HILBERT FUNCTIONS OF GENERAL HYPERSURFACE RESTRICTIONS AND LOCAL COHOMOLOGY FOR MODULES

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ABSTRACT

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In this thesis, we study invariants of graded modules over polynomial rings. In particular, we find bounds on the Hilbert functions and graded Betti numbers of certain modules. This area of research has been widely studied, and we discuss several well-known theorems and conjectures related to these problems. Our main results extend some known theorems from the case of homogeneous ideals of polynomial rings R to that of graded R-modules. In Chapters 2 & 3, we discuss preliminary material needed for the following chapters. This includes monomial orders for modules, Hilbert functions, graded Betti numbers, and generic initial modules.

In Chapter 4, we discuss x_n-stability of submodules M of free R-modules F, and use this stability to examine properties of lexsegment modules. Using these tools, we prove our first main result: a general hypersurface restriction theorem for modules. This theorem states that, when restricting to a general hypersurface of degree j, the Hilbert series of M is bounded above by that of $M^{\text{lex}} + x_n^j F$. In Chapter 5, we discuss Hilbert series of local cohomology modules. As a consequence of our general hypersurface restriction theorem, we give a bound on the Hilbert series of $H^i_{\mathfrak{m}}(F/M)$. In particular, we show that the Hilbert series of local cohomology modules of a quotient of a free module does not decrease when the module is replaced by a quotient by the lexicographic module M^{lex}.

The content of Chapter 6 is based on joint work with Gabriel Sosa. The main theorem is an extension of a result of Caviglia and Sbarra to polynomial rings with base field of any characteristic. Given a homogeneous ideal containing both a piecewise lex ideal and an ideal generated by powers of the variables, we find a lex ideal with the following property: the ideal in the polynomial ring generated by the piecewise lex ideal, the ideal of powers, and the lex ideal has the same Hilbert function and Betti numbers at least as large as those of the original ideal. This bound on the Betti numbers is sharp, and is a closer bound than what was previously known in this setting.

1. Introduction

1.1 Notation

Let $R = K[x_1, x_2, \ldots, x_n]$, a polynomial ring over a field K of arbitrary characteristic. For some results throughout this thesis, we will need to make an assumption on the characteristic of the field, and we will note this when necessary. Let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$, the homogeneous maximal ideal of R. Throughout this thesis, $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$ is a graded free R-module with basis elements $e_1, e_2, \ldots, e_r$ of degrees $\delta_1, \delta_2, \ldots, \delta_r$.

1.2 Hilbert Functions of Lexsegment Modules

Lexsegment modules are an important object in the study of many invariants in commutative algebra. Recall that the Hilbert function of a graded R-module M is the function $H : \mathbb{Z} \to \mathbb{Z}$ defined by $H(d) = \dim_K M_d$. A main result on the Hilbert functions of graded modules over a polynomial ring R is the following:

Theorem 1.2.1 *[28][26] If M is a graded R-submodule of F, then there is a lexsegment R-module $M^{lex} \subset F$ with the same Hilbert function as M.*

In the rank one case, that is, for homogeneous ideals of the polynomial ring R, this statement is a classical result of Macaulay [28]. The extension of Macaulay's Theorem to the case of graded R-submodules of F was proved in 1995 by Hulett [26]. Results showing Hilbert functions of homogeneous ideals of certain rings remain unchanged when replaced by a lexicographic ideal followed Macaulay's Theorem. One of the earliest related results is Clements and Lindström's Theorem:

Theorem 1.2.2 *[9] If I is a homogeneous R-ideal containing an ideal P generated by powers of the variables, then there is a lexsegment ideal $L \subset R$ such that I and $L + P$ have the same Hilbert function.*

A natural extension of this theorem is to determine whether this statement holds when I more generally contains a regular sequence of elements of R. This was proposed by Eisenbud, Green, and Harris as the following conjecture:

Conjecture 1.2.3 (EGH Conjecture) *[12], [13] If I is a homogeneous R-ideal containing a regular sequence $f_1, f_2, \ldots, f_r$ of degrees $e_1 \leq e_2 \leq \cdots \leq e_r$, for some $1 \leq r \leq n$ and $P = (x_1^{e_1}, x_2^{e_2}, \ldots, x_r^{e_r})$, then there is a lexsegment ideal $L \subset R$ such that I and $L + P$ have the same Hilbert function.*

The conjecture was proved to hold in a large number of cases by Caviglia and Maclagan [5]. They showed that the EGH Conjecture is true, assuming the gaps between the degrees $e_1, e_2, \ldots, e_r$ are large enough. Other results related to this conjecture have been shown by Abedelfatah [1], Caviglia and Kummini [4], Chong [8], Otwinowska [33], and Mermin and Peeva [31].

1.3 Hypersurface Restriction and Local Cohomology Modules

In Chapters 4 & 5 of this thesis, we study the extension to modules of some well-known results on extremal behavior of Hilbert functions. We first consider Hilbert functions of graded R-ideals and homogeneous R-modules when restricting to a general hypersurface. When working with general forms we will always assume that the field K is infinite. The earliest known result in this area is Green's Hyperplane Restriction Theorem:

Theorem 1.3.1 *[22] If I is a homogeneous R-ideal, g is a general linear form of R, then for all d,*

$$\dim_K(I + (g))_d \geq \dim_K(I^{lex} + (x_n))_d.$$

Green originally used this theorem to give an alternate proof of Macaulay's Theorem. Greco later extended this result to the case of graded submodules of free R-modules [21]. In another direction, Herzog and Popescu, and later Gasharov, showed that the inequality of Green's theorem holds when g is a general form of arbitrary degree:

Theorem 1.3.2 *[19], [24] If I is a homogeneous R-ideal, g is a general form of degree j, then for all d,*

$$\dim_K(I + (g))_d \geq \dim_K(I^{lex} + (x_n^j))_d.$$

Herzog and Popescu proved this theorem in the characteristic zero case, and Gasharov showed that it holds in arbitrary characteristic. Caviglia and Kummini proved the corresponding result, restricting to a general hypersurface, for embeddings of Hilbert functions in characteristic zero [3]. In Chapter 4, we use techniques of Caviglia and Kummini to prove a hypersurface restriction theorem for modules. We prove that the K-vector space dimensions of $(M + gF)_d$ for any general homogeneous element $g \in R_j$ are bounded below by those of $(M + x_n^j F)_d$ for each degree d, Theorem 4.3.2.

Our next main result concerns Hilbert functions of local cohomology modules. Sbarra considered local cohomology modules of quotient rings R/I, for homogeneous ideals I. He proved the following theorem:

Theorem 1.3.3 *[38] If I is a homogeneous R-ideal and I^{lex} is the lexsegment R-ideal with the same Hilbert function as I, then for all i and for all d,*

$$\dim_K H^i_{\mathfrak{m}}(R/I)_d \leq \dim_K H^i_{\mathfrak{m}}(R/I^{lex})_d.$$

Later, Caviglia and Sbarra studied local cohomology modules of embedded ideals. A consequence of their theorem was that Hilbert functions of local cohomology modules satisfy an inequality similar to the one of Theorem 1.3.3 in the lex-plus-powers setting. Their result is the following:

Theorem 1.3.4 *[7] If I is a homogeneous R-ideal containing an ideal P generated by powers of the variables and L is the lexsegment R-ideal such that $L+P$ has the same Hilbert function as I, then for all i and for all d,*

$$\dim_K H^i_{\mathfrak{m}}(R/I)_d \leq \dim_K H^i_{\mathfrak{m}}(R/(L+P))_d.$$

In Chapter 5, we extend some of the concepts used in the proof of Theorem 1.3.4 to the case of graded R-submodules $M \subset F$ to study the modules $H^i_{\mathfrak{m}}(F/M)$. We use these properties and our general hypersurface restriction theorem to prove Sbarra's theorem in this setting. That is, $\dim_K H^i_{\mathfrak{m}}(F/M)_d \leq \dim_K H^i_{\mathfrak{m}}(F/M^{\text{lex}})_d$ for all d, Theorem 5.2.10.

1.4 Graded Betti Numbers and Piecewise Lexsegment Ideals

Graded Betti numbers are another invariant that are widely studied in this setting. Recall that the graded Betti numbers, β_{ij}, are defined as K-vector space dimensions of Tor modules, $\text{Tor}_i^R(K,M)_j$. One of the first results in this area is a theorem of Bigatti, Hulett, and Pardue on the graded Betti numbers of homogeneous ideals of a polynomial ring:

Theorem 1.4.1 *[2],[25],[35] If I is a homogeneous R-ideal, then for all i,j,*

$$\beta_{ij}(I) \leq \beta_{ij}(I^{lex}).$$

This theorem was independently proved by Bigatti and Hulett in the characteristic zero case, and later shown to be true for arbitrary characteristic by Pardue. A natural question to ask following this result is whether the same inequality on the graded Betti numbers holds in the setting of the EGH Conjecture. This was conjectured by Evans:

Conjecture 1.4.2 (Lex-Plus-Powers Conjecture) *[17] If I is a homogeneous R-ideal containing an ideal of powers P, then for all i,j,*

$$\beta_{ij}(I) \leq \beta_{ij}(L+P),$$

where L is a lexsegment ideal such that I and $L+P$ have the same Hilbert function.

The Lex-Plus-Powers Conjecture is known to be true in very few cases, see for example results of Francisco [16] and Richert [36]. Mermin and Murai proved that the Lex-Plus-Powers Conjecture holds when $f_1, f_2, \ldots, f_r$ is a regular sequence of monomials. In this case, the Eisenbud-Green-Harris conjecture easily follows from Clements and Lindström's Theorem 1.2.2. The result of Mermin and Murai is the following theorem:

Theorem 1.4.3 *[30] Suppose I is a homogeneous R-ideal containing a regular sequence of monomials $f_1, f_2, \ldots, f_r$ for some $r \leq n$. Let $e_i = \deg f_i$, where $e_1 \leq e_2 \leq \cdots \leq e_r$. Then, there is a lexsegment R-ideal L such that I and $L + P$ have the same Hilbert function and $\beta_{ij}(I) \leq \beta_{ij}(L + P)$ for all i, j.*

In Chapter 6 of this thesis, we study graded Betti numbers of homogeneous ideals under a different set of assumptions. Suppose I is a homogeneous R-ideal containing $\widetilde{L} + P$, where $\widetilde{L}$ is a piecewise lex ideal. Piecewise lex ideals are R-ideals of the form $\widetilde{L} = L_{(1)}R + L_{(2)}R + \cdots + L_{(n)}R$, where each $L_{(i)}$ is a lexsegment ideal of the polynomial ring $R_{(i)} = K[x_1, x_2, \ldots, x_i]$. Shakin studied Hilbert functions and Betti numbers of homogeneous ideals in rings of the form $R/\widetilde{L}$. He showed that these rings satisfy both Macaulay's theorem and Bigatti, Hulett, and Pardue's inequality on the graded Betti numbers. These statements are summarized in the following theorem:

Theorem 1.4.4 *[40] If $\widetilde{L}$ is a piecewise lex ideal and I is a homogeneous R-ideal containing $\widetilde{L}$, then there is a lexsegment ideal $L \subset R$ such that I and $L + \widetilde{L}$ have the same Hilbert function. Furthermore, for all i, j, $\beta_{ij}(I) \leq \beta_{ij}(L + \widetilde{L})$.*

Inspired by these results and Clements and Lindström's Theorem, Caviglia and Sbarra [6] proved these statements hold over rings of the form $R/(\widetilde{L} + P)$, where $P = (x_1^{e_1}, x_2^{e_2}, \ldots, x_r^{e_r})$. That is, for any homogeneous ideal I containing $\widetilde{L} + P$, there is a lexsegment ideal $L \subset R$ such that I and $L + \widetilde{L} + P$ have the same Hilbert function. Additionally, for all i, j, $\beta_{ij}(I) \leq \beta_{ij}(L + \widetilde{L} + P)$, but unfortunately, this upper bound on the graded Betti numbers was only shown when $\operatorname{char}(K) = 0$. Our main theorem of this chapter states that the upper bound on the Betti numbers holds over fields K of arbitrary characteristic.

1.5 Outline

In Chapters 2 & 3, we begin by presenting the basic tools needed for the proofs of our main theorems in the following chapters. The main topic of Chapter 2 is monomial orders for modules. The two main monomial orders that we will use in this thesis are the lexicographic and reverse lexicographic orders on F. We discuss these orders in detail and explain how they are obtained by extending the corresponding monomial orders on R to the free module. We next introduce Gröbner bases for modules and Buchberger's algorithm, which will play a key role in the proof of the existence of the generic initial module, Theorem 3.4.3. Chapter 2 concludes with a discussion of weight orders and homogenization. The technique of homogenization will be used in our proof of the stability of generic initial modules, Proposition 3.5.4.

Chapter 3 opens with a discussion of Hilbert functions and graded Betti numbers for graded modules, which are the main invariants we study in this thesis. We present, without proof, Propositions 3.2.2 and 3.3.2, which are standard results in this area of study that allow us to move from a graded module M to its initial module $\mathrm{in}_{\succ}(M)$ in many of our proofs. The rest of Chapter 3 is dedicated to the topic of generic initial modules. We explore the structure of generic initial modules and show that they have a certain type of stability. This chapter concludes with Proposition 3.5.6, which gives a bound on the Hilbert function of a graded module, in terms of its generic initial module, when restricting to a general hypersurface. This statement is one of the key components of the proof of our general hypersurface restriction theorem for modules, Theorem 4.3.2.

In Chapter 4, we introduce stability of an R-module with respect to the variable x_n by extending the corresponding definition for homoogeneous ideals given in [3]. We then use x_n-stability to study the structure of lexsegment modules, specifically in

Proposition 4.2.8 and Lemma 4.2.9. These results lead to the proof of the main result of this chapter, a general hypersurface restriction theorem for modules, Theorem 4.3.2. We discuss local cohomology for modules in Chapter 5. We first reduce our problem to the setting of x_n-stable monomial submodules N of F, and then prove some useful properties of the Hilbert functions of local cohomology modules of F/N. This chapter culminates with our proof that, for graded R-modules $M \subset F$, the Hilbert functions of $H^i_{\mathfrak{m}}(F/M)$ are bounded above by those of $H^i_{\mathfrak{m}}(F/M^{\text{lex}})$, Theorem 5.2.10.

In the final chapter, we study the graded Betti numbers of homogeneous R-ideals containing a piecewise lex ideal $\widetilde{L}$ and an ideal P generated by powers of the variables. We discuss shifting and compression of homogeneous ideals, following Mermin and Murai [30], and prove that the operations defined in their work preserve $\widetilde{L} + P$. The main result of this chapter states that, for any homogeneous ideal I containing $\widetilde{L} + P$, there is a lexsegment ideal L such that I and $P + \widetilde{L} + L$ have the same Hilbert function, and the graded Betti numbers of I are bounded above by those of $P + \widetilde{L} + L$. We conclude with an example, demonstrating that our bound is sharp and is a closer bound than that of Mermin and Murai.

2. Monomial Orders and Gröbner Bases for Modules

2.1 Introduction

Monomial orders are essential in the study of invariants of graded modules. In this chapter, we will introduce monomial orders, along with Gröbner bases for modules. Using monomial orders, we will define specific classes of monomial modules that we will study throughout this thesis, specifically lexsegment modules, initial modules, and generic initial modules.

2.2 Monomial Orders for Modules

In this section, we will discuss monomial orders for modules. A standard reference for this material is Eisenbud [10]. We also reference Ene-Herzog [14].

Let $R = K[x_1, ..., x_n]$, a polynomial ring in n variables over a field K, and let $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$ a graded free R-module with basis $e_1, e_2, \ldots, e_r$. For each $i = 1, 2, \ldots, r$, let $\delta_i = \deg e_i$. After reordering, we can assume that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_r$. Furthermore, after shifting the module F, if necessary, we assume that all of the δ_i are nonnegative.

