SEQUENTIAL PROCEDURES FOR THE "SELECTION" PROBLEMS IN DISCRETE SIMULATION OPTIMIZATION

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Wenyu Wang

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

December 2019

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Dr. Hong Wan, Chair
School of Industrial Engineering
Dr. Andrew Liu
School of Industrial Engineering
Dr. Raghu Pasupathy
Department of Statistics
Dr. Susan Hunter
School of Industrial Engineering

Approved by:
Dr. Abhijit Deshmukh
Head of the School Graduate Program

To Bing and Nora.

ACKNOWLEDGMENTS

First and foremost, I wish to express my sincere gratitude to my advisor Hong Wan for her excellent guidance and support over the years. Her patience, humor, and immense knowledge have been a constant source of motivation, and I could not wish for a better or friendlier advisor. The continuous encouragement she provided was essential for the determination of the course of this work. I am indebted to her for the time and attention they devoted to this project.

I thank Raghu Pasupathy, Andrew Liu and Susan Hunter for sitting on my research committee and offering helpful comments. I also appreciate the rest of the faculty and staff at the School of Industrial Engineering and the School of Statistics whom I have had the great pleasure to work with and learn from.

My time at Purdue was made enjoyable in large part due to the many friends and groups that became a part of my life. It is a privilege to work among an outstanding and most easy-going group of Ph.D. students at Purdue. I praise the valuable insights into many aspects of graduate student life given by Tian Ni, Liang Wang, and Hong-Wei Tsui when I first came here. Thank you to my schoolmates Yaowu Hu, Yixuan Qiu, Yaowu Liu, and Yiming Wang, who never refuse to enjoy a moment of fellowship with me, be it in the after-school party, in the movie theaters, or during week-long trips on the road.

Last but certainly not least, I would like to thank my family for all their love and encouragement. For my parents who raised me with respect of science and supported me in all my pursuits. And most of all for my loving, supportive, encouraging, and patient wife Bing whose unwavering support during the final stages of this Ph.D. is so appreciated. Thank you.

TABLE OF CONTENTS

Page

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

ABBREVIATIONS

R&S	Ranking and Selection
SOR&S	Single-Objective Ranking and Selection
MOR&S	Multi-Objective Ranking and Selection
FS	Factor Screening
MOFS	Multi-Objective Factor Screening
SPRT	Sequential Probability Ratio Test
GSPRT	Generalized Sequential Probability Ratio Test
LR	Likelihood Ratio
GLR	Generalized Likelihood Ratio
MLE	Maximum Likelihood Estimator
FWERI	Type I Family-Wise Error Rate
FWERII	Type II Family-Wise Error Rate

ABSTRACT

Wang, Wenyu PhD, Purdue University, December 2019. Sequential Procedures for the "Selection" Problems in Discrete Simulation Optimization. Major Professor: Hong Wan.

The simulation optimization problems refer to the nonlinear optimization problems whose objective function can be evaluated through stochastic simulations. We study two significant discrete simulation optimization problems in this thesis: Ranking and Selection (R&S) and Factor Screening (FS). Both R&S and FS are the "selection" problems defined upon a finite set of candidate systems or factors. They vary mainly in their objectives: the R&S problems is to find the "best" system(s) among all alternatives; whereas the FS is to select important factors that are critical to the stochastic systems.

In this thesis, we develop efficient sequential procedures for these two problems. For the R&S problem, we propose fully-sequential procedures for selecting the "best" systems with a guaranteed probability of correct selection (PCS). The main features of the stated methods are: (1) **a Bonferroni-free model**, these procedures overcome the conservativeness of the Bonferroni correction and deliver the exact probabilistic guarantee without overshooting; (2) **asymptotic optimality**, these procedures achieve the lower bound of average sample size asymptotically; (3) **an indifference-zone-flexible formulation**, these procedures bridge the gap between the indifference-zone formulation and the indifference-zone-free formulation so that the indifference-zone parameter is not indispensable but could be helpful if provided. We establish the validity and asymptotic efficiency for the proposed procedure and conduct numerical studies to investigates the performance under multiple configurations.

We also consider the multi-objective R&S (MOR&S) problem. To the best of our knowledge, the procedure proposed is the first frequentist approach for MOR&S. These procedures identify the Pareto front with a guaranteed probability of correct selection (PCS). In particular, these procedures are fully sequential using the test statistics built upon the Generalized Sequential Probability Ratio Test (GSPRT). The main features are: 1) **an objective-dimension-free model**, the performance of these procedures do not deteriorate as the number of objectives increases, and achieve the same efficiency as KN family procedures for single-objective ranking and selection problem; 2) **an indifference-zone-flexible formulation**, the new methods eliminate the necessity of indifference-zone parameter while makes use of the indifference-zone information if provided. A numerical evaluation demonstrates the validity efficiency of the new procedure.

For the FS problem, our objective is to identify important factors for simulation experiments with controlled Family-Wise Error Rate. We assume a Multi-Objective first-order linear model where the responses follow a multivariate normal distribution. We offer three fully-sequential procedures: Sum Intersection Procedure (SUMIP), Sort Intersection Procedure (SORTIP), and Mixed Intersection procedure (MIP). SUMIP uses the Bonferroni correction to adjust for multiple comparisons; SORTIP uses the Holms procedure to overcome the conservative of the Bonferroni method, and MIP combines both SUMIP and SORTIP to work efficiently in the parallel computing environment. Numerical studies are provided to demonstrate the validity and efficiency, and a case study is presented.

1. INTRODUCTION

The Simulation Optimization (SO) problem refers to the optimization problem specified by a stochastic simulation. Specifically, the objective function of SO is stochastic and the goal is to find the optimal set of simulation inputs that lead to the best simulation output. It is used extensively to analyze and optimize complex systems in which closed-form analytical solutions do not exist due to the inherent complexities and randomnesses. For such problems, simulation allows one to accurately model the dynamics of the complex, stochastic systems and evaluate the performances using simulated outputs. Therefore, SO is widely applicable to a variety areas from transportation to health care, manufacturing, and supply chain management. For details on these and other applications of SO, see [1] for a library of SO problems. For overviews of SO methods, see, e.g. [2–5].

This thesis studies two discrete SO problems: the Ranking and Selection (R&S) problem, and the Factor Screening (FS) problem. Both of these two problems are defined on a finite set of decision variables and can be seen as the "selection" problems: RS is the "selection-of-the-best" problem to identify the best system(s) among all alternatives; FS is the "selection-of-the-important" problem to screen out all variables that are critical to the simulation system. Moreover, they both rely heavily on statistical methods. Indeed, this thesis models both problems as sequential hypothesis testings and applies the Sequential Probability Ratio Test (SPRT) methods; although these two problems differ a lot in their assumptions and objectives.

The organization of this thesis is as follows. In Chapter 2, we propose two frameworks for the R&S problem, give the validity and efficiency results, and provide implementations under normality assumptions. Chapter 3 extends the scope

to the Multi-Objective Ranking and Selection problems. Two procedures to select the Pareto front is proposed. Theoretical analysis and numerical evaluation are also provided. In Chapter 4, we propose several procedures for the Factor Screening problem, and evaluate the procedures numerically.

2. BONFERRONI-FREE SEQUENTIAL ELIMINATION PROCEDURES FOR RANKING AND SELECTION

2.1 Introduction

The Ranking and Selection (R&S) problem is to select the best system among a finite set of alternatives. In this paper, the best system refers to the system with the highest expected outcome. It is applied when the decision makers want to compare different alternatives whereas the expected performances are not available and can only be assessed through stochastic simulations. Typical applications of R&S procedures include inventory management that the decision maker needs to choose the optimal inventory level to minimize inventory costs; portfolio management that the investor compares the performances of various investment strategies, and so forth.

Ranking and Selection can be formulated as an unconstrained discrete simulation optimization problem

$$\arg\max_{i\in\Omega} \ \mu_i^\star = \mathbb{E}[X_i] \tag{2.1}$$

where $\Omega = \{1, ..., K\}$ is a finite set of index, $\mu_i^\star$ is the unknown quantity and can only be evaluated through stochastic observations obtained from simulation. In this paper, we assume systems are independent, i.e., there is no topological or structural relationship among systems.

There is a substantial body of literature on R&S procedures. Most R&S procedures fall into two categories: (a) the fixed-precision approaches that achieve a pre-specified *probability of correct selection* (*PCS*); (b) the fixed-budget approaches that efficiently allocate a finite computing budget among alternatives in order to optimize an objective, i.e., maximize the posterior *PCS*. For the fixed-budget ap-

proaches, the Optimal Computing Budget Allocation (OCBA) algorithm in [6] is one of the most widely applied and studied algorithms. See [7–9] for thorough overviews on R&S procedures.

In this paper, we focus on the fixed-precision approaches that aims to attain a target *PCS*. For this objective, the fixed-precision approaches often take one of three formulations: (a) the subset-selection formulation, (b) the indifference-zone formulation, and (c) the indifference-zone-free formulation. The subset-selection formulation [10, 11] guarantees to select a subset of alternatives that includes the best system with high probability. The indifference-zone formulation guarantees the selection of the best system with a pre-specified *PCS* when an indifference-zone $\delta > 0$ is assumed such that the mean performance of the best system is larger than all other systems by at least δ, see [12–15]. If there are systems within the indifference-zone distance from the best system, then the procedures achieve the *probability of good selection* (*PGS*) that guarantees to select a system within the indifference-zone distance from the best system. Leading procedures using the indifference-zone formulation includes KN procedure [13], KN++ procedure [14], BIZ procedure [16], and so forth. Recently, Fan and Hong designed a new formulation in [17,18], the indifference-zone-free formulation, to deliver the *PCS* guarantee without the indifference-zone setting. Under either the indifference-zone formulation or the indifference-zone-free formulation, procedures often run sequentially and terminate after eliminating all inferior systems. In addition, the probably approximately correct (PAC) selection [19] have also been studied in this stream of works.

Among the aforementioned formulations, there is a gap between the indifference-zone formulation and the indifference-zone-free formulation. As noted in [18], the correct specification of the indifference-zone parameter may significantly reduce the sample size. However, the indifference-zone formulation is valid only when the indifference-zone parameter is correctly specified, whereas the indifference-zone-free formulation does not require an indifference zone parameter. Therefore,

we are motivated to design a flexible formulation that delivers the *PCS* guarantee with or without the indifference-zone parameter, and, can take advantage of the indifference-zone information if it is provided.

Another troublesome problem is the usage of Bonferroni correction to adjust the error probability for multiple comparisons. Leading procedures, such as KN family procedures, divide the probability of correct selection by a factor of $K - 1$, where K is the total number of systems, and therefore the implementations of R&S procedures are expensive for the R&S problems with a large number of alternatives. A significant breakthrough is the BIZ procedure proposed in [16]. Under the indifference-zone formulation and the normality assumption, the BIZ procedure sequentially eliminates inferior systems without the Bonferroni correction. The numerical analysis confirms the BIZ outperforms other leading methods on the large-scale R&S problems.

To address the aforementioned challenges, we propose two new R&S procedures with the following characteristics:

1. **A Bonferroni-free model**. Many other leading procedures use Bonferroni correction to adjust the error probability when conducting multiple pairwise comparisons [13, 14, 17, 18]. In this paper, we model the R&S problem as a single comparison problem: a multiple hypotheses testing, to avoid the Bonferroni correction.

2. **An indifference-zone-flexible formulation**. The newly proposed procedures admit the indifference-zone parameter if provided while remaining valid even if the indifference-zone information is not specified. This new formulation is an extension of the indifference-zone-free formulation to eliminate the necessity of the indifference-zone parameter. Moreover, it utilizes indifference-zone information to eliminate inferior systems faster and preciser.

The technical contribution of this paper is the proposal of a set of likelihood methods for the studies of the R&S problems and, the method can be easily ex-

tended to the simulation optimization. In our proposed procedures, we model the R&S problem as a multiple-hypotheses testing problem and apply the Generalized Sequential Probability Ratio Test (GSPRT) methods to construct the continuation region.

