficiently small $\Delta > 0$ such that for any $0 < \delta \le \Delta$,

$$\psi(v_R^0, y_0 + \delta) - \psi(v_R^0, y_0) > 0.$$

Therefore,

$$
\begin{aligned}
0 &< \mathbb{E}[\psi(v_R^0, y_0 + (\Delta \wedge K)) - \psi(v_R^0, y_0 \wedge (y_0 + K))] \\
&= \mathbb{E}[\psi(v_R^0 \wedge (y_0 + K), (y_0 + \Delta) \wedge (y_0 + K)) - \psi(v_R^0 \wedge (y_0 + K), y_0 \wedge (y_0 + K))].
\end{aligned}
$$

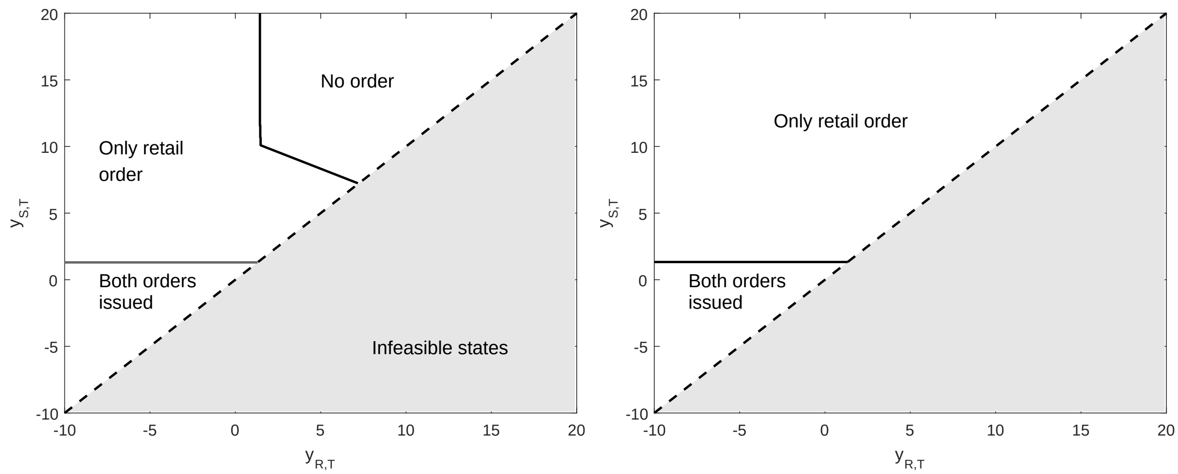
This contradicts the fact that (v_R^0, y_0) maximizes $\{\mathbb{E}[\psi(v_R \wedge (y_0 + K), v_S \wedge (y_0 + K))] : y_R \leq v_R \leq v_S, v_S \geq y_0\}$. Thus, we must have (v_R^0, y_0) maximizes $\{\psi(v_R, v_S) : y_R \leq v_R \leq v_S, v_S \geq y_0\}$. Now let $\bar{v}_R(v_S)$ be the maximizer of $\{\psi(v_R, v_S), v_R \leq v_S\}$. From the proof of Theorem 1, $\psi(\bar{v}_R(v_S), v_S)$ is unimodal in v_S and its maximum over $v_S \geq y_0$ is obtained when $v_S = y_0$. Thus, we must have $v_S^*(y_R, y_0 + \delta) = y_0 + \delta$. □

Theorem 2 (The Threshold Policy) *In each period t, there exist thresholds $(\bar{y}_{R,t}(y_{S,t}), \bar{y}_{S,t}(y_{R,t}))$ such that the downstream issues an order to the upstream if and only if $y_{R,t} < \bar{y}_{R,t}(y_{S,t})$, and the upstream facility produces if and only if $y_{S,t} \leq \bar{y}_{S,t}(y_{R,t})$. Moreover, $\bar{y}_{R,t}(\cdot)$ and $\bar{y}_{S,t}(\cdot)$ are decreasing.*

Proof of Theorem 2. As in the proof of Theorem 1, define $v_S = y_S + q_M$ and $v_R = y_R + q_R$. The optimal (v_R, v_S) maximizes ϕ defined in (2.21). Applying Lemma 4(i)&(iv) to ϕ, the optimality of the threshold policy follows immediately. Moreover, by Lemma 4(ii)&(iii), the thresholds are decreasing. □

The optimal policy characterized in Theorem 2 is a two-dimensional threshold policy. An example is demonstrated in the left panel of Figure 2.2. Under this policy, for each given value of upstream [downstream] inventory position $y_{S,t}$ [$y_{R,t}$], there exists a threshold $\bar{y}_{R,t}(y_{S,t})$ [$\bar{y}_{S,t}(y_{R,t})$], such that a positive order is issued by the downstream (upstream) if and only if its inventory position is below the threshold. The two threshold curves $(\bar{y}_{R,t}(y_{S,t}), \bar{y}_{S,t}(y_{R,t}))$ are generally nonlinear, and they together define the regions in which two, one or no order is issued as shown in the left panel of Figure 2.2.

In the left panel of Figure 2.2 there is a maximum amount Z for which the leftover inventory can be salvaged. This terminal value is not separable in the final inventory positions. In the existing literature, a separable terminal value is often assumed (see,



Note. $t = T = 1$, $c = 5, h_R = 4, h_W = 1, h_M = 0.2, b = 9, p = 20$, $D \sim$ unif$[0.5, 1.5]$, and $Pr\{K = 0.5\} = Pr\{K = 2\} = 0.5$. In left panel, $V_{T+1}(y_R, y_S) = -20y_R^- + \mathbb{E}[\min\{3y_R^+ + 2(y_S - y_R), Z\}]$, where $Z \sim$ unif$[18, 22]$ represents a random maximum salvage value. In right panel, $V_{T+1}(y_R, y_S) = -20y_R^- + 3y_R^+ + 2(y_S - y_R)$.

Fig. 2.2.: The optimal policy

e.g., [4, 28, 35]). In this case, the optimal policy reduces to an echelon base-stock policy, as demonstrated in the right-panel of Figure 2.2.

Corollary 1 (The Base-stock Policy) *When the terminal value function $V_{T+1}(\cdot, \cdot)$ is concave and separable, the optimal value function is separable, i.e, $V_t(y_{R,t}, y_{S,t}) = V_t^R(y_{R,t}) + V_t^S(y_{S,t})$ for each t. The thresholds in Theorem 2 become*

$$\bar{y}_{R,t}(y_{S,t}) = (\hat{y}_{S,t} \vee \hat{y}_t \vee y_{S,t}) \wedge \hat{y}_{R,t} \quad \textit{and} \quad \bar{y}_{S,t} \equiv \bar{y}_{S,t}(y_{R,t}) = \hat{y}_{S,t} \vee \hat{y}_t,$$

where $\hat{y}_{R,t} = \arg\max_v \psi^R(v)$, $\hat{y}_{S,t} = \arg\max_v \psi^S(v)$ and $\hat{y}_t = \arg\max_v\{\psi^R(v) + \psi^S(v)\}$ for $\psi^R(v) = h_M v - h_W v + E[V_{t+1}^R(v - D_t)]$ and $\psi^S(v) = -cv - h_M v + \mathbb{E}[V_{t+1}^S(v - D_t)]$. Moreover, the optimal downstream order quantity is $(\bar{y}_{R,t}(y_{S,t}) - y_{R,t})^+$ and the optimal upstream production quantity is $(\bar{y}_{S,t}(y_{R,t}) - y_{S,t})^+$.

Proof of Corollary 1. By assumption $V_{T+1}(\cdot,\cdot)$ is separable. Suppose that $V_{t+1}(\cdot,\cdot)$ is separable, i.e, $V_{t+1}(y_R, y_S) = V^R_{t+1}(y_R) + V^S_{t+1}(y_S)$. From the proof of Theorem 1, the optimal (v_R, v_S) must maximize

$$\phi(y_S, v_R, v_S) = \mathbb{E}[\psi^R(v_R \wedge (y_S + K_t)) + \psi^S(v_S \wedge (y_S + K_t))],$$

where

$$\psi^R(v) = h_M v - h_W v + \mathbb{E}[V^R_{t+1}(v - D)], \tag{2.22}$$

$$\psi^S(v) = -cv - h_M v + \mathbb{E}[V^S_{t+1}(v - D)], \tag{2.23}$$

subject to the constraints $v_R \geq y_R$ and $v_S \geq y_S \vee v_R$.

Define $\hat{y}_{R,t} = \arg\max_y \psi^R(y)$, $\hat{y}_{S,t} = \arg\max_y \psi^S(y)$, and $\hat{y}_t = \arg\max_y\{\psi^R(y) + \psi^S(y)\}$. Since ψ^R and ψ^S are concave, there are only two possible cases: (i) $\hat{y}_{S,t} \geq \hat{y}_t \geq \hat{y}_{R,t}$ and (ii) $\hat{y}_{S,t} \leq \hat{y}_t \leq \hat{y}_{R,t}$.

In case (i), $v^*_{R,t}(y_R, y_S) = \hat{y}_{R,t} \vee y_R$ and $v^*_{S,t}(y_R, y_S) = \hat{y}_{S,t} \vee y_S$. Therefore,

$$\begin{aligned}
&\phi(y_R, y_S, v^*_{R,t}(y_R, y_S), v^*_{S,t}(y_R, y_S)) \\
=\;& \mathbb{E}[\psi^R((\hat{y}_{R,t} \vee y_R) \wedge (y_S + K_t)) + \psi^S((\hat{y}_{S,t} \vee y_S) \wedge (y_S + K_t))] \\
=\;& \psi^R(y_R)\mathbb{I}_{\{y_R \geq \hat{y}_{R,t}\}} + \mathbb{E}[\psi^R(\hat{y}_{R,t} \wedge (y_S + K_t))\mathbb{I}_{\{y_R < \hat{y}_{R,t}\}} + \psi^S((\hat{y}_{S,t} \vee y_S) \wedge (y_S + K_t))] \\
=\;& (\psi^R(y_R) - \psi^R(\hat{y}_{R,t}))\mathbb{I}_{\{y_R \geq \hat{y}_{R,t}\}} + \mathbb{E}[\psi^R(\hat{y}_{R,t} \wedge (y_S + K_t)) + \psi^S((\hat{y}_{S,t} \vee y_S) \wedge (y_S + K_t))].
\end{aligned}$$

In deriving the last equation, we have use the fact that $y_R > \hat{y}_{R,t}$ implies $y_S + K_t > \hat{y}_{R,t}$ almost surely because $y_S \geq y_R$. It is clear that the right-hand side of the above equation is separable in (y_R, y_S).

In case (ii), $v^*_{R,t}(y_R, y_S) = (\hat{y}_t \vee y_S \wedge \hat{y}_{R,t}) \vee y_R$ and $v^*_{S,t}(y_R, y_S) = \hat{y}_t \vee y_S$. Therefore,

$$\begin{aligned}
&\phi(y_R, y_S, v^*_{R,t}(y_R, y_S), v^*_{S,t}(y_R, y_S)) \\
=\;& \mathbb{E}[\psi^R((\hat{y}_t \vee y_S \wedge \hat{y}_{R,t}) \vee y_R \wedge (y_S + K_t)) + \psi^S((\hat{y}_t \vee y_S) \wedge (y_S + K_t))] \\
=\;& \mathbb{E}[\psi^R(y_R \wedge (y_S + K_t))\mathbb{I}_{\{y_R \geq \hat{y}_{R,t}\}} + \psi^R((\hat{y}_t \vee y_S \wedge \hat{y}_{R,t}) \wedge (y_S + K_t))\mathbb{I}_{\{y_R < \hat{y}_{R,t}\}} \\
&+ \psi^S((\hat{y}_t \vee y_S) \wedge (y_S + K_t))] \\
=\;& (\psi^R(y_R) - \psi^R(\hat{y}_{R,t}))\mathbb{I}_{\{y_R \geq \hat{y}_{R,t}\}} \\
&+ \mathbb{E}[\psi^R((\hat{y}_t \vee y_S \wedge \hat{y}_{R,t}) \wedge (y_S + K_t)) + \psi^S((\hat{y}_t \vee y_S) \wedge (y_S + K_t))].
\end{aligned}$$

In deriving the last equation, we have use the fact that $y_R > \hat{y}_{R,t}$ and $\hat{y}_{R,t} \geq \hat{y}_t$ implies $y_S > \hat{y}_{R,t} \geq \hat{y}_t$. It is clear that the right-hand side of the above equation is separable in (y_R, y_S).

Combining the two cases, we deduce the optimality of a base-stock policy. Moreover, substituting the above expressions of the optimal ϕ into (2.20) and (2.21), we have $V_t(y_R, y_S) = V_t^R(y_R) + V_t^S(y_S)$, where

$$\begin{aligned} V_t^R(y_R) &= -\mathbb{E}[H_R(y_R - D_t - D_{t+1})] + h_W y_R \\ &\quad +(\psi^R(y_R) - \psi^R(\hat{y}_{R,t}))\mathbb{I}_{\{y_R \geq \hat{y}_{R,t}\}}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} V_t^S(y_S) &= p\mathbb{E}[D_t] + cy_S + \mathbb{E}[\psi^R((\hat{y}_t \vee y_{S,t} \wedge \hat{y}_{R,t}) \wedge (y_S + K_t)) \\ &\quad +\psi^S((\hat{y}_{S,t} \vee \hat{y}_t \vee y_S) \wedge (y_S + K_t))]. \end{aligned} \tag{2.25}$$

Hence, we conclude the proof. □

With a separable terminal value, the thresholds become base-stock levels, up to which the orders are determined. The quantity $\hat{y}_{R,t}$ and $\hat{y}_{S,t}$ are, respectively, the ideal post-order inventory position for the downstream and upstream. The quantity $\hat{y}_t$ is the ideal system inventory level when everything produced is shipped to the downstream and thus the post order inventory positions are the same for the upstream and downstream. To implement this policy, one first determines the target system inventory level, i.e., the base stock level $\hat{y}_{S,t} \vee \hat{y}_t$. A production order is issued to bring the post-order system inventory to the level of $\hat{y}_{S,t} \vee \hat{y}_t \vee y_{S,t}$. The order at the downstream then determines whether system inventory should be redistributed, depending on whether ideal downstream level $\hat{y}_{R,t}$ exceeds the post-order system inventory $\hat{y}_{S,t} \vee \hat{y}_t \vee y_{S,t}$. Thus, the base-stock level at the downstream depends on the upstream inventory position $y_{S,t}$ in general.

2.5.2 Comparative Statics

To obtain additional insights into the optimal policy, we analyze the responses of the optimal decisions to the changes of system parameters. For ease of exposition, we assume linear holding/backlogging cost functions, i.e., $H_R(x) = h_R x^+ + sx^-$ for $h_R, s > 0$ and $H_M(x) = h_M x$, and a separable $V_{T+1}(\cdot,\cdot)$. In this case, it is optimal to follow a base-stock policy as suggested in Corollary 1.

Proposition 1 (Comparative Statics: Random Parameters) *In two otherwise identical systems, indexed by a and b, with spreadable $V_{T+1}(\cdot,\cdot)$, the optimal base-stock levels satisfy*

$$\bar{y}^a_{S,t} \leq \bar{y}^b_{S,t} \quad \textit{and} \quad \bar{y}^a_{R,t}(y_{S,t}) \leq \bar{y}^b_{R,t}(y_{S,t}).$$

if $K^a >_{st} K^b$ or $D^a <_{st} D^b$.

Proof of Proposition 1. The result can be obtained in a similar way as Proposition 2. □

It is intuitive to target higher stock levels when the demand becomes larger and the production capacity becomes smaller (in the stochastic sense) because of the role of inventories in mitigating potential shortages. However, due to capacitated production, higher target stock levels do not necessarily imply higher on-hand stock levels. To see this, we observe from Table 2.1 that an increase in the production capacity can have different effect on inventories at different locations. In particular, when the system inventory increases with an increased production capacity, a large portion of the increased stock is allocated to the downstream. The upstream stock level, however, may increase or decrease. When the production capacity is very limited compared with the demand, as soon as production completes, the products are immediately shipped to the downstream to meet the demand. When there is an ample production capacity relative to the demand, the upstream facility can always produce on-demand without worrying about future shortage in fulfilling the downstream order. As a result, the upstream only holds inventory when the mean production capacity is around the mean demand.

In Table 2.2 we further examine the effect of capacity and demand uncertainty on the distribution of the inventory. Even though the mean demand equals the mean

Table 2.1.: The effect of the production capacity on the average inventories.

μ_K	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.1	2.3	2.5	2.7
$\mathbb{E}[X_R]$	-3.67	-1.89	-0.34	0.58	0.94	1.08	1.15	1.19	1.22	1.23	1.24
$\mathbb{E}[W]$	0.67	0.84	0.95	0.98	0.99	0.99	0.99	0.99	0.99	0.99	0.99
$\mathbb{E}[X_M]$	0.00	0.02	0.11	0.24	0.18	0.05	0.00	0.00	0.00	0.00	0.00
$\mathbb{E}[Y_S]$	-3.00	-1.03	0.73	1.80	2.11	2.12	2.14	2.18	2.21	2.22	2.23

Notes. $\mathbb{E}[X_R] = \mathbb{E}[\sum_{t=1}^{T} X_{R,t}]/T$, $\mathbb{E}[X_M] = \mathbb{E}[\sum_{t=1}^{T} X_{M,t}]/T$, $\mathbb{E}[W] = \mathbb{E}[\sum_{t=1}^{T} W_t]/T$, and $\mathbb{E}[Y_S] = \mathbb{E}[X_R] + \mathbb{E}[X_M] + \mathbb{E}[W]$. Model parameters are $T = 20$, $c = 5$, $h_R = h_W = 4$, $h_M = 0.2$, $s = 9$, $p = 20$, $D \sim unif[0, 2]$, $K \sim unif[\mu_k - 0.5, \mu_k + 0.5]$, $V_{T+1}(y_R, y_S) = -20y_R^-$, and the initial inventory levels are all zeros.

supply in the case of limited capacity, the system is frequently experiencing shortage due to the uncertainties, resulting in negative average inventories. An increased uncertainty, whether from the upstream production capacity or from the downstream customer demand, leads to an increased average backlog. Both sources of uncertainty also induce an increased upstream stock level and a reduced in-transit quantity. However, the reason behind such behavior is not the same. When the demand becomes more unpredictable, more inventory needs to be prepared to hedge against the demand risk. In view of the higher holding cost at the downstream, more inventory is held at the upstream. When the production becomes highly fluctuating, the upstream tends to place large production orders in order to obtain a large output in the case of high capacity, which makes up for the shortage in the case of low capacity. As a result, the upstream holds more inventory when capacity fluctuation becomes wider.

When the production capacity becomes ample, an increased uncertainty from either supply or demand leads to an increased system inventory (i.e., $\mathbb{E}[Y_S]$). An increased demand uncertainty induces a uniform increase of stock levels at all places. With an increased capacity uncertainty, however, more inventory is retained at the upstream as opposed to being kept at the downstream. We also note that the in-transit order is insensitive to the capacity uncertainty when the capacity is ample.

This suggests a stable material flow from the upstream to the downstream to meet the customer demand, which is independent of the production capacity.

Table 2.2.: The effect of uncertainties on the average inventories.

	Limited Capacity				Ample Capacity			
	$\mathbb{E}[X_R]$	$\mathbb{E}[W]$	$\mathbb{E}[X_M]$	$E[Y_S]$	$\mathbb{E}[X_R]$	$\mathbb{E}[W]$	$\mathbb{E}[X_M]$	$E[Y_S]$
$\Delta_D = 0$	-0.965	0.929	0.004	-0.031	0.693	0.961	0.000	1.655
0.25	-0.966	0.929	0.006	-0.031	0.801	0.974	0.000	1.775
0.5	-0.982	0.925	0.013	-0.044	0.887	0.978	0.000	1.865
0.75	-1.013	0.918	0.028	-0.067	0.966	0.983	0.027	1.975
1	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
$\Delta_K = 0$	-1.025	0.916	0.031	-0.078	1.044	0.988	0.028	2.059
0.25	-1.033	0.915	0.035	-0.083	1.039	0.988	0.048	2.076
0.5	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
0.75	-1.101	0.904	0.075	-0.121	1.005	0.988	0.217	2.210
1	-1.161	0.897	0.112	-0.153	0.972	0.988	0.342	2.302

Notes. $\mathbb{E}[X_R] = \mathbb{E}[\sum_{t=1}^{T} X_{R,t}]/T$, $\mathbb{E}[X_M] = \mathbb{E}[\sum_{t=1}^{T} X_{M,t}]/T$, $\mathbb{E}[W] = \mathbb{E}[\sum_{t=1}^{T} W_t]/T$, and $\mathbb{E}[Y_S] = \mathbb{E}[X_R] + \mathbb{E}[X_M] + \mathbb{E}[W]$. $D \sim unif[1-\Delta_D, 1+\Delta_D]$, and $K \sim unif[\mu_K - \Delta_K, \mu_K + \Delta_K]$, where $\mu_K = 1$ in the limited capacity case and $\mu_K = 1.6$ in the ample capacity case. Other model parameters are the same as in Table 2.1.

Proposition 2 (Comparative Statics: Cost Parameters) *In two otherwise identical systems, indexed by a and b, with spreadable $V_{T+1}(\cdot,\cdot), i = a, b$, the optimal base-stock levels satisfy*

$$\bar{y}^a_{S,t} \leq \bar{y}^b_{S,t} \quad \textit{and} \quad \bar{y}^a_{R,t}(y_{S,t}) \leq \bar{y}^b_{R,t}(y_{S,t}).$$

if $c^a > c^b$, $h^a_M > h^b_M$, $h^a_W > h^b_W$, $h^a_R > h^b_R$ or $s^a < s^b$.

Proof of Proposition 2. We prove the result for two otherwise identical systems with $h^a_R > h^b_R$. The results for other parameters can be obtained similarly. We shall also note that the derivation below proves a stronger result for nonstationary systems with $h^a_{R,t} > h^b_{R,t}$ for each t.

We use induction to show that $V^{R,a}_t(\cdot) - V^{R,b}_t(\cdot)$ and $V^{S,a}_t(\cdot) - V^{S,b}_t(\cdot)$ are both decreasing. This is clearly true for period $T+1$ because both V^R_{T+1} and V^S_{T+1} does not depend on h_R. Suppose this is true for period $t+1$. Then from (2.22) and (2.23), we deduce that $\Delta_{\psi^j}(\cdot) = \psi^{j,a}(\cdot) - \psi^{j,b}(\cdot), j \in \{R, S\}$, and $\Delta_\psi(\cdot) = \psi^a(\cdot) - \psi^b(\cdot)$, where $\psi^i(\cdot) = \psi^{R,i}(\cdot) + \psi^{S,i}(\cdot), i \in \{a, b\}$, are decreasing. Because $\hat{y}^i_{j,t} = \arg\max_y \psi^{j,i}(y), j \in \{R, S\}$, and $\hat{y}^i_t = \arg\max_y \psi^i(y)$,$i \in \{a, b\}$, we must have

$$\hat{y}^a_{R,t} \leq \hat{y}^b_{R,t}, \quad \hat{y}^a_{S,t} \leq \hat{y}^b_{S,t}, \quad \hat{y}^a_t \leq \hat{y}^b_t.$$

Thus, $\bar{y}^a_{S,t} \leq \bar{y}^b_{S,t}$ and $\bar{y}^a_{R,t}(y_{S,t}) \leq \bar{y}^b_{R,t}(y_{S,t})$.

Now define $\gamma^{R,i}(y) = (\psi^{R,i}(y) - \psi^{R,i}(\hat{y}^i_{R,t}))\mathbb{I}_{\{y > \hat{y}^i_{R,t}\}}$. Then

$$\begin{aligned}
\gamma^{R,a}(y) - \gamma^{R,b}(y) &= (\psi^{R,a}(y) - \psi^{R,a}(\hat{y}^a_{R,t}))\mathbb{I}_{\{y>\hat{y}^a_{R,t}\}} - (\psi^{R,b}(y) - \psi^{R,b}(\hat{y}^a_{R,t}))\mathbb{I}_{\{y>\hat{y}^b_{R,t}\}} \\
&= \begin{cases} 0 & \text{if } y \leq \hat{y}^a_{R,t}, \\ \psi^{R,a}(y) - \psi^{R,a}(\hat{y}^a_{R,t}) & \text{if } \hat{y}^a_{R,t} < y \leq \hat{y}^b_{R,t}, \\ \Delta_{\psi^R}(y) - \psi^{R,a}(\hat{y}^a_{R,t}) + \psi^{R,b}(\hat{y}^a_{R,t}) & \text{if } y > \hat{y}^b_{R,t}. \end{cases}
\end{aligned}$$

Because $\psi^{R,i}$ is concave and $\Delta_{\psi^R}(\cdot) = \psi^{R,a}(\cdot) - \psi^{R,b}(\cdot)$ is decreasing, it follows that $\gamma^{R,a}(\cdot) - \gamma^{R,b}(\cdot)$ is decreasing in y. From (2.24), we deduce that $V^{R,a}_t(\cdot) - V^{R,b}_t(\cdot)$ is decreasing.

Next define $\gamma^{S,i}(y) = \psi^{R,i}((\hat{y}^i_t \vee y \wedge \hat{y}^i_{R,t}) \wedge (y+k)) + \psi^{S,i}((\hat{y}^i_{S,t} \vee \hat{y}^i_t \vee y) \wedge (y+k))$ for $k \geq 0$, $i \in \{a, b\}$. We have four cases to consider:

(i) If $\hat{y}^a_{S,t} \leq \hat{y}^a_t \leq y^a_{R,t}$ and $\hat{y}^b_{S,t} \leq \hat{y}^b_t \leq y^b_{R,t}$, then

$$\begin{aligned}
&\gamma^{S,a}(y) - \gamma^{S,b}(y) \\
= &\begin{cases} \psi^a(\hat{y}^a_t \wedge (y+k)) - \psi^b(\hat{y}^b_t \wedge (y+k)) & \text{if } y \leq \hat{y}^a_t, \\ \psi^a(y) - \psi^b(\hat{y}^b_t \wedge (y+k)) & \text{if } \hat{y}^a_t < y \leq \hat{y}^a_{R,t} \wedge \hat{y}^b_t, \\ \psi^{R,a}(\hat{y}^a_{R,t}) + \psi^{S,a}(y) - \psi^b(\hat{y}^b_t \wedge (y+k)) & \text{if } y^a_{R,t} \leq \hat{y}^b_t \text{ and } \hat{y}^a_{R,t} < y \leq \hat{y}^b_t, \\ \Delta_\psi(y) & \text{if } y^a_{R,t} > \hat{y}^b_t \text{ and } \hat{y}^b_t < y \leq \hat{y}^a_{R,t}, \\ \psi^{R,a}(\hat{y}^a_{R,t}) + \psi^{S,a}(y) - \psi^b(y) & \text{if } \hat{y}^a_{R,t} \vee \hat{y}^b_t < y \leq \hat{y}^b_{R,t}, \\ \psi^{R,a}(\hat{y}^a_{R,t}) - \psi^{R,b}(\hat{y}^b_{R,t}) + \Delta_{\psi^S}(y) & \text{if } y > \hat{y}^b_{R,t}, \end{cases}
\end{aligned}$$

Note that ψ^i and $\psi^{j,i}$,$i \in \{a, b\}$ and $j \in \{R, S\}$ are concave. A concave function is increasing before its maximum and decreasing after its maximum. Therefore, it is easy to check each case above to conclude that $\gamma^{S,a}(\cdot) - \gamma^{S,b}(\cdot)$ is decreasing.

(ii) If $\hat{y}^a_{S,t} \geq \hat{y}^a_t \geq \hat{y}^a_{R,t}$ and $\hat{y}^b_{S,t} \geq \hat{y}^b_t \geq \hat{y}^b_{R,t}$, then

$$\gamma^{S,a}(y) - \gamma^{S,b}(y) = \psi^{R,a}(\hat{y}^a_{R,t} \wedge (y+k)) - \psi^{R,b}(\hat{y}^b_{R,t} \wedge (y+k)) + \begin{cases} \psi^a(\hat{y}^a_{S,t} \wedge (y+k)) - \psi^b(\hat{y}^b_{S,t} \wedge (y+k)) & \text{if } y \leq \hat{y}^a_{S,t}, \\ \psi^a(y) - \psi^b(\hat{y}^b_{S,t} \wedge (y+k)) & \text{if } \hat{y}^a_{S,t} < y \leq \hat{y}^b_{S,t}, \\ \Delta_\psi(y) & \text{if } y > \hat{y}^b_{S,t}. \end{cases}$$

Hence the right-hand side is decreasing in y.