The set of monomials of the polynomial ring R will be denoted by $\mathrm{Mon}(R)$. An element of $\mathrm{Mon}(R)$ can be written as $\underline{x}^{\underline{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $a_i \geq 0$ for all i. The total degree of a monomial $\underline{x}^{\underline{a}} \in R$ is $|\underline{a}| = \sum_{i=1}^{n} a_i$. Similarly, the set of monomials of the free R-module F will be denoted by $\mathrm{Mon}(F)$. An element of $\mathrm{Mon}(F)$ has the form $\underline{x}^{\underline{a}} e_j$, where $\underline{x}^{\underline{a}} \in \mathrm{Mon}(R)$. The total degree of a monomial $\underline{x}^{\underline{a}} e_j \in F$ is $|\underline{a}| + \delta_j$. For any degree d, the graded component of R of total degree d is denoted by R_d. Similarly, F_d consists of the homogeneous elements of F of total degree d.

Definition 2.2.1 *(see [10, Section 15.2]) A* **monomial order** *on F is a total order $\succ$ on the set $Mon(F)$ satisfying the following two conditions:*

(a) $nm \succ m$ for every $m \in Mon(F)$ and $1 \neq n \in Mon(R)$; and

(b) if $m, m' \in Mon(F)$ with $m \succ m'$, then $nm \succ nm'$ for all $n \in Mon(R)$.

Notice that in the rank one case, that is, when $r = 1$, this definition agrees with the usual definition of a monomial order on the ring R. Given a monomial order $\succ_R$ on the polynomial ring R, we will discuss two methods to extend $\succ_R$ to a monomial order on the free R-module F. Then, in the examples that follow, we will recall some standard monomial orders on R and discuss their extensions to orders on $\mathrm{Mon}(F)$.

Definition 2.2.2 *Suppose $\succ_R$ is a monomial order on R.*

1. *Define a total order $\succ_F$ on $Mon(F)$ so that for all $\underline{x}^{\underline{a}}e_j, \underline{x}^{\underline{b}}e_k \in Mon(F)$, $\underline{x}^{\underline{a}}e_j \succ_F \underline{x}^{\underline{b}}e_k$ if $\underline{x}^{\underline{a}} \succ_R \underline{x}^{\underline{b}}$, or if $\underline{x}^{\underline{a}} = \underline{x}^{\underline{b}}$ and $j < k$. Then $\succ_F$ is a* **term over position (TOP)** *monomial order on F.*

2. *Define a total order $\succ_F$ on $Mon(F)$ so that for all $\underline{x}^{\underline{a}}e_j, \underline{x}^{\underline{b}}e_k \in Mon(F)$, $\underline{x}^{\underline{a}}e_j \succ_F \underline{x}^{\underline{b}}e_k$ if $j < k$, or if $j = k$ and $\underline{x}^{\underline{a}} \succ_R \underline{x}^{\underline{b}}$. Then $\succ_F$ is a* **position over term (POT)** *monomial order on F.*

Example 2.2.3 *The lexicographic order, $\succ_{lex,R}$, on the polynomial ring R is the monomial order defined so that $\underline{x}^{\underline{a}} \succ_{lex,R} \underline{x}^{\underline{b}}$ if the first nonzero component of the vector $\underline{a} - \underline{b}$ is positive. The* **lexicographic order**, *$\succ_{lex}$, on F is the monomial order defined as follows: for monomials $\underline{x}^{\underline{a}}e_j$, $\underline{x}^{\underline{b}}e_k \in F$, $\underline{x}^{\underline{a}}e_j \succ_{lex} \underline{x}^{\underline{b}}e_k$ if either $j = k$ and $\underline{x}^{\underline{a}} \succ_{lex,R} \underline{x}^{\underline{b}}$ in R, or if $j < k$. Notice that the lexicographic order on F is a POT order.*

Example 2.2.4 *The reverse lexicographic order, $\succ_{rlex,R}$, on R is the monomial order defined so that $\underline{x}^{\underline{a}} \succ_{rlex,R} \underline{x}^{\underline{b}}$ if either $|\underline{a}| > |\underline{b}|$, or if $|\underline{a}| = |\underline{b}|$ and the last nonzero component of the vector $\underline{a} - \underline{b}$ is negative. The* **TOP-reverse lexicographic order**, *$\succ$, on F is the monomial order defined so that $\underline{x}^{\underline{a}}e_j \succ \underline{x}^{\underline{b}}e_k$ if $|\underline{a}| + \delta_j > |\underline{b}| + \delta_k$, or if $|\underline{a}| + \delta_j = |\underline{b}| + \delta_k$ and $\underline{x}^{\underline{a}} \succ_{rlex,R} \underline{x}^{\underline{b}}$ in R, or if $\underline{x}^{\underline{a}} = \underline{x}^{\underline{b}}$ and $j < k$.*

Now we present a specific example of the lexicographic and TOP-reverse lexicographic orders. In this example, we order the monomials of degree two of a free module over a polynomial ring in two variables.

Example 2.2.5 *Let* $R = K[x, y]$ *and* $F = Re_1 \oplus Re_2$ *with* $\delta_1 = \delta_2 = 0$*. Notice that* $\dim_K F_2 = 6$*. The monomials of degree two of* F*, ordered lexicographically are:*

$$x^2e_1 \succ_{lex} xye_1 \succ_{lex} y^2e_1 \succ_{lex} x^2e_2 \succ_{lex} xye_2 \succ_{lex} y^2e_2.$$

Recall that the lexicographic and reverse lexicographic orders agree on the monomials of degree two in R*. In the case of* F*, the lexicographic and TOP-reverse lexicographic orders are not the same on* $Mon(F_2)$*. The monomials of degree two, ordered using the TOP-reverse lexicographic order are:*

$$x^2e_1 \succ x^2e_2 \succ xye_1 \succ xye_2 \succ y^2e_1 \succ y^2e_2.$$

The reverse lexicographic order on F is not defined as a POT or a TOP order. Instead, we have the following definition:

Definition 2.2.6 *The* ***reverse lexicographic order****,* $\succ_{rlex}$ *on* F *is defined as follows:* $\underline{x}^{\underline{a}}e_j \succ_{rlex} \underline{x}^{\underline{b}}e_k$ *if* $|\underline{a}| + \delta_j > |\underline{b}| + \delta_k$*, or if* $|\underline{a}| + \delta_j = |\underline{b}| + \delta_k$ *and the last nonzero entry of* $\underline{a} - \underline{b}$ *is negative, or if* $|\underline{a}| + \delta_j = |\underline{b}| + \delta_k$ *and* $\underline{a} = \underline{b}$ *and* $j < k$*.*

In the next example, we compare the TOP-reverse lexicographic and reverse lexiocographic orders on the monomials of degree 2 in F, where the basis elements of F have different degrees. These orders do not coincide in this setting.

Example 2.2.7 *As in the previous example, consider* $R = K[x, y]$ *and* $F = Re_1 \oplus Re_2$*. Now suppose* $\delta_1 = 0$ *and* $\delta_2 = 1$*. Then,* $\dim_K F_2 = 5$*. The monomials of degree two, ordered lexicographically are:*

$$x^2e_1 \succ_{lex} xye_1 \succ_{lex} y^2e_1 \succ_{lex} xe_2 \succ_{lex} ye_2.$$

In this case, the TOP-reverse lexicographic order on the monomials of degree two is the same as the lexicographic order. On the other hand, we can order these monomials with respect to the reverse lexicographic order on F:

$$xe_2 \succ_{rlex} x^2e_1 \succ_{rlex} ye_2 \succ_{rlex} xye_1 \succ_{rlex} y^2e_1.$$

The lexicographic order plays a central role in our study of Hilbert functions of modules in the subsequent chapters via the following object.

Definition 2.2.8 *For any degree d, a* **lexsegment** *$W \subset F_d$ is a K-vector space generated by the first $\dim_K W$ monomials of degree d in F with respect to the lexicographic order on the module. An R-module $L \subset F$ is a* **lexsegment module** *if for each d, the K-vector space L_d is a lexsegment of F.*

Example 2.2.9 *In the setting of Example 2.2.5, let $M = I^1e_1 \oplus I^2e_2 \subseteq F$, where I^1, I^2 are the R-ideals $I^1 = (x^2, xy, y^3)$ and $I^2 = (x^3)$. Then:*

$$M_1 = 0$$
$$M_2 = Kx^2e_1 \oplus Kxye_1$$
$$M_3 = R_3e_1 \oplus Kx^3e_2$$
$$M_d = R_de_1 \oplus R_{d-3}x^3e_2, \text{ for all } d \geq 4.$$

Hence, by definition, M is a lexsegment R-module.

Example 2.2.10 *In the setting of Example 2.2.7, let $M = I^1e_1 \oplus I^2e_2 \subseteq F$, where $I^1 = (x^2, xy, y^3)$ and $I^2 = (x^2)$. Then:*

$$M_1 = 0$$
$$M_2 = Kx^2e_1 \oplus Kxye_1$$
$$M_3 = R_3e_1 \oplus Kx^2e_2$$
$$M_d = R_de_1 \oplus R_{d-3}x^2e_2, \text{ for all } d \geq 4.$$

Since each homogeneous component of M is a lexsegment, then M is a lexsegment module.

As we can see in these examples, the homogeneous components of a lexsegment module contain elements in a basis component Re_j only if all monomials of $Re_1, Re_2, \dots, Re_{j-1}$ are already present. This is a general property that all lexsegments will satisfy. We will take advantage of this structure of lexsegments in the proof the general hypersurface restriction theorem for modules in Chapter 4, so we state it in the following proposition.

Proposition 2.2.11 *Let W be a lexsegment K-vector subspace of F_d for any degree d. Since the lexicographic order is a position over term order, then W can be written in the form $W = R_{d-\delta_1}e_1 \oplus R_{d-\delta_2}e_2 \oplus \cdots \oplus R_{d-\delta_{j-1}}e_{j-1} \oplus W^j e_j$ for some $1 \leq j \leq r$, where W^j is a lexsegment of $R_{d-\delta_j}$.*

Proof Let $me_j \in W$ be the largest monomial with respect to lexicographic order that is contained in the K-vector space $W \subset F_d$. Such a monomial exists since the lexicographic order is a total order on $\mathrm{Mon}(F_d)$ and $\dim_K W < \infty$. By definition of the lexsegment, $ne_i \in W$ for all $ne_i \in \mathrm{Mon}(F)$ with $me_j \succ_{\mathrm{lex}} ne_i$. Now since $\succ_{\mathrm{lex}}$ is a position over term order, this means that W contains all monomials $ne_i \in F_d$ such that either $i < j$, or $i = j$ and $m \succ_{\mathrm{lex}} n$. ■

Remark 2.2.12 *Notice that if $L \subset F$ is a lexsegment module, then $L = L^1e_1 \oplus L^2e_2 \oplus \cdots \oplus L^r e_r$, where each $L^i \subset R$ is a lexsegment ideal.*

2.3 Gröbner Bases for Modules

In this section, we will discuss Gröbner bases and Buchberger's algorithm for modules. We use Buchberger's algorithm in our proof of the existence of the generic initial module in the next chapter, see Theorem 3.4.3. We refer the reader to Eisenbud [10, Chapter 15] for further discussion of the topics in this section.

Let M be an R-submodule of the free module F and $\succ$ any monomial order on F. If $f = \sum_{i=1}^{r} f^i e_i \in F$, the initial term of f with respect to $\succ$ is defined to be $\max_i \{\mathrm{in}_{\succ}(f^i e_i)\}$. The **initial module** of M with respect to $\succ$ is the R-module generated by the set of all $\text{in}_\succ(f)$ such that $f \in M$. Notice that $\text{in}_\succ(M)$ is a monomial submodule of F.

Definition 2.3.1 *For a module $M \subset F$ and any monomial order $\succ$ on M, a* ***Gröbner basis*** *for M is a set of elements $f_1, f_2, \ldots, f_m \in M$ such that*

$$in_\succ(M) = \langle in_\succ(f_1), in_\succ(f_2), \ldots, in_\succ(f_m) \rangle .$$

As in the case of homogeneous ideals of a polynomial ring, the initial module of M is not in general the same as the module generated by the initial terms of a system of generators of M. Although we do have upper-semicontinuity in general, which we will discuss in the next chapter of this thesis, Theorem 3.3.2.

Let $f, g_1, g_2, \ldots, g_s$ be nonzero elements of F, and $\succ$ a monomial order on F. There are elements $p_1, p_2, \ldots, p_s$ in R and $f' \in F$ such that

$$f = \sum_{i=1}^{s} p_i g_i + f', \tag{2.1}$$

where f' contains no monomials of the set $\{\text{in}_\succ(g_i) | 1 \leq i \leq s\}$ and $\text{in}_\succ(f) \succeq \text{in}_\succ(p_i g_i)$ for each $i = 1, 2, \ldots, s$. The equation (2.1) is a standard expression for f and f' is its remainder. In order to write f as a standard expression in terms of $g_1, g_2, \ldots, g_s$, we use the Division Algorithm, see [10, Section 15.3] for the details of the algorithm.

We now introduce **Buchberger's Algorithm** for computing a Gröbner basis of a module. Suppose $M \subset F$ is a submodule generated by elements $f_1, f_2, \ldots, f_s$, and let $\succ$ be a monomial order on F. For each $i = 1, 2, \ldots, s$, let $\text{in}_\succ(f_i) = m_i e_{l_i}$. For each j, k such that $l_j = l_k$, compute the S-pair:

$$S(f_j, f_k) = \frac{\text{lcm}(m_j, m_k)}{a_j m_j} f_j - \frac{\text{lcm}(m_j, m_k)}{a_k m_k} f_k$$

where $a_j, a_k \in K$ are the coefficients of f_j and f_k. Let r_{jk} be the remainder of $S(f_j, f_k)$ with respect to $f_1, f_2, \ldots, f_s$. If $r_{jk} = 0$ for all pairs j, k such that $l_j = l_k$, then $f_1, f_2, \ldots, f_s$ is a Gröbner basis for M. Otherwise, let $r_1, r_2, \ldots, r_l$ be the nonzero remainders and repeat the above process for $f_1, f_2, \ldots, f_s, r_1, r_2, \ldots, r_t$. This algorithm will terminate in a finite number of steps with a Gröbner basis for M, [10, Algorithm 15.9].

Example 2.3.2 *Suppose $R = K[x, y]$, $F = Re_1 \oplus Re_2$, and $\delta_1 = 0$, $\delta_2 = 1$. Let $\succ$ be the lexicographic order on F, and let M be the submodule of F generated by the elements $f_1 = x^2e_1 + ye_2$ and $f_2 = xy^2e_1 + xye_2$. Then, $in_\succ(f_1) = x^2e_1$ and $in_\succ(f_2) = xy^2e_1$. We compute the S-pair:*

$$S(f_1, f_2) = y^2f_1 - xf_2 = (y^3 - x^2y)e_2.$$

Notice that a remainder of $S(f_1, f_2)$ with respect to f_1, f_2 is itself since neither $in(f_1)$ nor $in(f_2)$ divides the initial term of $S(f_1, f_2)$. We repeat the process with the polynomials f_1, f_2, f_3, where $f_3 = (y^3 - x^2y)e_2$. But there are no more S-pairs to compute, since $in_\succ(f_3) = x^2ye_2$. Hence, the algorithm is finished. We have found that f_1, f_2, f_3 form a Gröbner basis for M, and $in_\succ(M)$ is generated by the monomials x^2e_1, xy^2e_1, x^2ye_2.

Buchberger's algorithm will enable us to construct the Zariski open set that defines the generic initial ideal. This statement and proof are given in the next chapter.

2.4 Weight Orders and Homogenization

A **weight order** on a free R-module F is a partial order on the monomials of F, defined by a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n_{\geq 0}$ and an r-tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_r) \in \mathbb{R}^r_{\geq 0}$. For any $m = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}e_j \in \text{Mon}(F)$, the weight of m is $\left(\sum_{i=1}^{n} \omega_i\alpha_i\right) + \tau_j$. If $n = x_1^{\beta_1}x_2^{\beta_2}\cdots x_n^{\beta_n}e_\ell$ is another monomial of F, then $m \succ n$ with respect to the weight (ω, τ) if $\left(\sum_{i=1}^{n} \omega_i\beta_i\right) + \tau_\ell > \left(\sum_{i=1}^{n} \omega_i\alpha_i\right) + \tau_j$. We will say that the weight of a term $cm = cx_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}e_j$ is the weight of the monomial m without the coefficient c.

For any element $f \in F$, the initial part of f with respect to the weight order (ω, τ) is the sum of all terms of f, hence with coefficients, that have the largest weight of all of the terms of f. The initial module of $M \subset F$ with respect to (ω, τ) is the R-submodule of F, $\text{in}(M)$, generated by all initial terms of elements of M. Since a weight order is a partial order, then $\text{in}(M)$ is not necessarily a monomial module.