The classic *Sequential Probability Ratio Test* (SPRT) was proposed by Wald [20,21] in response to the need of efficient testing tools. The SPRT is a sequential procedure that does not fix the sample size. Let $x_1, x_2, ...$ be i.i.d. random numbers generated from the same distribution f. The classic SPRT tests a simple null hypothesis $H_0 : f = f_0$ versus a simple alternative hypothesis $H_1 : f = f_1$. It calculates the likelihood ratio (LR) statistic $\Lambda_n = \prod_i^n (f_1(x_i)/f_0(x_i))$ at each turn, and terminates when the LR statistic exits the continuation region. [20,21] show that, to control the Type I and Type II error probabilities at α_1 and α_2, respectively, the continuation region is formed by two time-invariant boundaries $(A, B) = (\frac{\alpha_2}{1-\alpha_1}, \frac{1-\alpha_2}{\alpha_1})$. They further prove that SPRT is optimal in terms of minimizing the average sample size among all tests that control type I and type II error probabilities at α_1 and α_2 levels [21].

A natural generalization of the SPRT method is to test two or more composite hypothesis. Consider a multiple hypotheses testing problem

$$H_i \ : \ \theta \in \Theta_i \ i = 1, 2, ..., K \tag{2.2}$$

where Θ_i are disjoint subsets of the parameter space Θ. For multiple hypotheses testing problems, the generalized likelihood ratio (GLR) instead of the simple likelihood ratio is used as the testing statistic. Note that there are different forms of the GLR statistic in literature. The GLR statistic, in essence, "self-tunes" to information about the true θ over the course of the test. Various continuation regions are designed and proved to achieve asymptotically optimality under their corresponding conditions [22–27]. In this thesis, we adopt the adaptive GLR statistic proposed in [25,28] with the following form

$$\Lambda_{i,n}^{(1)} = \frac{\sup_{\theta \in \Theta_i} \prod_{m=1}^n f(x_m|\theta)}{\prod_{m=1}^n f(x_m|\hat{\theta}_{m-1})} \tag{2.3}$$

where $\{x_m,\ m = 1, ..., n\}$ are observations and $\{\hat{\theta}_m,\ m = 1, ..., n\}$ are the MLE of θ of the first m observations. See Section 5.4 in [29] for more discussions on this GLR statistic.

The rest of this section is organized as follows. We consider a R&S problem under normality assumption with known variance and propose Procedure 1 and Procedure 2 in Section 2.2. In Section 2.3, we consider the situation under normality assumption with unknown variance. In both sections, we show that the proposed procedures are valid to deliver the *PCS* guarantee. Simulation experiments are conducted in Section 2.4. We end the paper with conclusions and discussions in Section 2.5.

2.2 Normal Distribution with Known Variance

We consider a R&S problem to select the best system from K candidates. Let us denote $\Omega = \{1, ..., K\}$ as the set of all candidate systems. In this section, we assume that random observations of system i, $\{X_{i,n}, n = 1, 2, ..\}$, are from normal distribution, $N(\mu_i^\star, \sigma_i^2)$, where σ_i^2 is known beforehand and $\mu_i^\star$ is the unknown true mean value of the alternative i. For for sake of simplicity, we assume that only independent and identical random observations $\{X_{i,n}, n = 1, 2, ..\}$ are available from system i.

Throughout this section, we denote $\boldsymbol{\mu}^\star = [\mu_1^\star, ..., \mu_K^\star]$ as the vector of true mean values, and $\boldsymbol{\mu} = [\mu_1, ..., \mu_K]$ as the dummy vector of mean values with $\boldsymbol{\mu} \in \mathbb{U} \subseteq \mathbb{R}^K$, where $\mathbb{U}$ represents the set of any possible value of $\boldsymbol{\mu}$. We assume that $\mathbb{U}$ is a compact set. Let us denote $\delta > 0$ as the indifference-zone parameter. If the indifference-zone is specified, then the mean of the best system is better than all other systems by at least δ. Correspondingly, let us denote $\mathbb{U}_i$ as the parameter space that the system i is the best among all systems, i.e., $\mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i \geq \mu_j + \delta, \forall\ j \neq i\}$. An indifference-zone space $\mathbb{U}_0$ contains all possible scenarios that $\boldsymbol{\mu}$ does not satisfy the indifference-zone requirement. Therefore, $\{\mathbb{U}_0, \mathbb{U}_1, \mathbb{U}_2, ..., \mathbb{U}_K\}$

are disjoint subspaces of the decision space $\mathbb{U} \subseteq \mathbb{R}^K$, i.e., $\mathbb{U} \triangleq \cup_{i=0}^{K} \mathbb{U}_i$. If the indifference-zone parameter is not provided, then we have parameter space, $\mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i > \mu_j, \forall i \neq j\}$ as well as the indifference zone space $\mathbb{U}_0 = \{\boldsymbol{\mu} | \mu_i = \mu_j > \mu_k, \exists i \text{ such that } \forall i \neq j \neq k\}$, so that $\mathbb{U} = \cup_{i=0}^{K} \mathbb{U}_i$ is still a compact space.

The objective is to select the "best" system with the guaranteed *PCS*, i.e., selecting the true best system with probability at least $1 - \alpha$. Meanwhile, we aim to design the procedure that uses samples as minimum as possible. For such objectives, we model the R&S problem as a multiple-hypotheses testing problem. The multiple hypotheses testing problem is to determine whether the system i is the best, i.e., whether the parameter subspace, $\mathbb{U}_i$, contains the true mean vector $\boldsymbol{\mu}^\star$,

$$H_i : \boldsymbol{\mu}^\star \in \mathbb{U}_i, \;\; i = 1, ..., K \tag{2.4}$$

The main difference between our modeling approach and other approaches, i.e., KN family procedures, is that we use a single test to find the best system; whereas KN family approaches require $K - 1$ two-ended hypotheses tests. Therefore KN family procedures need to adjust the error probability for multiple comparisons using the Bonferroni correction. By doing so, our proposed methods deliver the *PCS* guarantee without the Bonferroni correction.

Let us denote $g_i(\cdot|\mu_i)$ as the density function of system i under μ_i. Thus, the random samples, $\{X_{i,n}, n = 1, 2, 3...\}$, are generated from the density $g_i(\cdot|\mu_i^\star)$, i.e., the density function with $\mu_i^\star$. Throughout this section, we assume following conditions hold:

A1 Systems have different mean values, i.e., $\mu_i^\star \neq \mu_j^\star$ for $i \neq j$. The best system is better than other systems by at least δ, where δ is the indifference-zone parameter; if the indifference-zone is not specified, then we have the degenerated indifference zone parameter $\delta > 0$;

A2 Systems are independent, i.e., $X_{i,n}$ and $X_{j,n}$ are independent for all $i \neq j$;

A3 The parameter space $\mathbb{U}$ is compact;

A4 $g_i(\cdot|\mu_i)$ is the Normal density function of system $i \in \Omega$ for a given μ_i;

A5 $\rho_i(\mu_i, \mu_i') > 0$ for all $\mu_i \neq \mu_i'$, where $\rho_i(\mu_i, \mu_i')$ is the Kullback-Leibler divergence from $g(\cdot|\mu_i)$ to $g(\cdot|\mu_i')$:

$$\rho_i(\mu_i, \mu_i') = \mathbb{E}_{g_i(\cdot|\mu_i)}[\log g_i(\cdot|\mu_i) - \log g_i(\cdot|\mu_i')] \tag{2.5}$$

where $\mathbb{E}_{g_i(\cdot|\mu_i)}$ represents taking the expectation with respect to $g_i(\cdot|\mu_i)$;

A6 $\sum_{m=1}^{\infty} \mathbb{E}_{g_i(\cdot|\mu_i)}[\rho_i(\mu_i, \hat{\mu}_{i,m})] < \infty$, where $\hat{\mu}_{i,m}$ represents the MLE estimator of μ_i with the first m observations.

For the sake of brevity, we define $f(\cdot|\boldsymbol{\mu}) = \prod_{i=1}^{K} g_i(\cdot|\mu_i)$ as the joint density function, and thus the Kullback-Leibler divergence from $\boldsymbol{\mu} = [\mu_1, ..., \mu_K]$ to $\boldsymbol{\mu}' = [\mu_1', ..., \mu_K']$ is defined by,

$$\rho(\boldsymbol{\mu}, \boldsymbol{\mu}') = \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\log \frac{f(\cdot|\boldsymbol{\mu})}{f(\cdot|\boldsymbol{\mu}_i')}] \tag{2.6}$$

where $\mathbb{E}_{f(\cdot|\boldsymbol{\mu})}$ represents taking the expectation with respect to $f(\cdot|\boldsymbol{\mu})$.

Lemma 2.2.1 *Under Condition A2, A5, and A6,* $\rho(\boldsymbol{\mu}, \boldsymbol{\mu}') > 0$ *and* $\sum_{m=1}^{\infty} \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_m)] < \infty$, *where* $\hat{\boldsymbol{\mu}}_m$ *is the MLE of* $\boldsymbol{\mu}$ *using the first m observations..*

Proof From Condition A2 and A5, we have

$$\rho(\boldsymbol{\mu}, \boldsymbol{\mu}') = \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\sum_{i=1}^{K} \big(\log g_i(\cdot|\mu_i) - \log g_i(\cdot|\mu_i')\big)] \tag{2.7}$$

$$= \sum_{i=1}^{K} \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\log g_i(\cdot|\mu_i) - \log g_i(\cdot|\mu_i')] \tag{2.8}$$

$$= \sum_{i=1}^{K} \mathbb{E}_{g_i(\cdot|\mu_i)}[\log g_i(\cdot|\mu_i) - \log g_i(\cdot|\mu_i')] = \sum_{i=1}^{K} \rho_i(\mu_i, \mu_i') > 0 \tag{2.9}$$

The last equality holds since Condition A2 states that $g_i(\cdot|\mu_i)$ for $i \in \Omega$ are independent density kernels.

From Condition A2 and A6, we have

$$\sum_{m=1}^{\infty} \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_m)] = \sum_{m=1}^{\infty} \mathbb{E}_{f(\cdot|\boldsymbol{\mu})}[\sum_{i=1}^{K} \rho_i(\mu_i, \hat{\mu}_{i,m})] \tag{2.10}$$

$$= \sum_{m=1}^{\infty} \sum_{i=1}^{K} \mathbb{E}_{g_i(\cdot|\mu_i)}[\rho_i(\mu_i, \hat{\mu}_{i,m})] < \infty \tag{2.11}$$

■

Condition A1 assumes that the indifference-zone parameter is provided correctly and avoids the situations when two or more systems tie with each other. Condition A2 requires no dependencies among alternatives so that the early eliminations of inferior systems do not compromise the validity of the GLR statistic. It can be replaced by the less-restricted conditional-independent condition to allow the usage of common random numbers. Condition A3 and A4 assume density functions are of normal distributions with a compact parameter space. Condition A5 ensures the validity of the likelihood ratio method. Condition A6 states that the MLE $\hat{\boldsymbol{\mu}}_m$ converges to the true parameter $\boldsymbol{\mu}$ fast enough so that, for a sufficiently large enough n, $\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_n) \approx 0$. It is worth noting that it is satisfied for many distributions, i.e., Gaussian, Poisson, Bernoulli, etc, where $\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_m) \leq C|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_m|^2$ [29].

2.2.1 The Procedure

The first procedure is demonstrated in Procedure 1. It is in a fully sequential manner. In particular, at each iteration, Procedure 1 uses all samples collected up to the current iteration to calculate the generalized likelihood ratio (GLR) [29] statistic for each system, and eliminate weak systems when their GLR statistics hit the elimination boundary. The procedure terminates when only one candidate is left.

Let us denote $\mathcal{S}_{cand,n}$ as the set of systems that are not eliminated at the nth iteration. The procedure starts with considering all systems as candidates of the best system, $\mathcal{S}_{cand,1} = \Omega$, and keeps rejecting systems from $\mathcal{S}_{cand,n}$ until there is

only one system left. $\mathcal{S}_{cand,n}$ also represents the set of systems to be sampled at the nth iteration. As to the systems removed from $\mathcal{S}_{cand,n}$, the procedure will not sample them anymore; nonetheless, their samples are still used to calculate the GLR statistics after eliminations.