(iii) If $\hat{y}^a_{S,t} \leq \hat{y}^a_t \leq \hat{y}^a_{R,t}$ and $\hat{y}^b_{S,t} \geq \hat{y}^b_t \geq \hat{y}^b_{R,t}$, then

$$\begin{aligned} & \gamma^{S,a}(y) - \gamma^{S,b}(y) \\ = \; & -\psi^{R,b}(\hat{y}^b_{R,t} \wedge (y+k)) \\ & + \begin{cases} \psi^a(\hat{y}^a_t \wedge (y+k)) - \psi^{S,b}(\hat{y}^b_{S,t} \wedge (y+k)) & \text{if } y \leq \hat{y}^a_t, \\ \psi^a(y) - \psi^{S,b}(\hat{y}^b_{S,t} \wedge (y+k)) & \text{if } \hat{y}^a_t < y \leq \hat{y}^a_{R,t}, \\ \psi^{R,a}(\hat{y}^a_{R,t} \wedge (y+k)) + \psi^{S,a}(y) - \psi^{S,b}(\hat{y}^b_{S,t} \wedge (y+k)) & \text{if } \hat{y}^a_{R,t} < y \leq \hat{y}^b_{S,t}, \\ \psi^{R,a}(\hat{y}^a_{R,t} \wedge (y+k)) + \Delta_{\psi^S}(y) & \text{if } y > \hat{y}^b_{S,t}. \end{cases} \end{aligned}$$

To see the function is decreasing in the first case, we note that when $y + k < \hat{y}^a_t < \hat{y}^b_{R,t} < \hat{y}^b_{S,t}$, the right-hand side reduces to $\Delta_\psi(y+k)$, which is decreasing in y. It is then easy to check that the right-hand side is decreasing in other cases.

(iv) If $\hat{y}^a_{S,t} \geq \hat{y}^a_t \geq \hat{y}^a_{R,t}$ and $\hat{y}^b_{S,t} \leq \hat{y}^b_t \leq \hat{y}^b_{R,t}$, then

$$\gamma^{S,a}(y) - \gamma^{S,b}(y) = \psi^{R,a}(\hat{y}^a_{R,t} \wedge (y+k)) + \begin{cases} \psi^{S,a}(\hat{y}^a_{S,t} \wedge (y+k)) + \psi^b(\hat{y}^b_t \wedge (y+k)) & \text{if } y \leq \hat{y}^a_{S,t} \\ \psi^{S,a}(y) - \psi^b(\hat{y}^b_t \wedge (y+k)) & \text{if } \hat{y}^a_{S,t} < y \leq \hat{y}^b_t \\ \psi^{S,a}(y) - \psi^b(y) & \text{if } \hat{y}^b_t < y \leq \hat{y}^b_{R,t} \\ \Delta_{\psi^S}(y) - \psi^{R,b}(\hat{y}^b_{R,t} \wedge (y+k)) & \text{if } y > \hat{y}^b_{R,t}. \end{cases}$$

Following the same arguments as in previous cases, we deduce that the right-hand side is decreasing in y.

Since $\gamma^{S,a}(\cdot) - \gamma^{S,b}(\cdot)$ is decreasing, it is immediate from (2.25) that $V^{S,a}_t(\cdot) - V^{S,b}_t(\cdot)$ is decreasing. Hence, we conclude the proof. $\square$

Proposition 2 suggests that the target stock levels become lower when it is more expensive to produce or to hold inventories, or when the backorder penalty is smaller. In Table 2.3, we observe that the average inventory positions reveal consistent response to the cost parameters as the target stock levels. However, the distribution of the inventory at different places in the system may reveal local patterns. In particular, while the system inventory (i.e., $\mathbb{E}[Y_S]$) becomes smaller when any inventory cost (i.e., h_R, h_M or h_W) increases , the downstream [upstream] carries more inventory when it becomes more expensive to hold inventory at the upstream [downstream]. We also observe that the upstream facility may hold less inventory when the backorder cost (i.e., s) increases. This happens when the production capacity is limited and thus an increased backorder cost induces a larger inventory distributed to the downstream.

We shall note that though we state Proposition 1 and Proposition 2 for stationary system parameters, both results hold for nonstationary systems (see the proofs of these propositions). The next proposition explores additional properties of the optimal policy under stationary environment.

Proposition 3 (Comparative Statics: Stationary Systems) *When* $V_{T+1}(y_R, y_S) = V^R_{T+1}(y_R) + V^S_{T+1}(y_S)$ *and system parameters are all stationary,*

$$\bar{y}_{S,t} \geq \bar{y}_{S,t+1} \quad \text{and} \quad \bar{y}_{R,t}(y_S) \geq \bar{y}_{R,t+1}(y_S).$$

Moreover, if $\mathbb{E}[V^R_{T+1}(y_2 - D)] - \mathbb{E}[V^R_{T+1}(y_1 - D)] \geq h_W(y_2 - y_1) - \mathbb{E}[H_R(y_2 - D^{(3)})] + \mathbb{E}[H_R(y_1 - D^{(3)})]$ *for any* $y_1 < y_2$*, then*

$$\hat{y}_{R,t} = F^{-1}_{D^{(3)}}\left(\frac{s + h_M}{s + h_R}\right) \text{ for } t = 1, 2, \ldots, T-1,$$

where $D^{(k)}$ *is the k-fold convolution of demand* $D =^d D_t$ *and* $F^{-1}_{D^{(k)}}$ *is the inverse of the distribution of* $D^{(k)}$*.*

Proof of Proposition 3. The first part of the proposition follows in a similar way as Proposition 2 when the time horizon T changes. We focus on deriving the second part. From the proof of Corollary 1, $\hat{y}_{R,t}$ is the unconstrained maximizer of

$$\psi^R_t(v) = h_M v - h_W v + \mathbb{E}[V^R_{t+1}(v - D)].$$

Table 2.3.: The effect of the cost parameters on the average inventories.

	Limited Capacity				Ample Capacity			
	$\mathbb{E}[X_R]$	$\mathbb{E}[W]$	$\mathbb{E}[X_M]$	$E[Y_S]$	$\mathbb{E}[X_R]$	$\mathbb{E}[W]$	$\mathbb{E}[X_M]$	$E[Y_S]$
$s=7$	-1.073	0.909	0.057	-0.108	0.910	0.982	0.100	1.993
8	-1.065	0.910	0.053	-0.103	0.975	0.985	0.107	2.067
9	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
10	-1.053	0.911	0.047	-0.094	1.072	0.991	0.125	2.188
11	-1.048	0.912	0.045	-0.091	1.114	0.993	0.129	2.236
$h_R=1$	-1.014	0.913	0.017	-0.084	1.517	0.999	0.004	2.520
2	-1.031	0.912	0.029	-0.090	1.291	0.994	0.060	2.345
3	-1.045	0.911	0.040	-0.094	1.144	0.991	0.091	2.225
4	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
5	-1.070	0.910	0.059	-0.102	0.931	0.986	0.135	2.052
$h_M=\mathbf{0.2}$	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
1	-1.050	0.910	0.026	-0.114	1.030	0.988	0.000	2.018
1.8	-1.045	0.910	0.012	-0.123	1.030	0.988	0.000	2.018
2.6	-1.042	0.910	0.003	-0.128	1.030	0.988	0.000	2.018
3.4	-1.040	0.910	0.000	-0.130	1.030	0.988	0.000	2.018
$h_W=1$	-1.054	0.915	0.052	-0.087	1.043	0.999	0.118	2.159
2	-1.056	0.914	0.051	-0.091	1.037	0.995	0.118	2.150
3	-1.057	0.912	0.050	-0.095	1.032	0.991	0.117	2.141
4	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
5	-1.060	0.909	0.049	-0.102	1.022	0.985	0.115	2.122
$c=4$	-1.057	0.912	0.051	-0.094	1.032	0.991	0.117	2.141
4.5	-1.058	0.911	0.050	-0.096	1.030	0.990	0.117	2.136
5	-1.058	0.911	0.050	-0.098	1.027	0.988	0.116	2.132
5.5	-1.059	0.910	0.049	-0.100	1.024	0.986	0.116	2.126
6	-1.060	0.909	0.049	-0.102	1.022	0.985	0.115	2.122

Notes. $\mathbb{E}[X_R] = \mathbb{E}[\sum_{t=1}^T X_{R,t}]/T$, $\mathbb{E}[X_M] = \mathbb{E}[\sum_{t=1}^T X_{M,t}]/T$, $\mathbb{E}[W] = \mathbb{E}[\sum_{t=1}^T W_t]/T$, and $\mathbb{E}[Y_S] = \mathbb{E}[X_R] + \mathbb{E}[X_M] + \mathbb{E}[W]$. $\mu_K = 1$ for the limited capacity case and $\mu_K = 1.6$ for the ample capacity case. Other model parameters are the same as in Table 2.1.

Substituting (2.24) in the above gives

$$\begin{aligned} \psi_t^R(v) &= h_M v - \mathbb{E}[H_R(v - D^{(3)})] - h_W \mathbb{E}[D] \\ &\quad + \mathbb{E}[(\psi_{t+1}^R(v-D) - \psi_{t+1}^R(\hat{y}_{R,t+1}))\mathbb{I}_{\{v-D \geq \hat{y}_{R,t+1}\}}]. \end{aligned} \tag{2.26}$$

Define $\hat{y} \equiv F_{D^{(3)}}^{-1}((s+h_M)/(s+h_R))$. Because ψ_{t+1}^R is concave and it is maximized at $\hat{y}_{R,t+1}$, it is immediate that ψ_t^R is maximized at $\hat{y}$ if $\hat{y}_{R,t+1} \geq \hat{y}$. Thus, we only need to show $\hat{y}_{R,T} \geq \hat{y}$. This follows from the fact that ψ_T^R is concave and $\mathbb{E}[V_{T+1}^R(y_2 - D)] - \mathbb{E}[V_{T+1}^R(y_1 - D)] \geq h_W(y_2 - y_1) - \mathbb{E}[H_R(y_2 - D^{(3)})] + \mathbb{E}[H_R(y_1 - D^{(3)})]$ for any $y_1 < y_2$. $\square$

In a stationary system, the base-stock levels are decreasing over time. This is because a unit produced earlier has a better chance to be sold before the end of the horizon. This observation is consistent with its counterpart in single-location systems (see,.e.g., [3]).

Proposition 3 further suggests that when the terminal value is highly sensitive to the inventory positions (relative to the difference between the holding cost at the downstream and that in transit), the ideal inventory position at the downstream can be derived using a newsvendor formula. The ideal inventory position at the downstream depends on the backorder penalty as well as the comparison between the holding costs at the upstream and at the downstream. When holding inventory is significantly cheaper at the upstream than at the downstream, a lower inventory position should be maintained at the downstream. Instead, a larger amount of inventory is kept at the upstream to save on inventory holding.

2.6 Extensions

In this section, we discuss two important extensions of our model, one with general production functions and one with multiple locations.

2.6.1 General Production Functions

Our previous analysis assumes that the production output is constrained by a random production capacity. In general, the input and output relationship can exhibit different various relationships depending on the input material flow, technology deployed and labor productivity. The study by [21] suggests that most of the production functions studied in the existing literature are stochastically linear in midpoint.

Thus, in this subsection, we consider a general production function $P(q_M)$ that has a stochastically linear in midpoint transformation.

The inventory dynamics at the downstream described in (2.12) now becomes

$$Y_{R,t+1} = (q_{R,t} + y_{R,t}) \wedge (y_{S,t} + P(q_{M,t})) - D_t, \tag{2.27}$$

and the system inventory dynamics described in (2.13) becomes

$$Y_{S,t+1} = y_{S,t} + P(q_{M,t}) - D_t. \tag{2.28}$$

Lemma 5 *Suppose* $\{A(u) : u \in \mathcal{U} \in \mathbf{R}\} \in SL(mp)$*. For* $u \in \mathcal{U}$ *and* $v \in \mathcal{V} \in \mathcal{R}$*,* $B(u,v) =^d v \wedge A(u)$ *can be transformed into functions that are stochastically linear in midpoint.*

Proof of Lemma 5. Take v_1, v_2, u_1, u_2 with $\bar{v} = (v_1 + v_2)/2$ and $\bar{u} = (u_1 + u_2)/2$. Let $\mu_1 = \mathbb{E}[v_1 \wedge A(u_1)]$ and $\mu_2 = \mathbb{E}[v_2 \wedge A(u_2)]$. Because $\{A(u) : u \in \mathcal{U}\} \in SL(mp)$, there exists $A_1 =^d A(u_1)$ and $A_2 =^d A(u_2)$ such that $\bar{A} =^d (A_1 + A_2)/2 \leq_{cv} A(\bar{u})$. Let $\bar{B} =^d (v_1 \wedge A_1 + v_2 \wedge A_2)/2$. Because the minimum function is concave, we have

$$\bar{B} \leq_{st} \bar{v} \wedge \bar{A}.$$

Then for any $b \leq \bar{v}$, $\bar{F}_{\bar{B}}(b) \leq \bar{F}_{\bar{A}}(b)$, where $\bar{F}_X$ is the survival function of random variable X. This implies that, for any $b \leq \bar{v}$,

$$\int_{-\infty}^{b} \bar{F}_{\bar{B}}(w)\mathrm{d}w \leq \int_{-\infty}^{b} \bar{F}_{\bar{A}}(w)\mathrm{d}w \leq \int_{-\infty}^{b} \bar{F}_{A(\bar{u})}(w)\mathrm{d}w.$$

The last inequality implies that $\bar{A} \leq_{cv} A(\bar{u})$.

Now let $\bar{\mu} = (\mu_1 + \mu_2)/2$. It is clear that $\bar{u} = \mathbb{E}[\bar{B}]$. Define $\hat{v}(u) = \inf\{v : \mathbb{E}[v \wedge A(\bar{u})] \geq \mu\}$. Because $\bar{B} \leq_{st} \bar{v} \wedge \bar{A}$, where $\hat{v}(u) \leq \bar{v}$. In other words, we have

$$\int_{-\infty}^{b} \bar{F}_{\bar{B}}(w)\mathrm{d}w \leq \int_{-\infty}^{b} \bar{F}_{A(\bar{u})}(w)\mathrm{d}w \text{ for } b \leq \hat{v}(\bar{\mu}).$$

Thus, In other words, $\bar{B} \leq_{cv} \hat{v}(\bar{\mu}) \wedge A(\bar{u})$. Thus, there exists a transformation of $B(u,v)$ that is stochastically linear in midpoint. $\square$

According to Lemma 5, we can conclude that both the downstream inventory in (2.27) and the system inventory in (2.28) can be transformed into a stochastic function that is linear in midpoint provided that is the case for the production function $P(q_M)$.

When both (2.27) and (2.28) satisfy the single-crossing property, a sufficient condition for stochastic linearity in midpoint, we can directly apply Lemma 2 to show that the profit function is concave and submodular in the inventory levels. Thus, the analysis in §2.4 continues to hold. The difference here is that even when the terminal value function $V_{T+1}(\cdot, \cdot)$ is concave and separable, the threshold policy for production order does not reduce to a base-stock policy.

2.6.2 Multi-Location Serial Systems

In this subsection, we discuss how our analysis can be extended to multi-location systems. There are N locations indexed by $n \in \{1, 2, \dots, N\}$. The manufacturing facility is located at N, the retail store is located at 1. After the completion of production, products are transported through intermediate locations $(N-1), \dots, 3, 2$ in sequence before reaching the store to meet the customer demand.

At the beginning of each period t, the on-hand inventory level x_n is reviewed at location $n \in \{1, 2, \dots, N\}$, a delivery of $w_{n+1,t-1}$ from the upstream location is expected to be delivered at the end of the period. The firm needs to determine the order quantity $q_{n,t}$ for each location $n \in \{1, 2, \dots, N\}$. Upon receiving the order from the downstream location $q_{n-1,t}$, location schedules a shipment $w_{n,t}$ based on the available stock at the end of period t and the shipment reaches the downstream location $n-1$ at the end of the next period (i.e., period $t+1$). Thus, the shipment dispatched stage n at the end of period t is

$$w_{n,t} = q_{n-1,t} \wedge (x_{n,t} + w_{n+1,t-1}) \text{ for } n \in \{2, \dots, N\}$$

with $w_{1,t} = D_t$ and $w_{N+1,t-1} = q_{N,t} \wedge K_t$. Then the dynamics of the inventory can be written as

$$x_{n,t+1} = x_{n,t} + w_{n+1,t-1} - w_{n,t}.$$

Using a similar transformation used for our base model, we define recursively the inventory positions as $y_{0,t} = 0$ and

$$y_{n,t} = y_{n-1,t} - x_{n,t} + w_{n+1,t-1}, \text{ for } n \in \{1, 2, \ldots, N\}.$$

Then we can rewrite the in-transit orders and on-hand inventory levels as

$$\begin{aligned} w_{n,t} &= (y_{n-1,t} + q_{n-1,t}) \wedge y_{n,t} - y_{n-1,t}, \\ x_{n,t+1} &= (y_{n,t} - y_{n-1,t} - q_{n-1,t})^+. \end{aligned}$$

With these relations, we can derive the dynamics of the inventory positions as

$$\begin{aligned} Y_{N,t+1} &= (y_{N,t} + q_{N,t}) \wedge (y_{N,t} + K_t) - D_t, \\ Y_{N-1,t+1} &= (y_{N-1,t} + q_{N-1,t}) \wedge (y_{N,t} + K_t) - D_t, \\ Y_{n,t+1} &= (y_{n,t} + q_{n,t}) \wedge y_{n+1,t} - D_t, \quad n \in \{1, 2, \ldots, N-2\}. \end{aligned}$$

Let $v_{n,t} = y_{n,t} + q_{n,t}$ denote the post-order inventory position and use $\mathbf{x_t}$ denote a vector of $(x_{1,t}, x_{2,t}, \ldots, x_{n,t})$. Assume that the holding and backorder cost at the store is a convex function $H_1(\cdot)$, the holding cost at location n is h_n and the holding cost for in-transit order shipped from location n is $h_{W,n}$ for $n \in \{2, 3, \ldots\}$. Then, we can write the dynamic programming equation as

$$\begin{aligned} \hat{J}_t(\mathbf{y}_t, \mathbf{v}_t) &= p\mathbb{E}[D_t] - c(\mathbb{E}[v_{N,t} \wedge (y_{N,t} + K)] - y_{N,t}) - \mathbb{E}[H_1(y_{1,t} - D_t - D_{t+1})] \\ &\quad - \sum_{n=2}^{N-1} h_n (y_{n,t} - v_{n-1,t})^+ - h_N \mathbb{E}[v_{N,t} \wedge (y_{N,t} + K_t) - v_{N-1,t} \wedge (y_{N,t} + K_t)] \\ &\quad - h_{W,N}\mathbb{E}[v_{N-1,t} \wedge (y_{N,t} + K_t) - y_{N-1,t}] - \sum_{n=2}^{N-1} h_{W,n}(v_{n-1,t} \wedge y_{n,t} - y_{n-1,t}) \\ &\quad + \mathbb{E}[V_{t+1}(\mathbf{Y}_{t+1})], \end{aligned}$$

where

$$V_t(\mathbf{y}_t) = \max\{\hat{J}_t((\mathbf{y}_t, \mathbf{v}_t)) : \mathbf{v}_t \geq \mathbf{y}_t, v_{1,t} \leq v_{2,t} \leq \cdots \leq v_{N,t}\}.$$

We assume that the terminal value $V_{T+1}(\mathbf{y}_{T+1})$ is concave and submodular.

One can use the same stochastic linearity transformation as the one in the two-location model to make the post-order inventory position at locations N and $N-1$ stochastically linear in midpoint. It is easy to see that Lemma 2 applies to the N-dimensional random functions $\mathbf{Y}_{t+1}$ and thus the joint concavity of $\mathbb{E}[V_{t+1}(\mathbf{Y}_{t+1})]$ in $(\mathbf{y}_t, \mathbf{v}_t)$ can be established. Thus, the analysis in §2.4 can be extended to treat the problem with multiple locations.

We can further show that when the terminal value is separable in the ending inventory positions, the value function V_t becomes separable. Consequently, an echelon base-stock policy becomes optimal in this case.

2.7 Summary

In this chapter, we analyze a two-location system in which the upstream production facility experiences random capacities and the downstream store faces random demands. Instead of decomposing the profit function widely used to treat multi-echelon systems, our approach builds on the notions of stochastic functions, in particular, the stochastic linearity in midpoint and the directional concavity in midpoint, which establishes the concavity and submodularity of the profit functions.

In general, it is optimal to follow a two-level state-dependent threshold policy such that an order is issued at a location if and only if the inventory position of that location is below the corresponding threshold. When the salvage values of the ending inventories are linear, the profit function becomes decomposable in the inventory positions at different locations and the optimal threshold policy reduces to the echelon base-stock policy. The optimal policy will serve as our coordination benchmark in the next chapter.

3. DECENTRALIZED DYNAMIC COORDINATION WITH SUPPLY UNCERTAINTY

3.1 Synopsis

In Chapter 2, we consider a two-stage supply chain problem and identify the optimal management policy of the material flows from a centralized perspective. In this chapter, we consider each stage makes the ordering decision based on self interests. The goal of our analysis in this chapter is to identify an approach to coordinate the dynamic decisions between the two locations under limited information visibility. The optimal policy we derive in Chapter 2 serves as our coordination benchmark.

In studies about the decentralized decision making in supply chain management, most of the literature assumes single-period setting. In the dynamic setting, the coordination becomes much more challenging to achieve and most simple contracts which can achieve coordination in the single-period setting fail to do so. We consider two dynamic settings in this chapter. In the first setting, the retail store does not know the distribution of the stochastic supply function and the inventory level at the manufacturing facility; in the second, the manufacturing facility does not know the demand distribution while the retail store does not have supply information.

In the first scenario, we show that the retail store and manufacturing facility can be dynamically coordinated with proper transfer contracts defined on the local inventories and their own value function in the dynamic recursion. In the second scenario, we identify the optimal transfer contracts to achieve the dynamic coordination and propose an iterative algorithm to compute the contracts in the decentralized setting.

The remainder of this chapter is organized as follows. In Section 3.2, we review the related literature. In Section 3.3, we study the first scenario problem and characterize the transfer contracts to achieve coordination. In Section 3.4, we study the second scenario problem and analyze the optimal contracts under which coordination can be achieved. In Section 3.5, we develop an iterative algorithm to compute the optimal transfer contracts and show the convergence of the algorithm. In Section 3.6, we analyze the convergence rate under different numerical settings and the profit splits under different fixed payments. In Section 3.7, we conclude this chapter.

3.2 Literature Review

Our study is related to three streams of literature. The first stream is multi-echelon inventory management, which we have reviewed in Section 2.2. We review the other two streams of literature in this section.

Supply chain coordination. A rich body of research in the filed of operations management has studied the coordination issue in supply chain. [38, 39] provide excellent surveys. Most of the literature focuses on designing different contracts to achieve coordination in a single-period or infinite-horizon setting. For example, [40] study the simple linear transfer payments in a Nash equilibrium framework; [41] consider the setting of single supplier and multiple retailers and achieve coordination in a wholesale pricing contract; [42] study the revenue-sharing contracts; [43] study the coordination between the production and marketing division in a firm. The paper by [44] considers an assembly system with one manufacturer and two suppliers.

Few papers consider the supply chain coordination under supply uncertainty. When modeling supply uncertainty, most papers assume a random yield model. [45] consider designing contracts to achieve coordination in a single-period setting with a single supplier whose production process follows a random yield model. [46] study the coordination in a specific industry: influenza vaccination, in which the vaccination production follows a random yield model. [47] and [48] study the coordination of assembly system in a single-period setting with random component yields. Comparing to these papers, we consider random capacity to model the supply uncertainty in a multi-period setting.

There are relatively few papers considering coordination under a multi-period setting, in which coordination becomes more challenging. [49] study the optimal policy of the vendor-managed inventory model under a contract called (z, Z) in practice. Under the (z, Z) contract, the supplier pays a penalty if the retailer's inventory level is less than z or more than Z. [50] consider a two-stage serial supply chain with deterministic capacity limit at each stage in a Markov equilibrium framework, and the equilibrium solution is a modified echelon base-stock policy. A key difference between our model and theirs is that we consider a random supply capacity. The main focus of the papers by [49] and [50] is to characterize the policy under their decentralized settings, and our aim of analysis is to identify an approach to achieve coordination.

It is worth mentioning that some literature in this stream adopt principal-agent framework to induce truth revelation. [5] study multi-period principal-agent problem, where the state of the system can only be observed by the agent. [51] analyze a supply chain with one retailer and one supplier, and the supplier can not observe the retailer's inventory level. [52] consider the situation when the supplier's unit production cost is unknown to the retailer, and characterize the optimal contract. In our model, in an essential contrast, the private knowledge is a function (i.e., the distribution). Applying the principal-agent approach would require the principal to specify the set of possible distributions and a distribution over this set. Such a knowledge structure is very complex and difficult to specify in practice. Instead, we first identify the optimal contracts to achieve coordination and then propose an iterative algorithm to compute the contracts under the decentralized setting.

Dynamic decentralized control in resource allocation. Our problem is also related to the dynamic resource allocation among different decentralized agents, which can be dated back to the papers by [53], [54] in the economics literature. We limit our attention to the most relevant papers here. [55] study the allocation of shared resources between multiple agents. Each agent receives requests for the shared resources. They design a transfer price contract so that the overall decentralized system profit is the same as the centralized optimal profit. [56] consider a similar problem in a queueing system between a pricing agent and a service agent. The pricing agent makes the dynamic pricing decision to control the arrival rate, and the service agent controls the service rate of the system. They also characterize the transfer contracts under which the decentralized system total profit is maximized. In our second scenario problem, we use the similar idea to design the transfer contracts to achieve coordination. However, our model is more complicated in that multiple agents solve their own dynamic programming sequentially, and the profit and decision of one agent are based on the optimal policy of the other agent, which makes it more challenging to characterize the optimal transfer contracts.

3.3 Coordination When the Supply Information is Unknown

3.3.1 Problem Formulation

We consider a two-stage supply chain planning problem over T-period. The upstream division is a manufacturing facility (he) and the downstream division is a retail store (she). The dynamics of the system is depicted in Figure 3.1.

Retail store's problem. At the beginning of each period t, the inventory level $x_{R,t}$ at the store is reviewed. An order $q_{R,t}$ is issued to the manufacturing facility. We denote $y_{R,t}$ as the order-up-to level, i.e., $y_{R,t} = x_{R,t} + q_{R,t}$. The retail store faces an exogenous demand D_t with distribution $F_{D_t}(\cdot)$ during period t. The demand is satisfied through the on-hand inventory $x_{R,t}$. Any unmet demand is fully backordered and leftover inventory is carried over to the next period. The delivered order from the manufacturing facility arrives at the end of period t. We assume **the retail store does not know the manufacturing facility's inventory level and production plan**, and believes that the inventory level after delivery would be a stochastic function $\tilde{S}(x_{R,t}, y_{R,t})$ given her initial inventory is $x_{R,t}$ and order-up-to level is $y_{R,t}$. The distribution of $\tilde{S}(x_{R,t}, y_{R,t})$ is based on the retail store's belief and might be erroneous.

We do not impose any assumptions on the supply function $\tilde{S}(x_{R,t}, y_{R,t})$, and we later show that our results hold for any form of $\tilde{S}(x_{R,t}, y_{R,t})$.

The inventory at the beginning of period $t + 1$ updates to

$$\tilde{X}_{R,t+1} = \tilde{S}(x_{R,t}, y_{R,t}) - D_t. \tag{3.1}$$

The retail price for the product is p dollars per unit, and the store incurs a holding/backlogging cost $H_R(\cdot)$. We assume that $H_R(x) : \mathbb{R} \to \mathbb{R}^+$ is continuous and convex, with $H_R(0) = 0$, $\lim_{x\to\pm\infty} H_R(x) = \infty$. Retail store pays the manufacturing facility a fixed amount b dollars for each delivery. Retail store also incurs a penalty $r_t(y_{R,t}, s)$ to the manufacturing facility if the order-up-to level is $y_{R,t}$ and the actual inventory position after receiving the delivery from the manufacturing facility is s.

The one-period retail profit is given by

$$\tilde{\pi}_{R,t}(x_{R,t}, y_{R,t}) = p\mathbb{E}[D_t] - \mathbb{E}[H_R(x_{R,t} - D_t)] + \mathbb{E}[r_t(y_{R,t}, \tilde{S}(x_{R,t}, y_{R,t}))] - b. \tag{3.2}$$

Let $\tilde{V}_{R,t}(x_{R,t})$ denote the retail store's optimal expected profit function in period t when the inventory level is $x_{R,t}$. Then the optimality equation can be written as

$$\tilde{V}_{R,t}(x_{R,t}) = \max\{\tilde{J}_{R,t}(x_{R,t}, y_{R,t}) : y_{R,t} \geq x_{R,t}\}, \tag{3.3}$$

where

$$\tilde{J}_{R,t}(x_{R,t}, y_{R,t}) = \tilde{\pi}_{R,t}(x_{R,t}, y_{R,t}) + \mathbb{E}[\tilde{V}_{R,t+1}(\tilde{X}_{R,t+1})]. \tag{3.4}$$

We assume a concave terminal value function $R^1_{T+1}(x_{R,T+1})$. In other words, the marginal terminal value of inventory at the retail store is subject to diminishing returns.