Example 2.4.1 *Let* $R = K[x,y]$*,* $F = Re_1 \oplus Re_2$*, and* $\delta_1 = 0$*,* $\delta_2 = 2$*. Let* $\omega = (1,1)$ *and* $\tau = (1,1)$*. Let* $f = (x^3 + y^2)e_1 + ye_2$*. We compute the weight of each monomial of* f *with respect to the weight order* (ω, τ)*:*

$$(\omega,\tau)(x^3e_1) = 3,\ (\omega,\tau)(y^2e_1) = 2,\ (\omega,\tau)(ye_2) = 3.$$

Notice that this weight order computes the degree of each monomial. Hence, the initial part of f *with respect to the weight order* (ω,τ) *is* $in(f) = x^3e_1 + ye_2$*. We see from this example that, in the case of weight orders, an initial module is not necessarily a monomial module.*

Definition 2.4.2 *A weight order* (ω,τ) *is a* ***degree weight order*** *if for any monomials* m *and* n *in* F*,* $\deg(m) > \deg(n)$ *implies that* $m \succ n$ *with respect to* (ω,τ)*.*

If R is $\mathbb{Z}^s$-graded, recall that an R-module M is $\mathbb{Z}^s$-graded, or multigraded by $\mathbb{Z}^s$, if $M = \bigoplus_{v \in \mathbb{Z}^s} M_v$ so that $R_vM_w \subseteq M_{v+w}$ for all $v, w \in \mathbb{Z}^s$.

Example 2.4.3 *The free* R*-module* F *has a* $\mathbb{Z}^2$*-grading by assigning* $\deg x_i = (1,0)$ *for* $i = 1,2,\ldots,n-1$*,* $\deg e_i = (\delta_i, 0)$*, and* $\deg x_n = (0,1)$*. Hence,* F *can be decomposed as:*

$$F = \bigoplus_{(a,b)\in\mathbb{Z}^2} \overline{F}_a x_n^b,$$

where $\overline{F} \cong F/x_nF$ *as* $K[x_1,\ldots,x_{n-1}]$ *modules, and* $\overline{F}_a$ *consists of all homogeneous elements of* $\overline{F}$ *of total degree* a*. This multigrading of* F *will be especially important towards our work in Chapter 4, specifically for Definition 4.2.1.*

Let t be an indeterminate over R and define $\widetilde{R} = R[t]$ and $\widetilde{F} = \bigoplus_{j=1}^{r} \widetilde{R}\widetilde{e_j}$ a free multigraded $\widetilde{R}$-module with basis $\widetilde{e_1}, \widetilde{e_2}, \ldots, \widetilde{e_r}$ of degrees (δ_j, τ_j). Given a weight vector (ω,τ), the **homogenization** of an element $f = \sum c_{\underline{a},j}\underline{x}^{\underline{a}}e_j \in F$ is:

$$\widetilde{f} = t^{D(f)} \sum c_{\underline{a},j}(t^{-\omega\cdot\underline{a}}\underline{x}^{\underline{a}})(t^{-\tau_j}e_j)$$

where $D(f) = \max\{\omega\cdot\underline{a} + \tau_j \mid c_{\underline{a},j} \neq 0\}$ [32, Definition 8.25].

Example 2.4.4 *In the setting of Example 2.4.1, that is,* $R = K[x, y]$, $F = Re_1 \oplus Re_2$, $\delta_1 = 0$, $\delta_2 = 2$, *and* $f = (x^3 + y^2)e_1 + ye_2$, *we consider the homogenization of* f *with respect to the weight order* (ω, τ), *where* $\omega = (1, 1)$ *and* $\tau = (1, 1)$. *For this element,* $D(f) = 3$. *Hence, the homogenization of* f *is:*

$$\widetilde{f} = t^3(t^{-3}x^3e_1 + t^{-2}y^2e_1 + t^{-3}ye_2) = x^3e_1 + ty^2e_1 + ye_2.$$

Since the weight of a monomial with respect to this weight order (ω, τ) *is just its total degree, then this homogenization gives a homogeneous element of* $\widetilde{F}$.

Definition 2.4.5 *Let* $M \subset F$ *be a graded* R*-submodule, and* (ω, τ) *a weight vector on* F. *The* ***homogenization*** $\widetilde{M}$ *of* M *is the* $\widetilde{R}$*-submodule of* $\widetilde{F}$ *generated by the set of all* $\widetilde{f}$ *such that* $f \in M$.

Homogenization with respect to a weight order is a standard tool in proving uppersemicontinuity of Betti numbers of modules. We will use this type of homogenization in our proof of the general hypersurface restriction theorem for modules in Chapter 4. In particular, this statement will allow us to compare the Hilbert series of a module and its generic initial module, when restricting to a general hypersurface, see Section 3.5 for this statement and its proof.

3. Hilbert Functions and Graded Betti Numbers for Modules

3.1 Introduction

In this chapter, we will discuss Hilbert functions, graded Betti numbers, and generic initial modules. Hilbert functions and graded Betti numbers are widely studied invariants in commutative algebra. In particular, the problem of transforming an ideal into another that has the same Hilbert function and graded Betti numbers greater than or equal to those of the original ideal is one of interest to many researchers. One of the earliest results in this direction is Macaulay's Theorem 1.2.1, which states that if $R = K[x_1, ..., x_n]$ is a polynomial ring over a field K, then there exists a lexsegment ideal realizing the Hilbert function of any homogeneous ideal of R. Later, Bigatti, Hulett, and Pardue proved that lexsegments ideals attain the highest Betti numbers among all ideals having the same Hilbert function, see Theorem 1.4.1.

3.2 Hilbert Functions

Hilbert functions and Hilbert series are frequently studied in commutative algebra. In this section, we will introduce these invariants and discuss a well-known result regarding Hilbert functions of initial modules of graded R-modules $M \subset F$. Throughout this chapter, we use the same notation as in Chapter 2. Let $R = K[x_1, ..., x_n]$ be a polynomial ring in n variables over a field K. Some results of this chapter will require that K is an infinite field, and we will include this assumption only when necessary. Let $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$, a graded free R-module with basis $e_1, e_2, \ldots, e_r$ of degrees $0 \leq \delta_1 \leq \delta_2 \leq \ldots \leq \delta_r$, as described in Section 2.2. For all ordered pairs (i, j), $1 \leq i, j \leq r$, let $\Delta_{ij} = \delta_i - \delta_j$.

For any finitely generated graded R-module M, we write $M = \bigoplus_d M_d$, where M_d is the K-vector space of all elements of M of total degree d. The **Hilbert function** of M is the function $\mathbb{Z} \to \mathbb{Z}$ defined as $d \mapsto \dim_K M_d$. Notice that, since M is finitely generated over R, each of the K-vector spaces M_d are finite dimensional. The **Hilbert series** of M is $H(M) = \sum_d (\dim_K M_d)t^d$. Hence, the d^{th} coefficient of the series $H(M)$ is the value of the Hilbert function at d.

In the following example, we compute the Hilbert series of a graded R-module. We will return to this example throughout the chapter.

Example 3.2.1 *Let $R = \mathbb{Q}[x, y, z]$, $F = Re_1 \oplus Re_2$, where $\deg e_i = 0$, and let M be the R-submodule of F generated by the elements $(x^3 + xy^2)e_1, x^2ze_1 + x^3e_2, yz^2e_2$. Then the Hilbert series of M is:*

$$H(M) = \frac{3t^3 - t^8}{(1-t)^3}.$$

Hence, the Hilbert function is $(0, 0, 0, 3, 9, 18, 30, 45, \ldots)$, where the $i^{th}-$component of this ordered tuple is the dimension of the K-vector space M_i.

We will use the following proposition to reduce to the case of monomial modules in Lemma 5.2.5. This is a crucial step in the proof of our main theorem on Hilbert functions of local cohomology modules, Theorem 5.2.10. This statement can be found in [10, Chapter 15].

Proposition 3.2.2 *Let $M \subseteq F$ be a graded R-module and $\succ$ a monomial order on F. Then, M and $in_\succ(M)$ have the same Hilbert function.*

Example 3.2.3 *Let M be the graded R-module in Example 3.2.1. If $\succ$ is the reverse lexicographic order on the monomials of F, then a Gröbner basis for M is $yz^2e_2, (x^3 + xy^2)e_1, x^2ze_1 + x^3e_2, x^2yz^3e_1, xy^3z^3e_1$. Hence, the initial ideal of M with respect to $\succ$ is $in_\succ(M) = (yz^2e_2, x^3e_1, x^3e_2, x^2yz^3e_1, xy^3z^3e_1)$. The Hilbert series of $in_\succ(M)$ is*

$$H(in_\succ(M)) = \frac{3t^3 - t^8}{(1-t)^3}.$$

Notice that this is the same as the Hilbert series of M that we computed in the previous example.

3.3 Graded Betti Numbers

Along with Hilbert functions, graded Betti numbers are another widely studied invariant in commutative algebra. In this section, we discuss these invariants, along with two examples, and state a well-known result relating graded Betti numbers of M and of its initial module with respect to a monomial order. A reference for graded Betti numbers is [11, Chapter 1].

Definition 3.3.1 *Let M be an R-submodule of F. Then, the* **graded Betti numbers** *of M over R are defined as follows:*

$$\beta_{ij}(M) = \dim_K Tor_i^R(K, M)_j.$$

We organize the graded Betti numbers of a module in a Betti table. The entry of the Betti table in the j^{th} row and i^{th} column is $\beta_{i,j+i}$. That is, the Betti table for a module with graded Betti numbers β_{ij} is:

	0	1	2	3	
0	β_{00}	β_{11}	β_{22}	β_{33}	$\cdots$
1	β_{01}	β_{12}	β_{23}	β_{34}	$\cdots$
2	β_{02}	β_{13}	β_{24}	β_{35}	$\cdots$
3	β_{03}	β_{14}	β_{25}	β_{36}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The following proposition is crucial for many proofs involving graded Betti numbers of graded modules. In particular, this statement will play a key role in our proof of Theorem 6.4.5.

Proposition 3.3.2 *[10] Suppose M is a graded R-submodule of the free module F and $\succ$ is a monomial order on F. Then, for all i, j*

$$\dim_K Tor_i^R(K, M)_j \leq \dim_K Tor_i^R(K, in_\succ(M))_j.$$

In particular, this result implies that the graded Betti numbers of a module M are bounded above by those of its initial module. We demonstrate this inequality in the following example. We use Macaulay2 [20] to compute the Betti numbers of a graded module and its initial module with respect to the reverse lexicographic order on F.

Example 3.3.3 *Let $R = \mathbb{Q}[x, y, z]$, $F = Re_1 \oplus Re_2$, where $\deg e_i = 0$, and let M be the R-submodule of F generated by the elements $(x^3 + xy^2)e_1, x^2ze_1 + x^3e_2, yz^2e_2$, as in Example 3.2.1. The Betti table for the F/M is:*

	0	1	2
0	*2*	-	-
1	-	-	-
2	-	*3*	-
3	-	-	-
4	-	-	-
5	-	-	-
6	-	-	*1*

On the other hand, the Betti table for $F/in_{\succ}(M)$ is:

	0	1	2	3
0	*2*	-	-	
1	-	-	-	
2	-	*3*	-	
3	-	-	-	
4	-	-	*1*	
5	-	*1*	*1*	
6	-	*1*	*1*	

3.4 Generic Change of Coordinates

Let $GL_n(K)$ be the group of invertible $n \times n$-matrices with entries in K. Recall that any $\beta = (\beta_{ij}) \in GL_n(K)$ acts on the variables $x_1, x_2, \ldots, x_n$ by $\beta x_i = \sum_j \beta_{ij} x_j$ for all $i = 1, 2, \ldots, n$. This action extends K-linearly to an action on the polynomial ring R as $\beta g(x_1, x_2, \ldots, x_n) = g(\beta x_1, \beta x_2, \ldots, \beta x_n)$ for any $g(x_1, x_2, \ldots, x_n) \in R$. This is the standard action of $GL_n(K)$ on the polynomial ring R, used to define the generic initial ideal. We now extend this action to a K-linear action on the free R-module F as follows: for any element $f = f_1e_1 + f_2e_2 + \cdots + f_re_r \in F$, define $\beta f = (\beta f_1)e_1 + (\beta f_2)e_2 + \cdots + (\beta f_r)e_r$.

Now let $GL(F)$ be the subgroup of $\mathrm{Aut}(F)$, consisting of the graded R-automorphisms of F. Hence, for any $\alpha = (\alpha_{ij}) \in GL(F)$ and $f = f_1e_1 + f_2e_2 + \cdots + f_re_r \in F$, $\alpha f = f_1(\alpha e_1) + f_2(\alpha e_2) + \cdots + f_r(\alpha e_r)$ and $\alpha e_i = \sum_j \alpha_{ij} e_j$ for all $i = 1, 2, \ldots, r$. Notice that the nonzero entries of the matrix α are homogeneous polynomials of R. The entry in the i^{th} row and j^{th} column, α_{ij}, is 0 whenever Δ_{ij} is negative. If $\Delta_{ij} \geq 0$ and $\alpha_{ij} \neq 0$, then $\alpha_{ij} \in R_{\Delta_{ij}}$.

Remark 3.4.1 *Let $\succ$ be a monomial order on F. For each $i = 1, 2, \ldots, r$, let $d_i = \dim_K F_{\delta_i}$. Then, F_{δ_i} has a K-vector space basis of monomials $\{m_{i1}, m_{i2}, \ldots, m_{id_i}\}$, ordered so that $m_{i1} \succ m_{i2} \succ \cdots \succ m_{id_i}$. Since the elements of $GL(F)$ are graded homomorphisms, we can represent the images of each basis element e_i under such a map as a K-linear combination of the monomials $\{m_{i1}, m_{i2}, \ldots, m_{id_i}\}$. That is, if $\beta = (\beta_{ij}) \in GL(F)$ is any change of coordinates, then there are scalars $b_{ij} \in K$ so that $\beta(e_i) = \sum_{j=1}^{d_i} b_{ij} m_{ij}$ for all $i = 1, 2, \ldots, r$.*

We have a homomorphism $\phi : GL_n(K) \to Aut(GL(F))$ defined by $\phi(\beta) = \phi_\beta : \alpha \mapsto \beta \circ \alpha \circ \beta^{-1}$. Notice that $\phi_\beta \in Aut(GL(F))$ since for any $\alpha \in GL(F)$, $\beta \circ \alpha \circ \beta^{-1}$ is an invertible homomorphism of $GL(F)$. The map ϕ gives an action of $GL_n(K)$ on $GL(F)$, so we can define the semidirect product $\mathcal{G}(F) = GL(F) \rtimes GL_n(K)$ with respect to ϕ, as described in Chapter 1, Example 5 of [34]. Notice that elements of $\mathcal{G}(F)$ have the form (α, β), for $\alpha \in GL(F)$ and $\beta \in GL_n(K)$. Multiplication of elements $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{G}(F)$ is defined by $(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1 \phi_{\beta_1}(\alpha_2), \beta_1\beta_2)$

Recall that the Borel subgroup $B_n(K) \subset GL_n(K)$ is defined to be the group of all invertible upper triangular matrices. That is, $B_n(K) = \{(\alpha_{ij}) \in GL_n(K) \,|\, \alpha_{ij} = 0 \text{ when } i > j\}$. To define the Borel subgroup of $\mathcal{G}(F)$, first let $B(F) = \{\beta = (\beta_{ij}) \in GL(F) \,|\, \beta_{ij} = 0 \text{ when } i > j\}$. Notice that $B(F)$ consists of the upper triangular matrices in $GL(F)$. In particular, any $\beta \in B(F)$ sends a basis element e_i to a homogeneous R-linear combination of the elements $e_1, e_2, \ldots, e_i$. The **Borel subgroup** of $\mathcal{G}(F)$ is defined to be $\mathcal{B}(F) = B(F) \rtimes B_n(K)$.