Remark 2.2.1 *We denote $\mathcal{S}_{cand,n}$ as the set of systems that are not rejected yet, and $\mathcal{S}_{inferior,n}$ as the set of systems that are already rejected.*

The generalized likelihood ratio (GLR) statistics for Procedure 1 have the following form. For the system i,

$$\Lambda_{i,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_n(\boldsymbol{\mu})}{\pi_n} \tag{2.12}$$

where $L_n(\boldsymbol{\mu})$ is the likelihood of all observations collected up to the nth iteration:

$$L_n(\boldsymbol{\mu}) = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\mu_i) \tag{2.13}$$

and π_n in Equation 2.12 is the adaptive maximum likelihoods using $\hat{\mu}_{i,m-1}$, the MLE of $\mu_i^\star$ based on the first $m-1$ observations on the system i, as the estimate of $\mu_i^\star$ to calculate the density at the mth round.

$$\pi_n = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\hat{\mu}_{i,m-1}) \tag{2.14}$$

We provide the calculations of $\Lambda_{i,n}^{(1)}$ for normal distribution with known variance in Section 2.2.2.

Intuitively, $\Lambda_{i,n}^{(1)}$ measures *how likely the system i is the best system*. The smaller the $\Lambda_{i,n}^{(1)}$, the less likely that the system i is the best system. By this intuition, Procedure 1 eliminates the systems when their GLR statistics are lower than a threshold.

The behavior of Procedure 1 is illustrated in Figure 2.1, in which we consider an example with $K = 4$ alternatives. In this example, the observations of the system i are from the normal distribution $N(-i, 20)$. The 1st system is the best. The procedure sequentially drops inferior candidates when their GLR statistics hit the elimination boundary. It concludes when there is only one system left.

Algorithm 1 Procedure 1 for R&S problems

Inputs:

1: $PCS = 1 - \alpha$ ▷ Desired PCS level

2: $\Omega = \{1, ..., K\}$ ▷ Pool of candidate systems

3: $g_i(\cdot|\mu_i)$ for $i \in \Omega$ ▷ Normal density functions under μ_i

Procedure:

1: generate n_0 samples for each system in Ω;

2: set $n = n_0$;

3: set $\mathcal{S}_{inferior,n+1} = \emptyset$ and $\mathcal{S}_{cand,n+1} = \Omega$;

4: **repeat**

5: set $n = n + 1$;

6: generate 1 more sample for all systems in $\mathcal{S}_{cand,n}$;

7: set $\mathcal{S}_{cand,n+1} = \mathcal{S}_{cand,n}$ and $\mathcal{S}_{inferior,n+1} = \mathcal{S}_{inferior,n}$;

8: **for** $i \in \mathcal{S}_{cand,n}$ **do**

9: calculate the GLR statistic $\Lambda_{i,n}^{(1)}$ for the system i;

10: **if** the $\Lambda_{i,n}^{(1)} \leq \alpha$ **then**

11: set $\mathcal{S}_{cand,n+1} = \mathcal{S}_{cand,n+1}/\{i\}$ and $\mathcal{S}_{inferior,n+1} = \mathcal{S}_{inferior,n+1} \cup \{i\}$;

12: **end if**

13: **end for**

14: **until** $|\mathcal{S}_{cand,n+1}| = 1$ ▷ Terminate when only one system is left in $\mathcal{S}_{cand,n+1}$

15: **return** $\mathcal{S}_{cand,n+1}$

To start the procedure, the candidate pool $\mathcal{S}_{cand,1}$ contains all 4 alternatives. At the nth iteration, we generate one more sample for each system in the candidate pool $\mathcal{S}_{cand,n}$ and calculate and plot the GLR statistics. In the plot, at $n = 1$, the alternative 4 is eliminated from the candidate pool since its GLR statistic hits the elimination boundary. We keep sampling the remaining three systems and calculating the GLR statistics until the second elimination happens at $n = 12$ when the GLR statistic of the alternative 3 hits the elimination boundary. After that, the GLR statistics are plotted for the system 1 and 2, until the GLR statistic of system 2 crosses the elimination boundary. Since all systems are excluded except the system 1, the procedure ends and returns the alternative 1 as the best.

To show the asymptotic efficiency of Procedure 1, i.e., the asymptotic sample size before the procedure terminates, we introduce Procedure 2 which provides a lower bound of the asymptotic efficiency of Procedure 1.

The main difference between Procedure 1 and Procedure 2 is the choice of GLR statistics, where the GLR statistics used in Procedure 2 utilize the pairwise comparisons.

Let us denote $\Lambda^{(2)}_{i,j,n}$ as the generalized likelihood ratio of the event that the system i is better than the system j,

$$\Lambda^{(2)}_{i,j,n} \triangleq \frac{\sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} L_n(\boldsymbol{\mu})}{\pi_n} \tag{2.15}$$

where the numerator of $\Lambda^{(2)}_{i,j,n}$ is maximized over the parameter space of $\boldsymbol{\mu}$ that $\mu_i > \mu_j$, $\mathbb{U}_{i,j} = \{\boldsymbol{\theta}|\mu_i > \mu_j\}$.

The GLR statistic used in Procedure 2 is the minimum of all $\Lambda^{(2)}_{i,j,n}$,

$$\Lambda^{(2)}_{i,\cdot,n} = \min_{j\neq i} \Lambda^{(2)}_{i,j,n} \tag{2.16}$$

The statistic, $\Lambda^{(2)}_{i,\cdot,n}$, is essentially the worst pairwise comparison between the system i and all other systems. We have the following theorem regarding $\Lambda^{(1)}_{i,n}$ and $\Lambda^{(2)}_{i,\cdot,n}$.

Theorem 2.2.1 *Under the preceding definitions of* $\Lambda^{(1)}_{i,n}$ *and* $\Lambda^{(2)}_{i,\cdot,n}$, $\Lambda^{(2)}_{i,\cdot,n} \geq \Lambda^{(1)}_{i,n}$.

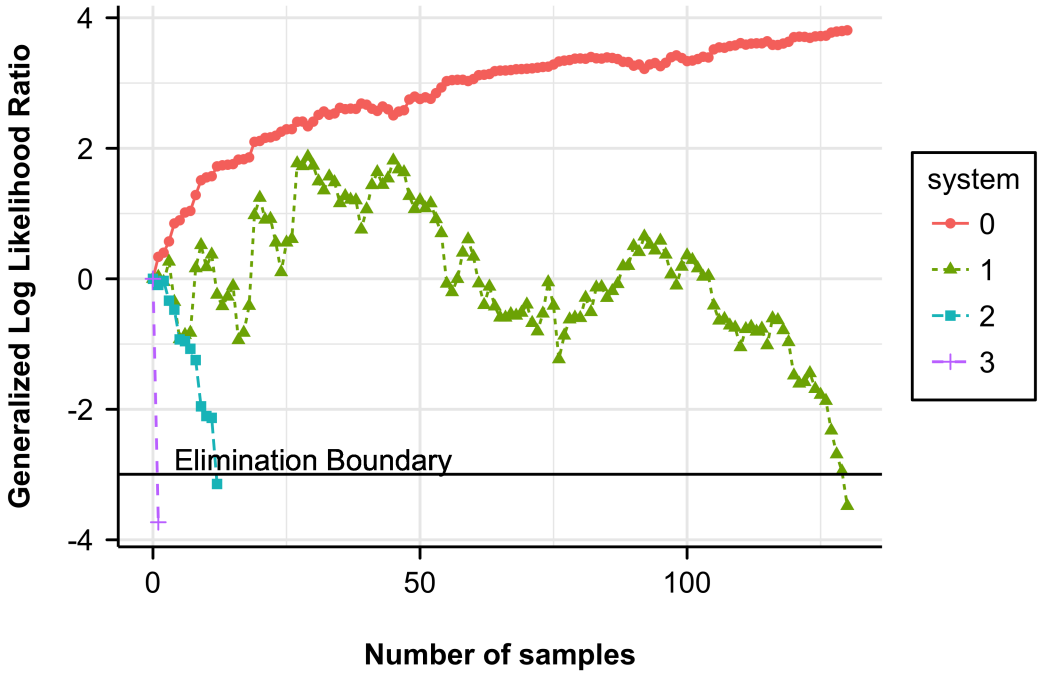


Figure 2.1.: Illustration of the Procedure 1 with $K = 4$ alternatives. The elimination boundary, $\log \alpha$, is plotted as a horizontal line. The procedure follows the log of the GLR statistic $\log \Lambda_{i,n}^{(1)}$ and screens out alternatives as they hit the elimination boundary at $\log \alpha$. The procedure terminates when there is only one system left in the candidate pool. And this remaining system is reported as the best system.

Remark 2.2.2 *This theorem states that $\Lambda_{i,\cdot,n}^{(2)}$ is bounded below by $\Lambda_{i,n}^{(1)}$ for any possible realizations. Thus, Procedure 2, which uses $\Lambda_{i,\cdot,n}^{(2)}$ as the GLR statistic, provides the lower bound of efficiency for Procedure 1, which uses $\Lambda_{i,n}^{(1)}$ as the GLR statistic.*

Proof Recall that the numerator of $\Lambda_{i,n}^{(1)}$ in Procedure 1 is the maximum likelihood under the condition that the system i is the best system. Since

$$\mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i > \mu_l, \forall l \neq i\} \subseteq \{\boldsymbol{\mu} | \mu_i > \mu_j\} = \mathbb{U}_{i,j}, \forall j \neq i \tag{2.17}$$

therefore, for any possible realization of the samples, we have, $\Lambda^{(2)}_{i,j,n} \geq \Lambda^{(1)}_{i,n}$ for all j, and

$$\Lambda^{(2)}_{i,\cdot,n} = \min_{j \neq i} \Lambda^{(2)}_{i,j,n} \geq \Lambda^{(1)}_{i,n} \tag{2.18}$$

■

Algorithm 2 Procedure 2 for R&S problems

Inputs:

1: $PCS = 1 - \alpha$ ▷ Desired PCS level

2: $\Omega = \{1, ..., K\}$ ▷ Pool of candidate systems

3: $g_i(\cdot|\mu_i)$ for $i \in \Omega$ ▷ Normal density functions under μ_i

Procedure:

1: generate n_0 samples for each system in Ω;

2: set $n = n_0$;

3: set $\mathcal{S}_{inferior,n+1} = \emptyset$ and $\mathcal{S}_{cand,n+1} = \Omega$;

4: **repeat**

5: set $n = n + 1$;

6: generate 1 more sample for all systems in $\mathcal{S}_{cand,n}$;

7: set $\mathcal{S}_{cand,n+1} = \mathcal{S}_{cand,n}$ and $\mathcal{S}_{inferior,n+1} = \mathcal{S}_{inferior,n}$;

8: **for** $i \in \mathcal{S}_{cand,n}$ **do**

9: calculate the GLR statistic $\Lambda^{(2)}_{i,\cdot,n}$ for the alternative i;

10: **if** the $\Lambda^{(2)}_{i,\cdot,n} \leq \alpha$ **then**

11: set $\mathcal{S}_{cand,n+1} = \mathcal{S}_{cand,n+1}/\{i\}$ and $\mathcal{S}_{inferior,n+1} = \mathcal{S}_{inferior,n+1} \cup \{i\}$;

12: **end if**

13: **end for**

14: **until** $|\mathcal{S}_{cand,n+1}| = 1$ ▷ Terminate when only one system is left in $\mathcal{S}_{cand,n+1}$

15: **return** $\mathcal{S}_{cand,n+1}$

2.2.2 The Implementation

This section considers the implementations of $\Lambda_{i,n}^{(1)}$ in Procedure 1. We consider two cases: (1) normal distribution with equal known variance; and (2) normal distribution with heterogeneous known variance. For both cases, we assume the knowledge of the indifference-zone parameter δ. When the indifference-zone is not specified, the implementations set $\delta = 0$.

Case 1: Normal Distribution with Equal Known Variance

Let $\{X_{i,n}, n = 1, 2, 3, ...\}$ be the i.i.d. random observations of the system i. Thus, $\{X_{i,n}\}$ are normally distributed with unknown mean $\mu_i^\star$ and equal known variance σ^2.