Manufacturing facility's problem. At the beginning of each period t, the order quantity $q_{R,t}$ from the retail store is received, and the inventory level $x_{M,t}$ is reviewed. We assume the manufacturing facility can also observe the inventory level at the retail store. The decision is an input quantity $q_{M,t}$ for production. Because the production capacity K_t is random with distribution $F_{K_t}(\cdot)$, the output from production at the end of the period is the minimum of the input and the realized capacity, i.e., $q_{M,t} \wedge K_t$ (where $a \wedge b = \min\{a, b\}$). Upon production completion, the stock level at the facility becomes $x_{M,t} + q_{M,t} \wedge K_t$ and a shipment is sent to the retail store. The shipment quantity is the minimum of the order quantity and the available stock, i.e., $q_{R,t} \wedge (x_{M,t} + q_{M,t} \wedge K_t)$. We denote $y_{S,t}$ as the production-up-to level of the system total inventory, i.e., $y_{S,t} = x_{M,t} + x_{R,t} + q_{M,t}$. The inventory at the beginning of period $t+1$ updates to

$$X_{M,t+1} \;=\; (y_{S,t} \wedge (x_{M,t} + x_{R,t} + K_t) - y_{R,t})^+, \tag{3.5}$$

where $a^+ = \max\{a, 0\}$. Since all the information is available to the manufacturing facility, he also knows the retail store's inventory at the beginning of period $t+1$ updates to

$$X_{R,t+1} \;=\; y_{R,t} \wedge y_{S,t} \wedge (x_{M,t} + x_{R,t} + K_t) - D_t. \tag{3.6}$$

It turns out that working with the system total inventory level $x_{S,t} = x_{M,t} + x_{R,t}$ allows for analytical convenience. The dynamics of the system inventory level is

$$X_{S,t+1} = y_{S,t} \wedge (x_{S,t} + K_t) - D_t. \tag{3.7}$$

The manufacturing cost is c dollars per unit. Manufacturing facility also incurs a holding cost of h_S dollars per unit for all the inventory in the system. The one-period profit is given by

$$\begin{aligned}\tilde{\pi}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t}) \;&=\; b - c\mathbb{E}[(y_{S,t} - x_{S,t}) \wedge K_t] - h_S x_{S,t} \\ &\quad -\mathbb{E}[r_t(y_{R,t}, y_{R,t} \wedge y_{S,t} \wedge (x_{S,t} + K_t))]. \end{aligned} \tag{3.8}$$

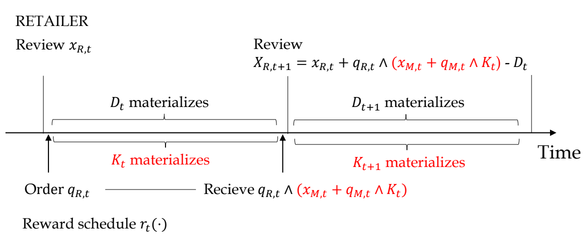
Let $\tilde{V}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t})$ denote the optimal expected profit function in period t when the retail inventory level is $x_{R,t}$, and the system inventory level is $x_{S,t}$, and the retail order-up-to level is $y_{R,t}$. Then the optimality equation can be written as

$$\tilde{V}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t}) = \max\{\tilde{J}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t}) : y_{S,t} \geq x_{S,t}\}, \tag{3.9}$$

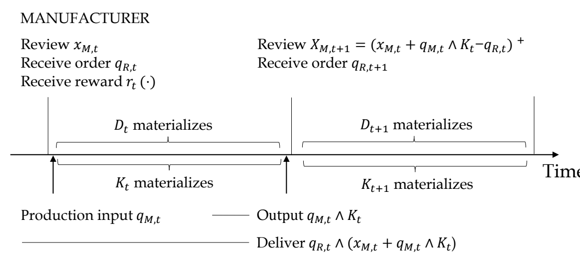
where

$$\begin{aligned}&\tilde{J}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t})\\ =\ &\tilde{\pi}_{M,t}(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t}) + \mathbb{E}[\tilde{V}_{M,t+1}(X_{R,t+1}, X_{S,t+1}, y_{R,t+1}(X_{R,t+1})].\end{aligned} \quad (3.10)$$

We assume a concave terminal value function $R^2_{T+1}(x_{S,T+1})$ at period $T+1$. In our analysis, we use a superscript $*$ to denote the quantities under an optimal solution. For ease of exposition, we may drop the time index from the subscript when it does not cause confusions.



(a) Retail store sequence of events.



(b) Manufacturing sequence of events.

Fig. 3.1.: Sequence of events when supply information is unknown.

3.3.2 Analysis of the Optimal Transfer Contracts

In this subsection, we show that the decentralized system described in Section 3.3.1 can be coordinated under our optimal transfer contracts.

Preliminary Results

Before characterizing the coordination mechanism in this section, we first briefly review the centralized problem; see Chapter 2 for more details. The one-period system profit is given by

$$\begin{aligned}&\pi_t(x_{R,t}, x_{S,t}, y_{S,t})\\ =\ &p\mathbb{E}[D_t] - c\mathbb{E}[(y_{S,t} - x_{S,t}) \wedge K_t] - \mathbb{E}[H_R(x_{R,t} - D_t)] - h_S x_{S,t}.\end{aligned} \quad (3.11)$$

Let $V_t(x_{R,t}, x_{S,t})$ denote the centralized optimal expected profit function in period t when the inventory level at the retail store is $x_{R,t}$ and the system inventory level is $x_{S,t}$. Then the optimality equation can be written as

$$V_t(x_{R,t}, x_{S,t}) = \max\{J_t(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t}) : y_{R,t} \geq x_{R,t}, y_{S,t} \geq x_{S,t}\}, \tag{3.12}$$

where

$$J_t(x_{R,t}, x_{S,t}, y_{R,t}, y_{S,t}) = \pi_t(x_{R,t}, x_{S,t}, y_{S,t}) + \mathbb{E}[V_{t+1}(X_{R,t+1}, X_{S,t+1})]. \tag{3.13}$$

We assume a terminal value function $V_{T+1}(x_{R,t+1}, x_{S,t+1}) = R^1_{T+1}(x_{R,t+1}) + R^2_{T+1}(x_{S,t+1})$ and both $R^1_{T+1}(\cdot)$ and $R^2_{T+1}(\cdot)$ are concave. An important structural property of the centralized problem which we derive in Chapter 2 is that the centralized optimal value function is concave and separable in the echelon inventory level and the optimal policy is a base-stock policy provided that the terminal value function is concave and separable. We state this property in the next proposition, and it lays a foundation for the design of our decentralized coordination mechanism.

Proposition 4 *The value function V_t is concave and separable in (x_R, x_S) for each t, i.e., $V_t(x_R, x_S) = V^1_t(x_R) + V^2_t(x_S)$, and*

$$\begin{aligned} V^1_t(x_R) &= \max\{J^1_t(x_R, y_R) : y_R \geq x_R\}, & (3.14)\\ V^2_t(x_S) &= \max\{J^2_t(x_S, y_S; \tilde{y}_R) : y_S \geq x_S\}, & (3.15) \end{aligned}$$

where

$$\begin{aligned} J^1_t(x_R, y_R) &= p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[V^1_{t+1}(y_R - D)], & (3.16)\\ J^2_t(x_S, y_S; \tilde{y}_R) &= -c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S + \mathbb{E}[V^1_{t+1}(\tilde{y}_R \wedge y_S \wedge (x_S + K) - D)] \\ &\quad -\mathbb{E}[V^1_{t+1}(\tilde{y}_R - D)] + \mathbb{E}[V^2_{t+1}(y_S \wedge (x_S + K) - D)], & (3.17) \end{aligned}$$

and $\tilde{y}_R = \arg\max\{\mathbb{E}[V^1_{t+1}(y - D)] : y \in \mathbf{R}\}$.

Proof. From Corollary 1 in Chapter 2, we know that the optimal production order-up-to level $y^*_S = \tilde{y}_S \vee x_S$, where $\tilde{y}_S$ is a constant in each period t. The optimal replenishment order-up-to level $y_R^* = \tilde{y}_R \vee x_R \wedge y_S^*$. We can rewrite the optimal profit function $V_t(x_R, x_S)$ as

$$\begin{aligned}
& V_t(x_R, x_S) \\
= \;& p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[V_{t+1}^1(\tilde{y}_R \vee x_R \wedge (x_S + K) - D)] \\
& + \mathbb{E}[V_{t+1}^1(y_S^* \wedge (x_S + K) - D) - V_{t+1}^1(\tilde{y}_R \wedge (x_S + K) - D)]\mathbb{I}_{\{y_S^* < \tilde{y}_R\}} \\
& - c\mathbb{E}[(y_S^* - x_S) \wedge K] - h_S x_S + \mathbb{E}[V_{t+1}^2(y_S^* \wedge (x_S + K) - D)] \\
= \;& p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[V_{t+1}^1(\tilde{y}_R \vee x_R - D)] \\
& + \mathbb{E}[V_{t+1}^1(\tilde{y}_R \wedge y_S \wedge (x_S + K) - D) - V_{t+1}^1(\tilde{y}_R - D)] \\
& - c\mathbb{E}[(y_S^* - x_S) \wedge K] - h_S x_S + \mathbb{E}[V_{t+1}^2(y_S^* \wedge (x_S + K) - D)],
\end{aligned}$$

from which we can easily derive V_t^1 and V_t^2. □

Optimal Transfer Contracts

We first give the conjecture about the optimal transfer contract and the optimality of the contract will be established afterwards. The optimal transfer contract satisfies

$$r_t(y_R, s) = \mathbb{E}[\tilde{V}_{R,t+1}(y_R - D) - \tilde{V}_{R,t+1}(s - D)]. \tag{3.18}$$

This contract defines the cash transfer between the retail store and the manufacturing facility. From the retail store's perspective, the transfer contract $r_t(y_R, s)$ is the amount to compensate for the loss of future value caused by insufficient supply from the manufacturing facility. In next theorem, we establish the optimality of the contract $r_t(y_R, s)$.

Theorem 3 *Let $\tilde{V}_{R,t}(x_R)$ and $\tilde{V}_{M,t}(x_R, x_S, y_R)$ be the value functions defined in (3.3) and (3.9) under the transfer contract (3.18). Then,*

$$\tilde{V}_{R,t}(x_R) + \tilde{V}_{M,t}(x_R, x_S, y_R^*(x_R)) = V_t(x_R, x_S),$$

where $y_R^*(x_R)$ is the retail optimal order-up-to level, and $V_t(x_R, x_S)$ is the centralized optimal value function defined in (3.12).

Proof of Theorem 3. We prove the result using induction. By assumption, the result holds for period $T+1$. Assume $V_{R,t+1}(x_R) = V^1_{t+1}(x_R)$ and $V_{M,t+1}(x_R, x_S, y^*_{R,t+1}(x_R))$ $= V^2_{t+1}(x_S)$, then

$$V_{R,t+1}(x_R) + V_{M,t+1}(x_R, x_S, y^*_{R,t+1}(x_R)) = V_{t+1}(x_R, x_S).$$

Under the contract (3.18), we can rewrite the retail objective function $J_{R,t}$ defined in (3.23) as

$$J_{R,t}(x_R, y_R) = p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[V_{R,t+1}(y_R - D)]. \tag{3.19}$$

It is straightforward to observe that (3.19) and the centralized objective function J^1_t defined in (3.16) are the same, provided that $V_{R,t+1} = V^1_{t+1}$, therefore $V_{R,t} = V^1_t$. Let $\tilde{y}_R = \arg\max\{\mathbb{E}[V_{R,t+1}(y - D)] : y \in \mathbf{R}\}$, then $y^*_R(x_R) = \tilde{y}_R \vee x_R$. The objective function of the manufacturing facility under the contract (3.18) can be written as

$$\begin{aligned}
& J_{M,t}(x_R, x_S, y^*_R(x_R), y_S) \\
= \quad & -c\mathbb{E}[(y_S - x_S) \wedge K_t] - h_S x_S + \mathbb{E}[V_{R,t+1}(y^*_R(x_R) \wedge y_S \wedge (x_S + K) - D)] \\
& -\mathbb{E}[V_{R,t+1}(y^*_R(x_R) - D)] + \mathbb{E}[V_{M,t+1}(X_{R,t+1}, X_{S,t+1}, y^*_{R,t+1}(X_{R,t+1}))].
\end{aligned}$$

Note that the only difference between $J_{M,t}(x_R, x_S, y^*_R(x_R), y_S)$ and $J^2_t(x_S, y_S; \tilde{y}_R)$ in (3.17) is that $y^*_R(x_R) = \tilde{y}_R \vee x_R$ is replaced by $\tilde{y}_R$ in (3.17). Thus we have $J_{M,t}(x_R, x_S,$ $y^*_R(x_R), y_S) = J^2_t(x_S, y_S; \tilde{y}_R)$ when $\tilde{y}_R \geq x_R$, and when $\tilde{y}_R < x_R$, $J_{M,t}(x_R, x_S, y^*_R(x_R), y_S)$ and $J^2_t(x_S, y_S; \tilde{y}_R)$ are still the same by observing the relation that $y_S \geq x_S \geq x_R$. Therefore $V_{R,t}(x_R) + V_{M,t}(x_R, x_S; y^*_R(x_R)) = V_t(x_R, x_S)$. This completes the induction argument. □

3.4 Coordination When No Party has Full Information

In this section, we consider the case in which the demand distribution is unknown to the manufacturing facility as well as the retail store does not know the capacity distribution and the inventory level at the manufacturing facility.

3.4.1 Problem Formulation

In our notation, we use a superscript $\sim$ to denote the quantities from the erroneous belief. For example, we denote the random capacity from the retail store's belief as $\tilde{K}_t$ and assume that $\tilde{K}_t$ follows a distribution $F_{\tilde{K}_t}(\cdot)$, which might not be the same as the true capacity distribution $F_{K_t}(\cdot)$. The demand from the manufacturing facility's belief is denoted as $\tilde{D}_t$, and it is assumed to follow a distribution $F_{\tilde{D}_t}$. Beyond the costs and revenues described in Section 3.3.1, we need to define two more transfer payments to achieve coordination in this setting. Suppose the initial inventory level at the beginning of period t is (x_R, x_S).

- $w_t(x_R, x_S, i)$ denotes the reward schedule from the manufacturing facility to retail store if the total inventory in the system at the end of period t (after demand realization) is i.

- $u_t(x_R, x_S, y)$ is the reward schedule from the retail store to manufacturing facility if the total inventory in the system after production (before demand realization) is y.

Retail store's problem. From the retail store's perspective, the inventory dynamics are

$$\tilde{X}_{R,t+1} = \tilde{S}(x_{R,t}, y_{R,t}) - D_t, \tag{3.20}$$

$$\tilde{X}_{S,t+1} = \tilde{I}(x_{R,t}, y_{R,t}) - D_t, \tag{3.21}$$

where $\tilde{S}(\cdot)$ and $\tilde{I}(\cdot)$ are based on the retail store's belief and might be different from the true dynamics. Here we assume that the retail store only observes the inventory level at her site.

Let $V_{R,t}(x_{R,t})$ denote the optimal expected profit function in period t when the inventory level is $x_{R,t}$. Then the optimality equation can be written as

$$V_{R,t}(x_{R,t}) = \max\{\hat{J}_{R,t}(x_{R,t}, y_{R,t}) : y_{R,t} \geq x_{R,t}\}, \tag{3.22}$$

where

$$
\begin{aligned}
& \hat{J}_{R,t}(x_{R,t}, y_{R,t}) \\
= \;& p\mathbb{E}[D_t] - \mathbb{E}[H_R(x_{R,t} - D_t)] + \mathbb{E}[r_t(x_{R,t}, y_{R,t}, \tilde{S}(x_{R,t}, y_{R,t}))] \\
& + \mathbb{E}[w_t(x_R, \tilde{X}_S, \tilde{I}(x_{R,t}, y_{R,t}) - D_t)] - \mathbb{E}[u_t(x_R, \tilde{X}_S, \tilde{I}(x_{R,t}, y_{R,t}))] \\
& + \mathbb{E}[V_{R,t+1}(\tilde{X}_{R,t+1})] - b. \qquad (3.23)
\end{aligned}
$$

The terminal value function $R^1_{T+1}(x_{R,T+1})$ is the same as in Section 3.3.1.

Manufacturing facility's problem. From the manufacturing facility's perspective, the inventory dynamics are

$$
\begin{aligned}
\hat{X}_{R,t+1} &= y_{R,t}(x_{R,t}) \wedge y_{S,t} \wedge (x_{S,t} + K_t) - \tilde{D}_t, \qquad (3.24) \\
\hat{X}_{S,t+1} &= y_{S,t} \wedge (x_{S,t} + K_t) - \tilde{D}_t, \qquad (3.25)
\end{aligned}
$$

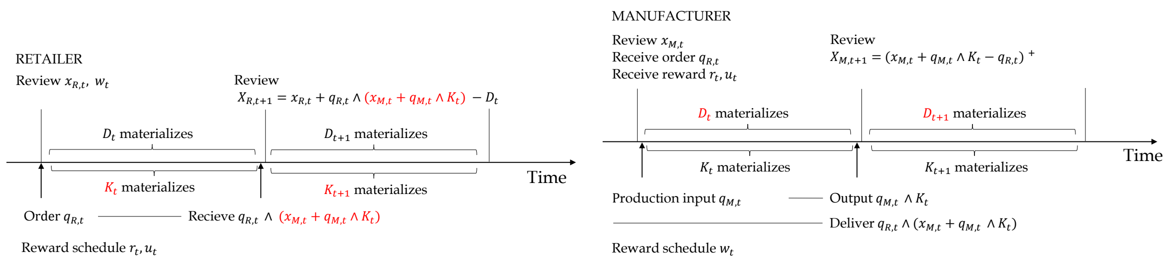
where $y_{R,t}(x_{R,t})$ is the retail order-up-to level. Let $V_{M,t}(x_{R,t}, x_{S,t})$ denote the optimal expected profit function in period t when the inventory level at the retail store is $x_{R,t}$ and system inventory level is $x_{S,t}$. The optimality equation can be written as

$$
V_{M,t}(x_{R,t}, x_{S,t}) = \max\{\hat{J}_{M,t}(x_{R,t}, x_{S,t}, y_{S,t}) : y_{S,t} \geq x_{S,t}\}, \qquad (3.26)
$$

where

$$
\begin{aligned}
& \hat{J}_{M,t}(x_{R,t}, x_{S,t}, y_{S,t}) \\
= \;& -c\mathbb{E}[(y_{S,t} - x_{S,t}) \wedge K_t] - h_S x_{S,t} - \mathbb{E}[r_t(y_{R,t}(x_{R,t}), y_{R,t}(x_{R,t}) \wedge y_{S,t} \wedge (x_{S,t} + K_t))] \\
& - \mathbb{E}[w_t(x_R, x_S, y_{S,t} \wedge (x_{S,t} + K_t) - \tilde{D}_t)] + \mathbb{E}[u_t(x_R, x_S, y_{S,t} \wedge (x_{S,t} + K_t))] \\
& + \mathbb{E}[V_{M,t+1}(\hat{X}_{R,t+1}, \hat{X}_{S,t+1})] + b. \qquad (3.27)
\end{aligned}
$$

The terminal value function $R^2_{T+1}(x_{S,T+1})$ is assumed to be the same as in Section 3.3.1.



(a) Retail store sequence of events. (b) Manufacturing sequence of events.

Fig. 3.2.: Sequence of events when both distributions are unknown.

3.4.2 Analysis of the Optimal Contracts

We first give the conjecture about the optimal transfer contracts, which should satisfy

$$r_t(y_R, s) = \mathbb{E}[V_{R,t+1}(y_R - D) - V_{R,t+1}(s - D)], \tag{3.28}$$

$$w_t(x_R, x_S, i) = \mathbb{E}[V_{M,t+1}(\hat{X}_{R,t+1}(x_R, x_S), i)], \tag{3.29}$$

$$u_t(x_R, x_S, y) = \mathbb{E}[w_t(x_R, x_S, y - D)]. \tag{3.30}$$

Existence

In this section, we want to check whether a set of optimal contracts satisfying conditions (3.28) - (3.30) exists. In Section 3.3.2, we show that the centralized optimal value function $V_t(x_R, x_S)$ is separable in each state, i.e., $V_t(x_R, x_S) = V_t^1(x_R) + V_t^2(x_S)$; see Equation (3.14) and (3.15) for details. If the following dynamic programming equations are defined, for $t = 1, \ldots, T$,

$$\begin{aligned} \hat{V}_{R,t}(x_R) &= \max\{p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[V_{t+1}^1(y_R - D)] : \\ &\quad y_R \geq x_R\}. \end{aligned} \tag{3.31}$$

$$\begin{aligned} \hat{V}_{M,t}(x_R, x_S) &= \max\{-c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S - \mathbb{E}[V_{t+1}^1(y_R^*(x_R) - D)] \\ &\quad + \mathbb{E}[V_{t+1}^1(y_R^*(x_R) \wedge y_S \wedge (x_S + K) - D) \\ &\quad + V_{t+1}^2(y_S \wedge (x_S + K) - D)] : y_S \geq x_S\}, \end{aligned} \tag{3.32}$$

where $y_R^*(x_R)$ is the optimal replenishment order policy solved from (3.31). It is straightforward to see that these are the same dynamic programming equations as (3.14) and (3.15), therefore

$$\begin{aligned}\hat{V}_{R,t}(x_R) &= V_t^1(x_R),\\ \hat{V}_{M,t}(x_R, x_S) &= V_t^2(x_S).\end{aligned}$$

Suppose the transfer contracts (3.28) - (3.30) are defined as

$$r_t(y_R, s) = \mathbb{E}[\hat{V}_{R,t+1}(y_R - D) - \hat{V}_{R,t+1}(s - D)], \tag{3.33}$$

$$w_t(x_R, x_S, i) = \mathbb{E}[\hat{V}_{M,t+1}(\hat{X}_{R,t+1}(x_R, x_S), i)], \tag{3.34}$$

$$u_t(x_R, x_S, y) = \mathbb{E}[w_t(x_R, x_S, y - D)]. \tag{3.35}$$

In next proposition, we show that the solutions of (3.31) and (3.32) are the value functions for the decentralized problem (3.22) - (3.27) under the contracts (3.33) - (3.35).

Proposition 5 *Let the value functions $V_{R,t}(x_R)$ and $V_{M,t}(x_R, x_S)$ be defined by the decentralized problem (3.22) - (3.27) under the contracts (3.33) - (3.35), then*

$$\begin{aligned}V_{R,t}(x_R) &= \hat{V}_{R,t}(x_R),\\ V_{M,t}(x_R, x_S) &= \hat{V}_{M,t}(x_R, x_S),\end{aligned}$$

where $\hat{V}_{R,t}(x_R)$ and $\hat{V}_{M,t}(x_R, x_S)$ is the solution of (3.31) and (3.32). In addition, the decentralized total channel profit $V_{R,t}(x_R) + V_{M,t}(x_R, x_S)$ equals to the centralized optimal profit $V_t(x_R, x_S)$.

Proof of Proposition 5. We prove the result using induction. Assume $V_{R,t+1}(x_R) = \hat{V}_{R,t+1}(x_R) = V^1_{t+1}(x_R)$ and $V_{M,t+1}(x_R, x_S) = \hat{V}_{M,t+1}(x_R, x_S) = V^2_{t+1}(x_S)$, which are true for $t = T$. $\hat{J}_{R,t}(x_R, y_R)$ defined in (3.23) can be rewritten as

$$
\begin{aligned}
& \hat{J}_{R,t}(x_R, y_R) \\
= \; & p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[\hat{V}_{R,t+1}(y_R - D) - \hat{V}_{R,t+1}(\tilde{S}(x_R, y_R) - D)] \\
& +\mathbb{E}[\hat{V}_{M,t+1}(\hat{X}_{R,t+1}(x_R, X_S), \tilde{I}(x_R, y_R) - D)] \\
& -\mathbb{E}[\hat{V}_{M,t+1}(\hat{X}_{R,t+1}(x_R, X_S), \tilde{I}(x_R, y_R) - D)] + \mathbb{E}[V_{R,t+1}(\tilde{S}(x_R, y_R) - D] \\
= \; & p\mathbb{E}[D] - \mathbb{E}[H_R(x_R - D)] + \mathbb{E}[\hat{V}_{R,t+1}(y_R - D)],
\end{aligned} \tag{3.36}
$$

which implies $V_{R,t}(x_R) = \hat{V}_{R,t}(x_R) = V^1_t(x_R)$. $\hat{J}_{M,t}(x_R, x_S, y_S)$ defined in (3.27) can be rewritten as

$$
\begin{aligned}
& \hat{J}_{M,t}(x_R, x_S, y_S) \\
= \; & -c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S - \mathbb{E}[\hat{V}_{R,t+1}(y_R(x_R) - D)] \\
& +\mathbb{E}[\hat{V}_{R,t+1}(y_R(x_R) \wedge y_S \wedge (x_S + K) - D)] - \mathbb{E}[\hat{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - \tilde{D})] \\
& +\mathbb{E}[\hat{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - D)] + \mathbb{E}[V_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - \tilde{D})] \\
= \; & -c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S - \mathbb{E}[\hat{V}_{R,t+1}(y_R(x_R) - D)] \\
& +\mathbb{E}[\hat{V}_{R,t+1}(y_R(x_R) \wedge y_S \wedge (x_S + K) - D)] + \mathbb{E}[V^2_{t+1}(y_S \wedge (x_S + K) - D)],
\end{aligned}
$$

where $y_R(x_R)$ is the retail optimal ordering policy solved from (3.36). It is easy to see that $y_R(x_R)$ is the same as the centralized ordering policy. Comparing $\hat{J}_{M,t}(x_R, x_S, y_S)$ with (3.32), we can immediately conclude that $V_{M,t}(x_R, x_S) = \hat{V}_{M,t}(x_R, x_S) = V^2_t(x_S)$. This completes the induction argument. □

According to Proposition 5, the decentralized total channel profit under the transfer contracts (3.33) - (3.35) is the same as the optimal centralized total profit, and this result is insensitive to the mis-specification of each site's unknown information. However, to compute the optimal transfer contracts, we must know the centralized decomposed value functions V^1_t and V^2_t, which do not exist in the decentralized problem. To tackle this challenge, we propose an iterative approach in next section.

3.5 Iterative Algorithm

In this section, we propose an iterative algorithm to compute the optimal transfer contracts defined in Section 3.4 in the decentralized setting. We further demonstrate that the decentralized channel profit is guaranteed to converge to the optimal centralized profit through our iterative algorithm. Figure 3.3 provides an illustration for our iterative algorithm.

3.5.1 Algorithm Description

Algorithm 1

Initialize: Set $n = 0$, $V^0_{R,t}(x_R) = 0$, $V^0_{M,t}(x_R, x_S) = 0$, $r^0_t(y_R, s) = 0$, $w^0_t(x_R, x_S, i) = 0$ *and* $u^0_t(x_R, x_S, y) = 0$.

Iteration n: Given transfer contracts $w^n_t(x_R, x_S, i)$ *and* $u^n_t(x_R, x_S, y)$.

- *Retail store sets* $r^n_t(y, s) = \mathbb{E}[V^n_{R,t+1}(y-D) - V^n_{R,t+1}(s-D)]$, *and solves her decentralized problem (3.22)-(3.23) based on* w^n_t *and* u^n_t. *The optimal value function* $V^n_{R,t}(x_R)$ *is computed, and the optimal order policy* $y^n_{R,t} = \arg\max\{\hat{J}^n_{R,t}(x_R, y_R) : y_R \geq x_R\}$ *is specified and communicated to the manufacturing facility. The transfer contract* $r^n_t(y, s) = \mathbb{E}[V^n_{R,t+1}(y - D) - V^n_{R,t+1}(s - D)]$ *is also passed to the manufacturing facility.*

- *Manufacturing facility takes the retail order* $y^n_{R,t}$, *and solves his decentralized problem (3.26)-(3.27) based on transfer contracts* r^n_t, u^n_t *and* w^n_t. *The optimal value function* $V^n_{M,t}(x_R, x_S)$ *is computed, and the optimal production policy* $y^n_{S,t} = \arg\max\{\hat{J}^n_{M,t}(x_R, x_S, y_S) : y_S \geq x_S\}$ *is specified. His transfer contract is updated as*

$$w^{n+1}_t(x_R, x_S, i) = \mathbb{E}[V^n_{M,t+1}(\hat{X}_{R,t+1}(x_R, x_S), i)].$$

- *Stop if a satisfactory level of precision* $\epsilon > 0$ *is reached, e.g.*

$$\sup_{t,x_R,x_S} |V^n_{R,t}(x_R) - V^{n-1}_{R,t}(x_R)| + |V^n_{M,t}(x_R, x_S) - V^{n-1}_{M,t}(x_R, x_S)| < \epsilon.$$

Otherwise, manufacturing facility communicates his updated transfer contract $w_t^{n+1}(x_R, x_S, i)$ *to retail store. Retail store communicates her updated transfer contract, i.e.,*

$$u_t^{n+1}(x_R, x_S, y) = \mathbb{E}[w_t^{n+1}(x_R, x_S, y - D)],$$

to manufacturing facility. Set n *to* $n+1$ *and repeat.*

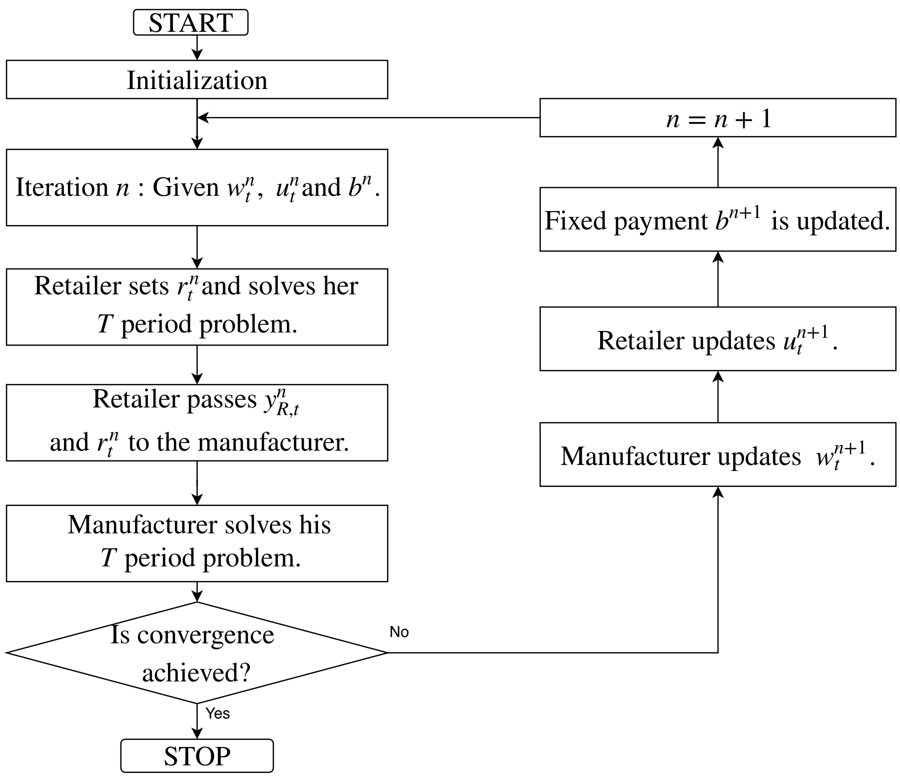


Fig. 3.3.: Illustration of iterative algorithm.