In order to define the generic initial module, we first need to extend the scalars. We start by defining new indeterminates. Let $\chi_0 = \{y_{ij} | 1 \leq i, j \leq n\}$. For $i = 1, 2, \ldots, r$, let $\chi_i = \{z_{ij} | 1 \leq j \leq \dim_K F_{\delta_i}\}$. Define $\overline{K} = K(\chi_0, \chi_1, \chi_2, \ldots, \chi_r)$, the field of fractions of the polynomial ring $K[\chi_0, \chi_1, \ldots, \chi_r]$. Let $\overline{R} = R \otimes_K \overline{K}$ and $\overline{F} = F \otimes_R \overline{R}$. For any submodule $M \subset F$, let $\overline{M}$ denote the image of M in the $\overline{R}$-module $\overline{F}$. Let $\overline{\gamma} \in \mathcal{G}(\overline{F})$ be the change of coordinates defined so that $\overline{\gamma}(x_i) = \sum_{j=1}^{n} y_{ij} x_j$ and $\overline{\gamma}(e_i) = \sum_{j=1}^{d_i} z_{ij} m_{ij}$, where the monomials m_{ij} are as in Remark 3.4.1. We say that $\overline{\gamma}$ is the **generic change of coordinates** of F.

Definition 3.4.2 *Let $\succ$ be any monomial order on F. For a module $M \subset F$, the* **generic initial module** *of M with respect to $\succ$ is $gin_{\succ}(M) = in_{\succ}(\overline{\gamma}(\overline{M})) \cap F$.*

The generic initial ideal was first introduced by Galligo [18]. He proved the following theorem in the case when K is any infinite field of characteristic zero, $F = R$, M is an R-ideal, and $\text{gin}_{\succ}(M)$ is the generic initial ideal of M with respect to a monomial order $\succ$ on R. The statement for modules has been shown by Pardue [35], and we give an alternate proof.

Theorem 3.4.3 *Suppose $|K| = \infty$. For any submodule $M \subset F$, there exists a nonempty Zariski open set $U \subset \mathcal{G}(F)$ such that for all $\gamma \in U$, $gin_{\succ}(M) = in_{\succ}(\gamma(M))$.*

Proof Let $f_1, f_2, \ldots, f_s$ be a generating set of M, hence, as elements of $\overline{F}$, $f_1, f_2, \ldots, f_s$ form a generating set of $\overline{M}$. Then, $\overline{\gamma}(f_1), \overline{\gamma}(f_2), \ldots, \overline{\gamma}(f_s)$ generate $\overline{\gamma}(\overline{M})$. Using Buchberger's algorithm, we can obtain a Gröbner basis for $\overline{\gamma}(\overline{M})$. Let $\mathcal{F} \subset K[\chi_0, \chi_1, \ldots, \chi_r]$ be the set of all nonzero numerators and denominators of elements of $\overline{K}$ that show up as coefficients of any polynomial at any step of the computation of the Gröbner basis. This set is finite since Buchberger's algorithm ends after a finite number of steps.

Let $\mathcal{U} = \{p \in \mathbb{A}^{n^2}(K) \times \mathbb{A}^{d_1}(K) \times \mathbb{A}^{d_2}(K) \times \cdots \times \mathbb{A}^{d_r}(K) \,|\, g(p) \neq 0 \ \forall g \in \mathcal{F}\}$. Then $\mathcal{U}$ is a Zariski open set, and $\mathcal{U}$ is nonempty since $|K| = \infty$. For $p \in \mathcal{U}$, we write

$$p = (p_{1,1}, p_{1,2}, \ldots, p_{1,n}, p_{2,1}, \ldots, p_{n,n}, q_{1,1}, q_{1,2}, \ldots, q_{1,d_1}, q_{2,1}, \ldots, q_{r,d_r}).$$

We specialize the generic change of coordinates $\overline{\gamma}$ by using the following substitution: $y_{ij} \mapsto p_{i,j}$ and $z_{ij} \mapsto q_{i,j}$. We denote this substitution by γ_p, and in this way, γ_p is identified as an element of $\mathcal{U}$.

Following the same computations we used to find a Gröbner basis for $\overline{\gamma}(M)$, we can use Buchberger's algorithm for $\gamma_p(f_1), \gamma_p(f_2), \ldots, \gamma_p(f_s)$ to obtain a Gröbner basis for $\gamma_p(M)$. In fact, if $g_1, g_2, \ldots, g_t \in \overline{F}$ is the Gröbner basis we found for $\overline{\gamma}(\overline{M})$, then $(g_1)_p, (g_2)_p, \ldots, (g_t)_p$ is a Gröbner basis for $\gamma_p(M)$. Since $\text{in}_\succ(g_i) = \text{in}_\succ(g_i)_p$ for all $i = 1, 2, \ldots, t$, then $\text{gin}_\succ(M) = \text{in}_\succ(\overline{\gamma}(\overline{M})) \cap F = \text{in}_\succ(\gamma_p(M))$ for all $\gamma_p \in \mathcal{U}$. ■

In the proof of Theorem 4.3.2, we replace M by its generic initial module with respect to the reverse lexicographic order. A property of generic initial modules that enables us to do this is stated in the following remark:

Remark 3.4.4 *[10] Let $\succ$ be a monomial order on F, and M a graded submodule of F. Then $\dim_K M_d = \dim_K gin_\succ(M)_d$ for each d. That is, M and $gin_\succ(M)$ have the same Hilbert function.*

Example 3.4.5 *Let $R = K[x, y, z]$ and $I = (x^2 + yz, xy + z^2)$. If $\succ$ is the reverse lexicographic order on the monomials of R, then the initial ideal of I with respect to $\succ$ is $in_\succ(I) = (x^2, xy, y^2z)$ and the generic initial ideal of I with respect to $\succ$ is $gin_\succ(I) = (x^2, xy, y^3)$.*

Example 3.4.6 *Let $R = \mathbb{Q}[x, y, z]$, $F = Re_1 \oplus Re_2$, where $\deg e_i = 0$, and let M be the R-submodule of F generated by the elements $(x^3 + xy^2)e_1, x^2ze_1 + x^3e_2, yz^2e_2$, as in Example 3.2.1. If $\succ$ is the reverse lexicographic order on F, then the generic initial module of M with respect to $\succ$ is $gin_{\succ}(M) = \langle x^2ye_2, x^3e_1, x^3e_2, x^2y^2e_1, xy^4e_2, xy^5e_1, y^7e_2\rangle$. The Hilbert function of $gin_{\succ}(M)$ is the same as that of M. Furthermore, the Betti table for $F/gin_{\succ}(M)$ is:*

	0	1	2	3
0	2	-	-	
1	-	-	-	
2	-	3	1	
3	-	1	1	
4	-	1	1	
5	-	1	1	
6	-	1	1	

3.5 Stability of Generic Initial Modules

In this section, we will discuss an important property of generic initial modules, namely stability. Different types of stability will be crucial to our proof of the general hypersurface restriction theorem for modules in Chapter 4 of this thesis and in the content of Chapter 6. For the following definition, recall the Borel subgroup discussed in the previous section.

Definition 3.5.1 *[35] An R-module $M \subset F$ is **Borel-fixed** if $\gamma M = M$ for all $\gamma \in \mathcal{B}(F)$.*

Definition 3.5.2 *[35] A monomial R-module $M = I^1e_1 \oplus I^2e_2 \oplus \cdots \oplus I^re_r \subset F$ is **strongly stable** if the following two conditions hold:*

(i) $\mathfrak{m}^{\delta_j - \delta_i} I^i \subseteq I^j$ for all $j < i$; and

(ii) whenever $x_i m$ is a monomial of M, then $x_j m \in M$ for all $j < i$.

The following remark is a frequently used result in the case when the characteristic of the field K is 0, see [35, Section 3].

Remark 3.5.3 *If a monomial R-module $M \subset F$ is strongly stable, then it is Borel-fixed. The converse holds, assuming $char(K) = 0$.*

We will discuss a weaker form of stability in the next chapter, which will play an important role in our proof of the general hypersurface restriction theorem for modules, specifically in Proposition 4.2.8.

We will now prove that $\mathrm{gin}_{\succ}(M)$ is Borel-fixed, under certain conditions on the monomial order $\succ$. We assume that $\succ$ is any monomial order on F satisfying the following conditions: $e_1 \succ e_2 \succ \cdots \succ e_r$, and for all $j = 1, 2, \ldots, r$, $x_1e_j \succ x_2e_j \succ \cdots \succ x_ne_j$.

For any degree d, we write $F_d = \langle m_1, m_2, \ldots, m_{i_d} \rangle$, where m_i are monomials of degree d and $i_d = \dim_K F_d$. Furthermore, we arrange $m_1, m_2, \ldots, m_{i_d}$ so that $m_1 \succ m_2 \succ \cdots \succ m_{i_d}$. Then, for any K-vector space $V \subset F_d$, $V = \langle v_1, v_2, \ldots, v_s \rangle$, where $s = \dim_K V$. We write each element $v_i \in V$ as a K-linear combination of the monomials $m_1, m_2, \ldots, m_{i_d}$: $v_i = \sum_{j=1}^{i_d} a_{ij}m_j$ and let $A = (a_{ij})$, the $s \times i_d$-matrix with entries in K. Finally, for each $i = 1, 2, \ldots, i_d$, define $r_i(V)$ to be the rank of the first i columns of A.

Theorem 3.5.4 *[35, Proposition 4] For any graded submodule $M \subset F$ and monomial order $\succ$ on F, the generic initial module $gin_{\succ}(M)$ is Borel-fixed.*

Proof By extending the field if necessary, we assume $|K| = \infty$. By 3.4.3, there is a nonempty Zariski open set $\mathcal{U} \subset \mathcal{G}$ such that for all $\gamma \in \mathcal{U}$, $\mathrm{gin}_{\succ}(M) = \mathrm{in}_{\succ}(\gamma M)$. We will show that $\mathrm{gin}_{\succ}(M)_d$ is Borel-fixed for each d. Let $W = \mathrm{gin}_{\succ}(M)_d$. Assume that W is not Borel-fixed. Then, there is an element $b \in \mathcal{B}$ such that $bW \neq W$. In particular, $\mathrm{in}_{\succ}(bW) \neq W$. Since b is an element of the Borel subgroup, then for all $m \in W$, $b(m) = m + \sum a_{ij}m_i$, where each m_i is a monomial of F_d with $m_i \succ m$. Hence, $r_j(bW) \geq r_j(W)$ for all $1 \leq j \leq i_d$. On the other hand, since taking the generic initial module with respect to $\succ$ maximizes ranks, then $r_j(W) \geq r_j(bW)$. Hence, we have equality, and therefore $\text{gin}_{\succ}(M)$ is Borel-fixed. ■

From this theorem and the fact that Borel-fixed modules are strongly stable in characteristic zero, Remark 3.5.3, we obtain the following corollary.

Corollary 3.5.5 *When char$(K) = 0$, if $M \subset F$ is a graded R-submodule, then for any monomial order $\succ$ on F, $gin_{\succ}(M)$ is strongly stable.*

Proposition 3.5.6 *Let $h \in R_j$ be a general form of degree j, $M \subset F$ a graded submodule, and $\succ$ a monomial order on F, then*

$$H(F/(M + hF)) \leq H(F/(gin_{\succ}(M) + hF)).$$

Proof Let $U_1 \subset \mathcal{G} = GL_n(K) \rtimes GL(F)$ be a Zariski open set such that for all $\gamma_1, \gamma_2 \in U_1$, $\text{in}_{\succ}(\gamma_1 M) = \text{in}_{\succ}(\gamma_2 M)$ as in Theorem 3.4.3. Let $s = \dim_K R_j$. By extending the field and using Buchberger's algorithm [10, Algorithm 15.9] to compute a Gröbner basis for $\text{gin}_{\succ}(M) + h'F$, h' being a generic form of degree j, we can find a nonempty Zariski open set $U_2 \subset \mathbb{P}^{s-1}(K)$ such that for all $h \in U_2$, the Hilbert function of $F/(\text{gin}_{\succ}(M) + hF))$ is constant. Similarly, there is a Zariski open set $U_3 \subset \mathbb{P}^{s-1}(K)$ such that for all $h \in U_3$, $H(F/(M + hF))$ is constant, and furthermore $H(F/(M + hF))$ is minimal among forms of degree j. Finally, let $U_4 \subset \mathcal{G}$ be a Zariski open set such that for all $(\alpha, \beta, h) \in U_4$, the Hilbert function of $F/((\alpha, \beta)M + hF)$ is constant and minimal. Let $W = (U_1 \times \mathbb{P}^s(K)) \cap (\mathcal{G} \times U_2) \cap (\mathcal{G} \times U_3) \cap U_4$. Hence, W is a nonempty Zariski open set. Write (ω, τ) for the weight vector representing the reverse lexicographic order on all monomials of F of degree less than a sufficiently larger integer, where $\omega \in \mathbb{N}^n$ gives the reverse lexicographic order on a subset of monomials of R and $\tau \in \mathbb{N}^r$ is the ordering on the basis elements $e_1, e_2, \ldots, e_r$.

For any $(\alpha, \beta, h) \in W$, we let $\widetilde{(\alpha, \beta)}M$ be the module obtained by homogenizing $(\alpha, \beta)M$ with respect to (ω, τ). The specialization $\widetilde{(\alpha, \beta)M}|_{t=0} = \text{in}_{\succ}((\alpha, \beta)M)$, which is the generic initial module of M by choice of (α, β). Furthermore, for all nonzero $c \in K$, $\widetilde{(\alpha, \beta)}M|_{t=c} = (D_c, \sigma_c)M$, where $D_c \in GL_n(K)$ is the diagonal $n \times n$ matrix whose entry in the i^{th} row and i^{th} column is c^{ω_i}, and $\sigma_c \in GL(F)$ is the automorphism sending e_i to $c^{\omega_i} e_i$ for all $i = 1, 2, \ldots, r$. Notice that for any $c \neq 0$, $H((D_c, \sigma_c)M + hF) = H(\sigma_c(M) + D_c^{-1}(h)F)$. Choose c general with respect to the property that $H(F/(\widetilde{(\alpha, \beta)M}|_{t=c} + hF)) \leq H(F/(\widetilde{(\alpha, \beta)M}|_{t=0} + hF))$. Then:

$$
\begin{aligned}
H(F/(\mathrm{gin}_{\succ}(M) + hF)) &= H(F/(\widetilde{(\alpha, \beta)M}|_{t=0} + hF)) \geq H(F/(\widetilde{(\alpha, \beta)M}|_{t=c} + hF)) \\
&= H(F/((D_c, \sigma_c)M + hF)) = H(F/(\sigma_c(M) + D_c^{-1}(h)F)) \geq H(F/((\alpha, \beta)M + hF)) \\
&= H(F/(M + (\alpha, \beta)^{-1}(hF))) \geq H(F/(M + hF)).
\end{aligned}
$$

■

4. General Hypersurface Restriction Theorem for Modules

4.1 Introduction

Green's Hyperplane Restriction Theorem 1.3.1 states that for any homogeneous ideal I of a polynomial ring R in n variables over an infinite field K and for any general linear form $g \in R_1$, the Hilbert function of $I + (g)$ is bounded below by the Hilbert function of $I^{\text{lex}} + (x_n)$. This theorem inspired Herzog and Popescu [24] and Gasharov [19] to prove the same inequality holds when g is a general form of any degree and x_n is replaced by $x_n^{\deg(g)}$. In characteristic zero, Caviglia and Kummini [4] prove an analogous result for embeddings of Hilbert functions when restricting to a general hypersurface.

In another direction, Greco [21] has shown that Green's Hyperplane Restriction Theorem holds for graded submodules of free R-modules. Our main theorem is an extension of these results. Assuming K is an infinite field, $R = K[x_1, x_2, \ldots, x_n]$, $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$, $M \subset F$ a graded R-submodule, and $h \in R_j$ a general form of degree j, then for every d, we have:

$$\dim_K(F/(M + hF))_d \leq \dim_K(F/(M^{\text{lex}} + x_n^j F))_d.$$

In this chapter, we study properties of lexsegment modules and stability with respect to the variable x_n. We will use these properties to prove the general hypesurface restriction theorem for modules.