Recall that the Procedure 1 models the R&S problem as a multiple hypotheses testing problem, in which the ith hypothesis corresponds to testing whether $\mu_i^\star$ is the largest among the mean vector $\boldsymbol{\mu}^\star = [\mu_1^\star, ...\mu_K^\star]$,

$$H_i : \boldsymbol{\mu}^\star \in \mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i \geq \mu_j + \delta, \ \forall j \neq i\} \tag{2.19}$$

Remark 2.2.3 *We assume the indifference-zone parameter, δ, is specified. If the indifference-zone information is not provided, the hypothesis H_i and the parameter space $\mathbb{U}_i$ have the degenerated form as follows*

$$H_i : \boldsymbol{\mu}^\star \in \mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i > \mu_j, \ \forall j \neq i\} \tag{2.20}$$

The GLR statistic, $\Lambda_{i,n}^{(1)}$, has the form of

$$\Lambda_{i,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu} \in \mathbb{U}_i} L_n(\boldsymbol{\mu})}{\pi_n} \tag{2.21}$$

The key to obtain the GLR statistic, $\Lambda_{i,n}^{(1)}$, is to calculate: (a) $\hat{\boldsymbol{\mu}}_n$ the MLE estimator of $\boldsymbol{\mu}$ and, (b) the constrained optimal solution, $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = \arg\sup_{\boldsymbol{\mu} \in \mathbb{U}_i} L_n(\boldsymbol{\mu})$. Under the normality assumption, $\hat{\boldsymbol{\mu}}_n$ is as follows

$$\hat{\mu}_{j,n} = \frac{\sum_{m=1}^{n_j} X_{j,m}}{n_j}, \text{ with } n_j = \min\{n, \tau_j^{(1)}\} \tag{2.22}$$

where $\tau_j^{(1)}$ is the elimination time of the system j in Procedure 1. Stopping time $\tau_j^{(1)}$ is introduced since, in Procedure 1, if a system is eliminated, it will not be sampled in the future iterations, however, the samples of the eliminated system generated before its termination are still in used.

As to the calculation of $\hat{\mu}_n^{\mathbb{U}_i}$, we have the following theorem:

Theorem 2.2.2 *Under the normality assumption with equal known variances, denote* $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_n(\boldsymbol{\mu})$ *as the constrained MLE of mean vector* $\boldsymbol{\mu}$ *achieved within the parameter space* $\mathbb{U}_i$ *at the nth iteration.* $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = [\hat{\mu}_{1,n}^{\mathbb{U}_i}, \hat{\mu}_{2,n}^{\mathbb{U}_i}, ...\hat{\mu}_{K,n}^{\mathbb{U}_i}]$ *is calculated as follows,*

$$\hat{\mu}_{j,n}^{\mathbb{U}_i} = \begin{cases} \hat{\mu}_{\cdot,n}^{\mathbb{U}_i} + \delta, & \text{if } j = i \\ \hat{\mu}_{\cdot,n}^{\mathbb{U}_i}, & \text{if } \hat{\mu}_{j,n} > \hat{\mu}_{i,n} - \delta \\ \hat{\mu}_{j,n}, & \text{if } \hat{\mu}_{j,n} \leq \hat{\mu}_{i,n} - \delta \end{cases} \tag{2.23}$$

where $\hat{\mu}_{\cdot,n}^{\mathbb{U}_i}$ *is the average of observations from all systems whose MLEs are higher than the MLE of the system i.*

$$\hat{\mu}_{\cdot,n}^{\mathbb{U}_i} = \frac{\sum_{m=1}^{n}\sum_{j\in\mathcal{S}_{cand,m}} X_{j,m}\cdot 1_{\hat{\mu}_{j,n}\geq\hat{\mu}_{i,n}} - n\delta}{\sum_{m=1}^{n}\sum_{j\in\mathcal{S}_{cand,m}} 1_{\hat{\mu}_{j,n}\geq\hat{\mu}_{i,n}}} \tag{2.24}$$

Remark 2.2.4 *The intuition of the theorem is as follows. If the unconstrained MLE* $\hat{\mu}_{i,n} \geq \hat{\mu}_{j,n} + \delta$*, then the constrained solution,* $\hat{\mu}_{j,n}^{\mathbb{U}_i}$*, coincides with the unconstrained MLE* $\hat{\mu}_{j,n}$*; otherwise, the* $\hat{\mu}_{j,n}^{\mathbb{U}_i}$ *is adjusted so that* $\hat{\mu}_{j,n}^{\mathbb{U}_i}$ *is less than* $\hat{\mu}_{i,n}^{\mathbb{U}_i}$ *by* δ*.*

Proof Under the normality assumption with equal known variance σ^2, $L_n(\boldsymbol{\mu})$ and $g_j(X_{j,m}|\mu_j)$ are defined by

$$L_n(\boldsymbol{\mu}) = \prod_{m=1}^{n}\prod_{j\in\mathcal{S}_{cand,m}} g_j(X_{j,m}|\mu_j) \tag{2.25}$$

$$g_j(X_{j,m}|\mu_j) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{(X_{j,m}-\mu_j)^2}{2\sigma^2}\} \tag{2.26}$$

where $\mathcal{S}_{cand,n}$ contains systems that have not been eliminated by the nth iteration.

Given the definition of $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i}$, we have

$$\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_n(\boldsymbol{\mu}) \tag{2.27}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \prod_{m=1}^{n} \prod_{j\in\mathcal{S}_{cand,m}} g_j(X_{j,m}|\mu_j) \tag{2.28}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \prod_{m=1}^{n} \prod_{j\in\mathcal{S}_{cand,m}} \exp\{-\frac{(X_{j,m}-\mu_j)^2}{2\sigma^2}\} \tag{2.29}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \sum_{m=1}^{n} \sum_{j\in\mathcal{S}_{cand,m}} \big(-\frac{(X_{j,m}-\mu_j)^2}{2\sigma^2}\big) \tag{2.30}$$

This optimization problem is solvable by using Lagrangian method. Denote $\boldsymbol{\lambda} = [\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ...\lambda_K]$ as the Lagrangian multipliers. We have the following KKT conditions.

$$\frac{1}{\sigma^2}(\mu_i - \hat{\mu}_{i,n}) - \sum_{j\neq i}\lambda_j = 0 \tag{2.31}$$

$$\frac{1}{\sigma^2}(\mu_j - \hat{\mu}_{j,n}) + \lambda_j = 0 \tag{2.32}$$

$$\lambda_j \cdot (\mu_j - \mu_i + \delta) = 0, \ \ \forall j \neq i \tag{2.33}$$

$$\mu_i - \mu_j - \delta \geq 0, \ \ \forall j \neq i \tag{2.34}$$

$$\lambda_j \geq 0 \tag{2.35}$$

It is straightforward to verify that 2.24 satisfies the above conditions.

■

Case 2: Normal Distribution with Unequal Known Variance

Let $\{X_{i,n}, n = 1, 2, 3, ...\}$ be the i.i.d. observations of the system i. $\{X_{i,n}\}$ are normally distributed with unknown mean $\mu_i^\star$ and known variance σ_i^2. The GLR statistic $\Lambda_{i,n}^{(1)}$ has the form of

$$\Lambda_{i,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_n(\boldsymbol{\mu})}{\pi_n} \tag{2.36}$$

where, in this case, $L_n(\boldsymbol{\mu})$ and π_n are defined by

$$L_n(\boldsymbol{\mu}) = \prod_{m=1}^{n} \prod_{j \in \mathcal{S}_{cand,m}} g_j(X_{j,m}|\mu_j) \tag{2.37}$$

$$\pi_n = \prod_{m=1}^{n} \prod_{j \in \mathcal{S}_{cand,m}} g_j(X_{j,m}|\hat{\mu}_{j,m-1}) \tag{2.38}$$

$$g_j(X_{j,m}|\mu_j) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\{-\frac{(X_{j,m}-\mu_j)^2}{2\sigma_i^2}\} \tag{2.39}$$

Recall that $\mathcal{S}_{cand,n}$ is the candidate set that contains systems that have not been eliminated by the nth iteration, and $\hat{\mu}_{j,n}$ as the MLE of μ_j based on observations collected before and at the nth iteration. Moreover, denote $\tau_j^{(1)}$ as the elimination time of the system j, we have

$$\hat{\mu}_{j,n} = \frac{\sum_{m=1}^{n'} X_{j,m}}{n'}, \text{ with } n' = \min\{n, \tau_j^{(1)}\} \tag{2.40}$$

We have the following theorem to solve $\sup_{\boldsymbol{\mu} \in \mathbb{U}_i} L_n(\boldsymbol{\mu})$. The proof of this theorem is included in Appendix 2.6.2.

Theorem 2.2.3 *Under the normality assumption with unequal known variances, denote* $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = \arg\sup_{\boldsymbol{\mu} \in \mathbb{U}_i} L_n(\boldsymbol{\mu})$ *as the constrained MLE of mean vector* $\boldsymbol{\mu}$ *achieved within the parameter space* $\mathbb{U}_i$ *at the nth iteration.* $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = [\hat{\mu}_{1,n}^{\mathbb{U}_i}, \hat{\mu}_{2,n}^{\mathbb{U}_i}, ...\hat{\mu}_{K,n}^{\mathbb{U}_i}]$ *is calculated as follows,*

$$\hat{\mu}_{j,n}^{\mathbb{U}_i} = \begin{cases} \hat{\mu}_{\cdot,n}^{\mathbb{U}_i} + \delta, & \text{if } j = i \\ \hat{\mu}_{\cdot,n}^{\mathbb{U}_i}, & \text{if } \hat{\mu}_{j,n} > \hat{\mu}_{i,n} - \delta \\ \hat{\mu}_{j,n}, & \text{if } \hat{\mu}_{j,n} \leq \hat{\mu}_{i,n} - \delta \end{cases} \tag{2.41}$$

where $\hat{\mu}_{\cdot,n}^{\mathbb{U}_i}$ *is the average of observations from all systems whose MLEs are higher than the MLE of the system i.*

$$\hat{\mu}_{\cdot,n}^{\mathbb{U}_i} = \frac{\sum_{m=1}^{n} \sum_{j \in \mathcal{S}_{cand,m}} \frac{X_{j,m}}{\sigma_j^2} \cdot 1_{\hat{\mu}_{j,n} \geq \hat{\mu}_{i,n}} - \frac{n\delta}{\sigma_i^2}}{\sum_{m=1}^{n} \sum_{j \in \mathcal{S}_{cand,m}} \frac{1}{\sigma_j^2} 1_{\hat{\mu}_{j,n} \geq \hat{\mu}_{i,n}}} \tag{2.42}$$

2.2.3 The Validity

In this subsection, we show that Procedure 1 and Procedure 2 are valid to deliver the *PCS* guarantee. The first theorem states that Procedure 1 and Procedure 2 terminate in finite time with probability 1. The proof of the following theorem is included in Appendix 2.6.3.

Theorem 2.2.4 *Under the Conditions A1-A6, Procedure 1 and Procedure 2 terminate in finite time with probability* 1.

With the guarantee that both Procedure 1 and Procedure 2 terminate in finite time, we can further show that they deliver the *PCS* guarantee. Denote $PCS^{(1)}$ and $PCS^{(2)}$ as the probability of correct selection under Procedure 1 and Procedure 2, respectively.

Theorem 2.2.5 *Under the Conditions A1-A6, Procedure 1 and 2 select the best system with probability at least* $1-\alpha$*, i.e.,*

$$PCS^{(1)} \geq 1-\alpha,\ PCS^{(2)} \geq 1-\alpha \tag{2.43}$$

Proof We first show that Procedure 1 delivers the PCS guarantee.

Let us denote $\boldsymbol{\mu}^\star = [\mu_1^\star, \mu_2^\star, ..., \mu_K^\star]$ as the true mean vector. Without loss of generality, we assume that the system 1 is the best system. Thus, the true hypothesis is $H_1 : \boldsymbol{\mu}^\star \in \mathbb{U}_1$.