3.5.2 Convergence

In this section, we demonstrate that the iterated value function will converge to the centralized value function. We first give two preliminary results. These two results are proved in [56] and [57] respectively. The first is

$$|\max_u f(u) - \max_u g(u)| \leq \max_u |f(u) - g(u)|. \tag{3.37}$$

The second is

$$\sum_{s=t}^{T} \frac{(T-s)^{k-1}}{(k-1)!} \leq \frac{(T-t+1)^k}{k!}. \tag{3.38}$$

In proving the convergence, we impose two assumptions.

- We assume that the mean demand is finite, i.e., $\mathbb{E}[D] < \infty$. Thus we do not need to consider the situation when the state variables (x_R, x_S) go to infinity, and assume that $|x_R| < \infty$ and $|x_S| < \infty$.

- We assume the terminal value functions $R^1_{T+1}(\cdot) = R^2_{T+1}(\cdot) = 0$. Note that this assumption is only intended to improve the clarity of the proof, and all results hold for more general terminal value functions as long as both functions are concave.

Lemma 6 *The difference between the optimal retail value function in each iteration $V^n_{R,t}$ and the centralized decomposed value function V^1_t equals to the fixed payment b^n, i.e.,*

$$V^n_{R,t}(x_R) + b^n = V^1_t(x_R),\ n \geq 1.$$

Therefore, the optimal retail ordering policy in each iteration is the same as the centralized optimal ordering policy.

Proof of Lemma 6. The optimal retail value function in iteration n can be written as

$$V^n_{R,t}(x_R) = \max\{\hat{J}^n_{R,t}(x_R, y_R) : y_R \geq x_R\},$$

where

$$\begin{aligned}
\hat{J}^n_{R,t}(x_R, y_R) &= p\mathbb{E}[D_t] - \mathbb{E}[H_R(x_R - D_t)] + \mathbb{E}[r^n_t(y_R, \tilde{S}(x_R, y_R))] \\
&\quad + \mathbb{E}[w^n_t(x_R, X_S, \tilde{I}(x_R, y_R) - D)] - \mathbb{E}[u^n_t(x_R, X_S, \tilde{I}(x_R, y_R))] - b^n \\
&\quad + \mathbb{E}[V^n_{R,t+1}(\tilde{S}(x_{R,t}, y_{R,t}) - D_t)] \\
&= p\mathbb{E}[D_t] - \mathbb{E}[H_R(x_R - D_t)] - b^n + \mathbb{E}[V^n_{R,t+1}(y_R - D_t)] \\
&\quad + \mathbb{E}[V^{n-1}_{M,t+1}(\hat{X}_{R,t+1}, \tilde{I}(x_R, y_R) - D)] - \mathbb{E}[V^{n-1}_{M,t+1}(\hat{X}_{R,t+1}, \tilde{I}(x_R, y_R) - D)] \\
&= p\mathbb{E}[D_t] - \mathbb{E}[H_R(x_R - D_t)] - b^n + \mathbb{E}[V^n_{R,t+1}(y_R - D_t)].
\end{aligned}$$

Here $\tilde{I}(\cdot)$ is the retail store's belief of the system total inventory before demand realization. Note that retail store can form any erroneous belief, and the proof does not depend on any specific form of the belief. Comparing the above functions with (3.14) and (3.16), we can easily see that $V^n_{R,t}(x_R) + b^n = V^1_t(x_R)$ provided that the terminal value function $V^n_{R,T+1}(\cdot)$ equals to $R^1_{T+1}(\cdot)$. Therefore, the optimal retail order policy in iteration n would be the same as that of the centralized problem. $\square$

Lemma 7 *There exist constants $c_1, c_2 > 0$ such that $\forall n \geq 1$,*

$$\max_{x_R, x_S} |V^n_{M,t}(x_R, x_S) - V^{n-1}_{M,t}(x_R, x_S)| \leq \frac{c_1[c_2(T - t + 1)]^n}{c_2 n!}.$$

Therefore,

$$\lim_{n\to\infty} \|V^n_M - V^{n-1}_M\| = \lim_{n\to\infty} \max_{t\in\{1,\ldots,T\}, x_R, x_S} |V^n_{M,t}(x_R, x_S) - V^{n-1}_{M,t}(x_R, x_S)| = 0.$$

Proof of Lemma 7. Denote

$$\Delta V^n_{M,t} = \max_{x_R, x_S} |V^n_{M,t}(x_R, x_S) - V^{n-1}_{M,t}(x_R, x_S)|.$$

For the first iteration, since $V^0_{R,t}(x_R)$ and $V^0_{M,t}(x_R, x_S)$ are set to be zero, we have

$$\begin{aligned}
& V^1_{M,t}(x_R, x_S) \\
= \; & \max_{y_S} \sum_{i=t}^{T} -c\mathbb{E}[(y_{s,i} - X_{s,i}) \wedge K_i] - h_S \mathbb{E}[X_{S,i}] \\
& + \mathbb{E}[V^1_{R,i+1}(y^1_{R,i}(X_{R,i}) \wedge y_{S,i} \wedge (X_{S,i} + K_i) - D_i) - V^1_{R,i+1}(y^1_{R,i}(X_{R,i}) - D_i)].
\end{aligned}$$

Since $x_{S,i}$ is bounded in each period, every term in one-period profit is bounded. Let c_1 be the upper bound of one-period profit function, then

$$\Delta V^1_{M,t} = \max_{x_R, x_S} |V^1_{M,t}(x_R, x_S)| \leq c_1(T - t + 1).$$

Suppose the result is true for n, such that

$$\Delta V^n_{M,t} \quad = \quad \max_{x_R, x_S} |V^n_{M,t}(x_R, x_S) - V^{n-1}_{M,t}(x_R, x_S)| \leq \frac{c_1[c_2(T - t + 1)]^n}{c_2 n!}.$$

Then,

$$\begin{aligned}
& |V^{n+1}_{M,t}(x_R, x_S) - V^n_{M,t}(x_R, x_S)| \\
= & \Big| \max_{y_S} \big\{ \mathbb{E} \sum_{i=t}^{T} -c\mathbb{E}[(y_{s,i} - X_{s,i}) \wedge K_i] - h_S \mathbb{E}[X_{S,i}] + u^{n+1}_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i)) \\
& -r^n_i (y^n_{R,i}(X_{R,i}), y^n_{R,i}(X_{R,i}) \wedge y_{S,i} \wedge (X_{S,i} + K_i)) \\
& -w^{n+1}_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) \big\} \\
& - \max_{y_S} \big\{ \mathbb{E} \sum_{i=t}^{T} -c\mathbb{E}[(y_{s,i} - X_{s,i}) \wedge K_i] - h_S \mathbb{E}[X_{S,i}] + u^n_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i)) \\
& -r^{n-1}_i (y^{n-1}_{R,i}(X_{R,i}), y^{n-1}_{R,i}(X_{R,i}) \wedge y_{S,i} \wedge (X_{S,i} + K_i)) \\
& -w^n_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) \big\} \Big| \\
\leq & \max_{y_S} \Big| \mathbb{E} \sum_{i=t}^{T} u^{n+1}_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i)) - u^n_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i)) \\
& \quad - w^{n+1}_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) + w^n_i (X_{R,i}, X_{S,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) \Big| \\
= & \max_{y_S} \Big| \mathbb{E} \sum_{i=t}^{T} \big(V^n_{M,i+1}(\hat{X}_{R,i}, y_{S,i} \wedge (X_{S,i} + K_i) - D_i) - V^{n-1}_{M,i+1}(\hat{X}_{R,i}, y_{S,i} \wedge (X_{S,i} + K_i) - D_i) \big) \\
& \quad - \big(V^n_{M,i+1}(\hat{X}_{R,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) - V^{n-1}_{M,i+1}(\hat{X}_{R,i}, y_{S,i} \wedge (X_{S,i} + K_i) - \tilde{D}_i) \big) \Big|,
\end{aligned}$$

where the first inequality follows from (3.37) and Lemma 6. From Lemma 6, we know that $y^n_{R,i}(\cdot) = y^{n-1}_{R,i}(\cdot)$, therefore the state dynamics $\hat{X}_{R,i}$ and $\hat{X}_{S,i}$, for $t \leq i \leq T$, are the same in iteration $n+1$ and n, provided that the same production policy $y_{S,i}$ is followed. Next we look at the maximum of the difference.

$$\begin{aligned}\Delta V_{M,t}^{n+1} &\leq \sum_{i=t}^{T} c_2 \frac{c_1[c_2(T-i)]^n}{c_2 n!} \\ &\leq c_1 c_2^{n+1} \frac{(T-t+1)^{n+1}}{c_2(n+1)!} \\ &= \frac{c_1[c_2(T-t+1)]^{n+1}}{c_2(n+1)!},\end{aligned}$$

where the first inequality follows from the induction together with $c_2 = 2$, and the second inequality follows from (3.38). Taking the maximum over t gives

$$\|V_M^n - V_M^{n-1}\| \leq \max_{t\in\{1,\ldots,T\}} \frac{c_1[c_2(T-t+1)]^n}{c_2 n!} \leq \frac{c_1(c_2T)^n}{c_2 n!}.$$

Thus,

$$\lim_{n\to\infty} \|V_M^n - V_M^{n-1}\| = 0.$$

□

Theorem 4 *Let $\hat{V}_{M,t}(x_R, x_S)$ be the value function defined in (3.32). The sequence $V_{M,t}^n(x_R, x_S)$, which is computed by Algorithm 1, converges uniformly to $\hat{V}_{M,t}(x_R, x_S)$, i.e.,*

$$\lim_{n\to\infty} \|\hat{V}_{M,t}(x_R, x_S) - V_{M,t}^n(x_R, x_S)\| = 0.$$

Moreover,

$$\lim_{n\to\infty} \|V_t(x_R, x_S) - V_{R,t}^n(x_R) - V_{M,t}^n(x_R, x_S)\| = 0,$$

where $V_t(x_R, x_S)$ is the centralized optimal value function defined in (3.12).

Proof of Theorem 4. Take any $n_1, n_2 \in \mathbf{Z}^+$,

$$\begin{aligned} \|V_M^{n_1+n_2} - V_M^{n_1}\| &\leq \sum_{i=1}^{n_2} \|V_M^{n_1+i} - V_M^{n_1+i-1}\| \\ &\leq \sum_{i=1}^{n_2} \frac{c_1(c_2T)^{n_1+i}}{c_2(n_1+i)!} \\ &\leq \frac{c_1(c_2T)^{n_1+1}}{c_2(n_1+1)!} \sum_{i=0}^{\infty} \frac{(c_2T)^i}{i!} \\ &= \frac{c_1(c_2T)^{n_1+1}}{c_2(n_1+1)!} e^{-c_2T}. \end{aligned}$$

This implies $\lim_{n_1\to\infty}\|V_M^{n_1+n_2} - V_M^{n_1}\| = 0$. Therefore, V_M^n is a Cauchy sequence under the sup-norm, and there exists a function $\tilde{V}_M$ such that $\lim_{n\to\infty}\|\tilde{V}_M - V_M^n\| = 0$. Taking the limit of the dynamic programming equations (3.26) and (3.27) gives

$$\tilde{V}_{M,t}(x_R, x_S) = \max\{\tilde{J}_{M,t}(x_R, x_S, y_S) : y_S \geq x_S\},$$

where

$$\begin{aligned} & \tilde{J}_{M,t}(x_R, x_S, y_S) \\ = \ & -c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S - \mathbb{E}[V_{t+1}^1(y_R(x_R) - D)] \\ & +\mathbb{E}[V_{t+1}^1(y_R(x_R) \wedge y_S \wedge (x_S + K) - D)] - \mathbb{E}[\tilde{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - \tilde{D})] \\ & +\mathbb{E}[\tilde{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - D)] + \mathbb{E}[\tilde{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - \tilde{D})] \\ = \ & -c\mathbb{E}[(y_S - x_S) \wedge K] - h_S x_S - \mathbb{E}[V_{t+1}^1(y_R(x_R) - D)] \\ & +\mathbb{E}[V_{t+1}^1(y_R(x_R) \wedge y_S \wedge (x_S + K) - D)] + \mathbb{E}[\tilde{V}_{M,t+1}(\hat{X}_{R,t+1}, y_S \wedge (x_S + K) - D)]. \end{aligned}$$

It is easy to prove that $\tilde{V}_{M,t}(x_R, x_S) = \hat{V}_{M,t}(x_R, x_S) = V_t^2(x_S)$ by induction. Therefore, we prove that $\lim_{n\to\infty}\|\hat{V}_M - V_M^n\| = 0$.

From Lemma 6 and Proposition 5, we can immediately get the second result, i.e.,

$$\lim_{n\to\infty}\|V_t(x_R, x_S) - V_{R,t}^n(x_R) - V_{M,t}^n(x_R, x_S)\| = 0.$$

□

3.6 Numerical Studies

In this section, we evaluate convergence rate of our iterative algorithm described in Section 3.5 under different demand distributions and system parameters.

3.6.1 Both Mean and Variance are Known

In this subsection, we assume the mean and variance of the random demand are known for the manufacturing facility, and only the distributional form is unknown. We show the convergence in terms of the relative error, i.e.,

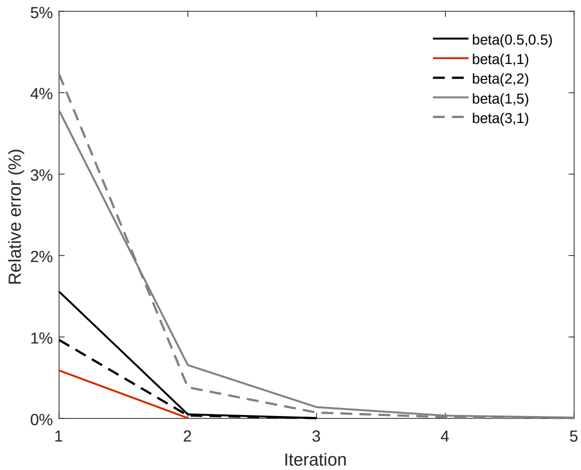
$$\text{Relative error at iteration n} = \max_{t,x_R,x_S} \left| \frac{V^n_{M,t}(x_R, x_S) - V^2_t(x_S)}{V^2_t(x_S)} \right|,$$

where $V^2_t(x_S)$ is the centralized decomposed value function, defined in (3.15). The convergence tolerance $\epsilon = 0.01\%$.

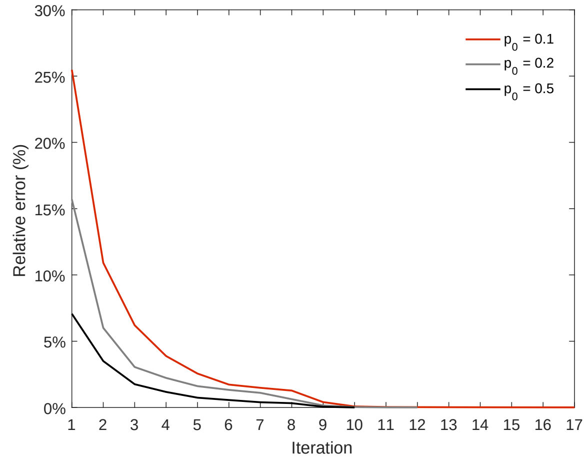
Impact of the true demand distribution

Figure 3.7 reports the convergence of our iterative algorithm when the planning horizon of the dynamic programming $T = 20$. We test the convergence rate when the true demand follows a scaled beta distribution or a two-point distribution. In Figure 3.7, we observe that it takes more iterations to converge when the true demand follows a two-point distribution. Meanwhile, the convergence rate is robust in different shapes of beta distributions, and the maximum number of iterations needed for convergence is 5. Below, we further examine the effect of the average demand, demand uncertainty and the random capacity on the convergence rate.

Impact of the average demand. Figure 3.5a shows that the relative error decreases as the average demand increases. This is mainly because the absolute value of the centralized decomposed value function V^2_t is larger as the mean demand increases. Since the relative error decreases, the number of iterations needed for convergence also tends to decrease.



(a) D: Beta distributions



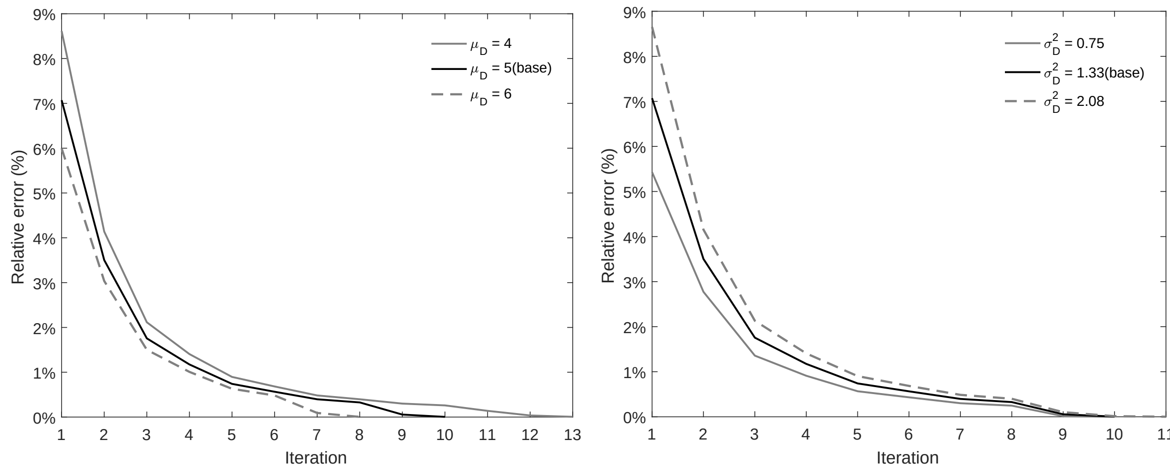
(b) D: Two-point distributions

Note. The planning horizon of the dynamic programming $T = 20$. We choose different forms of the true demand distribution by fixing $\mu_D = 5, \sigma_D^2 = 1.33$. In both subfigures, the erroneous demand $\tilde{D} \sim$ unif $[3, 7]$, and the random capacity $K \sim$ unif $[3, 4]$.

Fig. 3.4.: Impact of the true demand distribution

Impact of the demand uncertainty. Figure 3.5b shows that the relative error decreases as the variance of the demand distribution decreases. Intuitively, this is because a larger demand uncertainty magnify the mismatch between the true and erroneous demand distributions.

Impact of the mean capacity and capacity uncertainty. We also evaluate the convergence against the mean capacity μ_K and the capacity uncertainty σ_K. We find that the relative error and the number of iterations needed for convergence do not change much. Therefore, we can conclude that the number of iterations needed for convergence mainly depends on the difference between the true and erroneous demand distributions, which is consistent with our theoretical results in previous section.



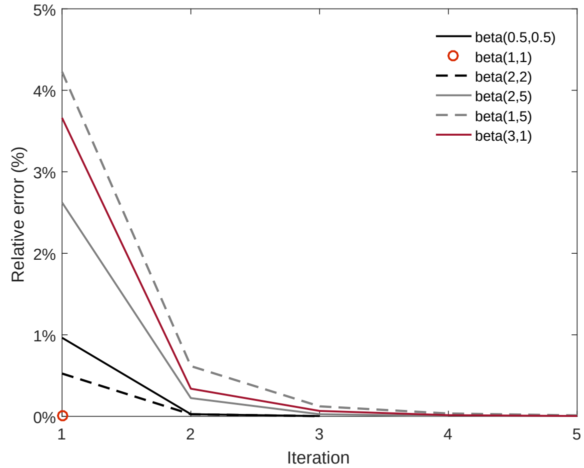
(a) Impact of the average demand (b) Impact of the demand variance

Note. The true demand D follows a two-point distribution, where $P(D = d_l) = P(D = d_h) = 0.5$, and $d_l = \mu_D - \sigma_D, d_h = \mu_D + \sigma_D$. The erroneous demand $\tilde{D} \sim$ unif $[\mu_D - 6\sigma_D^2, \mu_D + 6\sigma_D^2]$. Left subfigure: $\sigma_D^2 = 1.33$. Right subfigure: $\mu_D = 5$.

Fig. 3.5.: Impact of the fixed mean and variance

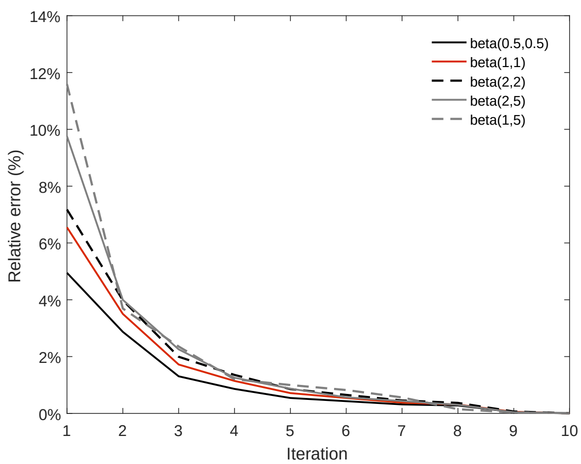
Impact of the erroneous demand distribution

In Figure 3.6 and 3.7, we report the effect of the form of erroneous demand distribution on the convergence assuming the true demand follows a uniform distribution and two-point distribution respectively. Interestingly, when the true demand follows a two-point distribution, assuming the erroneous demand also follows a two-point distribution may even cost more iterations to converge compared with assuming a beta distribution for the erroneous demand. This is because the difference between the true and erroneous demand distributions is larger when p_0 are different in the two-point distributions.

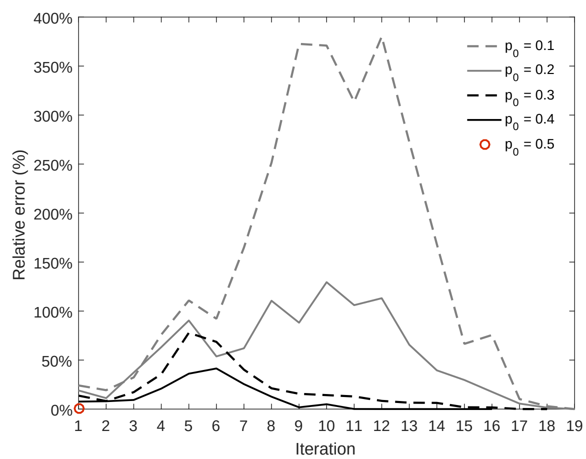


Note. The true demand $D \sim$ unif $[3, 7]$. We choose different forms of the erroneous demand distribution by fixing $\mu_D = 5, \sigma_D^2 = 1.33$.

Fig. 3.6.: Impact of the erroneous demand distribution



(a) $\tilde{D}$: beta distributions



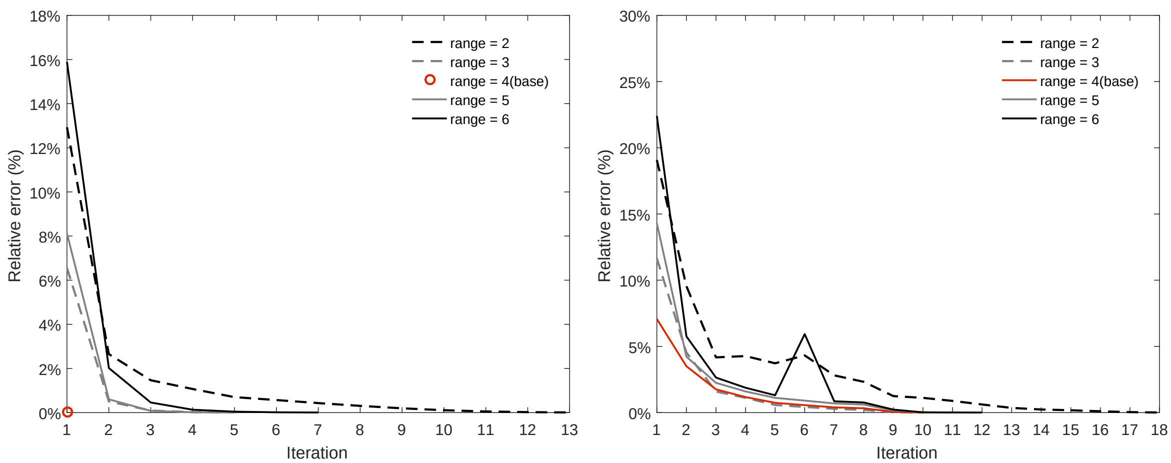
(b) $\tilde{D}$: two-point distributions

Note. The true demand D follows a two-point distribution, where $P(D = 3.85) = P(D = 5.15) = 0.5$. We choose different forms of the erroneous demand distribution by fixing $\mu_D = 5, \sigma_D^2 = 1.33$.

Fig. 3.7.: Impact of the erroneous demand distribution

3.6.2 Unknown Variance

In this subsection, we relax the previous assumption that the variance of the demand distribution is known for the manufacturing facility, and assume he only has the information about the average demand. Figure 3.8a and 3.8b show the convergence under different ranges of erroneous demand distribution when the true demand follows a uniform distribution and a two-point distribution respectively. We can observe that when the range deviates from the true range further in either direction, the relative error increases and it takes more iterations to converge.



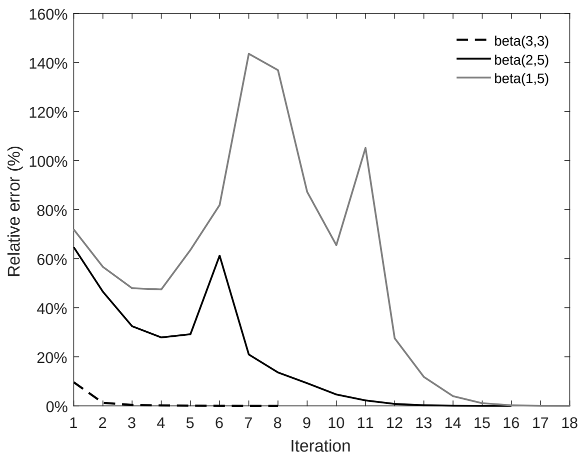
(a) True demand $D \sim \text{unif}[3, 7]$. (b) $P(D = 3.85) = P(D = 6.15) = 0.5$.

Note. The erroneous demand $\tilde{D} \sim \text{unif}\ [5 - 0.5 \times \text{range},\ 5 + 0.5 \times \text{range}]$.

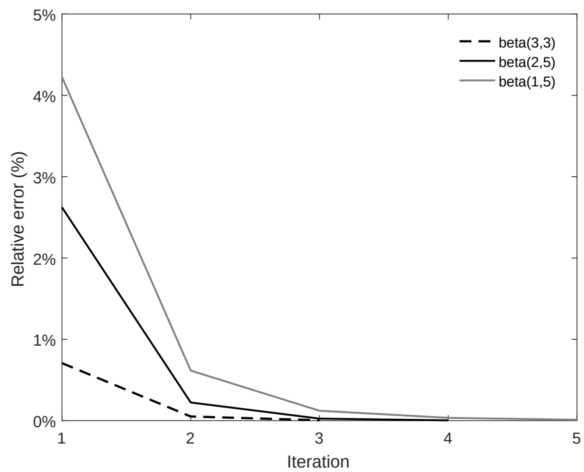
Fig. 3.8.: Impact of the range of the erroneous demand distribution

3.6.3 Only the Support of the Demand Distribution is Known

In this subsection, we consider the case when the manufacturing facility only knows the support of the random demand distribution, and both the mean and variance are unknown. Figure 3.10 and 3.9 show the convergence under different shapes of beta distributions when the true demand follows a two-point distribution and a uniform distribution respectively. As we observed in Section 3.6.1, the effect of the shapes of the erroneous demand distribution on the convergence is not significant when the mean and variance are known, however, this does not hold when only the support is known. Figure 3.10a and 3.9a show that the relative error may become large when the shape of the erroneous demand distribution changes.



(a) Support $[3, 7]$ is known.



(b) Benchmark: mean and variance are known.

Note. The true demand $D \sim \text{unif}[3, 7]$.