4.2 Lexsegments and Stability

Given R and F as above, we let $\overline{R}$ and $\overline{F}$ be $K[x_1, x_2, \ldots, x_{n-1}]$ and $\overline{R}e_1 \oplus \overline{R}e_2 \oplus \cdots \oplus \overline{R}e_r$ respectively. Recall that the free R-module F has a multigrading such that $\deg x_i = (1, 0)$ for $i = 1, 2, \ldots, n-1$, $\deg e_i = (\delta_i, 0)$, and $\deg x_n = (0, 1)$, see Example 2.4.3. Hence, any multigraded K-vector space $W = W^1e_1 \oplus W^2e_2 \oplus \cdots \oplus W^re_r \subseteq F_d$ can be written in the form:

$$W = \bigoplus_{i=0}^{d} V_{d-i}x_n^i$$

for some subspaces $V_{d-i} \subseteq \overline{F}_{d-i}$. We use this decomposition to define stability of a graded R-submodule of F with respect to the variable x_n.

Definition 4.2.1 *Let $W = W^1e_1 \oplus W^2e_2 \oplus \cdots \oplus W^re_r = \bigoplus_{i=0}^{d} V_{d-i}x_n^i \subseteq F_d$ be a multigraded K-vector space, where $V_{d-i} \subseteq \overline{F}_{d-i}$. Suppose the following two conditions hold:*

1. *$\overline{R}_1V_{d-i} \subseteq V_{d-i+1}$ for all $i \in \{0, 1, \ldots, d\}$,*
2. *$\mathfrak{m}^{\delta_j-\delta_i}W^j \subseteq W^i$ for all i, j such that $j > i$.*

*Then, W is called x_n**-stable**. A submodule $M \subseteq F$ is x_n**-stable** if the degree d component, $M_d \subseteq F_d$, is x_n-stable for each $d \in \mathbb{N}$.*

Remark 4.2.2 *When $r = 1$, we recover the definition of x_n-stablity for a multigraded K-vector subspace of R_d. That is, if $J = \bigoplus_{i=0}^{d} V_{d-i}x_n^i \subseteq R_d$, where $V_{d-i} \subseteq \overline{R}_{d-i}$, then J is x_n-stable if $\overline{R}_1V_{d-i} \subseteq V_{d-i+1}$ for all $i \in \{0, 1, \ldots, d\}$ [3, Definition 3.4]. Hence, for $r \geq 1$, the first condition of the definition of x_n-stable is equivalent to requiring that $W^j \subseteq R_{d-f_j}$ is x_n-stable for each $j \in \{1, 2, \ldots, r\}$.*

Example 4.2.3 *Recall from Remark 2.2.11 that every lexsegment K-vector subspace $W \subseteq F_d$ can be written in the form $W = R_{d-\delta_1}e_1 \oplus R_{d-\delta_2}e_2 \oplus \cdots \oplus R_{d-\delta_{j-1}}e_{j-1} \oplus W^je_j$ for some $1 \leq j \leq r$, where W^j is a lexsegment of $R_{d-\delta_j}$. From this characterization of lexsegments, it is clear that any lexsegment $W \subseteq F_d$ is an x_n-stable K-vector space.*

Lemma 4.2.4 *Let $M = \bigoplus_{j=1}^{r} I^je_j \subset F$, where each I^j is an R-ideal. Suppose M satisfies the property that $\mathfrak{m}^{\delta_j-\delta_i}I^j \subseteq I^i$ for all $1 \leq i < j \leq r$, for example, if M is an x_n-stable R-module. For each $j = 1, 2, \ldots, r$, let $L^j = (I^j)^{lex}$. Then, $\bigoplus_{j=1}^{r} L^je_j$ is x_n-stable.*

Proof The first condition of the definition of x_n-stability is satisfied since each L^j is a lexsegment, hence is x_n-stable by Example 4.2.3. Now, for each pair $1 \le k < \ell \le r$ and every degree $d > \delta_k$,

$$\dim_K(L^k)_{d-\delta_k} = \dim_K(I^k)_{d-\delta_k} \ge \dim_K(x_n^{\delta_\ell - \delta_k} I^\ell)_{d-\delta_k} \ge \dim_K(x_n^{\delta_\ell-\delta_k} L^\ell)_{d-\delta_k},$$

where the first inequality holds by the assumption that M satisfies the second condition of x_n-stability, and the second inequality follows from minimal growth of lexsegments. Since $L^k_{d-\delta_k}$ and $(x_n^{\delta_\ell-\delta_k} L^\ell)_{d-\delta_k}$ are lexsegments of $F_{d-\delta_k}$, then this inequality implies that $(x_n^{\delta_\ell-\delta_k} L^\ell)_{d-\delta_k} \subset L^k_{d-\delta_k}$. Thus, $x_n^{\delta_\ell-\delta_k} L^\ell \subset L^k$. Hence, the second condition of x_n-stability follows from x_n-stability of each L^j. ■

Example 4.2.5 *For any submodule $M \subset F$ and monomial order $\succ$ on F, the generic initial module $gin_\succ(M)$ is fixed by the Borel subgroup $\mathcal{B}(F) = B(F) \rtimes B_n(K)$ of $\mathcal{G}(F)$ [35, Proposition 4]. Hence, in characteristic zero, $gin_\succ(M)$ is an x_n-stable module. For arbitrary characteristic, $gin_\succ(M)$ only satisfies the second property of the definition of x_n-stability [35, Proposition 6].*

The following definition was originally described by Caviglia and Kummini [3, Definition 3.6] for multigraded K-vector spaces contained in a ring. We extend this definition to our setting of multigraded K-vector spaces contained in a free R-module.

Definition 4.2.6 *If $W = \bigoplus_{i=0}^{d} V_{d-i} x_n^i \subseteq F_d$ is a multigraded K-vector space, define the $(d+1)$-tuple*

$$d(W) = \Big(\sum_{i=0}^{j} \dim_K(V_{d-i}) \Big)_{j=0,1,\ldots,d}.$$

Let $\Lambda_{d,\ell}$ be the set of all $d(W)$ for all x_n-stable multigraded K-vector spaces $W \subseteq F_d$ with $\dim_K W = \ell$. The partial order on $\Lambda_{d,\ell}$ is given by pointwise inequality.

Remark 4.2.7 *If $\Lambda_{d,\overline{R}}$ is the set of all such $d_{\overline{R}}(W)$, then Caviglia and Kummini showed that, if $V_{d-i} \subset \overline{R}_{d-i}$ are lexsegments, and $d_{\overline{R}}(W)$ is minimal in $\Lambda_{d,\overline{R}}$, then W is a lexsegment of R_d [3][Lemma 3.7]. Furthermore, they showed that if W is a lexsegment, then $d_{\overline{R}}$ must be minimal in $\Lambda_{d,\overline{R}}$, among all $d_{\overline{R}}(Y)$ with $\dim_K Y = \dim_K W$. We prove the corresponding statements for F.*

Proposition 4.2.8 *If* $W = \bigoplus_{i=0}^{d} V_{d-i}x_n^i \subseteq F_d$ *is an* x_n*-stable* K*-vector space with* $\ell := \dim_K W$ *such that* $V_{d-i} \subset \overline{F}_{d-i}$ *are lexsegments for all* i *and* $d(W)$ *is minimal in* $\Lambda_{d,\ell}$*, then* W *is a lexsegment.*

Proof Let $\alpha \le d$ be the largest nonnegative integer such that $V_{d-\alpha} \neq 0$. Since each nonzero V_{d-i} is a lexsegment of $\overline{F}_{d-i}$, then for each $0 \le i \le \alpha$, we can write:

$$V_{d-i} = \left(\bigoplus_{j=1}^{k_i - 1} \overline{R}_{d-i-\delta_j} e_j \right) \oplus L^i e_{k_i}, \tag{4.1}$$

for some integers k_i with $1 \le k_i \le r$ and some nonzero lexsegments $L^i \subseteq \overline{R}_{d-i-\delta_{k_i}}$. By x_n-stability of W, notice that, for all $l \ge i$, $\overline{R}_{l-i} V_{d-l} \subseteq V_{d-i}$. Hence, we have a decreasing sequence of positive integers $k_0 \ge k_1 \ge \cdots \ge k_\alpha$, where $\alpha \le d$ is the largest index such that $V_{d-\alpha} \neq 0$. Let $k := \max\{k_i\} = k_0$.

We will prove that $W = R_{d-\delta_1} e_1 \oplus R_{d-\delta_2} e_2 \oplus \cdots \oplus R_{d-\delta_{k-1}} \oplus W^k e_k$, where $W^k \subseteq R_{d-\delta_k}$ is a nonzero vector space. Suppose W does not have this form. Let $\beta = \min\{j \,|0 \le j \le k-1, W^j \subsetneq R_{d-\delta_j}\}$, and let $m x_n^l e_\beta \in F_d$ be the largest monomial with respect to the lexicographic ordering that is not in $W^\beta e_\beta$, where m is a monomial in $\overline{R}$. Notice that, since V_{d-l} is a lexsegment, then $\beta \in \{k_l, k_l + 1\}$ by (4.1). Since $\beta \le k-1$, then this implies that $k_l \le k-1$. Let $h = \max\{i \,|k_i = k\}$. Since $\{k_i\}$ is a decreasing sequence, then we see that $h < l$. Let $m' x_n^h e_k$ be the smallest monomial in $V_{d-h} x_n^h$ with respect to the lexicographic ordering, where m' is a monomial of $\overline{R}$. Define

$$W' = \left(\bigoplus_{i \neq h,l} V_{d-i} x_n^i \right) \oplus \left(V_{d-h} - \{m' e_k\} \right) x_n^h \oplus \left(V_{d-l} + \{m e_\beta\} \right) x_n^l.$$

We first prove that W' is x_n-stable. To show that the second condition of x_n-stability is satisfied, we need only show that $x_n^{\delta_k - \delta_{k+1}} (W')^{k+1} \subseteq (W')^k$ and that $\mathfrak{m}^{\delta_{\beta-1} - \delta_\beta} (W')^\beta \subseteq (W')^{\beta-1}$, if $\beta \neq 1$. These statements follow trivially from the fact that $W^{k+1} = 0$ and $W^{\beta-1} = R_{d-\delta_{\beta-1}}$. To show that W' satisfies the first condition of x_n-stability, there are two cases.

If $l \neq h+1$, then we need to show two inclusions: $\overline{R}_1 V_{d-h-1} \subseteq V_{d-h} - \{m'e_k\}$ and $\overline{R}_1(V_{d-l} + \{me_\beta\}) \subseteq V_{d-l+1}$. By choice of h, we know that $k_{h+1} \leq k-1$, hence the first inclusion holds. For the second, it's enough to show that $x_i m x_n^{l-1} e_\beta \in W$ for all $i = 1, 2, \ldots, n-1$. But this is true by choice of $mx_n^l e_\beta$ since $x_i m x_n^{l-1} \succ_{lex} m x_n^l$ for all $i \leq n-1$.

If $l = h+1$, we only need to show that $\overline{R}_1(V_{d-l} + \{me_\beta\}) \subseteq V_{d-h} - \{m'e_k\}$. This statement follows as in the proof of the second inclusion of the case when $l \neq h+1$, since $e_\beta \neq e_k$. Hence, W' is x_n-stable, and furthermore $d(W') < d(W)$, which gives a contradiction. Therefore, W has the desired form.

For each $i \leq \alpha$, let $\widehat{V}_{d-i} e_k$ be the image of V_{d-i} in $\widehat{F} = F/(Re_1 \oplus Re_2 \oplus \ldots \oplus Re_{k-1}) = Re_k \oplus Re_{k+1} \oplus \ldots \oplus Re_r$. Then $W^k = \bigoplus_{i=0}^{d} \widehat{V}_{d-i} x_n^i$ is a multigraded K-vector subspace of $R_{d-\delta_k}$. Also, W^k is x_n-stable in $R_{d-\delta_k}$ by the first condition of the definition of x_n-stability for W. Each $\widehat{V}_{d-i}$ is a lexsegment of $\overline{R}_{d-\delta_k}$, since otherwise there would be $m, n \in \operatorname{Mon}(\overline{R}_{d-\delta_k})$ with $m \in \widehat{V}_{d-i}$ and $n \notin \widehat{V}_{d-i}$ such that $n \succ_{lex} m$, which implies that $me_k \in V_{d-i}$, $ne_k \notin V_{d-i}$, and $ne_k \succ_{lex} me_k$ in $\overline{F}_{d-i}$, contradicting that V_{d-i} is a lexsegment.

Finally, suppose $d_{\overline{R}}(W^k)$ is not minimal in $\Lambda_{d,\overline{R}}$. Then there is a multigraded x_n-stable K-vector space $Y^k \subseteq R$ with $\dim_K Y^k = \dim_K W^k$ such that $d_{\overline{R}}(Y^k) < d_{\overline{R}}(W^k)$. Let $Y = Re_1 \oplus Re_2 \oplus \ldots \oplus Re_{k-1} \oplus Y^k e_k$, then $d(Y) < d(W)$, which contradicts minimality of $d(W)$ in $\Lambda_{d,\ell}$. Thus, $d_{\overline{R}}(W^k)$ is minimal in $\Lambda_{d,\overline{R}}$, and so W^k is a lexsegment of $R_{d-\delta_k}$, see Remark 4.2.7. Therefore, W is a lexsegment in F_d. ∎

Lemma 4.2.9 *For any degree d and any positive integer $1 \leq \ell \leq |F_d|$, suppose $L \subset F_d$ is a lexsegment with $\dim_K L = \ell$. Then, $d(L)$ is minimal in $\Lambda_{d,\ell}$.*

Proof Suppose $d(L)$ is not minimal in $\Lambda_{d,\ell}$. Then there is an x_n-stable multigraded K-vector space $W = \bigoplus_{i=0}^{d} Y_{d-i} x_n^i \subseteq F_d$ such that $\dim_K(W) = \ell$ and $d(W) < d(L)$. If W is not a lexsegment, then using the construction of the proof of Proposition 4.2.8, there is an x_n-stable $W' \subset F$ such that $d(W') < d(W)$ and $\dim_K(W') = \ell$. If W' is not a lexsegment, we repeat this process, which terminates in a finite number of steps with a lexsegment K-vector space L' of length ℓ. Since there is exactly one lexsegment of F_d of length ℓ, then $L = L'$, and hence $d(L)$ is minimal in $\Lambda_{d,\ell}$. ■

4.3 The General Hypersurface Restriction Theorem for Modules

In this section and the next chapter, we will work with the Hilbert series of modules, since we have a partial order on the set of Hilbert series of graded R-submodules of F. Let $\mathcal{H}_F$ be the set of all graded R-submodules of F. Suppose $M, N \in \mathcal{H}_F$. Then, the partial order $\geq$ on $\mathcal{H}_F$ is defined as follows: $H(M) \geq H(N)$ if $H(M)_d \geq H(N)_d$ for all $d \in \mathbb{Z}$.

Proposition 4.3.1 *Let M be an x_n-stable R-submodule of the free module F. Then, for all j:*

$$H(F/(M + x_n^j F)) \leq H(F/(M^{lex} + x_n^j F)).$$

Proof Let $L = M^{\text{lex}}$. For all d, since M_d is x_n-stable and L_d is a lexsegment, we have decompositions: $M_d = \bigoplus_{i=0}^{d} W_{d-i} x_n^i$ and $L_d = \bigoplus_{i=0}^{d} V_{d-i} x_n^i$ for some K-vector spaces $W_{d-i} \subseteq \overline{F}_{d-i}$ and lexsegments $V_{d-i} \subseteq \overline{F}_{d-i}$.

Now, for each j and each degree d, the desired inequality $\dim_K(M + x_n^j F)_d \geq \dim_K(L + x_n^j F)_d$ is shown if we can prove the inequality $\sum_{i=0}^{j-1} \dim_K(W_{d-i}) \geq \sum_{i=0}^{j-1} \dim_K(V_{d-i})$. But this holds by minimality of $d(L_d)$ in Λ_d for each d, Lemma 4.2.9. ■

Finally, we reduce to the case of ideals and use Proposition 4.3.1 to prove the main theorem of this chapter.