Moreover, let us denote $\tau_i^{(1)} = \min\{n|\Lambda_{i,n}^{(1)} \leq \alpha\}$ as the elimination time of the alternative i in Procedure 1. For any arbitrary number $N > 0$, the probability of correct selection can be put as

$$\begin{aligned}
PCS^{(1)} \geq & P(CS \cap \text{Procedure 1 terminates before } N) \\
= & P(\text{Procedure 1 terminates before } N) P(CS|\text{Procedure 1 terminates before } N) \\
\geq & P(\text{Procedure 1 terminates before } N)(1 - P(\tau_1^{(1)} < N)) \\
\geq & P(\text{Procedure 1 terminates before } N)(1 - P(\tau_1^{(1)} < \infty))
\end{aligned}$$

The inequality holds for any $N > 0$. Using the property that Procedure 1 terminates in finite time, i.e., $\lim_{N\to\infty} P(\text{Procedure 1 terminates before } N) = 1$, we have

$$PCS^{(1)} \geq \lim_{N\to\infty} P(\text{Procedure 1 terminates before } N)(1 - P(\tau_1^{(1)} < \infty)) = 1 - P(\tau_1^{(1)} < \infty) \tag{2.44}$$

By definition, $\tau_1^{(1)} = \min\{n|\Lambda_{1,n}^{(1)} \leq \alpha\}$, where the GLR statistic, $\Lambda_{1,n}^{(1)}$, is calculated by

$$\Lambda_{1,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_1} L_n(\boldsymbol{\mu})}{\pi_n} \tag{2.45}$$

with

$$L_n(\boldsymbol{\mu}) = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\mu_i) \tag{2.46}$$

$$\pi_n = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\hat{\mu}_{i,m-1}) \tag{2.47}$$

Since, by assumption, $\boldsymbol{\mu}^\star \in \mathbb{U}_1$, we have

$$\Lambda_{1,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_1} L_n(\boldsymbol{\mu})}{\pi_n} \geq \frac{L_n(\boldsymbol{\mu}^\star)}{\pi_n} \tag{2.48}$$

Denote $\Gamma_n = \frac{\pi_n}{L_n(\boldsymbol{\mu}^\star)}$. Therefore, $\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\Gamma_n] = 1$ since

$$\mathbb{E}_{E_{f(\cdot|\boldsymbol{\mu}^\star)}}[\Gamma_n] = \int \Gamma_n \cdot \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(\cdot|\mu_i^\star) dX_{i,m} = \int \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(\cdot|\hat{\mu}_{i,m-1}) dX_{i,m} = 1 \tag{2.49}$$

This integration equals to 1 since $g_i(\cdot|\mu_i)$ are valid density functions.

We claim that Γ_n is a martingale since

$$\Gamma_n = \frac{\pi_{n-1}}{L_{n-1}(\boldsymbol{\mu}^\star|\cdot)} \times \prod_{i\in\mathcal{S}_{cand,n}} \frac{g_i(X_{i,n}|\hat{\mu}_{i,n-1})}{g_i(X_{i,n}|\mu_i^\star)} = \Gamma_{n-1} \times \prod_{i\in\mathcal{S}_{cand,n}} \frac{g_i(X_{i,n}|\hat{\mu}_{i,n-1})}{g_i(X_{i,n}|\mu_i^\star)} \tag{2.50}$$

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\Gamma_n|\Gamma_{n-1}] = \Gamma_{n-1} \times \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\prod_{i\in\mathcal{S}_{cand,n}} \frac{g_i(\cdot|\hat{\mu}_{i,n-1})}{g_i(\cdot|\mu_i^\star)}\right] \tag{2.51}$$

$$= \Gamma_{n-1} \int \prod_{i\in\mathcal{S}_{cand,n}} \frac{g_i(\cdot|\hat{\mu}_{i,n-1})}{g_i(\cdot|\mu_i^\star)} \cdot \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(\cdot|\mu_i^\star) dX_{i,m} = \Gamma_{n-1} \tag{2.52}$$

Using the fact that Γ_n is a non-negative martingale with $\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\Gamma_n] = 1$, we have the following inequality according to the Theorem of Doob's Inequality on Martingale [30],

$$P(\max_{n=1,\dots,N} \Gamma_n \geq \frac{1}{\alpha}) \leq \alpha \cdot \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\Gamma_n] = \alpha, \forall N = 1, 2, \dots \tag{2.53}$$

Moreover, since $\Lambda_{1,n}^{(1)} \geq 1/\Gamma_n$, we have

$$P(\min_{n=1,\dots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq P(\max_{n=1,\dots,N} \Gamma_n \geq \frac{1}{\alpha}) \leq \alpha \tag{2.54}$$

Therefore, $P(\tau_1^{(1)} < N) = P(\min_{n=1,\dots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq \alpha$ for any N, and

$$PCS^{(1)} \geq 1 - P(\tau_1^{(1)} < \infty) \geq 1 - \alpha \tag{2.55}$$

Next, we prove that Procedure 2 also delivers the PCS guarantee.

Let us denote $\tau_i^{(2)} = \min\{n | \Lambda_{i,\cdot,n}^{(2)} \leq \alpha\}$ as the elimination time of the alternative i in Procedure 2.

The $PCS^{(2)}$ is bounded below as follows

$$PCS^{(2)} \geq P(\text{Procedure 2 terminates before } N)(1 - P(\tau_1^{(2)} < \infty))$$

This inequality holds for any $N > 0$. Using the property that Procedure 2 terminates in finite time w.p.1, $\lim_{N\to\infty} P(\text{Procedure 2 terminates before } N) = 1$, we have

$$PCS^{(2)} \geq \lim_{N\to\infty} P(\text{Procedure 2 terminates before } N)(1 - P(\tau_1^{(2)} < \infty)) = 1 - P(\tau_1^{(2)} < \infty) \tag{2.56}$$

Using the inequality in 2.18, $\Lambda_{i,\cdot,n}^{(2)} \geq \Lambda_{i,n}^{(1)}$ for any n, we have

$$P(\tau_1^{(2)} < N) = P(\min_{n=1,\dots,N} \Lambda_{i,\cdot,n}^{(2)} \leq \alpha) \leq P(\min_{n=1,\dots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq \alpha, \ \forall N \tag{2.57}$$

where the last inequality is from the proof for Procedure 1.

Thus, $P(\tau_1^{(2)} < \infty) \leq \alpha$ and $PCS^{(2)} \geq 1 - \alpha$. ■

The preceding theorem states that the proposed procedure selects the best system with probability at least $1-\alpha$ when it terminates. Nonetheless, if Procedure 1 is stopped before the termination criteria is met, i.e., $|\mathcal{S}_{cand,n}| > 1$, it is still guaranteed that Procedure 1 selects a subset of systems that contains the best system with a probabilistic guarantee.

Lemma 2.2.2 *Under the settings of Procedure 1 and Procedure 2, denote $\mathcal{S}_{cand,n}$ as the subset of systems that have not been eliminated before the first n iterations. If Procedure 1 and Procedure 2 deliver the PCS guarantee as stated in Theorem 2.2.5, then, for any n, $\mathcal{S}_{cand,n}$ contains the best system with probability at least $1-\alpha$.*

Proof The proof of Theorem 2.2.5 states that $P(\tau_1^{(1)} < \infty) \leq \alpha$. Therefore, $P(\tau_1^{(1)} < n) \leq \alpha$ for all n. Thus, $P(\text{system } 1 \in \mathcal{S}_{cand,n}) < \alpha$. ■

2.2.4 The Asymptotic Efficiency

In this subsection, we demonstrate that the proposed procedures achieve asymptotic efficiency. Let us assume that the system 1 is the best system, i.e., $\mu_1^\star > \mu_i^\star$ for all $i = 2, ..., K$. Thus, we have $\boldsymbol{\mu}^\star \in \mathbb{U}_1$. Let us denote $\tau_i^{(1)}$ and $\tau_i^{(2)}$ as the number of iterations used by Procedure 1 and Procedure 2, respectively, to eliminate the system i. Let $\mathbb{U}_{i,1} = \{\boldsymbol{\mu} | \mu_i \geq \mu_1 + \delta\}$ be the parameter space that the system i is better than the system 1. The following theorem gives the asymptotic limits of the expectation of $\tau_i^{(1)}$ and $\tau_i^{(2)}$, as the error probability α approaches to 0. The proof of the following theorem is included in Appendix 2.6.4.

Theorem 2.2.6 *Under the Conditions A1-A6 and the settings of Procedure 1 and Procedure 2, the average sample size to eliminate the alternative i has the following asymptotic approximation*

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\tau_i^{(1)}] \leq \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\tau_i^{(2)}] \to \frac{\log \alpha^{-1}}{\inf_{\boldsymbol{\mu} \in \mathbb{U}_{i,1}} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})}(1 + o(1)) \tag{2.58}$$

as $\alpha \to 0$, for all $\boldsymbol{\mu}^\star \in \mathbb{U}_1$, $i = 1, ..., m$.

The theorem may be interpreted as follows. The numerator, $\log \alpha^{-1}$, is the amount of "information" to make an elimination decision. The denominator, $\inf_{\boldsymbol{\mu} \in \mathbb{U}_{i,1}} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})$, is the expected amount of "information" obtained from one round of observations. Therefore, the expected sample size to reject the system i is, roughly, $\log \alpha^{-1} / \inf_{\boldsymbol{\mu} \in \mathbb{U}_i} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})$.

Using Theorem 2.2.6, for the case of the normal distribution with equal known variance, we have the following asymptotic efficiency result.

Lemma 2.2.3 *Under the setting of Procedure 1 and Procedure 2, assume* $X_{i,n} \sim N(\mu_i^\star, \sigma^2)$. *Assume the system* 1 *is the best system, i.e.,* $\mu_1^\star > \mu_i^\star \ \forall i \neq 1$. *The asymptotic average sample size to eliminate the system i is*

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\tau_i^{(1)}] \leq \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\tau_i^{(2)}] \to \frac{-4\log(\alpha) \cdot \sigma^2}{(\mu_1^\star - \mu_i^\star + \delta)^2} \tag{2.59}$$

where $\delta \geq 0$ *is the indifference-zone parameter.*

Proof Given that $X_i \sim N(\mu_i^\star, \sigma^2)$, the Kullback-Leibler divergence from $\mu_i^\star$ to μ_i is

$$\rho_i(\mu_i^\star, \mu_i) = \mathbb{E}_{g_i(\cdot|\mu_i^\star)}[\log \frac{g_i(\cdot|\mu_i^\star)}{g_i(\cdot|\mu_i)}] = \frac{(\mu_i^\star - \mu_i)^2}{2\sigma^2} \tag{2.60}$$

Thus, the Kullback-Leibler divergence from $\boldsymbol{\mu}^\star$ to $\boldsymbol{\mu}$ is

$$\rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu}) = \sum_{i=1}^{K} \frac{(\mu_i^\star - \mu_i)^2}{2\sigma^2} \tag{2.61}$$

The infinitum of $\rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \mathbb{U}_{i,1}$ is achieved at

$$\inf_{\boldsymbol{\mu} \in \mathbb{U}_{1,i}} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu}) = \inf_{\mu_i \geq \mu_1 + \delta} \frac{(\mu_i^\star - \mu_i)^2}{2\sigma^2} + \frac{(\mu_1^\star - \mu_1)^2}{2\sigma^2} = \frac{(\mu_1^\star - \mu_i^\star + \delta)^2}{4\sigma^2} \tag{2.62}$$

Thus, we conclude the proof. ■

Table 2.1 compares the asymptotic average sample size of the KN++ procedure, IZ-free procedure, and Procedure 1 as functions of the true difference $\Delta \triangleq \mu_1^\star - \mu_i^\star$, the indifference-zone parameter δ, and the true variance σ^2. The theoretical results on the efficiencies of the KN++ procedure and IZ-free procedure are from [18].

Despite the usage of Bonferroni correction, we have two exciting findings from Table 2.1. First, Procedure 1 has of the same order of the IZ-free procedure when

the indifference-zone is not specified. Second, Procedure 1 utilizes the indifference zone much more efficient than the KN++ procedure. The indifference-zone parameter, if defined correctly, will significantly boost the efficiency of Procedure 1.

Table 2.1.: Asymptotic Average Sample Size of KN++ procedure, IZ-free procedure, and the Procedure 1. $\Delta \triangleq \mu_1^\star - \mu_i^\star$ is the true difference between system i and system 1, $\delta > 0$ is the indifference-zone parameter, and σ^2 is the variance; $f^2 = \log(\frac{k-1}{2\alpha})$ and c are parameters defined in the KN procedure and the IZ-free procedure, respectively. Please refer to [18] for the calculation of c.

Procedures	KN++	IZ-free	Procedure 1
Asymptotic Avg Sample Size	$\frac{2f^2\sigma^2}{(\Delta-\delta/2)\delta}$	$\frac{2c\sigma^2}{\Delta^2}$	$\frac{-4\log(\alpha)\cdot\sigma^2}{(\Delta+\delta)^2}$

2.3 Normal Distribution with Unknown Variance

In this section, we relax the assumption that the variances are known. We consider a R&S problem to select the system with the largest mean value from K candidates. Denote $\Omega = \{1, ..., K\}$ as the set of all candidate systems. The independent and identical random observations of system i, $\{X_{i,n}, n = 1, 2, ..\}$, are the only available information to infer the expected value, μ_i. In this section, we assume that observations of system i, $\{X_{i,n}, n = 1, 2, ..\}$, are of normal distribution, $N(\mu_i, \sigma_i^2)$, where both μ_i and σ_i^2 are unknown.