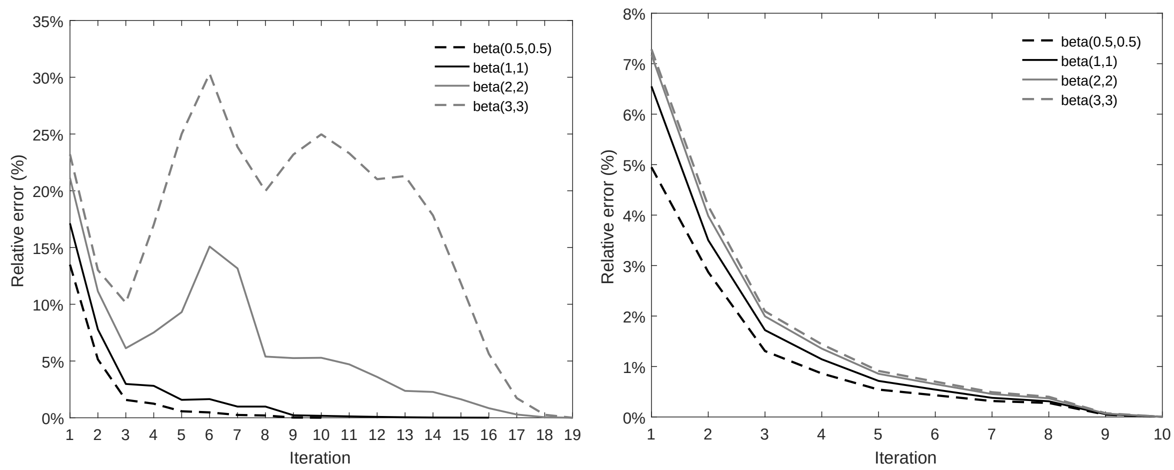
Fig. 3.9.: Impact of the erroneous demand distributions

3.6.4 Profit Share

In Figure 3.11, we show the profit share between the retail store and manufacturing facility using different fixed payments. We can observe that profits can be arbitrarily distributed for different fixed payments.

3.7 Summary

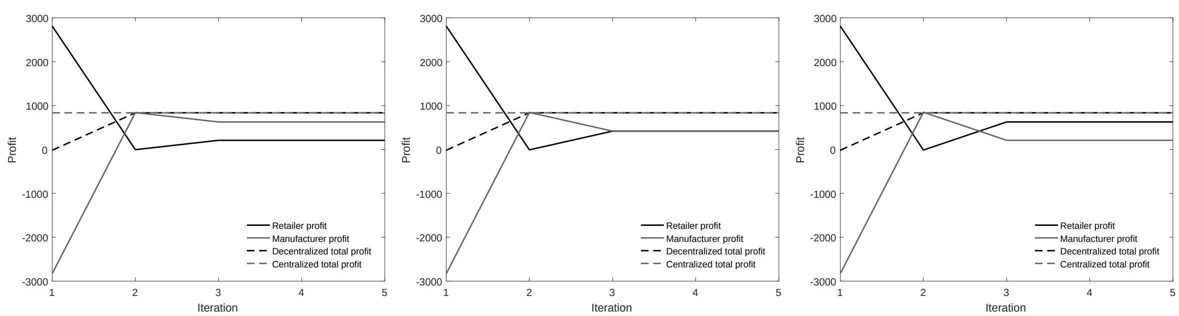
In this chapter, we examine the two-stage supply chain problem from the decentralized perspective. We consider two scenarios of information visibility. When the



(a) Support $[3.85, 6.15]$ and mean are known. (b) Benchmark: mean and variance are known.

Note. The true demand D follows a two-point distribution, where $P(D = 3.85) = P(D = 6.15) = 0.5$.

Fig. 3.10.: Impact of the erroneous demand distributions



(a) Retail share 25%. (b) Retail share 50%. (c) Retail share 75%.

Note. The true demand D follows a two-point distribution, where $P(D = 3.85) = P(D = 6.15) = 0.5$, and the erroneous demand $\tilde{D}$ follows beta(3,1). The profit reported in the figure is when $t = 1$ and $x_R = 0, x_M = 0$. The fixed payment in each iteration n follows $b^n = b^{n-1} + (1-r)V_{R,1}^{n-1}(0) - rV_{M,1}^{n-1}(0,0)$ and $b_0 = 0$, where r is the retail store's intended share of the total profit and $r = 25\%, 50\%, 75\%$ respectively from leftsubfigre to rightsubfigure.

Fig. 3.11.: Profit share

supply information is not available to the retail store, we design a transfer payment to achieve the centralized optimal profit. When the demand information is not available to the manufacturing facility as well as the retail store does not know the supply information, we characterize the optimal contracts under which coordination can be achieved, and propose an iterative algorithm to realize the coordination in the decentralized setting. The total profit under our algorithm is guaranteed to converge to the centralized optimal channel profit for any demand and capacity distribution functions. The coordinating mechanisms proposed here do not require sophisticated knowledge structure, making them more practical compared with conventional mechanisms.

4. CASE STUDY: NEW DIAGNOSTIC TEST EVALUATION

4.1 Synopsis

In the working paper [1], the authors develop a framework for evaluating the operational impact of the new diagnostic tests. This chapter mainly serves as a case study of their developed framework. We first describe the MDP model in their framework, and we present the structural properties of the optimal policy.

Since there are multiple classes of patients in the MDP model, the size of the state space becomes large as the number of patient classes increases. To overcome this curse of dimensionality, we describe the heuristic algorithm developed in [1], and we validate the numerical performance of their heuristic algorithm.

The remainder of this chapter is organized as follows. In Section 4.2, we review the related literature. In Section 4.3, we describe the underlying patient flow model and the test routing problem. In Section 4.4, we present the structural properties of the optimal policy. In Section 4.5, we describe a heuristic algorithm. We also validate the performance of the heuristic algorithm in this section. In Section 4.6, we conclude this chapter.

4.2 Literature Review

This study is related to four streams of literature.

Service operations management. A rich body of research in the filed of operations management has studied the problem of balancing the speed of the service versus the quality of the service. [58] provide an an excellent survey. [59] introduce discretion in task completion, which means the standard for a task completion is not clear.

A longer service time gives a higher service value. They show that the service time decreases as the congestion increases under the optimal policy. [60] study the payment and service strategy when a monopolist expert offers a service with discretionary task completion and the customers are strategic. [61] study the tradeoff between service quality and speed when customers are strategic. [62] model a gatekeeper who makes an initial diagnosis of the customer's problem and decides if to refer the customer to a specialist. They focus on the information asymmetry between the gatekeeper and the specialist and consider in a principal-agent framework. [63] extend the model to include queueing at both the gatekeeper and expert. They solve the optimal staffing levels and referral rates between gatekeeper and expert from a centralized perspective.

There are relatively few papers analyzing diagnostic decision making in healthcare. [64] study the staffing and service depth decision of a nurse triage line in which the patients decide to call the line or not based on their expectation of diagnostic accuracy and congestion. They focus on the equilibrium analysis between the provider and patients, and the diagnostic process is modeled as a Brownian motion in their paper. However, the model that we adopt in our case study focuses on the diagnostic process. [65] study the tradeoff between test accuracy and system congestion in a diagnostic process, in which the service provider conducts multiple tests to determine the customer's type. They consider dynamic decision of whether to run more tests or to stop the process and identify the customer's type. They find that the service provider should continue to run test on the customer as long as the belief of the customer type falls into an interval.

Infinite-server queue and state-dependent service rate. As suggested by the empirical results in [66], infinite server model with state-dependent service rate turns out to display a better fit with the empirical data in ED, therefore the service process of each station follows a same form in the model we adopt. Both infinite server queue and state-dependent service rate have been studied extensively in queueing literature. For example, [67] and [68] study the $M/M/\infty$ queue, and [69] and [70] study the $M/G/1$ queue with workload-dependent arrival and service rates. Note that the queueing model described in Section 4.3.1 can also be viewed as a processor sharing queue; see the paper by [71] for an overview of processor sharing queue. [72] study the $M/M/1$ queue with processor sharing discipline, and [73] explore the $G/M/1$ queue with processor sharing discipline. [74] investigate the problem of predicting the response times in $M/G/1$ processor-sharing queue. [75] provide an approximation for the waiting time distribution of $M/M/n$ processor-sharing queue.

Dynamic control of queueing systems. Another related stream of literature is dynamic control of queueing systems. Excellent surveys are provided by [76] and [77]. A rich body of research has studied the admission control, control of service rate and both. For example, [78] study the admission control of n job classes and multiple servers, and [79] consider the admission control for a two-station tandem queue loss model. [80] study a queueing system with removable servers and state-dependent arrival rates. They characterize the conditions that the optimal number of servers is increasing in the total number of customers. [81] consider a single-server queue with Poisson arrivals and state-dependent service rates. The objective is to minimize the long run average cost by controlling the service rate. There has been many extensions of the basic model, for example, [82] add pricing decision into the control of an $M/M/1$ queue, and [83] study the problem of pricing and admission control when the arrival and service rates are periodically varying and the customers are sensitive to the price for entering the system. Instead of making the decision of accepting or rejecting the arriving customer, [84] study the dynamic routing decision (which station should the arrival be sent to) in a two-station queueing network. They show the optimal routing decision is threshold type policy and can be characterized by a monotone switching curve in both finite horizon and long run average cost problems. Their model and queueing network are closely related to those described in Section 4.3, which also focus on the routing between two stations. A key difference between them is the service process for each station. The model we adopt use the infinite-server queue with state dependent service rate function, which is more general than the single-server setting with constant service rate in [84]. As mentioned in [85], extending the optimality of threshold routing policy in [84] to a multi-sever queueing system remains an open question. Therefore, it would be challenging to characterize the optimal routing policy in a multi-class, infinite-server setting. The survey paper by [85] provides an excellent overview of the results in controlled queueing system.

Clinical diagnostic decision making. Diagnostic decision making has been studied extensively in the medical literature. There are two common approaches to diagnosis. The first is non-analytic approach, like pattern recognition (e.g., [86], [87], [88], [89], [90], [91]). The second is the probabilistic approach that different tests are conducted to estimate the probability of having the condition (e.g., [92], [93], [94], [8]). Most of these studies are developed in a clinically controlled environment, without considering the interactions with existing workflows. The model described in Section 4.3 is related to the second approach, in which the probability of having the condition is updated after each test. Moreover, the operational effects is considered into the diagnostic decision making in the model. Another important set of questions related is evaluating the new diagnostic tests. [95] provides a general framework for clinical evaluation on diagnostic technologies, and the standards for determining the new test accuracy are well studied in [6], [7], [8].

4.3 Modeling Framework for Test Routing

In this section, we describe the underlying patient flow model and the test routing problem based on the framework developed by [1].

4.3.1 Queueing Model

J classes of patients are considered in the model, capturing different levels of patient risk of having a suspected disease. Without loss of generality, the pretest probability of having the disease is assumed to be the lowest for class 1 patients, and the highest for class J patients, i.e., $p_0^1 < p_0^2 < \cdots < p_0^J$, where p_0^j is the pretest probability for class j patients. There are two stations in the queueing model, in which the first station represents an "imperfect test", and the second station represents a "perfect test." Here, an imperfect test in the model means the test may give a false positive or a false negative result, while a perfect test does not. Each patient may go through station 1 or 2 or both, and then leave the system. Figure 4.1 illustrates the basic patient flow.

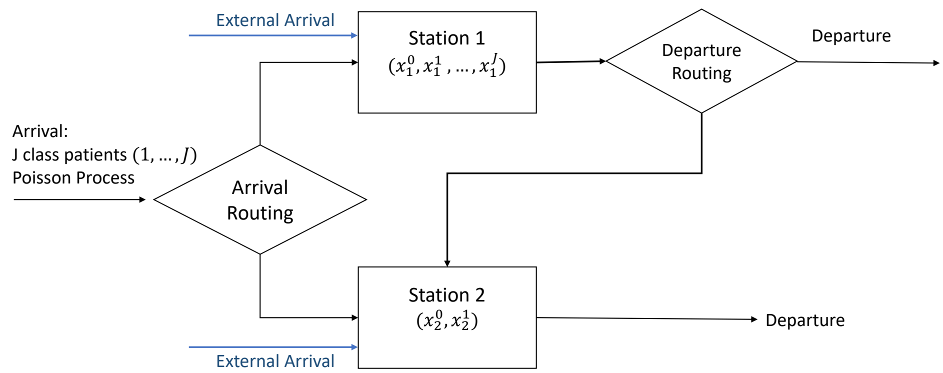


Fig. 4.1.: Basic patient flow.

Arrival process. Arrivals for each of the J classes form independent Poisson processes. The Poisson process is assumed to be time homogeneous for ease of exposition, but the modeling framework developed in [1] can be extended to incorporate time-varying arrival rate. For patient class j, the arrival rate is denoted as λ_j, $j = 1, \ldots, J$. In addition to the patients with the suspected disease, patients who may use each station for other reasons are also considered in the model. These exogenous arrivals to station i also follow a Poisson process with rate $\lambda_i^{\text{ex}}, i = 1, 2$.

Service process. In the framework developed by [1], each station is modeled as processor sharing queue, with the total service rate being characterized as $\mu_i(s_i)$, and s_i being the total number of patients in service at station i. A "uniform" sharing mechanism is considered. That is, each patient's service rate is $\mu_i(s_i)/s_i$. Two assumptions are made: (i) the total service rate $\mu_i(s_i)$ is non-decreasing in s_i; (ii) individual patient's service rate $\mu_i(s_i)/s_i$ is non-increasing in s_i.

Note that the processor sharing queue can be viewed as an infinite-server queue with state-dependent service rate function $\mu_i(s_i)/s_i$ for each patient. Infinite-server queues have been demonstrated in the literature as flexible and general enough to model complicated healthcare systems, while approximating the actual system performance reasonably well [66], [96]. In special cases where $\mu_i(s_i) = c$ as a constant, the processor sharing queue corresponds to an $M/M/1$ queue with service rate c; when $\mu_i(s_i) = cs_i$, the processor sharing queue corresponds to an $M/M/\infty$ queue with service rate c (for each patient).

4.3.2 MDP Formulation

Based on the developed framework in [1], we describe an infinite-horizon, discounted cost, continuous-time MDP model for the test routing decision problem. The state space, action space, cost structure, and objective function are introduced below.

State Space

The system state is captured with a $(J+3)$ - dimensional vector

$$\underline{x} = (x_1^0, x_1^1, \ldots, x_1^J, x_2^0, x_2^1).$$

- x_1^0 and x_2^0 denotes the number of patients from external arrivals in station 1 and station 2, respectively.
- x_1^j denotes the number of class j patients in station 1, $j = 1, \ldots, J$.
- x_2^1 denotes the total number of patients (excluding external arrivals) in station 2. Patient classes are not differentiated in station 2, the reason for which will be explained in Section 4.3.2 after we introduce the cost structure.

For notational convenience, we denote

$$\underline{x_1} = (x_1^0, x_1^1, \ldots, x_1^J), \quad \underline{x_2} = (x_2^0, x_2^1), \tag{4.1}$$

$$s_1 = \sum_{j=0}^{J} x_1^j, \quad s_2 = x_2^0 + x_2^1. \tag{4.2}$$

For analytical tractability, an upper bound for the number of patients in each station is imposed, denoted as (s_1^u, s_2^u). Once $s_i = s_i^u$, no more patients can be routed into that station. When the upper bounds are reached in both stations, the next arriving patient will be rejected with a penalty cost m^j if the arrival is from class j.

Action Space

The MDP action is triggered by a patient arrival or a patient departure (from station 1). At an arrival event, an *arrival routing* decision is made, that depends on which class j the arriving patient belongs to. The arrival action is given by

$$a_{\text{arr}}^j(\underline{x}) = \begin{cases} 0 & \text{if the arriving patient is rejected ,} \\ 1 & \text{if the arriving patient is routed to station 1,} \\ 2 & \text{if the arriving patient is routed to station 2.} \end{cases}$$

There are no routing decision for patients from the external arrivals, and they are admitted as long as there are available spots (i.e., $s_i < s_i^u$). In the case that $s_i = s_i^u$, the external patient will be rejected with a penalty of m_i^0.

When a class j patient departs from station 1, a *departure routing* decision is made about whether the patient needs additional testing at station 2. Because the test at station 1 is imperfect, the departure routing decision depends on the test result from station 1. The decision is denoted as $a_{\text{dep}}^{j,+}(\underline{x})$ ($a_{\text{dep}}^{j,-}(\underline{x})$) if the patient receives a positive (negative) test result at station 1. For $r = +, -$,

$$a_{\text{dep}}^{j,r}(\underline{x}) = \begin{cases} 1 & \text{if the patient is directly discharged from the system,} \\ 2 & \text{if the patient is routed to station 2.} \end{cases}$$

External patients finishing tests in each station and patients finishing tests in station 2 directly depart from the system.

Cost Structure

Two types of costs are considered: individual-level diagnostic cost and system-level congestion cost.

Diagnostic cost. Diagnostic cost is measured by the accuracy of the diagnostic result. Define $c_i^{\mathrm{FN}}, c_i^{\mathrm{TN}}, c_i^{\mathrm{FP}}, c_i^{\mathrm{TP}}$ as the cost associated with a false negative, true negative, false positive, and true positive result, respectively. The post-test probability needs to be specified as follows to calculate these diagnostic costs.

The test results in both stations could be either positive or negative, with probability $b_i^{j,+}$ being positive given the class j patient has the suspected disease, and $b_i^{j,-}$ being negative given the class j patient does not have the suspected disease, $i = 1, 2$. Based on the pretest probability p_0^j for class j patients and the test result at station i, the post-test probability is updated as the following: If the test result is positive,

$$\tilde{p}_i^{j,+} = \frac{p_0^j \times b_i^{j,+}}{p_0^j \times b_i^{j,+} + (1 - p_0^j) \times (1 - b_i^{j,-})}, \; i = 1, 2; \; j = 1, \ldots, J. \tag{4.3}$$

If the test result is negative,

$$\tilde{p}_i^{j,-} = \frac{p_0^j \times (1 - b_i^{j,+})}{p_0^j \times (1 - b_i^{j,+}) + (1 - p_0^j) \times (b_i^{j,-})}, \; i = 1, 2; \; j = 1, \ldots, J. \tag{4.4}$$

Thus, the expected diagnostic cost for class j patient after a test at station i is

$$c_i^{j,+} = \tilde{p}_i^{j,+} c_i^{\mathrm{TP}} + (1 - \tilde{p}_i^{j,+}) c_i^{\mathrm{FP}}, \tag{4.5}$$

$$c_i^{j,-} = \tilde{p}_i^{j,-} c_i^{\mathrm{FN}} + (1 - \tilde{p}_i^{j,-}) c_i^{\mathrm{TN}}. \tag{4.6}$$

For station 2, we have that $b_2^{j,+} = b_2^{j,-} = 1$ and $\tilde{p}_2^{j,+} = 1$ ($\tilde{p}_2^{j,-} = 0$) if the test result is positive (negative). Thus, the expression of the expected cost after a test at station 2 can be simplified as

$$c_2^{j,+} = c_2^{\mathrm{TP}}, c_2^{j,-} = c_2^{\mathrm{TN}}.$$

The following assumption is made about the diagnostic costs.

Assumption 1 $c_i^{FN} \geq c_i^{TN}, c_i^{FP} \geq c_i^{TP}$. *For station 2,* $c_2^{TP} = c_2^{TN}$.

For the first and second inequalities in this assumption, it is assumed that the false-negative (false-positive) cost is higher than the cost of a true-negative (true-positive), based on medical literature. Making the second assumption that $c_2^{\mathrm{TP}} = c_2^{\mathrm{TN}}$ is to reduce the dimensionality by eliminating the need to differentiate by patient class in the state space at station 2. For notational convenience, the superscript is omitted in the station 2 diagnostic costs and denote $c_2^{\mathrm{TP}}, c_2^{\mathrm{TN}}$ as c_2.

Congestion cost. To capture the workload impact of routing patients into each station, the unit time holding cost $h_i(\underline{x_i})$ is considered. $h_i(\underline{x_i})$ is assumed to be increasing in each coordinate of $\underline{x_i}$. For example, $h_i(\underline{x_i}) = \sum_j h_j x_j$ corresponds to a linear holding cost setting; $h_i(\underline{x_i}) = \sum_j h_j x_j^2$ corresponds to quadratic holding cost for each class j patient.

Total cost at time t. The total cost at time t is given by

$$\begin{aligned} & g(\underline{X}(t)) \\ = \quad & h_1(\underline{X_1}(t)) + h_2(\underline{X_2}(t)) + c_2 Y_2(t) + \sum_{j=1}^{J} \left(Y_1^{j,+}(t) c_1^{j,+} + Y_1^{j,-}(t) c_1^{j,-} \right), \end{aligned} \tag{4.7}$$

where the first two terms capture the holding costs at each station, and the other terms capture the diagnostic cost at each station. The indicator $Y_2(t) = 1$ denotes that a test is finished at station 2 at time t (and 0 otherwise); the indicator $Y_1^{j,+}(t) = 1$ ($Y_1^{j,-}(t) = 1$) denotes that a test is finished at station 1 at time t for a class j patient with positive (negative) result and this patient directly leaves the system.

Discounted cost formulation. The model is formulated as an infinite-horizon, discounted cost MDP. Specifically, given the state at time 0 is $\underline{x}(0)$, the optimal total discounted cost $V_\alpha(\underline{x}(0))$ follows

$$V_\alpha(\underline{x}(0)) \quad = \quad \inf_\pi \mathbb{E} \int_0^\infty e^{-\alpha t} g(\underline{X}(t)) dt,$$

where α is the discount rate.

Bellman Equation

Since the underlying system dynamics is in continuous time, the uniformization technique is applied to discretize the problem. Recall that the number of patients in station i is bounded by s_i^u, and the service rate function $\mu_i(\cdot)$ is assumed to be increasing, then there exists a finite upper bound $\bar{\mu}_i = \mu_i(s_i^u)$ for the service rate function $\mu_i(\cdot)$. Without loss of generality, we assume $\bar{\mu}_1 + \bar{\mu}_2 + \lambda + \lambda_1^{\text{ex}} + \lambda_2^{\text{ex}} + \alpha = 1$.

The optimality equation can be written using event-based dynamic programming [85], with event operators T_k. This approach is often useful for proving structural properties.

$$\begin{aligned} & V_\alpha(\underline{x}) \\ = \; & h_1(\underline{x_1}) + h_2(\underline{x_2}) + \lambda_1^{\text{ex}} T_{A_1^0} V_\alpha(\underline{x}) + \lambda_2^{\text{ex}} T_{A_2^0} V_\alpha(\underline{x}) + \mu_2(s_2) T_{D_2} V_\alpha(\underline{x}) \\ & + \sum_{j=1}^{J} \lambda^j T_{AR^j} V_\alpha(\underline{x}) + \mu_1(s_1) T_{D_1} V_\alpha(\underline{x}) \\ & + (\bar{\mu}_1 - \mu_1(s_1)) V_\alpha(\underline{x}) + (\bar{\mu}_2 - \mu_2(s_2)) V_\alpha(\underline{x}). \end{aligned} \tag{4.8}$$

Next, we specify each of the event operators, T_k. For a function $f : \mathbf{Z}_+^{J+3} \rightarrow \mathbf{R}$, $i = 1, 2, \; j = 1, \ldots, J$,

$$T_{A_i^0} f(\underline{x}) = \begin{cases} f(\underline{x}+e_i^0), & \text{if } s_i < s_i^u, \\ m_i^0 + f(\underline{x}), & \text{otherwise.} \end{cases} \tag{4.9}$$

$$T_{AR^j} f(\underline{x}) = \begin{cases} \min\{f(\underline{x}+e_1^j), f(\underline{x}+e_2^1)\}, & \text{if } s_1 < s_1^u, s_2 < s_2^u, \\ f(\underline{x}+e_1^j), & \text{if } s_1 < s_1^u, s_2 = s_2^u, \\ f(\underline{x}+e_2^1), & \text{if } s_1 = s_1^u, s_2 < s_2^u, \\ m^j + f(\underline{x}), & \text{if } s_1 = s_1^u, s_2 = s_2^u. \end{cases} \tag{4.10}$$

$$T_{D_1} f(\underline{x}) = \begin{cases} \sum_{j=1}^J x_1^j/s_1 (l_1^{j,+} T_{DR_1^{j,+}} f(\underline{x}) + l_1^{j,-} T_{DR_1^{j,-}} f(\underline{x})) & \\ \quad + x_1^0/s_1 T_{D_1^0} f(\underline{x}), & \text{if } s_1 > 0, \\ 0, & \text{if } s_1 = 0. \end{cases} \tag{4.11}$$

$$T_{D_2} f(\underline{x}) = \begin{cases} x_2^0/s_2 T_{D_2^0} f(\underline{x}) + x_2^1/s_2 T_{D_2^1} f(\underline{x}), & \text{if } s_2 > 0, \\ 0, & \text{if } s_2 = 0. \end{cases} \tag{4.12}$$

$$T_{DR_1^{j,r}} f(\underline{x}) = \begin{cases} \min\{c_1^{j,r} + f(\underline{x}-e_1^j), f(\underline{x}-e_1^j+e_2^1)\}, & \text{if } x_1^j > 0, s_2 < s_2^u, \\ c_1^{j,r} + f(\underline{x}-e_1^j), & \text{if } x_1^j > 0, s_2 = s_2^u, \\ 0, & \text{otherwise,} \end{cases} \tag{4.13}$$

$$(r = +, -).$$

$$T_{D_1^0} f(\underline{x}) = f\big((\underline{x}-e_1^0)^+\big). \tag{4.14}$$

$$T_{D_2^k} f(\underline{x}) = c_2 \mathbb{I}_{x_2^k > 0} + f\big((\underline{x}-e_2^k)^+\big),\ k = 0, 1. \tag{4.15}$$

Here, $e_i^j = (0, \ldots, 1, \ldots 0)$ is the unit vector with a 1 in the x_i^j coordinate and zero elsewhere. These unit vectors indicating adding or removing a single patient from their current position in the system. In (4.14) and (4.15), $f(\underline{x}^+) = f((x_1^0)^+, \ldots, (x_2^1)^+)$, where $(x_i^j)^+ = \max\{x_i^j, 0\}$. $l_i^{j,+}(l_i^{j,-})$ in (4.11) denotes the likelihood of the test result being positive (negative), which follows

$$l_i^{j,+} = p_0^j \times b_i^{j,+} + (1-p_0^j) \times (1-b_i^{j,-}),\ i = 1, 2,$$

and $l_i^{j,-} = 1 - l_i^{j,+}$.

The third to seventh terms in Equation (4.8) denote the action (including do nothing) when one of these events happens: external arrival to station i, denoted as $T_{A_i^0}$; service completion in station 2 with rate $\mu_2(s_2)$, denoted as T_{D_2}; arrival routing for a class j patient, denoted as T_{AR^j}; service completion in station 1 with rate $\mu_1(s_1)$, denoted as T_{D_1}. The last two terms in Equation (4.8) are dummy transitions.

4.4 Structural Properties

4.4.1 Threshold Policy in the Single-class Case

We present the proof for the optimality of a threshold policy when there is only one class of patients, i.e., $J = 1$. For ease of exposition, the external arrivals are omitted here. In this case, the state is the total number of patients in each station, denoted as (s_1, s_2) (see the definition of s_i in Equation (4.2)). The following assumptions are necessary to prove Theorem 5.

Assumption 2 *The service rate function $\mu_i(s_i)$ follows*

$$\mu_i(s_i) = \begin{cases} 0, & \text{if } s_i = 0, \\ \mu_i, & \text{otherwise.} \end{cases} \tag{4.16}$$

Assumption 3 *There is no upper limit for the number of patients in each station, i.e., $s_1^u = \infty$, $s_2^u = \infty$.*

Assumption 4 *The holding cost function $h_i(s_i)$ is increasing and convex in s_i, $i = 1, 2$.*

Before we present the formal proof, we need the following lemmas first.

Lemma 8 *Suppose $J = 1$, Assumption 3 is satisfied. Define operator T on function $f : Z_+^2 \to \mathbf{R}$ as*

$$Tf(\underline{x}) = \begin{cases} l_1^+ T_{DR_1^+} f(\underline{x}) + l_1^- T_{DR_1^-} f(\underline{x}) & \text{if } x_1 > 0; \\ f(\underline{x}) & \text{if } x_1 = 0, \end{cases}$$

where $T_{DR_1^r}$ *is defined in (4.13). If* $f(\underline{x}) \in I \cap Super \cap SuperC$ *and* $f(1,0) - f(0,0) \geq \max\{c_1^+, c_1^-\}$ *is satisfied, then* $Tf(\underline{x}) \in I \cap Super \cap SuperC$*.*

Proof. Note that when $x_1 > 0$ and Assumption 3 is satisfied, the operator $T_{DR_1^r}$ has the same form as $T_{CA(2)}$ defined in the Definition 5.2 in [85]. Specifically,

$$T_{DR_1^r} f(\underline{x}) = T_{CA(2)} f(\underline{x} - e_1), r = +, -.$$

From Theorem 7.2 in [85], $T_{CA(2)}$ can preserve the property of $I \cap Super \cap SuperC$, i.e., $T_{CA(2)} f(\underline{x}) \in I \cap Super \cap SuperC$ if $f(\underline{x}) \in I \cap Super \cap SuperC$. Therefore, we are left with proving the result when $x_1 = 0$. In this case,

$$\begin{aligned} Tf(\underline{x} + e_1) &= l_1^+ \min\{c_1^+ + f(\underline{x}), f(\underline{x} + e_2)\} + l_1^- \min\{c_1^- + f(\underline{x}), f(\underline{x} + e_2)\} \\ &\geq f(\underline{x}) \\ &= Tf(\underline{x}), \end{aligned}$$

where the first inequality comes from the induction that $f(\underline{x}) \in I$ and $c_1^+ \geq 0, c_1^- \geq 0$. Note that $Tf(\underline{x} + e_2) = f(\underline{x} + e_2) \geq f(\underline{x}) = Tf(\underline{x})$. Thus, we prove $Tf(\underline{x}) \in I$ when $x_1 = 0$.