Theorem 4.3.2 *Suppose M is an R-submodule of F and $h \in R_j$ is a general form of degree j. Then,*

$$H(F/(M + hF)) \leq H(F/(M^{lex} + x_n^j F)).$$

Proof Let $L = M^{\text{lex}}$, and let $\succ$ be the reverse lexicographic order on F. Since $H(M) = H(\text{gin}_\succ(M))$ by Remark 3.4.4 and because there is a unique lexsegment of

F_d with this Hilbert function, then $(\text{gin}_\succ(M))^{\text{lex}} = L$. Hence, by Proposition 3.5.6, we may replace M by $\text{gin}_\succ(M)$ to assume that M is a monomial module. So we can write $M = \bigoplus_{i=1}^{r} I^i e_i$, where each I^i is a monomial ideal. Furthermore, we can assume that M satisfies the second condition of x_n-stability by Example 4.2.5. Then:

$$H(F/(M+hF)) = H(\bigoplus_{i=1}^{r} R/(I^i+hR)e_i) = \sum_{i=1}^{r} H(R/(I^i+hR))(-\delta_i)).$$

Let $L^i = (I^i)^{\text{lex}}$. Then, $H(R/(I^i+hR)) \leq H(R/(L^i+x_n^j R))$ for all $i = 1, 2, \ldots, r$ by Theorem 1.3.2. Now $N = \bigoplus_{i=1}^{r} L^i e_i$ has the same Hilbert series as M, hence $N^{\text{lex}} = L$. Furthermore, N is x_n-stable by Lemma 4.2.4. Thus:

$$H(F/(M+hF)) \leq \sum_{i=1}^{r} H(R/L^i+x_n^j R)(-\delta_i) = H(F/(N+x_n^j F)) \leq H(F/(L+x_n^j F)),$$

where the last inequality holds by Proposition 4.3.1. ■

5. Hilbert Functions of Local Cohomology for Modules

5.1 Introduction

Another important topic of study concerns extremal behavior of Hilbert functions of local cohomology modules. Sbarra proved that, for $R = K[x_1, x_2, \ldots, x_n]$ with homogeneous maximal ideal $\mathfrak{m}$ and I a homogeneous R-ideal, the K-vector space dimensions of $H^i_{\mathfrak{m}}(R/I)_j$ are bounded above by those of $H^i_{\mathfrak{m}}(R/I^{\text{lex}})_j$, for all i and j, Theorem 1.3.3. A related result has been shown by Caviglia and Sbarra [7], when the homogeneous ideal I contains an ideal P generated by powers of the variables. They proved that the Hilbert function of the local cohomology modules of $R/(L+P)$, where $L+P$ is the Lex-Plus-Power ideal of I, is an upper bound for the Hilbert function of local cohomology modules of the quotient by I, Theorem 1.3.4.

We use Theorem 4.3.2 and other results on general hyperplane restriction from the previous chapter to show that the inequality of Sbarra for homogeneous R-ideals is true when extended to the case of graded submodules of a free R-module. We prove this result in Theorem 5.2.10 of this chapter. A different, unpublished proof of this theorem was previously given by Sbarra [37] in his thesis.

5.2 Local Cohomology for Modules

We continue to use the notation of previous chapters: $R = K[x_1, x_2, \ldots, x_n]$ is a polynomial ring in n-variables over a field K, $\mathfrak{m}$ is the homogeneous maximal ideal of R, and $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$ is a free R-module with $\delta_j = \deg(e_j)$ for each $j = 1, 2, \ldots, r$. Without loss of generality, we assume that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_r$.

A reference for the following definitions is [10, Appendix A3 & Section 15.10]. If $M \subset F$ is a graded R-module and $\mathbb{A}$ is an injective R-resolution of F/M, the i^{th}

local cohomology module of F/M with respect to $\mathfrak{m}$, denoted $H^i_{\mathfrak{m}}(F/M)$, is the cohomology of the complex $H^0_{\mathfrak{m}}(\mathbb{A})$, where

$$H^0_{\mathfrak{m}}(V) = \bigcup_{i\geq 0}(0 :_V \mathfrak{m}^i),$$

for all R-modules V. Also, recall that the **saturation** of M is the R-submodule of F:

$$M^{\text{sat}} = \bigcup_{i\geq 0}(M :_F \mathfrak{m}^i).$$

We will use the following preliminary remarks on local cohomology modules throughout this chapter.

Remark 5.2.1 *Notice that the 0^{th} local cohomology module of F/M is related to the saturation, M^{sat}, in the following way:*

$$H^0_{\mathfrak{m}}(F/M) = \bigcup_{i\geq 0}(0 :_{F/M} \mathfrak{m}^i) = \bigcup_{i\geq 0}(M :_F \mathfrak{m}^i)/M = M^{sat}/M.$$

Furthermore, we note that if $M \subseteq F$ is an x_n-stable submodule, then

$$M^{sat} = \bigcup_{i\geq 0}(M :_F \mathfrak{m}^i) = \bigcup_{i\geq 0}(M :_F x_n^i).$$

Remark 5.2.2 *Suppose $N = I^1e_1 \oplus I^2e_2 \oplus \cdots \oplus I^re_r \subset F$ is an R-module, where each I^j is an R-ideal. Then, for all i,*

$$N^{sat} = \bigoplus_{j=1}^{r}(I^j)^{sat}e_j.$$

Remark 5.2.3 *It is proved in [37] that for any graded R-submodule $M \subseteq F$ and $\succ$ a monomial order on F, for all i,*

$$H(H^i_{\mathfrak{m}}(F/M)) \leq H(H^i_{\mathfrak{m}}(F/in_{\succ}(M))).$$

Remark 5.2.4 *Suppose $N = I^1e_1 \oplus I^2e_2 \oplus \cdots \oplus I^re_r \subset F$ is an R-module, where each I^j is an R-ideal. Then, for all i, we have*

$$H^i_{\mathfrak{m}}(F/N) = \bigoplus_{j=1}^{r} H^i_{\mathfrak{m}}((R/I^j)(-\delta_j)).$$

In this section, we will prove a corollary of Theorem 4.3.2, which gives an upper bound on the Hilbert series of local cohomology modules. The bound is attained by the Hilbert series of local cohomology modules of the quotient of a free module by a lexsegment module. In proving this theorem, we may first reduce to the setting of x_n-stable submodules of F, studied in the previous chapter, via the following lemma.

Lemma 5.2.5 *For any graded submodule $M \subseteq F$, there is an x_n-stable submodule $N \subset F$ satisfying the conditions:*

1. *$N = (I')^1 e_1 \oplus (I')^2 e_2 \oplus \cdots \oplus (I')^r e_r$ for some R-ideals $(I')^1, (I')^2, \ldots, (I')^r$;*

2. *$H(N) = H(M)$;*

3. *$H(H^i_{\mathfrak{m}}(F/M)) \leq H(H^i_{\mathfrak{m}}(F/N))$.*

Proof Let $\succ$ be the reverse lexicographic order on F, as defined in 2.2.6. We may first assume M is a monomial module by replacing it with $\text{gin}_{\succ}(M) = I^1 e_1 \oplus I^2 e_2 \oplus \cdots \oplus I^r e_r$, if necessary. Notice that the Hilbert series of M and $\text{gin}_{\succ}(M)$ are the same by Proposition 3.2.2. Furthermore, by Remark 5.2.3, we have the inequality:

$$H(H^i_{\mathfrak{m}}(F/M)) \leq H(H^i_{\mathfrak{m}}(F/\text{gin}_{\succ}(M))).$$

Define $N = (I^1)^{\text{lex}} e_1 \oplus (I^2)^{\text{lex}} e_2 \oplus \cdots \oplus (I^r)^{\text{lex}} e_r$. Each $(I^j)^{\text{lex}}$ is an x_n-stable R-ideal and $H(N) = H(M)$. Then:

$$\begin{aligned} H(H^i_{\mathfrak{m}}(F/M)) &= \sum_{j=1}^{r} H(H^i_{\mathfrak{m}}((R/I^j)(-\delta_j))) \\ &\leq \sum_{j=1}^{r} H(H^i_{\mathfrak{m}}(R/(I^j)^{\text{lex}}(-\delta_j))) \\ &= H(H^i_{\mathfrak{m}}(F/N)), \end{aligned}$$

where the above inequality uses the corresponding result for homogeneous R-ideals, Theorem 1.3.3, and the equalities hold by Remark 5.2.4. Notice that N is x_n-stable by Example 4.2.5 and Lemma 4.2.4. ■

The proof of Theorem 5.2.10 proceeds by inducting on the number of variables of the polynomial ring. Hence, we must study what happens to the Hilbert series of local cohomology modules when we add one variable at a time. This motivates Proposition 5.2.9, which we will prove using the following lemmas regarding saturations and local cohomology modules. We introduce notation for this setting here. Let $\overline{R} = K[x_1, x_2, \dots, x_{n-1}]$, the polynomial ring in the first $n-1$ variables of R over K. We identify $\overline{R}$ with the quotient ring $R/(x_nR)$. When M is a submodule of the free R-module F, $\overline{M}$ denotes the image of M in the free $\overline{R}$-module $\overline{F} = \overline{R}e_1 \oplus \overline{R}e_2 \oplus \cdots \oplus \overline{R}e_r$.

Lemma 5.2.6

(a) For any graded R-submodule $M \subset F$:

$$\overline{M^{sat}}^{sat} = \overline{M}^{sat}.$$

(b) If M is x_n-stable, then $\overline{M}^{lex}$ and $\overline{M^{lex}}$ have the same saturation.

Proof The first statement follows from the fact that M and M^{sat} coincide in large degrees. To prove the second statement, it is clear that $\overline{M^{\text{lex}}}$ is a lexsegment submodule of the free $\overline{R}$-module $\overline{F}$. By 4.3.1, this means that for all degrees d, $\dim_K \overline{M^{\text{lex}}}_d \geq \dim_K \overline{M^{\text{lex}}}^{\text{lex}}_d = \dim_K \overline{M}^{\text{lex}}$. Uniqueness of lexsegments implies the inclusion $\overline{M}^{\text{lex}} \subset \overline{M^{\text{lex}}}$. Since M and M^{lex} have the same Hilbert series and are both x_n-stable, it follows that $\dim_K \overline{M}_d = \dim_K \overline{M^{\text{lex}}}_d$ for large enough values of d. Hence, $\dim_K \overline{M}^{\text{lex}}_d = \dim_K \overline{M^{\text{lex}}}_d$ for $d \gg 0$. Therefore, for $d \gg 0$, $\overline{M}^{\text{lex}}_d$ and $\overline{M^{\text{lex}}}_d$ are equal so that the modules have the same saturation. ■

Lemma 5.2.7 *Let $N \subset \overline{F}$ be an $\overline{R}$-submodule and $M = L^1e_1 \oplus L^2e_2 \oplus \cdots \oplus L^re_r \subset F$ an R-submodule such that each L^i is a lexsegment ideal of R.*

(a) For all degrees d, and for all i:

$$H(H^i_{\mathfrak{m}}(F/NF))_d = \sum_{k \geq d} H(H^{i-1}_{\overline{\mathfrak{m}}}(\overline{F}/N))_{k+1}.$$

(b) For all d, and for all $i \geq 1$:

$$H(H^i_{\mathfrak{m}}(F/M))_d = \sum_{k \geq d} H(H^{i-1}_{\overline{\mathfrak{m}}}(\overline{F}/\overline{M^{sat}}))_{k+1}.$$

(c) For all $i \geq 2$:

$$H(H^i_{\mathfrak{m}}(F/M)) = \Big(\sum_{k<0} t^k\Big) H(H^{i-1}_{\overline{\mathfrak{m}}}\overline{F}/\overline{M}^{sat})).$$

Proof

(a) The proof of this statement in the case of a homogeneous R-ideal I is done in [39]. The case of modules follows similarly.

(b) A corresponding statement for x_n-stable ideals of R was shown in [7, (3.7)]. In particular, if I is an x_n-stable R-ideal, then for all d and for all $i \geq 1$,

$$H(H^i_{\mathfrak{m}}(R/I))_d = \sum_{k \geq d} H(H^{i-1}_{\overline{\mathfrak{m}}}(\overline{R}/\overline{I^{\mathrm{sat}}}))_{k+1}.$$

By Remark 5.2.4,

$$H(H^i_{\mathfrak{m}}(F/M)) = \sum_{j=1}^{r} H(H^i_{\mathfrak{m}}(R/L^j)(-\delta_j)). \tag{5.1}$$

Furthermore, recall that lexsegment R-ideals are x_n-stable. Thus, applying the ring case to this setting gives

$$H(H^i_{\mathfrak{m}}(F/M) = \sum_{k \geq d} H(H^{i-1}_{\overline{\mathfrak{m}}}(\overline{R}/\overline{(L^j)^{\mathrm{sat}}})(-\delta_j))_{k+1} = \sum_{k \geq d} H(H^{i-1}_{\overline{\mathfrak{m}}}(\overline{F}/\overline{M^{\mathrm{sat}}}))_{k+1}.$$

The final equality holds by Remarks 5.2.2 & 5.2.4.

(c) If I is an x_n-stable R-ideal, then for all $i \geq 2$,

$$H(H^i_{\mathfrak{m}}(R/I)) = \Big(\sum_{k<0} t^k\Big) H(H^{i-1}_{\overline{\mathfrak{m}}}\overline{R}/\overline{I}^{\mathrm{sat}})),$$

see [7, (3.9)]. Using this statement for homogeneous ideals, along with (5.1) and Remark 5.2.2, we see that

$$\begin{aligned} H(H^i_{\mathfrak{m}}(F/M) &= \sum_{j=1}^{r} H(H^i_{\mathfrak{m}}(R/L^j)(-\delta_j)) \\ &= \Big(\sum_{k<0} t^k\Big) \sum_{j=1}^{r} H(H^{i-1}_{\overline{\mathfrak{m}}}\overline{R}/\overline{L^j}^{\mathrm{sat}})(-\delta_j)) \\ &= \Big(\sum_{k<0} t^k\Big) H(H^{i-1}_{\overline{\mathfrak{m}}}\overline{F}/\overline{M}^{\mathrm{sat}})). \end{aligned}$$

■

Lemma 5.2.8 *If $M \subset F$ is x_n-stable, $L = M^{lex}$, and $d \gg 0$, then for all k:*

$$\sum_{j=0}^{k} H(\overline{M^{sat}})_{d-j} \geq \sum_{j=0}^{k} H(\overline{L^{sat}})_{d-j}.$$

Proof For each degree d, we can write $M_d = \bigoplus_{i=0}^{d} W_{d-i}x_n^i$ and $L_d = \bigoplus_{i=0}^{d} V_{d-i}x_n^i$, for some K-vector spaces $W_{d-i}, V_{d-i} \subseteq \overline{F}_d$. Since both M and L are x_n-stable, then for $d \gg 0$,

$$N^{\mathrm{sat}} = (M_d)^{\mathrm{sat}} = M_d :_F x_n^{\infty} = \bigoplus_{i=0}^{d} W_{d-i}$$

$$L^{\mathrm{sat}} = (L_d)^{\mathrm{sat}} = L_d :_F x_n^{\infty} = \bigoplus_{i=0}^{d} V_{d-i}.$$

Hence, we conclude that $\overline{M^{\mathrm{sat}}}$ and $\overline{L^{\mathrm{sat}}}$ are the $\overline{R}$-modules generated by $\bigoplus_{i=0}^{d} W_{d-i}$ and $\bigoplus_{i=0}^{d} V_{d-i}$, respectively. Thus, $H((M + x_n^kF)/x_n^kF) = \sum_{j=0}^{k} H(\overline{M^{\mathrm{sat}}})_{d-j}$ and $H((L + x_n^kF)/x_n^kF) = \sum_{j=0}^{k} H(\overline{L^{\mathrm{sat}}})_{d-j}$, so that the desired inequality follows from Proposition 4.3.1. ■

The following proof of Proposition 5.2.9 closely follows the ideas of the proof of an analogous statement, which was shown for the local cohomology modules of quotient rings in the setting of embeddings [7, Theorem 3.1].