Throughout this section, we denote $\boldsymbol{\theta}^\star = [\theta_1^\star, ..., \theta_K^\star] = [[\mu_1^\star, \sigma_1^{2\star}], ..., [\mu_K^\star, \sigma_K^{2\star}]]$ as the vector of true parameters, including both mean values and variances, and $\boldsymbol{\theta} = [[\mu_1, \sigma_1^2], ..., [\mu_K, \sigma_K^2]]$ as the dummy vector of parameters with $\boldsymbol{\theta} \in \Theta \subseteq [\mathbb{R} \times \mathbb{R}_+]^K$, where Θ represents the set of any possible value of $\boldsymbol{\theta}$. We assume that Θ is a

compact set. Let us denote Θ_i as the parameter space that the system i is the best among all systems, i.e., $\Theta_i = \{\boldsymbol{\theta} | \mu_i > \mu_j, \forall\ j \neq i\}$. Therefore, $\{\Theta_i, i = 1, ..., K\}$ are subspaces of the entire decision space. If the indifference-zone parameter is provided, then $\Theta_i = \{\boldsymbol{\theta} | \mu_i \geq \mu_j + \delta, \forall i \neq j\}$. An indifference-zone set Θ_0 may be introduced so that $\Theta = \cup_{i=0}^{K} \Theta_i$ is still a compact space. Similar as Procedure 1, we formulate the R&S problem as a multiple hypotheses test to determine whether the system i is the best, i.e., whether the parameter subspace, Θ_i, contains the true parameter vector $\boldsymbol{\theta}^\star$,

$$H_i : \boldsymbol{\theta}^\star \in \Theta_i, \ \ i = 1, ..., K \tag{2.63}$$

Let us denote $g_i(\cdot|\theta_i) = g_i(\cdot|\mu_i, \sigma_i^2)$ as the density function of system i under $\theta_i = [\mu_i, \sigma_i^2]$. The random samples, $\{X_{i,n}, n = 1, 2, 3...\}$, are generated from the density $g_i(\cdot|\theta^\star)$, i.e., the density function with the true but unknown parameters, $\theta_i^\star = [\mu_i^\star, \sigma_i^{2\star}]$. Throughout this section, we assume following conditions hold:

B1 Systems have different mean values, i.e., $\mu_i^\star \neq \mu_j^\star$ for $i \neq j$. The best system is better than other systems by at least δ, where δ is the indifference-zone parameter; if the indifference-zone is not specified, then $\delta = 0$;

B2 Systems are independent, i.e., $\{X_{i,n}\}$ and $\{X_{j,n}\}$ are independent for all $i \neq j$;

B3 The parameter space Θ is compact;

B4 $g_i(\cdot|\theta_i)$ is the Normal density function for all $i \in \Omega$;

B5 $\rho_i(\theta_i, \theta_i') > 0$ for all $\theta_i \neq \theta_i'$, where $\rho_i(\theta_i, \theta_i')$ is the Kullback-Leibler divergence from θ_i to θ_i':

$$\rho_i(\theta_i, \theta_i') = \mathbb{E}_{g_i(\cdot|\theta_i)}[\log g_i(\cdot|\theta_i) - \log g_i(\cdot|\theta_i')] \tag{2.64}$$

where $\mathbb{E}_{g_i(\cdot|\theta_i)}$ represents taking the expectation with respect to $g_i(\cdot|\theta_i)$;

B6 $\sum_{m=1}^{\infty} \mathbb{E}_{g_i(\cdot|\theta_i)}[\rho_i(\theta_i, \hat{\theta}_{i,m})] < \infty$ where $\hat{\theta}_{i,m}$ represents the MLE estimator of θ_i with the first m observations.

Let us define the Kullback-Leibler divergence from $\boldsymbol{\theta} = [\theta_1, ..., \theta_K]$ to $\boldsymbol{\theta}' = [\theta'_1, ..., \theta'_K]$ as

$$\rho(\boldsymbol{\theta}, \boldsymbol{\theta}') = \mathbb{E}_{f(\cdot|\boldsymbol{\theta})}[\sum_{i=1}^{K} \log \frac{g_i(\cdot|\theta_i)}{g_i(\cdot|\theta'_i)}] \tag{2.65}$$

where $\mathbb{E}_{f(\cdot|\boldsymbol{\theta})}$ represents taking the expectation with respect to $f(\cdot|\boldsymbol{\theta}) = \prod_{i=1}^{K} g_i(\cdot|\theta'_i)$. Under the Condition B2, Conditions B5 and B6 can be easily extended to $\rho(\boldsymbol{\theta}, \boldsymbol{\theta}') > 0$ and $\sum_{m=1}^{\infty} \mathbb{E}_{E_{f(\cdot|\boldsymbol{\theta})}}[\rho(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_m)]$, where $\hat{\boldsymbol{\theta}}_m$ is the MLE of $\boldsymbol{\theta}$ using the first m observations.

Conditions B1 to B6 are almost identical to Conditions A1 to A6, except that the parameter space and parameter set are expanded to take account of the unknown variances. In fact, with the knowledge of σ_i^2, Conditions B1 to B6 implies Conditions A1 to A6.

2.3.1 The Procedure

To adopt Procedure 1 proposed in Section 2.2.1 for R&S problems under normality assumption with unknown variance, we need to calculate the corresponding GLR statistic as follows

$$\Lambda_{i,n}^{(1)} = \frac{\sup_{\boldsymbol{\theta} \in \Theta_i} L_n(\boldsymbol{\theta})}{\pi_n} \tag{2.66}$$

where $L_n(\boldsymbol{\theta})$ is the likelihoods of all observations collected up to the nth iteration

$$L_n(\boldsymbol{\theta}) = \prod_{m=1}^{n} \prod_{i \in \mathcal{S}_{cand,m}} g_i(X_{i,m}|\theta_i) \tag{2.67}$$

and π_n in Equation 2.12 is the adaptive maximum likelihoods using $\hat{\theta}_{i,m-1}$, the MLE of θ_i based on the first $m-1$ observations on the system i, as the estimate of θ_i to calculate the density at the mth round.

$$\pi_n = \prod_{m=1}^{n} \prod_{i \in \mathcal{S}_{cand,m}} g_i(X_{i,m}|\hat{\theta}_{i,m-1}) \tag{2.68}$$

However, the calculation of $\Lambda_{i,n}^{(1)}$ is too expensive due to the fact that there is no closed-form solution to the optimization problem $\Lambda_{i,n}^{(1)}$. Therefore, we suggests

to implement Procedure 2 to solve R&S problems under normality assumption with unknown variance. As stated in the previous section, the GLR statistic used in Procedure 2 is defined upon the pairwise comparisons. Denote $\Lambda^{(2)}_{i,j,n}$ as the generalized likelihood ratio of the event that the system i is better than the system j,

$$\Lambda^{(2)}_{i,j,n} \triangleq \frac{\sup_{\boldsymbol{\theta}\in\Theta_{i,j}} L_n(\boldsymbol{\theta})}{\pi_n} \tag{2.69}$$

where the parameter space of $\boldsymbol{\theta}$ that $\mu_i > \mu_j$, $\Theta_{i,j} = \{\boldsymbol{\theta}|\mu_i > \mu_j\}$.

The GLR statistic used in Procedure 2 is the minimum of all $\Lambda^{(2)}_{i,j,n}$,

$$\Lambda^{(2)}_{i,\cdot,n} = \min_{j\neq i} \Lambda^{(2)}_{i,j,n} \tag{2.70}$$

Although $\Lambda^{(2)}_{i,\cdot,n}$ is more conservative than $\Lambda^{(1)}_{i,n}$, $\Lambda^{(2)}_{i,\cdot,n}$ is much easier to calculate. Recall that the numerator of $\Lambda^{(1)}_{i,n}$ is the maximum likelihood under the condition that the system i is the best system. Since

$$\Theta_i = \{\boldsymbol{\theta}|\mu_i > \mu_l, \forall l \neq i\} \subseteq \{\boldsymbol{\theta}|\mu_i > \mu_j\} = \Theta_{i,j}, \forall j \neq i \tag{2.71}$$

We have, $\Lambda^{(2)}_{i,j,n} \geq \Lambda^{(1)}_{i,n}$ for all j. Thus,

$$\Lambda^{(2)}_{i,\cdot,n} \geq \Lambda^{(1)}_{i,n} \tag{2.72}$$

meaning that $\Lambda^{(2)}_{i,\cdot,n}$ is bounded below by $\Lambda^{(1)}_{i,n}$.

The calculation of $\Lambda^{(2)}_{i,\cdot,n}$ is based on a collection of pairwise comparisons between the system i and the system j, whose closed-form solution can be easily found.

2.3.2 The Implementation

We consider the implementation the GLR statistic $\Lambda^{(2)}_{i,j,n}$ in Procedure 2. It is worth mentioning that the optimization problem in $\Lambda^{(1)}_{i,n}$ is solvable under the normality assumption with unknown variance as the kernel function is convex. However, since Procedure 1 requires the calculation of $\Lambda^{(1)}_{i,n}$ for each system at each

iteration, if the optimization problem solving $\Lambda_{i,n}^{(1)}$ has no closed-form solution, then the computation burden could be heavy. Thus, we prefer the usage of $\Lambda_{i,\cdot,n}^{(2)}$ as an alternative to reduce the computation complexity.

Let $\{X_{i,n}, n = 1, 2, 3, ...\}$ be the i.i.d. observations of the alternative i. $\{X_{i,n}\}$ are normally distributed with unknown mean $\mu_i^\star$ and unknown variance $\sigma_i^{2\star}$. The GLR statistic $\Lambda_{i,j,n}^{(2)}$ has the form as

$$\Lambda_{i,j,n}^{(2)} = \frac{\sup_{\boldsymbol{\theta}\in\Theta_{i,j}} L(\boldsymbol{\theta})}{\pi_n} \tag{2.73}$$

where $\Theta_{i,j}$ is the parameter space that $\Theta_{i,j} = \{\boldsymbol{\theta}|\mu_i \geq \mu_j + \delta\}$.

Under the normality assumption with unknown variance σ^2, $L_n(\boldsymbol{\theta})$ and π_n are defined by

$$L_n(\boldsymbol{\theta}) = \prod_{m=1}^{n} \prod_{l\in\mathcal{S}_{cand,m}} g_l(X_{l,m}|\mu_l, \sigma_l^2) \tag{2.74}$$

$$\pi_n = \prod_{m=1}^{n} \prod_{l\in\mathcal{S}_{cand,m}} g_l(X_{l,m}|\hat{\mu}_{l,m-1}, \hat{\sigma}_{l,m-1}^2) \tag{2.75}$$

$$g_l(X_{l,m}|\mu_l, \sigma_l^2) = \frac{1}{\sqrt{2\pi\sigma_l^2}} \exp\{-\frac{(X_{l,m}-\mu_l)^2}{2\sigma_l^2}\} \tag{2.76}$$

where $\mathcal{S}_{cand,n}$ contains systems that are not eliminated at the nth iteration, and $\hat{\mu}_{l,n}$ and $\hat{\sigma}_{l,n}^2$ are the MLEs of μ_l and σ_l^2 at the nth iteration. In the proposed procedure, we denote τ_l as the elimination time of the system l, then we have

$$\hat{\mu}_{l,n} = \frac{\sum_{m=1}^{n'} X_{l,m}}{n_l}, \tag{2.77}$$

$$\hat{\sigma}_{l,n}^2 = \frac{\sum_{m=1}^{n_l}(X_{l,m}-\hat{\mu}_{l,n})^2}{n_l} \tag{2.78}$$

with $n_l = \min\{n, \tau_l\}$.