We next prove $Tf(\underline{x}) \in Super$ when $x_1 = 0$, which is equivalent to prove the relation $Tf(\underline{x} + e_1 + e_2) + Tf(\underline{x}) \geq Tf(\underline{x} + e_1) + Tf(\underline{x} + e_2)$. Since $T_{DR_1^+}$ and $T_{DR_1^-}$ have the same form, it is easy to observe that proving the above relation can be reduced to prove

$$T_{DR_1^+} f(\underline{x} + e_1 + e_2) + f(\underline{x}) \geq T_{DR_1^+} f(\underline{x} + e_1) + f(\underline{x} + e_2).$$

We need to consider two cases to prove the above relation. If $T_{DR_1^+} f(\underline{x} + e_1 + e_2) = c_1^+ + f(\underline{x} + e_2)$, then

$$\begin{aligned} T_{DR_1^+} f(\underline{x} + e_1 + e_2) + f(\underline{x}) &= c_1^+ + f(\underline{x} + e_2) + f(\underline{x}) \\ &\geq \min\{c_1^+ + f(\underline{x}), f(\underline{x} + e_2)\} + f(\underline{x} + e_2) \\ &= T_{DR_1^+} f(\underline{x} + e_1) + f(\underline{x} + e_2), \end{aligned}$$

where the first inequality follows from the minimum operator.

If $T_{DR_1^+} f(\underline{x}+e_1+e_2) = f(\underline{x}+2e_2)$, then

$$\begin{aligned} T_{DR_1^+} f(\underline{x}+e_1+e_2) + f(\underline{x}) &= f(\underline{x}+2e_2)+f(\underline{x}) \\ &\geq 2f(\underline{x}+e_2) \\ &\geq \min\{c_1^+ + f(\underline{x}), f(\underline{x}+e_2)\} + f(\underline{x}+e_2) \\ &= T_{DR_1^+} f(\underline{x}+e_1) + f(\underline{x}+e_2). \end{aligned}$$

To see the first inequality, we note that $f(\underline{x})$ is component-wise convex in x_1 and x_2 according to (6.2) in [85], which is $Super(i,j) \cap SuperC(i,j) \subset Cx(i)$. Thus, we prove $Tf(\underline{x}) \in Super$ when $x_1 = 0$.

The last part of the proof is to show $Tf(\underline{x}) \in SuperC$ when $x_1 = 0$, which is equivalent to prove $Tf(\underline{x}+e_2) - Tf(\underline{x}+e_1)$ is decreasing in x_1 and increasing in x_2. Since $T_{DR_1^+}$ and $T_{DR_1^-}$ have the same form, proving $T_{DR_1^+} f(\underline{x}+e_2) - T_{DR_1^+} f(\underline{x}+e_1) \geq T_{DR_1^+} f(\underline{x}+e_2+e_1) - T_{DR_1^+} f(\underline{x}+2e_1)$ implies $Tf(\underline{x}+e_2) - Tf(\underline{x}+e_1)$ is decreasing in x_1.

We need to consider two cases to prove the above relation. If $T_{DR_1^+} f(\underline{x}+2e_1) = c_1^+ + f(\underline{x}+e_1)$, then

$$\begin{aligned} & T_{DR_1^+} f(\underline{x}+e_2) + T_{DR_1^+} f(\underline{x}+2e_1) \\ = \; & f(\underline{x}+e_2) + c_1^+ + f(\underline{x}+e_1) \\ \geq \; & c_1^+ + f(\underline{x}+e_2) + c_1^+ + f(\underline{x}) \\ \geq \; & \min\{c_1^+ + f(\underline{x}+e_2), f(\underline{x}+2e_2)\} + \min\{c_1^+ + f(\underline{x}), f(\underline{x}+e_2)\} \\ = \; & T_{DR_1^+} f(\underline{x}+e_2+e_1) + T_{DR_1^+} f(\underline{x}+e_1), \end{aligned}$$

where the first inequality comes from the convexity of $f(\underline{x})$ in x_2 and the assumption that $f(1,0) - f(0,0) \geq \max\{c_1^+, c_1^-\}$, thus $f(1,x_2) - f(0,x_2) \geq c_1^+$ for any $x_2 \geq 0$.

If $T_{DR_1^+} f(\underline{x} + 2e_1) = f(\underline{x} + e_1 + e_2)$, then

$$\begin{aligned}
& T_{DR_1^+} f(\underline{x} + e_2) + T_{DR_1^+} f(\underline{x} + 2e_1) \\
= \; & f(\underline{x} + e_2) + f(\underline{x} + e_1 + e_2) \\
\geq \; & f(\underline{x} + e_2) + c_1^+ + f(\underline{x} + e_2) \\
\geq \; & \min\{c_1^+ + f(\underline{x} + e_2), f(\underline{x} + 2e_2)\} + \min\{c_1^+ + f(\underline{x}), f(\underline{x} + e_2)\} \\
= \; & T_{DR_1^+} f(\underline{x} + e_2 + e_1) + T_{DR_1^+} f(\underline{x} + e_1).
\end{aligned}$$

Similarly, proving $T_{DR_1^+} f(\underline{x} + 2e_2) - T_{DR_1^+} f(\underline{x} + e_1 + e_2) \geq T_{DR_1^+} f(\underline{x} + e_2) - T_{DR_1^+} f(\underline{x} + e_1)$ can imply that $Tf(\underline{x} + e_2) - Tf(\underline{x} + e_1)$ is increasing in x_2. We need to consider two cases to prove the above relation.

If $T_{DR_1^+} f(\underline{x} + e_1) = c_1^+ + f(\underline{x})$, then

$$\begin{aligned}
T_{DR_1^+} f(\underline{x} + 2e_2) + T_{DR_1^+} f(\underline{x} + e_1) &= f(\underline{x} + 2e_2) + c_1^+ + f(\underline{x}) \\
&\geq f(\underline{x} + e_2) + c_1^+ + f(\underline{x} + e_2) \\
&\geq T_{DR_1^+} f(\underline{x} + e_2) + T_{DR_1^+} f(\underline{x} + e_1 + e_2),
\end{aligned}$$

where the first inequality follows from the convexity of $f(\underline{x})$ in x_2.

If $T_{DR_1^+} f(\underline{x} + e_1) = f(\underline{x} + e_2)$, then

$$\begin{aligned}
T_{DR_1^+} f(\underline{x} + 2e_2) + T_{DR_1^+} f(\underline{x} + e_1) &= f(\underline{x} + 2e_2) + f(\underline{x} + e_2) \\
&\geq T_{DR_1^+} f(\underline{x} + e_2) + T_{DR_1^+} f(\underline{x} + e_1 + e_2),
\end{aligned}$$

where the inequality comes from the minimum operator in $T_{DR_1^+} f(\underline{x} + e_1 + e_2)$. Therefore, we can conclude the result in this lemma. $\square$

Lemma 9 *For the departure operator T on function $f : Z_+^2 \to \mathbf{R}$, defined as*

$$Tf(\underline{x}) = \begin{cases} c_2 + f(\underline{x} - e_2) & \text{if } x_2 > 0; \\ f(\underline{x}) & \text{if } x_2 = 0. \end{cases}$$

If $f(\underline{x}) \in I \cap Super \cap SuperC$ and $f(0,1) - f(0,0) \geq c_2$ is satisfied, then $Tf(\underline{x}) \in I \cap Super \cap SuperC$.

Proof. Note that when $x_2 > 0$, the results follow directly from induction. Thus, we focus on the proof when $x_2 = 0$. It is easy to see that $Tf(\underline{x} + e_1) \geq Tf(\underline{x})$ by induction, and

$$Tf(\underline{x} + e_2) = c_2 + f(\underline{x}) \geq Tf(\underline{x}),$$

where the inequality comes from the nonnegativity of c_2. Therefore, $Tf(\underline{x}) \in I$ when $x_2 = 0$.

It is easy to see that $Tf(\underline{x} + e_1 + e_2) + Tf(\underline{x}) = Tf(\underline{x} + e_1) + Tf(\underline{x} + e_2)$ when $x_2 = 0$, which implies $Tf(\underline{x}) \in Super$ when $x_2 = 0$.

The last part of the proof is to show $Tf(\underline{x}) \in SuperC$ when $x_2 = 0$, which consists of proving $Tf(\underline{x}+e_2) - Tf(\underline{x}+e_1)$ is decreasing in x_1 and increasing in x_2. Note that

$$\begin{aligned} Tf(\underline{x} + e_2) - Tf(\underline{x} + e_1) &= c_2 + f(\underline{x}) - f(\underline{x} + e_1) \\ &\geq c_2 + f(\underline{x} + e_1) - f(\underline{x} + 2e_1), \end{aligned}$$

where the inequality comes from the convexity of $f(\underline{x})$ in x_1.

Next is to prove that $Tf(\underline{x} + e_2) - Tf(\underline{x} + e_1)$ is increasing in x_2.

$$\begin{aligned} Tf(\underline{x} + 2e_2) - Tf(\underline{x} + e_1 + e_2) &= c_2 + f(\underline{x} + e_2) - c_2 - f(\underline{x} + e_1) \\ &\geq c_2 + f(\underline{x}) - f(\underline{x} + e_1) \\ &= Tf(\underline{x} + e_2) - Tf(\underline{x} + e_1), \end{aligned}$$

where the inequality comes from the convexity of $f(\underline{x})$ in x_1 and the assumption that $f(0,1) - f(0,0) \geq c_2$, thus $f(x_1, 1) - f(x_1, 0) \geq c_2$ for any $x_1 \geq 0$. $\square$

With the above three lemmas, we can now present the proof for the optimality of the threshold policy.

Theorem 5 (Threshold policy) *Suppose $J = 1$, Assumption 2, 3 and 4 are satisfied, and $h_1(1) \geq (1 - \lambda - \mu_2)\max\{c_1^+, c_1^-\}$, $h_2(1) \geq \alpha c_2$.*

1. *[Arrival routing.] For a given s_1, there exists a threshold $a(s_1)$, as a function of s_1, such that an arriving patient is routed to station 2 if and only if $s_2 \leq a(s_1)$.*

2. *[Departure routing.] For a given s_1, there exist thresholds $d^+(s_1)$ and $d^-(s_1)$ such that a patient with a positive (negative, respectively) test results at station 1 is routed to station 2 if and only if $s_2 \leq d^+(s_1)$ ($s_2 \leq d^-(s_1)$, respectively).*

3. *[Action structure.] $a(s_1)$ is increasing in s_1, and $d^+(s_1)$ and $d^+(s_1)$ are decreasing in s_1.*

Proof of Theorem 5. Let us first rewrite the optimality equation (4.8) when $J = 1$ and $\mu_i(s_i) = \mu_i, i = 1, 2$.

$$\begin{aligned} V_n(\underline{x}) &= h_1(s_1) + h_2(s_2) + \lambda T_{AR} V_{n+1}(\underline{x}) + \mu_1 T_{D_1} V_{n+1}(\underline{x}) + \mu_2 T_{D_2} V_{n+1}(\underline{x}) \\ &\quad + \mu_1 \mathbb{I}_{s_1=0} V_{n+1}(\underline{x}) + \mu_2 \mathbb{I}_{s_2=0} V_{n+1}(\underline{x}), \qquad (4.17) \end{aligned}$$

where T_{AR}, T_{D_1} and T_{D_2} are defined in (4.10), (4.11) and (4.12), respectively. Define $\mathcal{V}_1$ to be the set of functions $f \in \mathbf{R}^+$ on $\underline{x} = (s_1, s_2) \in Z_+^2$, such that:

1. $f(\underline{x}) \in I$, i.e., $f(\underline{x})$ is increasing in s_1 and s_2;

2. $f(\underline{x}) \in Super$, i.e., $f(\underline{x})$ is supermodular in (s_1, s_2);

3. $f(\underline{x}) \in SuperC$, i.e., $f(\underline{x} + e_2) - f(\underline{x} + e_1)$ is decreasing in s_1 and increasing in s_2.

Note that if $V_n(\underline{x}) \in \mathcal{V}_1$, the results of the switching curves can be derived accordingly. The result that the arrival routing switching curve $s_n(s_1)$ is increasing in s_1 can be immediately derived from the property of $SuperC$. According to (6.2) in [85], which is $Super(i,j) \cap SuperC(i,j) \subset Cx(i)$, we can get that $V_n(\underline{x})$ is also componentwise convex in s_1 and s_2 if $V_n(\underline{x}) \in \mathcal{V}_1$. Therefore $V_n(\underline{x} - e_1 + e_2) - V_n(\underline{x} - e_1)$ is increasing in s_2 because of the convexity in s_2, and increasing in s_1 due to the supermodularity in (s_1, s_2), which implies the departure routing switching curves $d_n(s_1, c)$ are decreasing in s_1. Thus, we are left with proving $V_n(\underline{x}) \in \mathcal{V}_1$ by induction.

Suppose $V_{n+1}(\underline{x}) \in \mathcal{V}_1$, $V_{n+1}(1,0) - V_{n+1}(0,0) \geq \max\{c_1^+, c_1^-\}$, and $V_{n+1}(0,1) - V_{n+1}(0,0) \geq c_2$. We next prove that $h_1(s_1) + h_2(s_2) \in \mathcal{V}_1$. The monotonicity in s_1

and s_2 immediately comes from our assumption that $h_i(s_i)$ is increasing in s_i, and the supermodularity follows from $h_1(s_1) + h_2(s_2)$ is separable in s_1 and s_2. For the proof of "$SuperC''$, note that

$$\begin{aligned} & h_1(\underline{x} + e_2) + h_2(\underline{x} + e_2) - h_1(\underline{x} + e_1) - h_2(\underline{x} + e_1) \\ = \; & h_2(s_2 + 1) - h_2(s_2) - (h_1(s_1 + 1) - h_1(s_1)), \end{aligned}$$

which is increasing in s_2 (respectively, decreasing in s_1) due to the convexity in s_2 (respectively, s_1), thus $h_1(s_1) + h_2(s_2) \in SuperC$, and consequently $h_1(s_1) + h_2(s_2) \in \mathcal{V}_1$.

Applying Lemma 8, 9, and Theorem 7.2 in [85], we get $V_n(\underline{x}) \in \mathcal{V}_1$. To finish the induction, it remains to show that $V_n(1, 0) - V_n(0, 0) \geq \max\{c_1^+, c_1^-\}$ and $V_n(0, 1) - V_n(0, 0) \geq c_2$.

$$\begin{aligned} & V_n(1, 0) - V_n(0, 0) \\ = \; & h_1(1) + \lambda \min\{V_{n+1}(2, 0), V_{n+1}(1, 1)\} - \lambda \min\{V_{n+1}(1, 0), V_{n+1}(0, 1)\} \\ & + \mu_1 l_1^+ \min\{V_{n+1}(0, 0) + c_1^+, V_{n+1}(0, 1)\} + \mu_1 l_1^- \min\{V_{n+1}(0, 0) + c_1^-, V_{n+1}(0, 1)\} \\ & - \mu_1 V_{n+1}(0, 0) + \mu_2 V_{n+1}(1, 0) - \mu_2 V_{n+1}(0, 0) \\ \geq \; & h_1(1) + (\lambda + \mu_2)(V_{n+1}(1, 0) - V_{n+1}(0, 0)) \\ \geq \; & \max\{c_1^+, c_1^-\}. \end{aligned}$$

To see the first inequality, we note that $V_{n+1}(\underline{x}) \in \mathcal{V}_1$, therefore $\min\{V_{n+1}(0, 0) + c_1^r, V_{n+1}(0, 1)\} - V_{n+1}(0, 0) \geq 0$, for $r = +, -$, and

$$\begin{aligned} & \min\{V_{n+1}(2, 0), V_{n+1}(1, 1)\} - \min\{V_{n+1}(1, 0), V_{n+1}(0, 1)\} \\ \geq \; & \min\{V_{n+1}(2, 0) - V_{n+1}(1, 0), V_{n+1}(1, 1) - V_{n+1}(0, 1)\} \\ \geq \; & V_{n+1}(1, 0) - V_{n+1}(0, 0). \end{aligned}$$

The proof for $V_n(0,1) - V_n(0,0) \geq c_2$ is similar.

$$\begin{aligned}
& V_n(0,1) - V_n(0,0) \\
= \;& h_2(1) + \lambda \min\{V_{n+1}(1,1), V_{n+1}(0,2)\} - \lambda \min\{V_{n+1}(1,0), V_{n+1}(0,1)\} \\
& +\mu_1(V_{n+1}(0,1) - V_{n+1}(0,0)) + \mu_2 c_2 \\
\geq \;& h_2(1) + (\lambda + \mu_1)(V_{n+1}(0,1) - V_{n+1}(0,0)) + \mu_2 c_2 \\
\geq \;& c_2.
\end{aligned}$$

Therefore we can conclude that for the optimal value function V_α, i.e.,

$$V_\alpha(\underline{x}) = \lim_{n\to\infty} V_n(\underline{x}),$$

we have $V_\alpha(\underline{x}) \in \mathcal{V}_1$. □

The optimal policy characterized in Theorem 5 is a threshold routing policy. Under this optimal policy, for a given number of patients in station 1, s_1, there exists threshold value $a(s_1)$ such that the arriving patient is routed to station 2 only when the number of patients in station 2, s_2, is less than the threshold $a(s_1)$. Similarly, there exist two threshold values $d^+(s_1)$ and $d^-(s_1)$ for the departure routing decision. These thresholds depend on the state s_1, which are referred as the switching curves. The arrival routing switching curve $a(s_1)$ is increasing in s_1. This implies that as s_1 increases, a new arrival is more likely to be routed to station 2, which is intuitive. The departure routing switching curves $d^+(s_1)$ and $d^+(s_1)$ are decreasing in s_1, which is less intuitive. A patient after station 1 test is less likely to get a second test at station 2 when s_1 is large, because under a large s_1, more future arrivals will be routed to station 2 (from the arrival routing), and the optimal departure action subsequently routes fewer departures to station 2 to alleviate the congestion.

Note that the assumptions in Theorem 5 are made to prove the optimality of the threshold policy. In next subsection, those assumptions are relaxed and we present the structural properties of a general multi-class case.

4.4.2 Dominance among Classes in the Multi-class Case

Note that the service rate function in Assumption 2 in Section 4.4.1 is a special case of the processor sharing queue. This is a necessary assumption because, even in a single-class problem, the optimality of the threshold routing policy remains an open question when using other service rate functions [85]. Thus, it would be even more challenging to characterize the optimal routing policy in the multi-class setting. Instead, we present a dominance among different classes of patients in the optimal policy in Theorem 6.

To begin with, $\mathcal{V}$ is defined to be the set of functions $f \in \mathbf{R}^+$ on $\underline{x} \in Z_+^{J+3}$, such that:

1. $f(\underline{x}) \in I(x_2^0) \cap I(x_2^1)$, i.e., $f(\underline{x})$ is increasing in x_2^0 and x_2^1;

2. $f(\underline{x}) \in UD(x_1^j)$, for $j = 0, \ldots, J-1$, i.e., $f(\underline{x} + e_1^j) \leq f(\underline{x} + e_1^{j+1})$.

Define $\tilde{\mathcal{V}}$ to be the set of functions $f \in \mathbf{R}^+$ on $\underline{x} \in Z_+^{J+3}$, such that:

1. $f(\underline{x}) \in I(x_2^0) \cap I(x_2^1)$, i.e., $f(\underline{x})$ is increasing in x_2^0 and x_2^1.

2. $f(\underline{x}) \in UI(x_1^j)$, for $j = 1, \ldots, J-1$, i.e., $f(\underline{x} + e_1^j) \geq f(\underline{x} + e_1^{j+1})$.

3. $f(\underline{x} + e_1^J) \geq f(\underline{x} + e_1^0)$.

Lemma 10 *Suppose the upper bounds s_1^u, s_2^u are finite.*

1. *Suppose $m^j \geq \sup_{\underline{x}} |f(\underline{x} + e_2^0) - f(\underline{x} + e_2^1)|$. If $f(\underline{x}) \in \mathcal{V}$, then $T_{AR^j} f(\underline{x}) \in \mathcal{V}$, $j = 1, \ldots, J$;*

2. *Suppose $m^j \geq \sup_{\underline{x}} |f(\underline{x} + e_2^0) - f(\underline{x} + e_2^1)|$. If $f(\underline{x}) \in \tilde{\mathcal{V}}$, then $T_{AR^j} f(\underline{x}) \in \tilde{\mathcal{V}}$, $j = 1, \ldots, J$,*

where T_{AR^j} is defined in (4.10).

Proof. We prove statement (1) using induction, and the proof of statement (2) is similar. To begin with, We prove that $T_{AR^j} \in I(e_2^1)$ and $T_{AR^j} \in I(e_2^0)$ for $j = 1, \ldots, J$. Depending on the state $\underline{x}_1$ and $\underline{x}_2$, we need to consider four cases.

Case 1: $s_1 < s_1^u$ and $s_2 + 1 < s_2^u$. For $k = 1, 2$,

$$\begin{aligned} T_{AR^j} f(\underline{x} + e_2^k) &= \min\{f(\underline{x} + e_1^j + e_2^k), f(\underline{x} + e_2^1 + e_2^k)\} \\ &\geq \min\{f(\underline{x} + e_1^j), f(\underline{x} + e_2^1)\} \\ &= T_{AR^j} f(\underline{x}), \end{aligned}$$

where the inequality comes from the induction that $f(\underline{x})$ is increasing in x_2^k and the minimum operator.

Case 2: $s_1 = s_1^u$ and $s_2 + 1 < s_2^u$. For $k = 1, 2$, using a similar induction argument as in the first case, we can immediately get

$$T_{AR^j} f(\underline{x} + e_2^k) = f(\underline{x} + e_2^1 + e_2^k) \geq f(\underline{x} + e_2^1) = T_{AR^j} f(\underline{x}).$$

Case 3: $s_1 < s_1^u$ and $s_2 + 1 = s_2^u$. For $k = 1, 2$, using similar arguments as in the first case, we have

$$T_{AR^j} f(\underline{x} + e_2^k) = f(\underline{x} + e_1^j + e_2^k) \geq \min\{f(\underline{x} + e_1^j), f(\underline{x} + e_2^1)\} = T_{AR^j} f(\underline{x}).$$

Case 4: $s_1 = s_1^u$ and $s_2 + 1 = s_2^u$. We prove $T_{AR^j} f(\underline{x}) \in I(e_2^1)$ first, which follows from

$$T_{AR^j} f(\underline{x} + e_2^1) = m^j + f(\underline{x} + e_2^1) \geq f(\underline{x} + e_2^1) = T_{AR^j} f(\underline{x}).$$

The inequality comes from the nonnegativity of m^j.

To prove $T_{AR^j} f(\underline{x}) \in I(e_2^0)$, we note that

$$T_{AR^j} f(\underline{x} + e_2^0) = m^j + f(\underline{x} + e_2^0) \geq f(\underline{x} + e_2^1) = T_{AR^j} f(\underline{x}),$$

where the inequality comes from the assumption that $m^j \geq \sup_{\underline{x}} |f(\underline{x}+e_2^0) - f(\underline{x}+e_2^1)|$. Therefore, we prove that $T_{AR^j} \in I(e_2^1) \cap I(e_2^0)$. Next step is to show $T_{AR^j} \in UD(x_1^k)$ for $k = 0, \ldots, J-1$. We need to consider four cases: $(1) s_1 + 1 < s_1^u$, $s_2 < s_2^u$;

(2)$s_1+1=s_1^u$, $s_2<s_2^u$; (3)$s_1+1<s_1^u$, $s_2=s_2^u$; (4)$s_1+1=s_1^u$, $s_2=s_2^u$. Each case follows via similar arguments as those in the proof of $T_{AR^j}\in I(e_2^1)\cap I(e_2^0)$, which we omit for brevity. $\square$

Lemma 11 *Define operator T on function $f: Z_+^{J+3}\to\mathbf{R}$ as $Tf(\underline{x})=\mu_2(s_2)T_{D_2}f(\underline{x})+(\bar{\mu}_2-\mu_2(s_2))f(\underline{x})$, where T_{D_2} is defined in (4.12).*

1. *If $f(\underline{x})\in\mathcal{V}$, then $Tf(\underline{x})\in\mathcal{V}$;*

2. *If $f(\underline{x})\in\tilde{\mathcal{V}}$, then $Tf(\underline{x})\in\tilde{\mathcal{V}}$,*

Proof. We exhibit the proof of statement (1), and that of statement (2) is similar. To prove statement (1), we first show that $Tf(\underline{x})$ is increasing in x_2^0. If $s_2>0$,

$$\begin{aligned}
& Tf(\underline{x}+e_2^0)-Tf(\underline{x})\\
= & \frac{\mu_2(s_2+1)}{s_2+1}\big((x_2^0+1)T_{D_2^0}f(\underline{x}+e_2^0)+x_2^1T_{D_2^1}f(\underline{x}+e_2^0)\big)+(\bar{\mu}_2-\mu_2(s_2+1))f(\underline{x}+e_2^0)\\
& -\frac{\mu_2(s_2)}{s_2}\big(x_2^0T_{D_2^0}f(\underline{x})+x_2^1T_{D_2^1}f(\underline{x})\big)+(\bar{\mu}_2-\mu_2(s_2))f(\underline{x})\\
\geq & \left(\frac{\mu_2(s_2+1)(x_2^0+1)}{s_2+1}-\frac{\mu_2(s_2)x_2^0}{s_2}\right)T_{D_2^0}f(\underline{x}+e_2^0)\\
& +\left(\frac{\mu_2(s_2+1)}{s_2+1}-\frac{\mu_2(s_2)}{s_2}\right)x_2^1T_{D_2^1}f(\underline{x})+(\mu_2(s_2)-\mu_2(s_2+1))f(\underline{x})\\
\geq & (\mu_2(s_2+1)-\mu_2(s_2))(T_{D_2^0}f(\underline{x}+e_2^0)-f(\underline{x}))\\
\geq & 0.
\end{aligned}$$

Note that $T_{D_2^0}f(\underline{x}+e_2^0)\geq T_{D_2^0}f(\underline{x})$ and $T_{D_2^1}f(\underline{x}+e_2^0))\geq T_{D_2^1}f(\underline{x})$ can be proved by induction, and thus we can get the first inequality. To get the second inequality, we first observe that

$$T_{D_2^0}f(\underline{x}+e_2^0)=c_2+f(\underline{x})\geq c_2\mathbb{I}_{x_2^1>0}+f((\underline{x}-e_2^1)^+)=T_{D_2^1}f(\underline{x}).$$

This observation together with the assumption that $\mu_2(s_2)/s_2$ is nonincreasing in s_2 suggest the second inequality. The third inequality comes from the assumption that $\mu_2(s_2)$ is nondecreasing in s_2 and the relation that $T_{D_2^0}f(\underline{x}+e_2^0)=c_2+f(\underline{x})\geq f(\underline{x})$.

If $s_2 = 0$, using similar arguments, we have

$$
\begin{aligned}
& Tf(\underline{x}+e_2^0) - Tf(\underline{x}) \\
= \ & \mu_2(1)T_{D_2^0}f(\underline{x}+e_2^0) + (\bar{\mu}_2 - \mu_2(1))f(\underline{x}+e_2^0) - \bar{\mu}_2 f(\underline{x}) \\
\geq \ & \mu_2(1)(T_{D_2^0}f(\underline{x}+e_2^0) - f(\underline{x})) + (\bar{\mu}_2 - \mu_2(1))(f(\underline{x}+e_2^0) - f(\underline{x})) \\
\geq \ & 0.
\end{aligned}
$$

Therefore we prove that $Tf(\underline{x})$ is increasing in x_2^0. Note that x_2^0 and x_2^1 are symmetric in the operator T, thus the proof of $Tf(\underline{x})$ is increasing in x_2^1 can be established in a similar way. The remaining proof of $Tf(\underline{x}) \in \mathcal{V}$ is to show $Tf(\underline{x}) \in UD(x_1^j)$, for $j = 0, \ldots, J-1$, which follows directly from the induction argument, and we omit the rest proof for brevity. $\square$

Lemma 12 *Suppose the expected diagnostic costs defined in (4.5) and (4.6) satisfy*

(i) $c_1^{j,+} \geq c_1^{j,-}$*;*

(ii) $l_1^{j,-}c_1^{j,-}$ *is increasing in* j*;*

(iii) $l_1^{j,+}c_1^{j,+} + l_1^{j,-}c_1^{j,-}$ *is increasing in* j*,*

for $j = 1, \ldots, J$*. Define operator* T *on function* $f : Z_+^{J+3} \to \mathbf{R}$ *as* $Tf(\underline{x}) = \mu_1(s_1)T_{D_1}f(\underline{x}) + (\bar{\mu}_1 - \mu_1(s_1))f(\underline{x})$*, where* T_{D_1} *is defined in (4.11). If* $f(\underline{x}) \in \mathcal{V}$*, then* $Tf(\underline{x}) \in \mathcal{V}$*.*

Proof. Recall the definition of T_{D_1} in (4.11), which is

$$
T_{D_1}f(\underline{x}) = \begin{cases} \sum_{j=1}^{J} x_1^j/s_1(l_1^{j,+}T_{DR_1^{j,+}}f(\underline{x}) + l_1^{j,-}T_{DR_1^{j,-}}f(\underline{x})) + x_1^0/s_1 T_{D_1^0}f(\underline{x}), & \text{if } s_1 > 0, \\ 0, & \text{if } s_1 = 0, \end{cases}
$$

where the operators $T_{DR_{j,+}}, T_{DR_{j,-}}$ and $T_{D_1^0}$ are defined in (4.13) and (4.14) respectively. It is easy to prove that $Tf(\underline{x}) \in I(x_2^0) \cap I(x_2^1)$ using induction, which we omit for brevity.