Proposition 5.2.9 *Suppose that for any $\overline{R}$-submodule $N \subset \overline{F}$ and for all i:*

$$H(H^i_{\overline{\mathfrak{m}}}(\overline{F}/N)) \leq H(H^i_{\overline{\mathfrak{m}}}(\overline{F}/N^{lex})).$$

Then, for all R-submodules $M \subset F$ and for all i:

$$H(H^i_{\mathfrak{m}}(F/M)) \leq H(H^i_{\mathfrak{m}}(F/M^{lex})).$$

Proof To prove this statement, we first reduce to the case where $M \subset F$ is an x_n-stable submodule satisfying the properties of Lemma 5.2.5. We proceed by inducting on the cohomological degree i. If $i = 0$, then consider, for every j, the following short-exact sequences of R-modules:

$$0 \longrightarrow F/(M :_F x_n^j) \xrightarrow{\cdot x_n^j} F/M \longrightarrow F/(M + x_n^j F) \longrightarrow 0,$$
$$0 \longrightarrow F/(M^{\text{lex}} :_F x_n^j) \xrightarrow{\cdot x_n^j} F/M^{\text{lex}} \longrightarrow F/(M^{\text{lex}} + x_n^j F) \longrightarrow 0.$$

Since $H(F/M) = H(F/M^{\text{lex}})$ and $H(F/(M + x_n^j F)) \leq H(F/(M^{\text{lex}} + x_n^j F)$ by Proposition 4.3.1, then additivity along short-exact sequences implies that $H(F/(M :_F x_n^j)) \geq H(F/(M^{\text{lex}} :_F x_n^j))$. Thus, for $j \gg 0$, we have:

$$H(H^0_{\mathfrak{m}}(F/M)) = H((M :_F x_n^j)/M) \leq H((M^{\text{lex}} :_F x_n^j)/M) = H(H^0_{\mathfrak{m}}(F/M^{\text{lex}})).$$

For cohomological degree $i = 1$, first notice that:

$$H^0_{\overline{\mathfrak{m}}}(\overline{F}/\overline{M^{\text{sat}}}) \cong (\overline{M^{\text{sat}}} :_{\overline{F}} \overline{\mathfrak{m}}^\infty)/\overline{M^{\text{sat}}} \cong \overline{M^{\text{sat}}}^{\text{sat}}/\overline{M^{\text{sat}}}.$$

Combining this isomorphism with Lemma 5.2.7(b) gives, for each d:

$$H(H^1_{\mathfrak{m}}(F/M))_d = \sum_{j \geq d} H(H^0_{\overline{\mathfrak{m}}}(\overline{F}/\overline{M^{\text{sat}}}))_{j+1} = \sum_{j \geq d} H(\overline{M^{\text{sat}}}^{\text{sat}})_{j+1} - \sum_{j \geq d} H(\overline{M^{\text{sat}}})_{j+1}. \tag{5.2}$$

By the same argument, setting $L = M^{\text{lex}}$, we see that:

$$H(H^1_{\mathfrak{m}}(F/L))_d = \sum_{j \geq d} H(\overline{L^{\text{sat}}}^{\text{sat}})_{j+1} - \sum_{j \geq d} H(\overline{L^{\text{sat}}})_{j+1}. \tag{5.3}$$

Now using both statements of Lemma 5.2.6, we find that:

$$\sum_{j \geq d} H(\overline{M^{\text{sat}}}^{\text{sat}})_{j+1} = \sum_{j \geq d} H(\overline{M}^{\text{sat}})_{j+1},$$
$$\sum_{j \geq d} H(\overline{L^{\text{sat}}}^{\text{sat}})_{j+1} = \sum_{j \geq d} H(\overline{L}^{\text{sat}})_{j+1} = \sum_{j \geq d} H((\overline{M}^{\text{lex}})^{\text{sat}})_{j+1}.$$

Hence, applying the assumption of the proposition with $N = \overline{M}$ and $i = 0$, along with the fact that $\overline{M}$ and $\overline{M}^{\text{lex}}$ have the same Hilbert series, we obtain the inequality:

$$\begin{aligned}\sum_{j \geq d} H(\overline{M^{\text{sat}}}^{\text{sat}})_{j+1} &= \sum_{j \geq d} H(\overline{M}^{\text{sat}})_{j+1} \\ &\leq \sum_{j \geq d} H((\overline{M}^{\text{lex}})^{\text{sat}})_{j+1} \\ &= H(\overline{L^{\text{sat}}}^{\text{sat}})_{j+1}.\end{aligned} \tag{5.4}$$

Finally, by Lemma 5.2.8, we also have the inequality for $k \gg 0$:

$$\begin{aligned}\sum_{j \geq d} H(\overline{M^{\text{sat}}})_{j+1} &= \sum_{l=0}^{k} H(\overline{M^{\text{sat}}})_{d+1+k-l} \\ &\geq \sum_{l=0}^{k} H(\overline{L^{\text{sat}}})_{d+1+k-1} \\ &= \sum_{j \geq d} H(\overline{L^{\text{sat}}})_{j+1}.\end{aligned} \tag{5.5}$$

Combining (5.2), (5.3), (5.4), and (5.5), we obtain the desired inequality in cohomological degree $i = 1$.

When $i \geq 2$, applying Lemma 5.2.7(c) to both sides of the desired inequality reduces the problem to showing:

$$H(H_{\overline{\mathfrak{m}}}^{i-1}(\overline{F}/\overline{M}^{\text{sat}})) \leq H(H_{\overline{\mathfrak{m}}}^{i-1}(\overline{F}/\overline{L}^{\text{sat}})).$$

By assumption, we again have:

$$H(H_{\overline{\mathfrak{m}}}^{i-1}(\overline{F}/\overline{M})) \leq H(H_{\overline{\mathfrak{m}}}^{i-1}(\overline{F}/\overline{M}^{\text{lex}})),$$

which, together with 5.2.6 gives the desired inequality. ■

Theorem 5.2.10 *If R is a polynomial ring over a field K with homogeneous maximal ideal $\mathfrak{m}$ and M is a submodule of a free R-module $F = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_r$, then $\forall i$:*

$$H(H^i_{\mathfrak{m}}(F/M)) \leq H(H^i_{\mathfrak{m}}(F/M^{lex})).$$

Proof By Lemma 5.2.5, we may assume $m = I^1e_1 \oplus I^2e_2 \oplus \cdots \oplus I^re_r$, where each I^j is an x_n-stable R-ideal. We prove this statement by induction on the number of variables in the polynomial ring R. Over the field K, we consider $F = Ke_1 \oplus Ke_2 \oplus \cdots \oplus Ke_r$. It's clear that $F/M \cong F/M^{\text{lex}}$, and the modules have the same cohomology. For the induction step, since the inequality holds in one less variable for all $\overline{R}$-submodules of $\overline{F}$, then the desired inequality also holds for $M \subseteq F$ by Proposition 5.2.9. ■

6. Betti Numbers of Piecewiselex Ideals

6.1 Introduction

The problem of transforming an ideal into another that has the same Hilbert function and graded Betti numbers greater than or equal to those of the original ideal is one of interest to many researchers. One of the earliest results in this direction is Macaulay's Theorem 1.2.1, which states that if $R = K[x_1, ..., x_n]$ is a polynomial ring over a field K, then there exists a lexsegment ideal realizing the Hilbert function of any homogeneous ideal of R. Later, Bigatti, Hulett, and Pardue showed that lexsegment ideals attain the highest Betti numbers among all ideals having the same Hilbert function, Theorem 1.4.1.

The two main conjectures in this area of research are the EGH Conjecture 1.2.3 and the Lex-Plus-Powers Conjecture 1.4.2. These conjectures propose that, for a homogeneous ideal I containing a homogeneous regular sequence $(f_1, ..., f_r)$ with degrees $e_1 \leq e_2 \leq \cdots \leq e_r$, there exists a lex-plus-powers ideal $L + P$ which has the same Hilbert function as I and graded Betti numbers at least as large as those of I, where $P = (x_1^{e_1}, ..., x_r^{e_r})$. A strong result related to these conjectures has been shown by Mermin and Murai, Theorem 1.4.3. They prove the Lex-Plus-Powers Conjecture holds when $(f_1, ..., f_r)$ is a regular sequence of monomials. Notice that, under this assumption, the Eisenbud-Green-Harris Conjecture easily follows from Clements and Lindström's Theorem 1.2.2.

A generalization of the Mermin-Murai result was shown by Caviglia and Sbarra [6]. In their article, the authors study homogeneous ideals I containing $P + \widetilde{L}$, where $\widetilde{L}$ is a **piecewise lex ideal**, that is, an ideal which is the sum of extensions to R of lexsegment ideals $L_i \subset K[X_1, ..., X_i]$. The quotient rings $R/(P + \widetilde{L})$ are known as Shakin rings.

Theorem 6.1.1 *[6, Theorem 3.4] Suppose $\widetilde{L}$ is a piecewise lex ideal, and I is a homogeneous R-ideal containing $P+\widetilde{L}$. Then, there is a lexsegment ideal L such that $P+\widetilde{L}+L$ and I have the same Hilbert function. Furthermore, when char$(K)=0$, $\beta_{ij}(I) \leq \beta_{ij}(P+\widetilde{L}+L)$ for all i,j.*

The main theorem of this chapter removes the assumption on the characteristic of the field K in this result. We use the following statement of Mermin and Murai to reduce to the case when I is a strongly-stable-plus-P ideal:

Proposition 6.1.2 *[30, Proposition 8.7] If I is a monomial ideal of R containing an ideal of powers $P=(x_1^{e_1}, x_2^{e_2}, \ldots, x_r^{e_r})$, then there is a strongly-stable-plus-P ideal B with the same Hilbert function as I such that, for all i,j, $\beta_{ij}(I) \leq \beta_{ij}(B)$.*

In Sections 6.2 and 6.3, we describe the operations used in [30] to replace the ideal I by a strongly-stable-plus-P ideal whose graded Betti numbers are an upper bound for those of the monomial ideal I. We prove that strongly stable ideals are fixed under these operations. Section 6.4 contains the proof of our main theorem using a result of Caviglia and Kummini [4] to reduce the problem to the characteristic zero case.

A large part of the content of this chapter is based on joint work of the author with Gabriel Sosa [27].

6.2 Shifting

Throughout this chapter, $R=K[x_1,...,x_n]$ is a polynomial ring over a field K, where char(K) is arbitrary, and $P=(x_1^{e_1},...,x_r^{e_r})$, for some positive integer $r \leq n$ and integers $2 \leq e_1 \leq e_2 \leq \cdots \leq e_r$. Furthermore, in this section, we will assume that I is a monomial ideal containing $P+J$ where J is a strongly stable ideal. We previously discussed strongly-stable modules, see Definition 3.5.2. We obtain the corresponding definition for ideals $J \subset R$ as a special case of the definition for modules. See Green [22] for more discussion of this class of ideals.

Definition 6.2.1 *An R-ideal J is strongly stable if it satisfies the combinatorial property that whenever $x_i m \in J$, then $x_j m \in J$ for all monomials m and for all $j < i$.*

In the proof of their main theorem, Mermin and Murai show that there exists a strongly-stable-plus-P ideal B with the same Hilbert function as the ideal I such that $\beta_{ij}(B) \geq \beta_{ij}(I)$ for all i, j, see Theorem 6.1.2. In this section and the one that follows, we recall the operations that Mermin and Murai use to construct the ideal B from I, and show that in addition to the properties above, we also have $J \subset B$.

We first discuss shifting of monomial ideals, which is a generalization of combinatorial shifting of simplicial complexes, introduced by Erdös, Ko, and Rado [15], see also [23, Chapter 11]. When I is a square-free monomial ideal, the definition of combinatorial shifting on simplicial complexes can be translated to a definition of combinatorial shifting of I. This definition is extended to all monomial ideals by Mermin and Murai [30]:

Definition 6.2.2 *Let I be a monomial ideal. Fix variables $a >_{lex} b$. The (a,b)-**shift** of I, denoted Shift$_{a,b}(I)$, is the K-vector space generated by monomials of the form:*

$$\left\{ \begin{array}{lcl} ma^sb^s & | & ma^sb^s \in I \\ ma^\ell b^s & | & ma^\ell b^s \in I \ \text{ or } \ ma^sb^\ell \in I \\ ma^sb^\ell & | & ma^\ell b^s \in I \ \text{ and } \ ma^sb^\ell \in I \end{array} \right\}$$

where the set is taken over all monomials m such that $a \nmid m$ and $b \nmid m$, and over all integers $0 \leq s < \ell$.

Notice that $\text{Shift}_{(a,b)}(I)$ is the ideal obtained from I by replacing $ma^sb^\ell \in I$ by the larger monomial, with respect to the lexicographic order, $ma^\ell b^s$, if $ma^\ell b^s$ is not in I.

Definition 6.2.3 *Let I be a monomial ideal, and fix variables $a >_{lex} b$ and $t \in \mathbb{Z}_{\geq 0}$. The (a,b,t)**-shift** of I, denoted Shift$_{a,b,t}(I)$, is the K-vector space generated by monomials of the form:*

$$\left\{ \begin{array}{lll} ma^s b^r & | & ma^s b^r \in I, r < t \\ ma^s b^{s+t} & | & ma^s b^{s+t} \in I \\ ma^\ell b^{s+t} & | & ma^\ell b^s + t \in I \ \text{ or } \ ma^s b^{\ell+t} \in I \\ ma^s b^{\ell+t} & | & ma^\ell b^s + t \in I \ \text{ and } ,ma^s b^{\ell+t} \in I \end{array} \right\}$$

where the set is taken over all monomials m such that $a \nmid m$ and $b \nmid m$, and over all integers $0 \leq s < \ell$.

Remark 6.2.4 *Notice that the (a,b)-shift of I is the special case of the (a,b,t)-shift when $t = 0$.*

Remark 6.2.5 *For $t \neq 0$, Shift$_{a,b,t}(I)$ does not fix ideals generated by powers of variables. Hence, when applying the shifting operation Shift$_{(a,b,t)}(I)$, when $t \neq 0$, Mermin and Murai replace I by Shift$_{(a,b,t)}(I) + P$. We will discuss this further in the next section.*

Now we will prove the main result of this section, that strongly stable ideals are fixed under the operation of (a,b,t)-shifting. This statement is vital to our proof of Proposition 6.3.6.

Proposition 6.2.6 *Let I be a monomial ideal containing $P+J$. Fix variables $a >_{lex} b$ and $t > 0$. Then, $J \subset$ Shift$_{a,b,t}(I)$.*

Proof Write $m = m'a^\alpha b^\beta \in J$, where $a \nmid m'$ and $b \nmid m'$. If $\beta \leq \alpha + t$, then it's clear that $m \in \text{Shift}_{a,b,t}(I)$. The only case where we need to use the assumption that J is strongly stable is when $\beta > \alpha + t$. Here, we need to show that $m'a^{\beta-t}b^{\alpha+t} \in I$. Let $N = \beta - (\alpha + t)$. Since J is strongly stable and $N > 0$, then $m \cdot \frac{a^N}{b^N} \in J \subset I$. We see that $m \cdot \frac{a^N}{b^N} = m'a^{\beta-t}b^{\alpha+t}$. Since both $m = m'a^\alpha b^{(\beta-t)+t} \in I$ and $m'a^{\beta-t}b^{\alpha+t} \in I$, it follows that $m \in \text{Shift}_{a,b,t}(I)$. ■

6.3 Compression

The second operation used to transform a monomial ideal I in the proof of Mermin and Murai is compression. This operation was used by Macaulay and Clements and Lindström in their proofs of Theorem 1.2.1 and Theorem 1.2.2, respectively.

Compression can be defined more generally with respect to a subset of variables $\mathcal{A} \subset \{x_1, x_2, \ldots, x_n\}$ of any cardinality. For our purposes, we define compression only in the case when $\mathcal{A}$ is a set consisting of two variables. Using this case, we will obtain an important property of strongly stable ideals. The following definition is described by Mermin in [29]:

Definition 6.3.1 *Let I be a monomial ideal, and fix variables $a >_{lex} b$. Write I as a direct sum of the form $I = \bigoplus_m mV_m$, where the sum is taken over all monomials m in $K[\{x_1, ..., x_n\} \setminus \{a, b\}]$ and V_m are $K[a, b]$-ideals. The $\{a, b\}$-***compression** *of I is the ideal $\bigoplus_m mN_m$, where $N_m \subset K[a, b]$ are the lexsegment ideals with the same Hilbert function as V_m. A monomial ideal I is called $\{a, b\}$-**compressed** if its $\{a, b\}$-compression is itself.*

Example 6.3.2 *Let $R = K[x, y, z]$ and $I = (xy^2, xyz, z^4)$. We compute the $\{y, z\}$-compression of I. We first decompose I as follows:*

$$I = 1 \cdot (z^4) \oplus x \cdot (y^2, yz, z^4) \oplus x^2 \cdot (y^2, yz, z^4) \oplus \cdots .$$

The $\{y, z\}$-compression is obtained from I by replacing each of the $K[y, z]$-ideals of this decomposition with the lexsegment $K[x, y]$-ideals with the same Hilbert function. Hence, the $\{y, z\}$-compression of I is:

$$J = 1 \cdot (y^4) \oplus x \cdot (y^2, yz, z^4) \oplus x^2 \cdot (y^2, yz, z^4) \oplus \cdots .$$

Thus, the $\{y, z\}$-compression of I is the ideal $J = (xy^2, xyz, y^4, xz^4)$.