Under Condition B2 that all systems are independent, $\Lambda_{i,j,n}^{(2)}$ can be written as

$$\Lambda_{i,j,n}^{(2)} = \frac{\sup_{\boldsymbol{\mu}_i\in\Theta_{i,j}} L(\boldsymbol{\theta})}{\pi_n} \tag{2.79}$$

$$= \prod_{m=1}^{n} \prod_{l\in\mathcal{S}_{cand,m}} \frac{g_l(X_{l,m}|\hat{\mu}_{l,n}, \hat{\sigma}_{l,n}^2)}{g_l(X_{l,m}|\hat{\mu}_{l,m-1}, \hat{\sigma}_{l,m-1}^2)} \cdot \prod_{m=1}^{n} \frac{g_i(X_{i,m}|\tilde{\mu}_{i,n}, \tilde{\sigma}_{i,n}^2)\cdot g_j(X_{j,m}|\tilde{\mu}_{j,n}, \tilde{\sigma}_{j,n}^2)}{g_i(X_{i,m}|\hat{\mu}_{i,m-1}, \hat{\sigma}_{i,m-1}^2)\cdot g_j(X_{j,m}|\hat{\mu}_{j,m-1}, \hat{\sigma}_{j,m-1}^2)} \tag{2.80}$$

where $[\tilde{\mu}_{i,n}, \tilde{\mu}_{j,n}, \tilde{\sigma}^2_{i,n}, \tilde{\sigma}^2_{j,n}]$ are the MLE of mean values (μ_i, μ_j) and variances (σ^2_i, σ^2_j) under the constrain $\mu_i > \mu_j$ at the nth iteration.

To speed up the calculation of $\Lambda^{(2)}_{i,j,n}$, in implementation, we approximate the $[\tilde{\mu}_{i,n}, \tilde{\mu}_{j,n}, \tilde{\sigma}^2_{i,n}, \tilde{\sigma}^2_{j,n}]$ with

$$[\tilde{\mu}_{i,n}, \tilde{\mu}_{j,n}] \approx \begin{cases} [\hat{\mu}_{i,n}, \hat{\mu}_{j,n}] & \text{if } \hat{\mu}_{i,n} \geq \hat{\mu}_{j,n} + \delta \\ [\frac{\hat{\mu}_{i,n}+\hat{\mu}_{j,n}+\delta}{2}, \frac{\hat{\mu}_{i,n}+\hat{\mu}_{j,n}-\delta}{2}] & \text{otherwise} \end{cases} \tag{2.81}$$

$$[\tilde{\sigma}^2_{i,n}, \tilde{\sigma}^2_{j,n}] \approx [\frac{\sum^n_{m=1}(X_{i,m} - \tilde{\mu}_{i,n})^2}{n}, \frac{\sum^n_{m=1}(X_{j,m} - \tilde{\mu}_{j,n})^2}{n}] \tag{2.82}$$

2.3.3 The Validity

In this subsection, we show that both Procedure 1 and Procedure 2 are valid to deliver the *PCS* guarantee, although Procedure 1 is difficult, if still possible, to implement in the case of unknown variance. The proofs of the following two theorems are included in Appendix 2.6.4.

Theorem 2.3.1 *Under the Conditions B1-B6, both Procedure 1 and Procedure 2 terminates in finite time with probability* 1.

With the guarantee that these two procedures terminate in finite time, we can further show that both procedures deliver the *PCS* guarantee.

Theorem 2.3.2 *Under the Conditions B1-B6, Procedure 1 selects the best system with probability at least* $1 - \alpha$*, i.e.,*

$$P(CS) \geq 1 - \alpha \tag{2.83}$$

Proof Let us denote $\boldsymbol{\theta}^\star = [[\mu^\star_1, \sigma^{2\star}_1], ..., [\mu^\star_K, \sigma^{2\star}_1]]$ as the true parameter. Without loss of generality, we assume that the system 1 is the best system. Thus, the true hypothesis is $H_1 : \boldsymbol{\theta}^\star \in \Theta_1$. Let us denote $\tau^{(1)}_i = \min\{n | \Lambda^{(1)}_{i,n} \leq \alpha\}$ as the elimination time of the alternative i in Procedure 1.

Using the same logic as in the proof of Theorem 2.2.5 and the result of Theorem 2.3.1, for arbitrary choice of $N > 0$, we have

$$P(CS) \geq \lim_{N\to\infty} P(\text{Procedure 1 terminates before } N)(1 - P(\tau_1^{(1)} < \infty)) = 1 - P(\tau_1^{(1)} < \infty) \tag{2.84}$$

The GLR statistic, $\Lambda_{1,n}^{(1)}$, is calculated by

$$\Lambda_{1,n}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\Theta_1} L_n(\boldsymbol{\theta})}{\pi_n} \tag{2.85}$$

with

$$L_n(\boldsymbol{\theta}) = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\theta_i) \tag{2.86}$$

$$\pi_n = \prod_{m=1}^{n} \prod_{i\in\mathcal{S}_{cand,m}} g_i(X_{i,m}|\hat{\theta}_{i,m-1}) \tag{2.87}$$

where $\hat{\theta}_{i,n}$ is the MLE of θ_i using the first n observations on system i.

Since $\boldsymbol{\theta}^\star \in \Theta_1$, we have

$$\Lambda_{1,n}^{(1)} = \frac{\sup_{\boldsymbol{\theta}\in\Theta_1} L_n(\boldsymbol{\theta})}{\pi_n} \geq \frac{L_n(\boldsymbol{\theta}^\star)}{\pi_n} \tag{2.88}$$

This integration equals to 1 since $g_i(X|\theta_i)$ are valid density functions.

Denote $\Gamma_n = \frac{\pi_n}{L_n(\boldsymbol{\theta}^\star|\cdot)}$. Using the exactly same arguments as in the proof of Theorem 2.2.5, we have Γ_n is a positive martingale with $\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[\Gamma_n] = 1$. Apply the Theorem of Doob's Inequality on Martingale,

$$P(\max_{n=1,\ldots,N} \Gamma_n \geq \frac{1}{\alpha}) \leq \alpha \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[\Gamma_n] = \alpha, \forall N = 1, 2, \ldots \tag{2.89}$$

Since $\Lambda_{1,n}^{(1)} \geq 1/\Gamma_n$, we have

$$P(\min_{n=1,\ldots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq P(\max_{n=1,\ldots,N} \Gamma_n \geq \frac{1}{\alpha}) \leq \alpha \tag{2.90}$$

Therefore, $P(\tau_1^{(1)} < N) = P(\min_{n=1,\ldots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq \alpha$ for any N, and

$$P(CS) \geq 1 - P(\tau_1^{(1)} < \infty) \geq 1 - \alpha \tag{2.91}$$

■

Procedure 2 has the same validity as the Procedure 1. The result is stated in the next theorem.

Theorem 2.3.3 *Under the Conditions B1-B6, Procedure 2 selects the best system with probability at least* $1-\alpha$*, i.e.,*

$$P(CS) \geq 1-\alpha \tag{2.92}$$

Proof Using the same notations and assumptions as in the proof of Theorem 2.3.2, let us denote $\tau_i^{(2)} = \min\{n|\Lambda_{i,\cdot,n}^{(2)} \leq \alpha\}$ as the elimination time of the alternative i in Procedure 2.

The $P(CS)$ is bounded below as follows

$$P(CS) \geq P(\text{Procedure 2 terminates before } N)(1-P(\tau_1^{(2)} < \infty))$$

This inequality holds for any $N > 0$. Using the property that Procedure 2 terminates in finite time w.p.1, $\lim_{N\to\infty} P(\text{Procedure 2 terminates before } N) = 1$. Thus, we have

$$P(CS) \geq \lim_{N\to\infty} P(\text{Procedure 2 terminates before } N)(1-P(\tau_1^{(2)} < \infty)) = 1-P(\tau_1^{(2)} < \infty) \tag{2.93}$$

Using the inequality in 2.72, $\Lambda_{i,\cdot,n}^{(2)} \geq \Lambda_{i,n}^{(1)}$ for any n, we have

$$P(\tau_1^{(2)} < N) = P(\min_{n=1,\ldots,N} \Lambda_{i,\cdot,n}^{(2)} \leq \alpha) \leq P(\min_{n=1,\ldots,N} \Lambda_{1,n}^{(1)} \leq \alpha) \leq \alpha,\ \forall N \tag{2.94}$$

where the last inequality is from the proof of the Theorem 2.3.2.

Thus, $P(\tau_1^{(2)} < \infty) \leq \alpha$ and $P(CS) \geq 1-\alpha$. ■

With the preceding theorem, Procedure 2 selects the best system with probability at least $1-\alpha$ when the proposed procedure terminates. Nonetheless, if Procedure 2 is stopped before the termination criteria is met, i.e., $|\mathcal{S}_{cand,n}| = 1$, it is still guaranteed that the set $|\mathcal{S}_{cand,n}|$ contains the best system with a probabilistic guarantee.

Lemma 2.3.1 *Under the settings of Procedure 2, denote $\mathcal{S}_{cand,n}$ as the subset of systems that are not eliminated during the first n iterations. If Procedure 2 delivers the PCS guarantee as stated in Theorem 2.3.3, then, for any n, $\mathcal{S}_{cand,n}$ retains the best system with probability at least $1-\alpha$.*

Proof The proof of Theorem 2.3.3 states that $P(\tau_1^{(2)} < \infty) \leq \alpha$. Therefore, $P(\tau_1^{(2)} < n) \leq \alpha$ for all n. ■

2.3.4 The Asymptotic Efficiency

In this subsection, we present the asymptotic efficiency result of Procedure 2. Denote $\tau_i^{(2)}$ as the sample sizes used by Procedure 2 to eliminate the alternative i. The following theorem gives the asymptotic limit of the average of $\tau_i^{(2)}$, as the error probability α approaches to 0. The proof of the following theorem is included in Appendix 2.6.6.

Theorem 2.3.4 *Under the Conditions B1-B6 and the settings of Procedure 2, the average sample size to eliminate the alternative i has the following asymptotic approximation*

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[\tau_i^{(2)}] \to \frac{\log \alpha^{-1}}{\inf_{\boldsymbol{\theta}\in\Theta_{i,1}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta})}(1+o(1)) \tag{2.95}$$

as $\alpha \to 0$, for all $\boldsymbol{\theta}^\star \in \Theta_1$, $i = 1, ..., m$.

2.4 Simulation Studies

In this section, we conduct numerical experiments to test the performance of the proposed procedures and compare it with other leading R&S procedures.

In the first example, we establish the efficiency of Procedure 1 by examining its average sample size under a R&S problem with 2 systems. In the second study, we assess the performance of Procedure 2 under different indifference-zone configurations. The third numerical study tests the proposed procedure under the context of heterogeneous variance.

2.4.1 Study 1: Validity and Efficiency

In this study, we evaluate the performance of Procedure 1. We consider the case of only two alternatives with normally distributed observations. Let $X_i \sim N(\mu_i^\star, 1)$, where $\mu_i^\star$ is the mean value of the system i. We examine how the configurations of PCS and $(\mu_1^\star, \mu_2^\star)$ impact the average sample size and compare the numerical results with the analytical lower bound provided in Theorem 2.2.6.

Table 2.2.: Study 1: the estimated PCS (Est. PCS) and the average sample size (Avg. SSize) of a R&S problem with $K = 2$ alternatives and observations from $N(\mu_i^\star, 1)$. Numbers are summarized from 10,000 independent macroreplications of Procedure 1 .