Next is to prove $Tf(\underline{x}) \in UD(j)$ for $j = 1, \ldots, J-1$, i.e., $Tf(\underline{x}+e_{j+1}) \geq Tf(\underline{x}+e_j)$. We first prove the result when $j = 2, \ldots, J-1$.

To compare each term between $Tf(\underline{x}+e_{j+1})$ and $Tf(\underline{x}+e_j)$, we first prove the following relation

$$\begin{aligned} & \frac{x_1^j}{s_1+1}\Big(l_1^{j,+}T_{DR_1^{j,+}}f(\underline{x}+e_{j+1}) + l_1^{j,-}T_{DR_1^{j,-}}f(\underline{x}+e_{j+1})\Big) \\ & +\frac{x_1^{j+1}+1}{s_1+1}\Big(l_1^{j+1,+}T_{DR_1^{j+1,+}}f(\underline{x}+e_{j+1}) + l_1^{j+1,-}T_{DR_1^{j+1,-}}f(\underline{x}+e_{j+1})\Big) \\ \geq\ & \frac{x_1^j+1}{s_1+1}\Big(l_1^{j,+}T_{DR_1^{j,+}}f(\underline{x}+e_j) + l_1^{j,-}T_{DR_1^{j,-}}f(\underline{x}+e_j)\Big) \\ & +\frac{x_1^{j+1}}{s_1+1}\Big(l_1^{j+1,+}T_{DR_1^{j+1,+}}f(\underline{x}+e_j) + l_1^{j+1,-}T_{DR_1^{j+1,-}}f(\underline{x}+e_j)\Big), \end{aligned} \tag{4.18}$$

which can be further broken down into proving the following three inequalities:

$$x_1^jT'f(\underline{x}+e_{j+1}) \geq x_1^jT'f(\underline{x}+e_j) \text{ for } T' = T_{DR_1^{j,+}} \text{ and } T_{DR_1^{j,-}}; \tag{4.19}$$

$$x_1^{j+1}T'f(\underline{x}+e_{j+1}) \geq x_1^{j+1}T'f(\underline{x}+e_j) \text{ for } T' = T_{DR_1^{j+1,+}} \text{ and } T_{DR_1^{j+1,-}}; \tag{4.20}$$

$$\begin{aligned} & l_1^{j+1,+}T_{DR_1^{j+1,+}}f(\underline{x}+e_{j+1}) + l_1^{j+1,-}T_{DR_1^{j+1,-}}f(\underline{x}+e_{j+1}) \\ & \geq l_1^{j,+}T_{DR_1^{j,+}}f(\underline{x}+e_j) + l_1^{j,-}T_{DR_1^{j,-}}f(\underline{x}+e_j). \end{aligned} \tag{4.21}$$

Applying the induction argument that $f(\underline{x}+e_{j+1}) \geq f(\underline{x}+e_j)$, we can obtain inequalities (4.19) and (4.20). We next show the proof of (4.21) when $s_1 > 0$, and it is easy to check (4.21) holds when $s_1 = 0$. We first consider the case when $s_2 < s_2^u$, and (4.21) can be rewritten as

$$\begin{aligned} & l_1^{j+1,+}\min\{f(\underline{x})+c_1^{j+1,+}, f(\underline{x}+e_{J+2})\} + l_1^{j+1,-}\min\{f(\underline{x})+c_1^{j+1,-}, f(\underline{x}+e_{J+2})\} \\ \geq\ & l_1^{j,+}\min\{f(\underline{x})+c_1^{j,+}, f(\underline{x}+e_{J+2})\} + l_1^{j,-}\min\{f(\underline{x})+c_1^{j,-}, f(\underline{x}+e_{J+2})\}. \end{aligned} \tag{4.22}$$

We need to consider two cases to prove this inequality.

Case 1: When the optimal departure routing decisions are $a_{\text{dep}}^{j,+}(\underline{x}+e_{j+1}) = 1$ and $a_{\text{dep}}^{j,-}(\underline{x}+e_{j+1}) = 1$,

$$\begin{aligned}
& l_1^{j+1,+}\min\{f(\underline{x})+c_1^{j+1,+}, f(\underline{x}+e_{J+2})\} + l_1^{j+1,-}\min\{f(\underline{x})+c_1^{j+1,-}, f(\underline{x}+e_{J+2})\} \\
= \; & l_1^{j+1,+}(f(\underline{x})+c_1^{j+1,+}) + l_1^{j+1,-}(f(\underline{x})+c_1^{j+1,-}) \\
\geq \; & f(\underline{x}) + l_1^{j,+}c_1^{j,+} + l_1^{j,-}c_1^{j,-} \\
\geq \; & l_1^{j,+}\min\{f(\underline{x})+c_1^{j,+}, f(\underline{x}+e_{J+2})\} + l_1^{j,-}\min\{f(\underline{x})+c_1^{j,-}, f(\underline{x}+e_{J+2})\},
\end{aligned}$$

where the first inequality comes from condition (iii), i.e., $l_1^{j,+}c_1^{j,+}+l_1^{j,-}c_1^{j,-}$ is increasing in j, and the second inequality comes from the minimum operator.

Case 2: Suppose $a_{\text{dep}}^{j,+}(\underline{x}+e_{j+1}) = 2$ and $a_{\text{dep}}^{j,-}(\underline{x}+e_{j+1}) = 1$.

$$\begin{aligned}
& l_1^{j+1,+}\min\{f(\underline{x})+c_1^{j+1,+}, f(\underline{x}+e_{J+2})\} + l_1^{j+1,-}\min\{f(\underline{x})+c_1^{j+1,-}, f(\underline{x}+e_{J+2})\} \\
= \; & l_1^{j+1,+}f(\underline{x}+e_{J+2}) + l_1^{j+1,-}(f(\underline{x})+c_1^{j+1,-}) \\
= \; & f(\underline{x}) + l_1^{j+1,+}(f(\underline{x}+e_{J+2})-f(\underline{x})) + l_1^{j+1,-}c_1^{j+1,-} \\
\geq \; & f(\underline{x}) + l_1^{j,+}(f(\underline{x}+e_{J+2})-f(\underline{x})) + l_1^{j,-}c_1^{j,-} \\
\geq \; & l_1^{j,+}\min\{f(\underline{x})+c_1^{j,+}, f(\underline{x}+e_{J+2})\} + l_1^{j,-}\min\{f(\underline{x})+c_1^{j,-}, f(\underline{x}+e_{J+2})\}
\end{aligned}$$

where the first inequality comes from the assumption that $l_1^{j+1,+} \geq l_1^{j,+}$ and the induction that $f(\underline{x})$ is increasing in x_2, which together gives the inequality $l_1^{j+1,+}(f(\underline{x}+e_{J+2})-f(\underline{x})) \geq l_1^{j,+}(f(\underline{x}+e_{J+2})-f(\underline{x}))$, and the assumption that $l_1^{j+1,-}c_1^{j+1,-}(-) \geq l_1^{j,-}c_1^{j,-}$.

When $a_{\text{dep}}^{j,+}(\underline{x}+e_{j+1}) = 2$ and $a_{\text{dep}}^{j,-}(\underline{x}+e_{j+1}) = 2$, proving (4.22) is straightforward. Since we assume that $c_1^{j+1,+} \geq c_1^{j+1,-}$, the case when $a_{\text{dep}}^{j,+}(\underline{x}+e_{j+1}) = 1$ and $a_{\text{dep}}^{j,-}(\underline{x}+e_{j+1}) = 2$ is not possible in the optimal policy. Thus, we prove (4.22) when $s_2 < s_2^u$. Similar reasoning as in case 1 can be used to show (4.22) holds when $s_2 = s_2^u$.

It is easy to prove the following inequalities by induction,

$$\begin{aligned}
&l_1^{k,+} T_{DR_{k,+}} f(\underline{x} + e_{j+1}) + l_1^{k,-} T_{DR_{k,-}} f(\underline{x} + e_{j+1}) \\
&\quad \geq l_1^{k,+} T_{DR_{k,+}} f(\underline{x} + e_j) + l_1^{k,-} T_{DR_{k,-}} f(\underline{x} + e_j), \\
&T_{D_1^0} f(\underline{x} + e_{j+1}) \geq T_{D_1^0} f(\underline{x} + e_j), \\
&(\bar{\mu}_1 - \mu_1(s_1)) f(\underline{x} + e_{j+1}) \geq (\bar{\mu}_1 - \mu_1(s_1)) f(\underline{x} + e_j),
\end{aligned}$$

where $k = 1, \ldots, J(\neq j, j+1)$. Therefore, combing (4.18), we can conclude $Tf(\underline{x}) \in UD(j)$ for $j = 2, \ldots, J-1$.

The last part is to prove $Tf(\underline{x}) \in UD(1)$, i.e., $Tf(\underline{x}+e_1) \geq Tf(\underline{x}+e_0)$. As in the proof of $Tf(\underline{x}) \in UD(j)$ for $j = 2, \ldots, J-1$, it suffices to show that

$$l_1^{1,+} T_{DR_1^{1,+}} f(\underline{x} + e_1) + l_1^{1,-} T_{DR_1^{1,-}} f(\underline{x} + e_1) \geq f(\underline{x}) = T_{D_1^0} f(\underline{x} + e_0),$$

where the inequality follows from the nonnegativity of the diagnostic costs $c_1^{1,+}$ and $c_1^{1,-}$ and the induction that $f(\underline{x})$ is increasing in x_2. Therefore, we prove $Tf(\underline{x}) \in UD(j)$ for $j = 1, \ldots, J-1$, and conclude the result that $Tf(\underline{x}) \in \mathcal{V}$. □

Theorem 6 *Suppose test sensitivity $b_1^{j,+}$ and specificity $b_1^{j,-}$ are independent of j, and the diagnostic costs satisfy*

(i) $c_1^{j,+} \geq c_1^{j,-}$, *for* $j = 1, \ldots, J$, *and*

(ii) $c_1^{FN}(1 - b_1^+) \geq c_1^{TN} b_1^-$, *and*

(iii) $c_1^{FN}(1 - b_1^+) + c_1^{TP} b_1^+ \geq c_1^{TN} b_1^- + c_1^{FP}(1 - b_1^-)$.

Then,

1. $a_{arr}^{j_1}(\underline{x}) \leq a_{arr}^{j_2}(\underline{x})$, *for* $1 \leq j_1 < j_2 \leq J$.
2. $a_{dep}^{j,+}(\underline{x}) \geq a_{dep}^{j,-}(\underline{x})$, *for* $1 \leq j \leq J$.

Proof of Theorem 6. First note that the second result of this theorem can be directly proved by $c_1^{j,+} \geq c_1^{j,-}$. To prove the first result, we need to prove $V_\alpha(\underline{x}) \in \mathcal{V}$ by induction, then the results can be implied by $V_\alpha(\underline{x}) \in UD(x_1^j)$ for $j = 0, \ldots, J-1$.

We first prove the following inequality using a sample path argument.

$$|V_\alpha(\underline{x}+e_2^0) - V_\alpha(\underline{x}+e_2^1)| < \infty. \tag{4.23}$$

We construct two processes on the same probability space. Process 1 begins at state $(\underline{x}+e_2^0)$ and follows the optimal policy π, while process 2 starts at state $(\underline{x}+e_2^1)$ and follows the same decision as process 1 for each event. Under this construction, process 1 and 2 are "parallel" to each other until two following events occur:

- Process 1 is in state $(\underline{x}_1, 1, x_2^1)$ and process 2 is in state $(\underline{x}_1, 0, x_2^1{+}1)$ or $(\underline{x}_1, 0, x_2^1)$. Process 1 sees a service completion in station 2 from an external patient, and it is not observed by process 2.

- Process 1 is in state $(\underline{x}_1, x_2^0{+}1, 0)$ or $(\underline{x}_1, x_2^0, 0)$ and process 2 is in state $(\underline{x}_1, x_2^0, 1)$. Process 2 sees a service completion in station 2 from a class 1 to J patient, and it is not observed by process 1.

After both events happen, two processes couple and behave identically from this time onward. Denote Δ as the difference in cost between two processes before coupling occurs, and we have $\mathbb{E}\Delta = V_\alpha^\pi(\underline{x}{+}e_2^1) - V_\alpha(\underline{x}{+}e_2^0)$. We observe that the cost differences for events 1 and 2 are $-c_2$ and c_2 (modulo discounting effect) respectively. Other than above two events, the cost difference only lies in the station 2 holding cost, which is obviously finite. This implies $\Delta < \infty$ pathwise. Therefore, we can conclude that

$$V_\alpha(\underline{x}+e_2^1) - V_\alpha(\underline{x}+e_2^0) \leq \mathbb{E}\Delta < \infty.$$

Using a similar argument, we can show $V_\alpha(\underline{x}+e_2^0) - V_\alpha(\underline{x}+e_2^1) < \infty$, and complete the proof for Equation (4.23).

Suppose $V_\alpha(\underline{x}) \in \mathcal{V}$. We first show that $T_{A_1^0} V_\alpha(\underline{x}) \in \mathcal{V}$. We omit the proof for $T_{A_1^0} V_\alpha(\underline{x}) \in I(x_2^0) \cap I(x_2^1)$ since it can be easily proved by induction. Next we prove that $T_{A_1^0} V_\alpha \in UD(x_1^j)$, for $j = 0, \ldots, J-1$. When $s_1 + 1 < s_1^u$, induction argument immediately gives

$$T_{A_1^0} V_\alpha(\underline{x}+e_1^{j+1}) = V_\alpha(\underline{x}+e_1^{j+1}+e_1^0) \geq V_\alpha(\underline{x}+e_1^j+e_1^0) = T_{A_1^0} V_\alpha(\underline{x}+e_1^j).$$

When $s_1+1=s_1^u$,

$$T_{A_1^0}V_\alpha(\underline{x}+e_1^{j+1}) = m_1^0 + V_\alpha(\underline{x}+e_1^{j+1}) \geq m_1^0 + V_\alpha(\underline{x}+e_1^{j}) = T_{A_1^0}V_\alpha(\underline{x}+e_1^{j}).$$

Thus we prove $T_{A_1^0}V_\alpha(\underline{x}) \in \mathcal{V}$.

Next we prove $T_{A_2^0}V_\alpha(\underline{x}) \in \mathcal{V}$. The proof of $T_{A_2^0}V_\alpha(\underline{x}) \in I(x_2^0)$ and $T_{A_2^0}V_\alpha(\underline{x}) \in UD(j)$ for $j=0,\ldots,J-1$ follow directly from induction argument, and we focus on proving $T_{A_2^0}V_\alpha(\underline{x}) \in I(x_2^1)$. When $s_2+1<s_2^u$, using induction that $V_\alpha(\underline{x}) \in I(x_2^1)$, we have

$$T_{A_2^0}V_\alpha(\underline{x}+e_2^1) = V_\alpha(\underline{x}+e_2^0+e_2^1) \geq V_\alpha(\underline{x}+e_2^0) = T_{A_2^0}V_\alpha(\underline{x}).$$

When $s_2+1=s_2^u$,

$$T_{A_2^0}V_\alpha(\underline{x}+e_2^1) = m_2^0 + V_\alpha(\underline{x}+e_2^1) \geq V_\alpha(\underline{x}+e_2^0) = T_{A_2^0}V_\alpha(\underline{x}).$$

Due to Equation (4.23), we can always find a large enough m_2^0 to ensure the above inequality.

The last part in our proof is to check the conditions in Lemma 12. Note that

$$l_1^{j,-}c_1^{j,-} = p_0^j(1-b_1^{j,+})c_1^{\mathrm{FN}} + (1-p_0^j)b_1^{j,-}c_1^{\mathrm{TN}}, \tag{4.24}$$

$$l_1^{j,+}c_1^{j,+} = p_0^j b_1^{j,+}c_1^{\mathrm{TP}} + (1-p_0^j)(1-b_1^{j,-})c_1^{\mathrm{FP}}. \tag{4.25}$$

Assume $b_1^{j,+}$ and $b_1^{j,-}$ are independent of j, then condition (ii) in this theorem gives $l_1^{j,-}c_1^{j,-}$ is increasing in j, and condition (iii) gives $l_1^{j,+}c_1^{j,+}+l_1^{j,-}c_1^{j,-}$ is increasing in j. Applying the results in Lemma 10, 11 and 12, we can conclude the results. □

For condition (i) to be satisfied, **the negative test result should be more accurate than a positive result** at station 1, which is the case for the D-dimer test in our study for the PE disease. In result (2) of the theorem, a patient is more likely to be discharged from the system after a negative test result at station 1, than that after a positive test result. This also explains result (1) of the theorem: if the negative test result is more accurate, then it is preferable to use the test on low risk patients to *rule out* the disease. In other words, if a low risk patient is routed to station 2 upon arrival, a high risk patient must also be routed to station 2.

In the next proposition, the opposite result is presented when the positive test result at station 1 is more accurate than a negative result. In this case, it is preferred to using the test on high risk patients to *confirm* the disease.

Lemma 13 *Suppose the expected diagnostic costs defined in (4.5) and (4.6) satisfy*

(i) $c_1^{j,+} \leq c_1^{j,-}$*;*

(ii) $l_1^{j,+} c_1^{j,+}$ *is increasing in* j*;*

(iii) $l_1^{j,+} c_1^{j,+} + l_1^{j,-} c_1^{j,-}$ *is decreasing in* j*,*

for $j = 1, \ldots, J$*. Define operator* T *on function* $f : Z_+^3 \rightarrow \mathbf{R}$ *as* $Tf(\underline{x}) = \mu_1(s_1) T_{D_1} f(\underline{x}) + (\bar{\mu}_1 - \mu_1(s_1)) f(\underline{x})$*, where* T_{D_1} *is defined in (4.11). If* $f(\underline{x}) \in \tilde{\mathcal{V}}$*, then* $Tf(\underline{x}) \in \tilde{\mathcal{V}}$*.*

Proof. The result can be proved in a similar way as in Lemma 12. □

Proposition 6 *Suppose* $b_i^{j,+}$ *and* $b_i^{j,-}$ *are independent of* j*, and the diagnostic costs satisfy*

(i) $c_1^{j,+} \leq c_1^{j,-}$ *for* $j = 1, \ldots, J$*, and*

(ii) $c_1^{TP} b_1^+ \leq c_1^{FP}(1 - b_1^-)$*, and*

(iii) $c_1^{TP} b_1^+ + c_1^{FN}(1 - b_1^+) \leq c_1^{FP}(1 - b_1^-) + c_1^{TN} b_1^-$*.*

Then,

1. $a_{arr}^{j_1}(\underline{x}) \geq a_{arr}^{j_2}(\underline{x})$*, for* $1 \leq j_1 < j_2 \leq J$*.*
2. $a_{dep}^{j,+}(\underline{x}) \leq a_{dep}^{j,-}(\underline{x})$*, for* $1 \leq j \leq J$*.*

Proof of Proposition 6. The results can be proved in a similar way as in Theorem 6 by applying the results in Lemma 10, 11 and 13. □

4.5 Heuristic Algorithms

Due to the curse-of-dimensionality, the MDP model described in Section 4.3.2 becomes computationally challenging to solve when $J \geq 3$. For example, when $J = 3$, $s_1^u = s_2^u = 20$, the size of the state space is $20^6 = 6.4 \times 10^7$. Even storing all $V_n(\underline{x})$ is difficult, let alone computing the optimal value function and finding the optimal policy. In the working paper [1], the authors develop a basic heuristic algorithm and an refinement based on the decomposition idea to address this challenge. We describe their basic heuristic algorithm in Section 4.5.1. In Section 4.5.2, we conduct numerical experiments to validate the performance of their heuristic algorithms.

4.5.1 Heuristic Description

The curse-of-dimensionality of the original MDP model is caused by the fact that the number of patients in station 1 for each of the J classes needs to be kept track of. To address this challenge, a heuristic algorithm is developed in [1] that obtains the arrival and departure routing decisions for each class j using a class-specific two-dimensional MDP, which only keeps track of the total number of patients in station 1 and station 2, (s_1, s_2). For the two-dimensional MDP of each class j, the routing of other classes is approximated by a static policy that routes class k patients to station 1 with a fixed probability p_r^k, referred to as *fixed-probability routing policy.*

Specifically, the heuristic gives the class j arrival and departure routing decisions, which comes from first solving $V^j(s_1, s_2)$ from

$$\begin{aligned}
& V^j(s_1, s_2) \\
= \quad & h_1(s_1\underline{q_1}) + h_2(s_2\underline{q_2}) + \lambda^j T_{AR^j} V^j(s_1, s_2) \\
& +(\lambda_1^{ex} + \sum_{k \neq j} \lambda^k p_r^k) T_{A_1^0} V^j(s_1, s_2) + (\lambda_2^{ex} + \sum_{k \neq j} \lambda^k (1 - p_r^k)) T_{A_2^0} V^j(s_1, s_2) \\
& +\mu_1(s_1) T_{\tilde{D}_1} V^j(s_1, s_2) + \mu_2(s_2) T_{\tilde{D}_2} V^j(s_1, s_2) \\
& +(\bar{\mu}_1 - \mu_1(s_1)) V^j(s_1, s_2) + (\bar{\mu}_2 - \mu_2(s_2)) V^j(s_1, s_2), \qquad (4.26)
\end{aligned}$$

and obtaining the arrival routing or departure routing action from

$$\tilde{a}^j_{\mathrm{arr}}(s_1,s_2)=\begin{cases}0 & \text{if } T_{AR^j}V^j(s_1,s_2)=m^j+V^j(s_1,s_2),\\ 1 & \text{if } T_{AR^j}V^j(s_1,s_2)=V^j(s_1+1,s_2),\\ 2 & \text{if } T_{AR^j}V^j(s_1,s_2)=V^j(s_1,s_2+1);\end{cases}$$

for $s_1>0$ and $r=+,-$,

$$\tilde{a}^{j,r}_{\mathrm{dep}}(s_1,s_2)=\begin{cases}1 & \text{if } T_{DR_1^{j,r}}V^j(s_1,s_2)=c_1^{j,r}+V^j(s_1-1,s_2),\\ 2 & \text{if } T_{DR_1^{j,r}}V^j(s_1,s_2)=V^j(s_1-1,s_2+1).\end{cases}$$

Here, in Equation (4.26), $s_1\underline{q_1}=(s_1q_1^0,\ldots,s_1q_1^J)$ and $s_2\underline{q_2}=(s_2q_2^0,s_2q_2^1)$.

The operators $T_{A_i^0}$ and T_{AR^j} remain the same as in (4.9) and (4.10), respectively. The main approximations for Equation (4.8) are replacing the departure operators T_{D_1} and T_{D_2} with the following two:

$$T_{\tilde{D}_1}f(s_1,s_2)=\begin{cases}\sum_{k=1}^J q_1^k(l_1^{k,+}T_{DR_1^{k,+}}f(s_1,s_2) \\ \quad +l_1^{k,-}T_{DR_1^{k,-}}f(s_1,s_2))+q_1^0T_{D_1^0}f(s_1,s_2), & \text{if } s_1>0,\\ 0, & \text{otherwise,}\end{cases}\tag{4.27}$$

$$T_{\tilde{D}_2}f(s_1,s_2)=\begin{cases}q_2^0T_{D_2^0}f(s_1,s_2)+q_2^1T_{D_2^1}f(s_1,s_2), & \text{if } s_2>0,\\ 0, & \text{otherwise,}\end{cases}\tag{4.28}$$

for a function $f:\mathbf{Z}_+^2\to\mathbf{R}$. In $T_{\tilde{D}_1}$ and $T_{\tilde{D}_2}$, the operators $T_{DR_1^{k,r}}, T_{D_1^0}, T_{D_2^0}$ and $T_{D_2^1}$ are defined in (4.11) - (4.14), respectively. p_r^k in (4.26) is an approximation for the probability that a class k arrival will be routed to station 1. q_1^j is an approximation for the proportion of class j patients in station 1, and q_2^0 (q_2^1) is an approximation for the proportion of exogenous patients (class 1 to J patients) in station 2. In the heuristic, the proportions $\underline{q_1} = (q_1^0, \ldots, q_1^J)$ and $\underline{q_2} = (q_2^0, q_2^1)$ are obtained from the routing probabilities $(p_r^1, \ldots, p_r^J)$ as follows.

$$q_1^0 = \frac{\lambda_1^{\text{ex}}}{\lambda_1^{\text{ex}} + \sum_{k=1}^J \lambda^k p_r^k}, \tag{4.29}$$

$$q_1^j = \frac{\lambda^j p_r^j}{\lambda_1^{\text{ex}} + \sum_{k=1}^J \lambda^k p_r^k}, \quad j = 1, \ldots, J, \tag{4.30}$$

$$q_2^0 = \frac{\lambda_2^{\text{ex}}}{\lambda_2^{\text{ex}} + \sum_{k=1}^J \lambda^k (1 - p_r^k)}, \tag{4.31}$$

$$q_2^1 = \frac{\sum_{k=1}^J \lambda^k (1 - p_r^k)}{\lambda_2^{\text{ex}} + \sum_{k=1}^J \lambda^k (1 - p_r^k)}. \tag{4.32}$$

This heuristic algorithm is developed in [1] based on the decomposition idea, and we briefly describe how the heuristic approximates the original MDP model below; see [1] for the detailed and formal explanations of the rationale of this decomposition idea.

For class j's two-dimensional MDP, it is essentially assumed that among the s_1 patients that are currently in station 1, there are $q_1^j s_1$ patients that are from class j, i.e.,$x_1^j \approx q_1^j s_1, j = 0, \ldots, J$, and among the s_2 patients in station 2, there are $q_2^0 s_2$ external patients and $q_2^1 s_2$ patients that are from class 1 to J. Therefore, the holding cost $h_1(\underline{x_1}) + h_2(\underline{x_2})$ in Equation (4.8) is approximated by $h_1(s_1 \underline{q_1}) + h_2(s_2 \underline{q_2})$ in Equation (4.26). This also explains the approximation for the departure operators $T_{\tilde{D}_1}$ and $T_{\tilde{D}_2}$.

In solving the heuristic for class j, a static routing policy with probability p_r^k is assumed to be used for future arrivals of other classes k, where $k \neq j$, i.e., a class k patients will be routed to station 1 with probability p_r^k, and to station 2 with probability $1 - p_r^k$. In other words, patients in these $J - 1$ other classes form exogenous arrival processes. In Section 4.5.2, we show that our heuristic performance is robust to different routing probabilities. Therefore a good approximation of $(p_r^1, \ldots, p_r^J)$ is not required for the heuristic algorithm to perform well. See [1] for the reasons why a fixed-probability routing policy is chosen in the approximation.

4.5.2 Validation

In this section, we compare the long-run average cost from the routing policies obtained from the heuristic algorithms developed in [1] and the optimal policy obtained from value iteration. We refer to the policy obtained from the heuristic algorithm described in Section 4.5.1 as *basic heuristic policy*. In [1], the authors also develop a refinement for the basic heuristic algorithm, which we refer to as *refined heuristic policy*. Due to the curse of dimensionality, we can only use value iteration to solve a three-class patient problem (i.e., $J = 3$) without arrivals of exogenous patients to station 1 (i.e., $\lambda_1^{\text{ex}} = 0$).

Table 4.1 reports the average cost under different system load conditions. The optimality gap for each heuristic policy is defined as *(heuristic policy cost - optimal cost) / optimal cost*. We can observe that the optimality gaps are less than 1.5% under all load conditions.

Robustness. Table 4.2 summarizes the impact of routing probabilities on the heuristic performance. We take the heavily loaded setting in Table 4.1, and change each class routing probability from 10% to 100%. We can observe that the optimality gaps in all cases of both basic heuristic and refined heuristic remain smaller than 1.5%.

Impact of other parameters. In Table 4.3, we compare the optimality gaps under a variety of system conditions, and we can observe that the optimality gaps of the basic heuristic policy are less than 3% under most settings, and the maximum optimality gap is 5.18%. The optimality gaps of the refined heuristic policy are less than 1.50% under all settings.