Remark 6.3.3 *We note that, as in the case of the (a, b, t)-shift, the $\{a, b\}$-compression does not fix an ideal of powers, which we can see in the previous example. Specifically, if b^{e_b} is a minimal generator of I, then b^{e_b} is not in the $\{a, b\}$-compression of*

I. Hence, in their construction, Mermin and Murai compress the monomial ideal generated by all minimal generators of I except for b^{e_b}.

Proposition 6.3.4 *[29, Proposition 3.8] If I is a strongly stable R-ideal, then I is (a,b)-compressed, for each pair of variables $a >_{lex} b$ of R.*

Proof Fix a pair of variables $a >_{\text{lex}} b$, and write $I = \bigoplus_m mV_m$, where the sum is taken over all $m \in \text{Mon}(K[\{x_1, x_2, \ldots, x_n\} \setminus \{a,b\}])$ and $V_m \subset K[a,b]$ are ideals. We will show that each V_m is a lexsegment ideal, using that I is strongly stable by assumption.

Suppose $a^\alpha b^\beta \in V_m$, $\beta \neq 0$. Since I is strongly stable and $ma^\alpha b^\beta \in I$, then $ma^{\alpha+1}b^{\beta-1} \in V_m$. Hence, $a^{\alpha+1}b^{\beta-1} \in V_m$. Repeating this argument, we see that the segment $a^{\alpha+\beta} >_{\text{lex}} a^{\alpha+\beta-1}b >_{\text{lex}} \cdots >_{\text{lex}} a^\alpha b^\beta$ is in the ideal V_m for each $a^\alpha b^\beta \in V_m$. Thus, V_m is a lexsegment ideal for each $m \in \text{Mon}(K[\{x_1, x_2, \ldots, x_n\} \setminus \{a,b\}])$, and therefore I is $\{a,b\}$-compressed. ∎

Proposition 6.3.5 *Let I be a monomial ideal containing $P+J$. Fix variables $a >_{lex} b$. Let I' be the ideal of R generated by all the minimal generators of I except for b^{e_b}, let T' be the $\{a,b\}$-compression of I', and let $T = T' + P$. Then, $J \subset T$.*

Proof As in the definition of $\{a,b\}$-compression, write $I' = \bigoplus_m mV_m$ with $m \in \text{Mon}(K[\{x_1, ..., x_n\} \setminus \{a,b\}])$ and $V_m \subset K[a,b]$. Let $T' = \bigoplus_m mN_m$ be the $\{a,b\}$-compression of I'. First, suppose b^{e_b} is not a minimal generator of I. In this case, $I' = I$, and therefore, T' is the $\{a,b\}$-compression of I. Since strongly stable ideals are $\{a,b\}$-compressed by Proposition 6.3.4, then $J \subset T'$.

If instead b^{e_b} is a minimal generator of I, let $m = m'a^\alpha b^\beta$ be a monomial in J with $a \nmid m'$, $b \nmid m'$. Clearly, if $\beta \geq e_b$, then $m \in P \subset T$. So we may assume $\beta < e_b$. Since J is strongly stable, then we have:

$$m = m'a^\alpha b^\beta <_{lex} m'a^{\alpha+1}b^{\beta-1} <_{lex} \cdots <_{lex} m'a^{\alpha+\beta} \in J.$$

Furthermore, all of these monomials are in I'. Thus,

$$a^\alpha b^\beta <_{lex} a^{\alpha+1}b^{\beta-1} <_{lex} \cdots <_{lex} a^{\alpha+\beta} \in V_{m'}.$$

These are the first monomials of degree $\alpha + \beta$ in $K[a, b]$, hence they are also elements of the lex ideal $N_{m'}$. In particular, this implies that $m \in T'$. ■

Finally, we discuss how the operations of shifting and compression are used in Mermin and Murai's proof of Proposition 6.1.2, and prove the main proposition of this chapter. Given a monomial ideal I, for pairs of variables $a >_{\text{lex}} b$, a strongly-stable-plus-P ideal B is constructed in finitely many steps by replacing I with any of the following ideals:

1. $\text{Shift}_{a,b}(I)$
2. $\text{Shift}_{a,b,t}(I) + P$
3. $T = T' + P$ as in Proposition 6.3.5.

Proposition 6.3.6 *If I is a monomial ideal containing $P + J$, then there exists a strongly-stable-plus-P ideal B with the same Hilbert function as I such that $\beta_{ij}(B) \geq \beta_{ij}(I)$ for all i, j and $J \subset B$.*

Proof By Proposition 6.1.2, there exists a strongly-stable-plus-P ideal B with the same Hilbert function as I and $\beta_{ij}(B) \geq \beta_{ij}(I)$ for all i, j. Furthermore, by Propositions 6.2.6 and 6.3.5, strongly stable ideals do not move under the operations used to construct the ideal B. Hence, when $J \subset I$, we also have $J \subset B$. ■

6.4 Piecewise Lex Ideals and the Main Result

In this section, we discuss piecewise lex ideals, which are a generalization of lexsegment ideals discussed earlier. We will then use our work of the previous sections of this chapter to prove our main theorem. We begin by reminding the reader of the definition of piecewise lex ideals introduced by Shakin [40].

Definition 6.4.1 *For each $1 \leq i \leq n$, let $R_{(i)}$ be the polynomial ring over K in the first i variables, that is, $R_{(i)} = K[x_1, x_2, \ldots, x_i]$. An ideal $\widetilde{L} \subset R$ is called a* **piecewise lex ideal** *if it can be written as a sum:*

$$\widetilde{L} = L_{(1)}R + L_{(2)}R + \ldots + L_{(n)}R$$

where each nonzero $L_{(i)}$ is a lexsegment ideal in the ring $R_{(i)}$ and not all $L_{(i)}$ are zero.

Example 6.4.2 *Let $R = K[x, y, z]$, $L_{(1)} = (x^4) \subset K[x]$, and $L_{(2)} = (x^3, x^2y, xy^2)$. Then, $\widetilde{L} = L_{(1)}R + L_{(2)}R$ is the R-ideal (x^4, x^3, x^2y, xy^2). From this example, we can see that a piecewise lex ideal need not be a lexsegment ideal. In particular, the monomials of $\widetilde{L}$ of degree 4 are $x^4, x^3y, x^3z, x^2y^2, x^2yz, xy^2$, which is not a lexsegment as the monomial x^2z^2 is not in $\widetilde{L}$.*

In the previous section, we showed that strongly stable ideals do not move under any of the three operations used to construct the ideal B. Now, we apply this to the situation in which $J = \widetilde{L}$ is a piecewise lex ideal.

Lemma 6.4.3 *If I is a monomial ideal containing $P + \widetilde{L}$, and B is the strongly-stable-plus-P ideal constructed as above, then $\widetilde{L} \subset B$.*

Proof Since $\widetilde{L}$ is a piecewise lex ideal, then $\widetilde{L} = L_{(1)}R + L_{(2)}R + \ldots + L_{(n)}R$, for some lexsegments $L_{(i)} \subset R_{(i)}$ and $1 \leq r \leq n$. We first show that $\widetilde{L}$ is strongly stable. If m is any monomial of $\widetilde{L}$, then $m \in L_{(i)}R$ for some i, hence can be written as $m = x_{i+1}^{\alpha_{i+1}} x_{i+2}^{\alpha_{i+2}} x_n^{\alpha_n} m'$, where $m' \in L_{(i)}$ and α_j are non-negative integers. Suppose x_j divides m.

There are two main cases. If $j \leq i$, then for all $k < j$, $\frac{x_k}{x_j} m' \in L_{(i)}$, hence $\frac{x_k}{x_j} m \in \widetilde{L}$. If instead $j \geq i + 1$, then for all $k < j$, $\frac{x_k}{x_j} x_{i+1}^{\alpha_{i+1}} x_{i+2}^{\alpha_{i+2}} x_n^{\alpha_n} \in R$, so that $\frac{x_k}{x_j} m \in L_{(i)}R$. Thus, $\widetilde{L}$ is strongly stable. By Propositions 6.2.6 and 6.3.5, $\widetilde{L}$ is fixed under the operations used to construct B, therefore $\widetilde{L} \subset B$. ∎

In the proof of the main theorem, we use the fact that the graded Betti numbers of a strongly-stable-plus-P ideal of R do not depend on the field K. This allows us to reduce to the characteristic zero case, where the statement of the theorem is known. We state this result here, with proof.

Lemma 6.4.4 *[4, Corollary 3.7] If B is a strongly-stable-plus-P ideal of R, for an ideal of powers of the variables $P = (x_1^{e_1}, x_2^{e_2}, \dots, x_r^{e_r})$, $r \leq n$, then the values of the graded Betti numbers of B do not depend on the field K.*

Proof The proof proceeds by induction on the number of variables of the polynomial ring. Let $\overline{R} = K[x_1, x_2, \dots, x_{n-1}]$, and for homogeneous ideals $I \subset R$, let $\overline{I}$ denote the image of I in $\overline{R}$, that is, its image modulo the variable x_n. We can write the monomial ideal B in the form $B = \bigoplus_{h \geq 0} V_h x_n^h$, where each V_i is an $\overline{R}$-ideal. There are two cases. If $r < n$, then, as $\overline{R}$-modules, we have

$$B/x_nB = V_0 \oplus \bigoplus_{h \geq 1} (V_h/V_{h-1})(-h).$$

On the other hand, if $r = n$, then as $\overline{R}$-modules,

$$B/x_nB = V_0 \oplus \bigoplus_{h=1}^{e_n - 2} (V_h/V_{h-1})(-h) \oplus \overline{R}/V_{e_n - 1},$$

where V_{e_n-1} is a nonzero ideal. In either case, V_0 is an $\overline{R}$ ideal which contains the image of the ideal P in the smaller polynomial ring $\overline{R}$. In fact, if we write $B = J + P$, where J is a strongly stable R-ideal, then $V_0 = \overline{J} + \overline{P}$. Thus, V_0 is a strongly-stable-plus-$\overline{P}$ ideal. Hence, by induction, its graded Betti numbers do not depend on the field K. Similarly, V_{e_n-1} is a strongly-stable-plus-$\overline{P}$ ideal, and so the statement follows for the graded Betti numbers of R/V_{e_n-1} by induction.

Furthermore, each V_h/V_{h-1} is a K-vector space, hence has Betti numbers determined by the Koszul complex. So the Betti numbers of each vector space V_h/V_{h-1} does not depend on the field K.

For all i, we have the following isomorphism as $\overline{R}$-modules:

$$\mathrm{Tor}_i^{\overline{R}}(K, B/x_nB) \cong \mathrm{Tor}_i^{\overline{R}}(K, V_0) + \mathrm{Tor}_i^{\overline{R}}(K, \oplus_h (V_h/V_{h-1})(-h)) + \mathrm{Tor}_i^{\overline{R}}(K, \overline{R}/V_{e_n-1}),$$

where the last term, $\mathrm{Tor}_i^{\overline{R}}(K, \overline{R}/V_{e_n-1})$, is zero in the case when $r < n$. Hence, the graded Betti numbers of $\overline{B}$ over $\overline{R}$ do not depend on the characteristic of K. Furthermore, for all i, $\mathrm{Tor}_i^{\overline{R}}(K, B/x_nB) \cong \mathrm{Tor}_i^R(K, B)$, see [4, Lemma 3.3]. Thus, the graded Betti numbers of B over R do not depend on the characteristic of the field K. ■

Theorem 6.4.5 *Let $I \subset R$ be a homogeneous ideal with $P + \widetilde{L} \subset I$. Then, there exists a lexsegment R-ideal L such that*

(i) $P + \widetilde{L} + L$ has the same Hilbert function as I.

(ii) $\beta_{ij}(I) \leq \beta_{ij}(P + \widetilde{L} + L)$ for all i, j.

Proof Let $\succ$ be the reverse lexicographic order on the ring R, as defined in Example 2.2.6. By uppersemicontinuity, Proposition 3.3.2, $\beta_{ij}(I) \leq \beta_{ij}(\mathrm{in}_\succ(I))$. Furthermore, $\mathrm{in}_\succ(I)$ has the same Hilbert function as I. Since P and $\widetilde{L}$ are monomial ideals, and by assumption $P + \widetilde{L} \subset I$, then $P + \widetilde{L}$ is also contained in $\mathrm{in}_\succ(I)$. Hence, without loss of generality, we assume I is a monomial ideal containing $P + \widetilde{L}$ by replacing I with $\mathrm{in}_\succ(I)$.

By Proposition 6.3.6, there is a strongly-stable-plus-P ideal B with the same Hilbert function as I and such that $\beta_{ij}(B) \geq \beta_{ij}(I)$. Furthermore, we have that $P + \tilde{L} \subset B$. In this situation, the graded Betti numbers, $\beta_{ij}(B)$, do not depend on $\mathrm{char}(K)$ by Lemma 6.4.4. Hence, we can assume $\mathrm{char}(K) = 0$. The characteristic zero result Theorem 6.1.1 gives a lex ideal L such that $P+\widetilde{L}+L$ has the same Hilbert function as B and $\beta_{ij}(P + \widetilde{L} + L) \geq \beta_{ij}(B)$ for all i, j. Again, the Betti numbers do not depend on the characteristic of the field, so the inequality also holds for $\mathrm{char}(K)$ arbitrary. ■

Notice that the bound in Theorem 6.4.5(ii) on the graded Betti numbers is sharp since the ideal $P + \widetilde{L} + L$ is a homogeneous ideal containing $P + \widetilde{L}$ and has the same Hilbert function as I by Theorem 6.4.5(i). The following example shows that, among all ideals that have the same Hilbert function and contain $P + \widetilde{L}$, our theorem gives a closer bound for the graded Betti numbers than that of Mermin and Murai. The computations for this example were done using Macaulay2 [20].

Example 6.4.6 *Let* $R = (\mathbb{Z}/2\mathbb{Z})[x,y,z,w]$, $P = (x^3, y^3, z^3, w^4)$ *and* $\widetilde{L} = (x^3, x^2y, xy^2, x^2z^2)$. *Let* I *be any homogeneous* R*-ideal such that* $P + \widetilde{L} \subset I$ *and* R/I *has Hilbert series* $1 + 4t + 10t^2 + 15t^3 + 15t^4 + 10t^5 + 2t^6$. *For example,* $I = P + \widetilde{L} + (zw^3)$ *is such an ideal. There are many different possible Betti tables for ideals satisfying the above conditions, but for our specific example of such an ideal* I*, the Betti table for* R/I *is:*

	0	1	2	3	4
0	1	-	-	-	-
1	-	-	-	-	-
2	-	5	3	-	-
3	-	3	4	1	-
4	-	-	3	2	-
5	-	-	9	13	4
6	-	-	-	2	2

Using Mermin and Murai's result, we obtain a lex-plus-powers ideal with the same Hilbert function as I*:*

$$P + (x^2y, x^2z, x^2w^2, xy^2z, xy^2w^2, xyz^2w, xyzw^3, xz^2w^3, y^2z^2w^2).$$

The Betti table for the quotient of R *by this lex-plus-powers ideal is:*

	0	1	2	3	4
0	1	-	-	-	-
1	-	-	-	-	-
2	-	5	3	1	-
3	-	3	7	4	1
4	-	2	9	8	2
5	-	3	14	16	5
6	-	-	1	3	2

Notice that the graded Betti numbers for the lex-plus-powers ideal are an upper bound for the graded Betti numbers of I. We obtain the best possible bound by using our result, since there is an ideal, namely $P + \widetilde{L} + L$, that has the largest possible graded Betti numbers under these assumptions. The ideal obtained from Theorem 6.4.5 with the same Hilbert function as I is:

$$P + \widetilde{L} + (x^2zw, x^2w^3, xyz^2w, xyzw^3, xz^2w^3, y^2z^2w^2).$$

The Betti table for $R/(P + \widetilde{L} + L)$ is:

	0	1	2	3	4
0	1	-	-	-	-
1	-	-	-	-	-
2	-	5	3	-	-
3	-	3	6	4	1
4	-	2	9	8	2
5	-	3	14	16	5
6	-	-	1	3	2

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