Table 4.1.: Long-run average daily costs of different policies

	Lightly loaded	Intermediately loaded	Heavily loaded
Optimal	6.90 ± 0.01	11.49 ± 0.02	15.74 ± 0.03
Basic heuristic (Sec. 4.5.1)	6.98 ± 0.01	11.65 ± 0.02	15.86 ± 0.04
Optimality gap	1.29%	1.33%	0.76%
Refined heuristic ([1])	6.92 ± 0.01	11.50 ± 0.02	15.75 ± 0.02
Optimality gap	0.31%	0.05%	0.09%

Notes. We set $J = 3, p_0^1 = 0.05, p_0^2 = 0.35, p_0^3 = 0.65$. For $j = 1,2,3, b_1^{j,+} = 0.9819, b_1^{j,-} = 0.4249, \lambda^j = 0.2$. $\lambda_1^{\mathrm{ex}} = 0, \lambda_2^{\mathrm{ex}} = 0.6, \mu_1(x) = 0.9x^{0.5}$. $c_1^{\mathrm{TP}} = c_1^{\mathrm{TN}} = c_2 = 0, c_1^{\mathrm{FP}} = 100, c_1^{\mathrm{FN}} = 800, h_1(\underline{x}) = 6s_1, h_2(\underline{x}) = 6s_2$. Routing probabilities for the heuristic policies are uniformly set as $(0.6, 0.6, 0.6)$. Left panel: $\mu_2(x) = 2.2$; Middle panel: $\mu_2(x) = 1.7$; Right panel: $\mu_2(x) = 1.45$. The number after the $\pm$ sign is the half-width of the corresponding 95% confidence interval.

Table 4.2.: Impact of routing probability

	p_r^1				p_r^2				p_r^3			
	0.1	0.3	0.6	1	0.1	0.3	0.6	1	0.1	0.3	0.6	1
Basic	0.80%	0.84%	0.76%	0.76%	0.93%	0.82%	0.76%	0.76%	1.03%	0.91%	0.76%	0.73%
Refined	0.22%	0.18%	0.09%	0.00%	0.24%	0.17%	0.09%	0.53%	0.96%	0.53%	0.09%	0.43%

Note. $\mu_1(x) = 0.9x^{0.5}, \mu_2(x) = 1.45$. In each panel, the other two classes routing probabilities are set as 0.6. Other parameters are the same as those in Table 4.1.

4.5.3 Scenarios where Refined Heuristic Policies Show a Large Improvement

In this section, we compare the performance of the heuristic policy and other simple routing policies under a numerical setting which is populated from the dataset in [1]. In particular, we compare the long-run average costs from the following policies.

- Refined heuristic policy. The policy is obtained from the refined heuristic algorithm proposed by [1].
- All first test. All suspected patients are sent to the first test upon arrival.

Table 4.3.: Optimality gaps under different parameter settings

		Basic heuristic (from Sec. 4.5.1)	Refined heuristic (from [1])
Change unit holding cost rate h	2	**2.83**%	0.19%
	4	1.47%	0.17%
	6 (base)	0.76%	0.09%
	8	0.48%	0.16%
	10	0.41%	0.20%
Change false-negative cost c_1^{FN}	400	0.44%	0.09%
	600	0.59%	0.10%
	800 (base)	0.76%	0.09%
	1000	1.08%	0.05%
	1200	1.23%	−0.01%
Change pre-test prior	Base setting	0.76%	0.09%
	Reduce medium prior	0.36%	0.06%
	Reduce high prior	1.23%	0.08%
Change patient mix of low-, medium-, and high-risk patients	Base setting	0.76%	0.09%
	More low-risk	0.70%	0.15%
	More medium-risk	0.78%	0.21%
	More high-risk	0.68%	0.06%
Change proportion between three classes patients and external arrivals	More external arrivals	0.31%	0.00%
	Base setting	0.76%	0.09%
	No external arrivals	**3.59**%	0.43%
Change station 2 service rate function	1.45 (base)	0.76%	0.09%
	$0.57x^{0.5}$	**1.92**%	0.01%
	$0.25x$	1.31%	0.00%
Change holding cost function	Linear (base)	0.76%	0.09%
	Linear (different h for each class)	**5.18**%	1.40%
	Quadratic	**2.03**%	0.24%

Note. $\mu_1(x) = 0.9x^{0.5}, \mu_2(x) = 1.45$. Other parameters in each panel are the same as those in Table 4.1.

- Class-dependent policy. All low- and medium-risk patients are routed to the first test upon arrival and high-risk patients are routed to the second test.
- Point threshold policy. There is a set of (y^1, y^2, y^3) such that a class j patient is routed to the second test if $s_2 < y^j$.

For the "all first test" and class-dependent policies, the departure routing decision is fixed. We consider two forms of threshold policies: with and without fixing the departure routing decisions. Fixing the departure routing means that a patient with a positive result from the first test is always sent to the second test, regardless of the cost or occupancy of the second test, whereas a patient with a negative test result is always sent home. This departure routing agrees with the current clinical practice of diagnosing the specific disease studied in [1].

The value we report in Table 4.4 is the benefit gained by using the refined heuristic policy, which can be calculated as:

$$\text{Benefit of heuristic} = \frac{V_\pi - V_{refine}}{V_{refine}}, \tag{4.33}$$

where V_{refine} is the long-run average cost under the refined heuristic policy, and V_π is the long-run average cost under the other policies.

In most parameter settings, the benefits gained from the refined heuristic policy against the threshold policy with fixed departure routing is the smallest, which is more than 5%. The benefits against the other three policies are more than 10%. Note that the policies under the fixed departure routing decisions perform better than the full threshold policy, which validates the effectiveness of the current clinical practice.

4.6 Summary

In this chapter, we provide a case study for the framework developed in [1]. We first describe the test routing problem in their framework. The optimal routing policy is a threshold policy when the service process at each station is an $M/M/1$ queue. For

Table 4.4.: Benefit gained by the refined heuristic policy

		Fix departure routing (Fix DR)			Relax DR
		All first test	Class-dependent	Threshold	Full threshold
Change unit holding cost rate h	2	6.33%	7.10%	6.43%	60.00%
	4	7.33%	8.59%	0.46%	34.09%
	6 (base)	10.76%	12.24%	6.43%	21.86%
	8	14.48%	16.11%	10.26%	14.94%
	10	18.51%	20.25%	14.29%	11.19%
Change false-negative cost c_1^{FN}	400	10.39%	12.05%	6.56%	22.86%
	600	10.52%	12.08%	6.43%	22.34%
	800(base)	10.76%	12.24%	6.43%	21.86%
	1000	10.95%	12.34%	6.38%	21.53%
	1200	10.99%	12.30%	6.19%	20.94%
Change station 2 service rate function	1.91 (base)	10.76%	12.24%	6.43%	21.86%
	$1.20x^{0.3}$	1.96%	2.08%	0.22%	17.95%
	$0.90x^{0.5}$	1.48%	1.32%	0.02%	6.86%
Change holding cost function	Linear (base)	10.76%	12.24%	6.43%	21.86%
	Quadratic	36.12%	42.56%	33.07%	44.84%

Note. For the base setting, we set $J = 3, p_0^1 = 0.05, p_0^2 = 0.15, p_0^3 = 0.20$. For $j = 1, 2, 3, b_1^{j,+} = 0.9819, b_1^{j,-} = 0.4249, \lambda^j = 0.217$. $\lambda_1^{\mathrm{ex}} = 0, \lambda_2^{\mathrm{ex}} = 1.06, \mu_1(x) = 0.96x, \mu_2(x) = 0.9x^{0.5}$. $c_1^{\mathrm{TP}} = c_1^{\mathrm{TN}} = c_2 = 0, c_1^{\mathrm{FP}} = 100, c_1^{\mathrm{FN}} = 800, h_1(\underline{x}) = 6s_1, h_2(\underline{x}) = 6s_2$. Routing probabilities for the heuristic policies are uniformly set as $(0.6, 0.6, 0.6)$. The quadratic holding cost function is set as $h_1(\underline{x}) = 0.6s_1^2$, $h_2(\underline{x}) = 0.6s_2^2$ in the last panel.

the general service rate functions, there exists a dominance among different classes in the optimal policy when the model parameters satisfy proper conditions.

To overcome the curse of dimensionality in the multi-class MDP model, we describe a heuristic algorithm proposed in [1]. We numerically validate the performance of the heuristic algorithm, and find that the largest optimality gap is less than 1.50% in all the experiments.

5. CONCLUSION AND DIRECTION FOR FUTURE RESEARCH

In this chapter, we conclude the findings of this dissertation and discuss several important future research directions.

In the first part of this dissertation, we model a two-stage supply chain using dynamic programming framework. We consider both centralized and decentralized control. Under centralized control, the problem is a nonconcave optimization problem because of the random supply. We extend the notion of stochastic linearity in midpoint developed by [21] to a multi-dimensional problem and transfer the problem of deciding the order quantity into one of deciding the distribution function of the supply. The objective function after transformation becomes concave, and the optimal value function is concave and submodular in the echelon inventory level. The optimal policy is a threshold policy. When the terminal value is separable in the inventory positions, the associated profit function becomes separable, and the threshold policy reduces to the echelon base-stock policy. We further show that our analysis can be extended when the upstream production follow a general stochastic function, which satisfies the single-crossing property for stochastic linearity in midpoint. Moreover, we show that our analysis can be carried out for serial systems with multiple locations.

The structural properties and the optimal policy under the centralized control are important for the contract design under the decentralized control. When the supply information is not available to the retail store, we find a transfer payment to coordinate the supply chain. The idea behind the transfer payment is to compensate the retail store for the loss of future value caused by insufficient supply from the manufacturing facility. In addition, the result is robust to any mis-specification of the supply capacity from the retail store. When the demand information is not available to the manufacturing facility as well as the supply information is not available to the retail store, we show there exist transfer contracts to achieve coordination. We further propose an iterative algorithm to compute the transfer contracts in our decentralized setting, and prove the convergence of the decentralized total profit to the optimal channel profit. The convergence does not depend on any information structure.

In summary, the developments from Chapter 2 and 3 expands our understanding of supply uncertainty in multi-stage supply chains in both centralized and decentralized systems. The coordinating mechanisms proposed here do not require sophisticated knowledge structure, making them more practical compared with conventional mechanisms.

In the last part of this dissertation, we provide a case study for the framework developed in [1]. We describe their MDP model for the test routing problem. The optimal policy is a threshold policy when the service process of each test corresponds to an $M/M/1$ queue. For the general setting, there exists a dominance among different classes: if the negative test result is more accurate, it is preferable to use the test on low-risk patients to rule out the disease; on the other hand, it is preferred to using the test on high-risk patients to confirm the disease if the positive test result is more accurate. We describe an efficient heuristic algorithm developed in [1] to overcome the curse of dimensionality of the MDP model. We validate the numerical performance of their heuristic algorithm and find that the largest optimality gap is less than 1.50% in all the numerical experiments conducted.

Next, we briefly discuss several interesting questions remain for further exploration.

Integrate dynamic pricing decision. For the first problem under centralized control, it would be interesting to add pricing decision to the model and analyze an integrated decision-making process. There is very limited work looking at both the dynamic pricing and inventory control under supply uncertainty, and most of these studies focus on a single-stage system (e.g., [3], [18], [19]).

Incentive issues in the coordination. For the first problem under decentralized control, we assume the retail store and manufacturing facility will follow the contract or iterative process to achieve coordination. An interesting question is to study whether each individual will deviate and manipulate the reported value function in the iterative process.

Continuous pretest probability. In our case study of the framework developed by [1], the pretest probability for each class is assumed to be the same. This assumption is mainly for analytical convenience. In reality, each class may be stratified based on a threshold for the pretest probability. For example, any patients with pretest probability lower than 5% are classified as low-risk patients. It would be interesting to see how to incorporate this into the MDP model or address this issue in the heuristic algorithm.

REFERENCES

REFERENCES

[1] J. Helm, P. Shi, S. Heese, and A. Mitchell, "An operational framework for the integration of medical research into clinical environments," *Working Paper*, 2019.

[2] F. W. Ciarallo, R. Akella, and T. E. Morton, "A periodic review, production planning model with uncertain capacity and uncertain demandoptimality of extended myopic policies," *Management Science*, vol. 40, no. 3, pp. 320–332, 1994.

[3] Q. Feng, "Integrating dynamic pricing and replenishment decisions under supply capacity uncertainty," *Management Science*, vol. 56, no. 12, pp. 2154–2172, 2010.

[4] A. J. Clark and H. Scarf, "Optimal policies for a multi-echelon inventory problem," *Management science*, vol. 6, no. 4, pp. 475–490, 1960.

[5] H. Zhang and S. Zenios, "A dynamic principal-agent model with hidden information: Sequential optimality through truthful state revelation," *Operations Research*, vol. 56, no. 3, pp. 681–696, 2008.

[6] D. F. Ransohoff and A. R. Feinstein, "Problems of spectrum and bias in evaluating the efficacy of diagnostic tests," *New England Journal of Medicine*, vol. 299, no. 17, pp. 926–930, 1978.

[7] P. F. Griner, R. J. Mayewski, A. I. Mushlin, and P. Greenland, "Selection and interpretation of diagnostic tests and procedures," *Annals of internal medicine*, vol. 94, no. 4 II, 1981.

[8] G. Guyatt, D. Rennie, M. Meade, D. Cook *et al.*, *Users' guides to the medical literature: a manual for evidence-based clinical practice.* AMA press Chicago, 2002, vol. 706.

[9] Q. Feng, Z. Ma, J. G. Shanthikumar, and P. Shi, "Multi-stage supply chain with supply uncertainty," *Working Paper*, 2018.

[10] R. Anupindi and R. Akella, "Diversification under supply uncertainty," *Management Science*, vol. 39, no. 8, pp. 944–963, 1993.

[11] J. M. Swaminathan and J. G. Shanthikumar, "Supplier diversification: effect of discrete demand," *Operations research letters*, vol. 24, no. 5, pp. 213–221, 1999.

[12] M. Dada, N. C. Petruzzi, and L. B. Schwarz, "A newsvendor's procurement problem when suppliers are unreliable," *Manufacturing & Service Operations Management*, vol. 9, no. 1, pp. 9–32, 2007.

[13] A. Federgruen and N. Yang, "Selecting a portifolio of suppliers under demand and supply risks." vol. 56, no. 4, pp. 916–936, 2008.

[14] ——, "Optimal supply diversification under general supply risks." vol. 57, no. 6, pp. 1451–1468, 2009.

[15] H. L. Lee and C. A. Yano, "Production control in multistage systems with variable yield losses," *Operations Research*, vol. 36, no. 2, pp. 269–278, 1988.

[16] M. Henig and Y. Gerchak, "The structure of periodic review policies in the presence of random yield," *Operations Research*, vol. 38, no. 4, pp. 634–643, 1990.

[17] Y. Wang and Y. Gerchak, "Periodic review production models with variable capacity, random yield, and uncertain demand," *Management science*, vol. 42, no. 1, pp. 130–137, 1996.

[18] Q. Li and S. Zheng, "Joint inventory replenishment and pricing control for systems with uncertain yield and demand," *Operations Research*, vol. 54, no. 4, pp. 696–705, 2006.

[19] X. Chao, H. Chen, and S. Zheng, "Joint replenishment and pricing decisions in inventory systems with stochastically dependent supply capacity," *European Journal of Operational Research*, vol. 191, no. 1, pp. 142–155, 2008.

[20] Q. Feng and R. Shi, "Sourcing from multiple suppliers for price-dependent demands," *Production and Operations Management*, vol. 21, no. 3, pp. 547–563, 2012.

[21] Q. Feng and J. G. Shanthikumar, "Supply and demand functions in inventory models." *Operations Research*, vol. 66, no. 1, pp. 77–91, 2018.

[22] A. Federgruen and P. Zipkin, "Computational issues in an infinite-horizon, multiechelon inventory model," *Operations Research*, vol. 32, no. 4, pp. 818–836, 1984.

[23] F. Chen and Y.-S. Zheng, "Lower bounds for multi-echelon stochastic inventory systems," *Management Science*, vol. 40, no. 11, pp. 1426–1443, 1994.

[24] K. H. Shang and J.-S. Song, "Newsvendor bounds and heuristic for optimal policies in serial supply chains," *Management Science*, vol. 49, no. 5, pp. 618–638, 2003.

[25] D. G. Lawson and E. L. Porteus, "Multistage inventory management with expediting," *Operations Research*, vol. 48, no. 6, pp. 878–893, 2000.

[26] D. D. Yao, "Optimal run quantities for an assembly system with random yields," *IIE transactions*, vol. 20, no. 4, pp. 399–403, 1988.

[27] W. Luo and K. Shang, "Joint inventory and cash management for multidivisional supply chains," *Operations Research*, vol. 63, no. 5, pp. 1098–1116, 2015.

[28] R. P. Parker and R. Kapuscinski, "Optimal policies for a capacitated two-echelon inventory system," *Operations Research*, vol. 52, no. 5, pp. 739–755, 2004.

[29] G. Janakiraman and J. A. Muckstadt, "A decomposition approach for a class of capacitated serial systems," *Operations Research*, vol. 57, no. 6, pp. 1384–1393, 2009.

[30] P. Glasserman and S. Tayur, "The stability of a capacitated, multi-echelon production-inventory system under a base-stock policy," *Operations Research*, vol. 42, no. 5, pp. 913–925, 1994.

[31] ——, "Sensitivity analysis for base-stock levels in multiechelon production-inventory systems," *Management Science*, vol. 41, no. 2, pp. 263–281, 1995.

[32] C. Speck and J. Van der Wal, "The capacitated multiechelon inventory system with serial structure: 2. an average cost approximation method," *Memorandum COSOR*, pp. 91–40, 1991.

[33] W. T. Huh, G. Janakiraman, and M. Nagarajan, "Capacitated serial inventory systems: sample path and stability properties under base-stock policies," *Operations Research*, vol. 58, no. 4-part-1, pp. 1017–1022, 2010.

[34] ——, "Capacitated multiechelon inventory systems: Policies and bounds," *Manufacturing & Service Operations Management*, vol. 18, no. 4, pp. 570–584, 2016.

[35] A. Angelus and W. Zhu, "Looking upstream: Optimal policies for a class of capacitated multi-stage inventory systems," *Production and Operations Management*, 2017.

[36] J. Hwang and M. R. Singh, "Optimal production policies for multi-stage systems with setup costs and uncertain capacities," *Management Science*, vol. 44, no. 9, pp. 1279–1294, 1998.

[37] M. Shaked and J. G. Shanthikumar, *Stochastic orders*. Springer Science & Business Media, 2007.

[38] G. P. Cachon, "Competitive supply chain inventory management," in *Quantitative models for supply chain management*. Springer, 1999, pp. 111–146.

[39] ——, "Supply chain coordination with contracts," *Handbooks in operations research and management science*, vol. 11, pp. 227–339, 2003.

[40] G. P. Cachon and P. H. Zipkin, "Competitive and cooperative inventory policies in a two-stage supply chain," *Management science*, vol. 45, no. 7, pp. 936–953, 1999.

[41] F. Bernstein, F. Chen, and A. Federgruen, "Coordinating supply chains with simple pricing schemes: The role of vendor-managed inventories," *Management Science*, vol. 52, no. 10, pp. 1483–1492, 2006.

[42] G. P. Cachon and M. A. Lariviere, "Supply chain coordination with revenue-sharing contracts: strengths and limitations," *Management science*, vol. 51, no. 1, pp. 30–44, 2005.

[43] Q. Li and D. Atkins, "Coordinating replenishment and pricing in a firm," *Manufacturing & Service Operations Management*, vol. 4, no. 4, pp. 241–257, 2002.

[44] F. Zhang, "Competition, cooperation, and information sharing in a two-echelon assembly system," *Manufacturing & Service Operations Management*, vol. 8, no. 3, pp. 273–291, 2006.

[45] S. Y. Tang and P. Kouvelis, "Pay-back-revenue-sharing contract in coordinating supply chains with random yield," *Production and Operations Management*, vol. 23, no. 12, pp. 2089–2102, 2014.

[46] S. E. Chick, H. Mamani, and D. Simchi-Levi, "Supply chain coordination and influenza vaccination," *Operations Research*, vol. 56, no. 6, pp. 1493–1506, 2008.

[47] H. Gurnani and Y. Gerchak, "Coordination in decentralized assembly systems with uncertain component yields," *European Journal of Operational Research*, vol. 176, no. 3, pp. 1559–1576, 2007.

[48] X. Yan, M. Zhang, and K. Liu, "A note on coordination in decentralized assembly systems with uncertain component yields," *European Journal of Operational Research*, vol. 205, no. 2, pp. 469–478, 2010.

[49] M. J. Fry, R. Kapuscinski, and T. L. Olsen, "Coordinating production and delivery under a (z, z)-type vendor-managed inventory contract," *Manufacturing & Service Operations Management*, vol. 3, no. 2, pp. 151–173, 2001.

[50] R. P. Parker and R. Kapuscinski, "Managing a noncooperative supply chain with limited capacity," *Operations research*, vol. 59, no. 4, pp. 866–881, 2011.

[51] H. Zhang, M. Nagarajan, and G. Sošić, "Dynamic supplier contracts under asymmetric inventory information," *Operations Research*, vol. 58, no. 5, pp. 1380–1397, 2010.

[52] M. Çakanyıldırım, Q. Feng, X. Gan, and S. P. Sethi, "Contracting and coordination under asymmetric production cost information," *Production and Operations Management*, vol. 21, no. 2, pp. 345–360, 2012.

[53] J. Marschak, "Elements for a theory of teams," *Management science*, vol. 1, no. 2, pp. 127–137, 1955.

[54] R. Radner, "Team decision problems," *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 857–881, 1962.

[55] H. Cai and A. E. Lim, "Decentralized control of a multi-agent stochastic dynamic resource allocation problem," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on.* IEEE, 2011, pp. 6400–6406.

[56] P. Li, A. E. Lim, and J. G. Shanthikumar, "Decentralized control of a stochastic multi-agent queueing system," *Automatic Control, IEEE Transactions on*, vol. 57, no. 11, pp. 2762–2777, 2012.

[57] H. Cai, "Decentralized control of stochastic dynamic systems with applications to resource allocation and portfolio management," 2012.

[58] N. Gans, G. Koole, and A. Mandelbaum, "Telephone call centers: Tutorial, review, and research prospects," *Manufacturing & Service Operations Management*, vol. 5, no. 2, pp. 79–141, 2003.

[59] W. J. Hopp, S. M. Iravani, and G. Y. Yuen, "Operations systems with discretionary task completion," *Management Science*, vol. 53, no. 1, pp. 61–77, 2007.

[60] L. G. Debo, L. B. Toktay, and L. N. Van Wassenhove, "Queuing for expert services," *Management Science*, vol. 54, no. 8, pp. 1497–1512, 2008.

[61] K. S. Anand, M. F. Paç, and S. Veeraraghavan, "Quality–speed conundrum: Trade-offs in customer-intensive services," *Management Science*, vol. 57, no. 1, pp. 40–56, 2011.

[62] R. A. Shumsky and E. J. Pinker, "Gatekeepers and referrals in services," *Management Science*, vol. 49, no. 7, pp. 839–856, 2003.

[63] S. Hasija, E. J. Pinker, and R. A. Shumsky, "Staffing and routing in a two-tier call centre," *International Journal of Operational Research*, vol. 1, no. 1-2, pp. 8–29, 2005.

[64] X. Wang, L. G. Debo, A. Scheller-Wolf, and S. F. Smith, "Design and analysis of diagnostic service centers," *Management Science*, vol. 56, no. 11, pp. 1873–1890, 2010.

[65] S. Alizamir, F. De Véricourt, and P. Sun, "Diagnostic accuracy under congestion," *Management Science*, vol. 59, no. 1, pp. 157–171, 2013.

[66] M. Armony, S. Israelit, A. Mandelbaum, Y. N. Marmor, Y. Tseytlin, G. B. Yom-Tov *et al.*, "On patient flow in hospitals: A data-based queueing-science perspective," *Stochastic Systems*, vol. 5, no. 1, pp. 146–194, 2015.

[67] T. Collings and C. Stoneman, "The m/m/ queue with varying arrival and departure rates," *Operations Research*, vol. 24, no. 4, pp. 760–773, 1976.

[68] A. El-Sherbiny, H. Zamani, N. Ismail, G. Mokaddis, M. Ghazal, A. El-Desokey, M. El-Sherbeny, E. Al-Esayeh, and O. Olotu, "Transient solution to an infinite server queue with varying arrival and departure rate," *Journal of Mathematics and Statistics*, vol. 6, no. 1, pp. 1–3, 2010.

[69] R. Bekker, S. C. Borst, O. J. Boxma, and O. Kella, "Queues with workload-dependent arrival and service rates," *Queueing Systems*, vol. 46, no. 3-4, pp. 537–556, 2004.

[70] R. Bekker, "Finite-buffer queues with workload-dependent service and arrival rates," *Queueing Systems*, vol. 50, no. 2-3, pp. 231–253, 2005.

[71] S. Yashkov, "Processor-sharing queues: Some progress in analysis," *Queueing systems*, vol. 2, no. 1, pp. 1–17, 1987.

[72] F. Guillemin and J. Boyer, "Analysis of the m/m/1 queue with processor sharing via spectral theory," *Queueing Systems*, vol. 39, no. 4, pp. 377–397, 2001.

[73] S. C. Borst, O. J. Boxma, J. A. Morrison, and R. N. Queija, "The equivalence between processor sharing and service in random order," *Operations Research Letters*, vol. 31, no. 4, pp. 254–262, 2003.

[74] W. Whitt, "Predicting response times in processor-sharing queues," *Analysis of Communication Networks: Call Centres, Traffic, and Performance*, vol. 28, p. 1, 2000.

[75] J. Braband, "Waiting time distributions for processor sharing queues with state-dependent arrival and service rates," in *International Conference on Modelling Techniques and Tools for Computer Performance Evaluation*. Springer, 1994, pp. 111–122.

[76] S. Stidham, "Optimal control of admission to a queueing system," *IEEE Transactions on Automatic Control*, vol. 30, no. 8, pp. 705–713, 1985.

[77] S. Stidham Jr and R. Weber, "A survey of markov decision models for control of networks of queues," *Queueing systems*, vol. 13, no. 1-3, pp. 291–314, 1993.

[78] B. L. Miller, "A queueing reward system with several customer classes," *Management Science*, vol. 16, no. 3, pp. 234–245, 1969.

[79] B. Zhang and H. Ayhan, "Optimal admission control for tandem queues with loss," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 163–167, 2013.

[80] W. Winston, "Optimality of monotonic policies for multiple-server exponential queuing systems with state-dependent arrival rates," *Operations Research*, vol. 26, no. 6, pp. 1089–1094, 1978.

[81] J. M. George and J. M. Harrison, "Dynamic control of a queue with adjustable service rate," *Operations research*, vol. 49, no. 5, pp. 720–731, 2001.

[82] B. Ata and S. Shneorson, "Dynamic control of an m/m/1 service system with adjustable arrival and service rates," *Management Science*, vol. 52, no. 11, pp. 1778–1791, 2006.

[83] S. Yoon and M. E. Lewis, "Optimal pricing and admission control in a queueing system with periodically varying parameters," *Queueing Systems*, vol. 47, no. 3, pp. 177–199, 2004.

[84] B. Hajek, "Optimal control of two interacting service stations," *IEEE Transactions on Automatic Control*, vol. 29, no. 6, pp. 491–499, 1984.

[85] G. Koole, *Monotonicity in Markov reward and decision chains: Theory and applications*. Now Publishers Inc, 2007, vol. 1.

[86] A. S. Elstein, L. S. Shulman, and S. A. Sprafka, "Medical problem solving an analysis of clinical reasoning," 1978.

[87] H. Schmidt, G. Norman, and H. Boshuizen, "A cognitive perspective on medical expertise: theory and implications," *Academic medicine*, vol. 65, no. 10, pp. 611–621, 1990.

[88] K. W. Eva, "What every teacher needs to know about clinical reasoning," *Medical education*, vol. 39, no. 1, pp. 98–106, 2005.

[89] G. R. Norman and L. R. Brooks, "The non-analytical basis of clinical reasoning," *Advances in health sciences education*, vol. 2, no. 2, pp. 173–184, 1997.

[90] G. R. Norman, "The epistemology of clinical reasoning: perspectives from philosophy, psychology, and neuroscience," *Academic Medicine*, vol. 75, no. 10, pp. S127–S133, 2000.

[91] D. Simel and D. Rennie, *The Rational Clinical Examination; Evidence-Based Clinical Diagnosis (JAMA & Archives Journals)*. McGraw-Hill, 2008.

[92] H. S. Barrows and G. C. Pickell, *Developing clinical problem-solving skills: a guide to more effective diagnosis and treatment*. Ww Norton, 1991.

[93] J. P. Kassirer, R. I. Kopelman, and J. B. Wong, *Learning clinical reasoning*. Williams & Wilkins Baltimore, 1991.

[94] G. Bordage, "Elaborated knowledge: a key to successful diagnostic thinking." *Academic medicine*, 1994.

[95] G. H. Guyatt, P. Tugwell, D. Feeny, R. Haynes, and M. Drummond, "A framework for clinical evaluation of diagnostic technologies." *CMAJ: Canadian Medical Association Journal*, vol. 134, no. 6, p. 587, 1986.

[96] W. Whitt and X. Zhang, "A data-driven model of an emergency department," *Operations Research for Health Care*, vol. 12, pp. 1–15, 2017.

VITA

VITA

Zhongjie Ma was born in Shanghai, P. R. China. He received his Bachelor of Science Degrees in Industrial Engineering from Shanghai Jiao Tong University. In 2013, he entered the Ph.D. program in Operations Management at Purdue University.