$\mu_1^\star - \mu_2^\star$	Target $PCS = 0.05$		Target $PCS = 0.025$		Target $PCS = 0.0125$	
	Avg. SSize	Est. PCS	Avg. SSize	Est. PCS	Avg. SSize	Est. PCS
2^3	6.006	1.000	6.030	1.000	6.148	1.000
2^2	6.474	1.000	7.010	1.000	7.632	1.000
2^1	10.002	1.000	12.272	1.000	14.380	1.000
2^0	24.848	0.998	31.754	0.997	38.412	0.998
2^{-1}	79.230	0.976	107.122	0.987	127.222	0.994
2^{-2}	306.512	0.960	395.592	0.976	457.404	0.988
2^{-3}	1001.188	0.934	1454.400	0.971	1808.962	0.985
2^{-4}	4053.468	0.950	5354.470	0.979	6960.056	0.988

We implement Procedure 1 with the GLR statistics calculated under the normal distribution with unknown mean and known variance. For each combination of the $(\mu_1^\star, \mu_2^\star)$ and the target PCS, we repeat the experiment for 10,000 times, and report the average sample size and the estimated PCS in Table 2.2. Figure 2.2 compares the average sample sizes with the analytical of Theorem 2.2.6, which

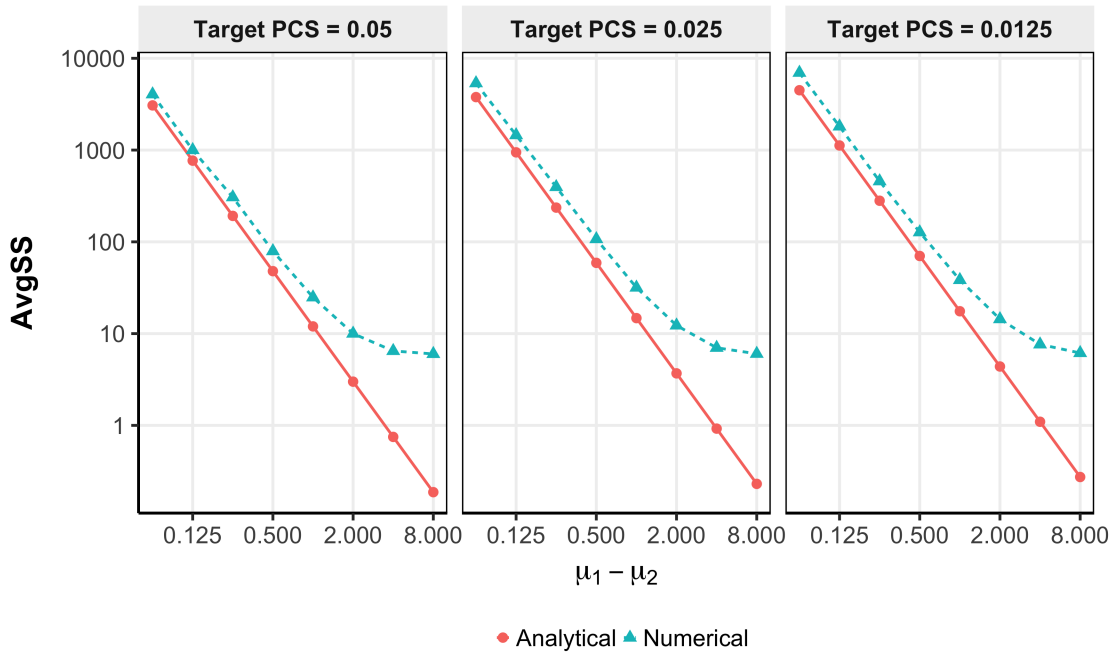


Figure 2.2.: Study 1: the log-log plot of average sample sizes (y-coordinate) against the true difference $\mu_1^\star - \mu_2^\star$. Orange dots are analytical average samples according to Theorem 2.2.6; blue triangles are average sample size from Table 2.2

states the average sample size asymptotically approaches to the limit $\frac{4 \cdot \log \alpha}{(\mu_1^\star - \mu_2^\star)^2}$ as $\alpha \to 0$.

From Table 2.2 and Figure 2.2, we have two interesting findings. First, Procedure 1 tends to overshoot the *PCS* when the when the gap between two alternatives, $\mu_1^\star - \mu_2^\star$, is large. As the gap decreases, the estimated *PCS* drops steadily to the level of the target *PCS*. It is because the slope of the expected GLR statistic is proportional to the square of the gap. The larger the gap, the more likely the GLR statistic overshoot the elimination boundary, and therefore, the higher achieved (estimated) *PCS*. As a result, the estimated *PCS* converges to the target *PCS* as the gap decreases. The second finding is that the average sample size used by Procedure 1 approaches to the theoretical lower bound as the gap decreases.

Still, there is a gap between the numerical and analytical results. There are two explanations. First of all, in the numerical evaluation, the GLR statistic is a discrete stochastic process; whereas, in theoretical analysis, the GLR statistic is approximated as continuous. Secondly, the analytical result assumes the knowledge of $\mu^\star$, the true mean of the distribution; this information, however, is unknown from the numerical evaluations.

2.4.2 Study 2: Indifference-Zone Parameter

This study examines the indifference-zone setting. We use the KN++ procedure and the IZ-free procedure as benchmarks. This study uses the same setting as the experiment in [17] and [18]. Part of the numerical results are cited from [18]. Consider a R&S problem with K alternatives, with $\mu_i = 1.5 - 0.5i$ and $\sigma_i^2 = 10$ for all $i = 1, ...K$. Note that the major difference between our proposed procedures and other procedures is that our procedures do not use the Bonferroni correction.

Let us denote the IZ parameter as δ. In the experiment, δ takes values from 0 to 0.5, where 0 means indifference-zone information is not available, and 0.5 is the true gap between the best system and all other systems. For $K = 20,\ 50,\ 100,\ 500$, we report the estimated PCS and average sample sizes over 1,000 independent evaluations in Table 2.3.

From Table 2.3, we obtain the following three findings related to the indifference-zone. Firstly, Procedure 1 outperforms the KN++ procedure and the IZ-free procedure regarding using fewer samples. Secondly, Procedure 1 is valid with or without the IZ parameter, and therefore it fills the gap between the indifference-zone formulation and the indifference-zone-free formulation.

2.4.3 Study 3: Unknown and Heterogeneous Variance

In this study, we investigate the efficiency of our proposed procedures in the context of unknown and heterogeneous variance.

Table 2.3.: Study 3: the estimated PCS (Est. PCS) and the average sample size (Avg. SSize) of a R&S problem with K alternatives and observations from $N(\mu_i = 1 - 0.5 * i, 10)$. Assume $PCS = 1 - \alpha = 0.95$ and $IZ = \delta$. Numbers are summarized from 1,000 independent macroreplications of Procedure 1 with $n_0 = 10$, the KN++ procedure [14], and the IZ-free procedure [18]

K		KN++ procedure				IZ-free	Procedure 1				
		$\delta = \frac{1}{2}$	$\delta = \frac{1}{4}$	$\delta = \frac{1}{8}$	$\delta = \frac{1}{16}$		$\delta = \frac{1}{2}$	$\delta = \frac{1}{4}$	$\delta = \frac{1}{8}$	$\delta = \frac{1}{16}$	$\delta = 0$
20	Est. PCS	0.99	1.00	1.00	1.00	0.99	0.964	0.996	0.999	0.998	0.999
	Avg. SSize	1371	3247	7045	14700	2816	801	1689	2065	2469	2516
50	Est. PCS	1.00	1.00	1.00	1.00	0.99	0.999	0.996	0.999	1.00	0.999
	Avg. SSize	2014	4454	9779	20700	3588	1346	2376	2884	3013	3059
100	Est. PCS	0.99	1.00	1.00	1.00	1.00	1.00	0.999	1.00	1.00	1.00
	Avg. SSize	2710	5506	11790	25140	4388	2304	3346	3844	3900	4137
500	Est. PCS	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	Avg SSize	7217	10770	18770	36280	9138	6658	7276	7919	8469	9385

We consider three configurations of variance: $\sigma^2 = 10$ (Equal Var.), $\sigma^2 = 10 \times (0.95 + 0.05i)$ (Increasing), and $\sigma^2 = 10/(0.95 + 0.05i)$ (Decreasing). We adopt all other configurations from last example, except we set $\delta = 0.5$ here. The results are reported in Table 2.4. From Table 2.4, Procedure 2 demonstrates validity and efficiency across all three configurations of variance.

2.5 Conclusion

In this paper, we propose two new procedures for ranking and selection. Under normality assumptions on distributions, the proposed procedures are designed to select the best system with a *PCS* guarantee. Unlike many other leading procedures in R&S, the new methods do not use the Bonferroni correction. They also bridge the gap between the indifference-zone formulation and the indifference-zone-free formulation so that the indifference-zone information is not indispensable and yet valuable if provided correctly.

2.6 Appendix: Technical Proofs

2.6.1 Technical Theorems

We need the following technical results for our proofs.

Theorem 2.6.1 *(Theorem 2.3.5 Part iii in [29])[Moment Inequality of Martingales] Let* $S_n = X_1 + X_2 + ... + X_n$, $n \geq 1$ *be a square integrable martingale with independent increments. Then for any* $0 < p \leq 2$ *and any stopping time* T*, there is a positive constant* C_p*, independent of* T *and* X*, such that*

$$\mathbb{E}\left[\left(\sup_{t \leq T} |S_t|\right)^p\right] \leq C_p \mathbb{E}[D_T]^{p/2} \tag{2.96}$$

where $D_t = Var[X_t]$.

The preceding theorem leads to the following lemma.

Table 2.4.: Study 4: the estimated PCS (Est. PCS) and the average sample size (Avg. SSize) with 95% confidence intervals of a R&S problem with K alternatives and observations from $N(\mu_i = 1 - 0.5 * i, \sigma_i^2)$. We use Procedure 2 with $n_0 = 10$. Assume the target $PCS = 0.95$. Numbers are summarized from 1,000 independent macroreplications.

K	Equal Var		Increasing		Decreasing	
	Avg. SSize	Est. PCS	Avg. SSize	Est. PCS	Avg. SSize	Est. PCS
10	637.521 ± 35.5199	0.948	672.394 ± 30.9655	0.957	628.667 ± 30.7411	0.962
20	886.165 ± 39.3117	0.948	974.374 ± 40.9527	0.970	876.176 ± 38.7272	0.964
50	1336.324 ± 40.2337	0.965	1439.249 ± 38.1211	0.956	1289.106 ± 33.8968	0.957
100	2388.974 ± 37.8915	0.951	2521.190 ± 37.6908	0.949	2448.217 ± 36.1007	0.957
500	8372.089 ± 38.3407	0.965	8940.924 ± 40.0802	0.958	8843.532 ± 37.0205	0.954

Lemma 2.6.1 *Let $S_n = X_1 + X_2 + ... + X_n$, $n \geq 1$ random walk. Where X_i are independent increments with $\mathbb{E}[X_i] = 0$. If $\mathbb{E}[X_i^2] < \infty$. Then there is a positive constant C, independent of n and X, such that*

$$\mathbb{E}[|S_n|] \leq C \cdot n^{1/2} \tag{2.97}$$

for all $n \geq 1$.

Proof Consider a constant n as the stopping time T in Theorem 2.6.1 and set $p = 1$. Therefore, we have

$$\mathbb{E}[|S_n|] \leq \mathbb{E}\left[\left(\sup_{t \leq T} |S_t|\right)\right] \leq C_1 \mathbb{E}[D_n]^{1/2} = C_1 (\sum_{m=1}^{n} Var[X_n])^{1/2} \leq C \cdot n^{1/2} \tag{2.98}$$

■

The following two theorems, Theorem 2.6.2 and Theorem 2.6.3, provide the convergent properties of the log of the GLR statistic $\Lambda_{i,j,n}^{(2)}$

$$\lambda_{i,j,n}^{(2)} = \log \Lambda_{i,j,n}^{(2)} \tag{2.99}$$

Theorem 2.6.2 *Under the Conditions A1-A6, assuming that the systems i and j, with $\mu_i^\star < \mu_j^\star$, are exempt from the elimination step, then $\frac{\lambda_{i,j,n}^{(2)}}{n}$ converges almost surely to $-\inf_{\boldsymbol{\mu} \in \mathbb{U}_{i,j}} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})$.*

Remark 2.6.1 *The proof of this theorem follows the proof of Theorem 5.4.2 in [29].*

Proof In the following proof, for the sake of brevity, we use τ_l to represent the stopping time of system l.

For any n, the log GLR can be written as

$$\lambda^{(2)}_{i,j,n} \triangleq \log \Lambda^{(2)}_{i,j,n} = \sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\mu}) - \log \pi_n \tag{2.100}$$

$$= \sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\mu_l) - \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) \tag{2.101}$$

$$= \sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \Bigg[\sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}/\{i,j\}} \log g_l(X_{l,m}|\mu_l) - \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}/\{i,j\}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) + \tag{2.102}$$

$$\sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\mu_l) - \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) \Bigg] \tag{2.103}$$

$$= \sum_{l\in\Omega/\{i,j\}} \Bigg[\sum_{m=1}^{\min\{n,\tau_l\}} [\log g_l(X_{l,m}|\hat{\mu}_{l,n}) - \log g_l(X_{l,m}|\hat{\mu}_{l,m-1})] \Bigg] + \tag{2.104}$$

$$\Bigg[\sup_{\mu_i>\mu_j} \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\mu_l) - \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) \Bigg] \tag{2.105}$$

where $\hat{\mu}_{l,n}$ represents the unconstrained MLE of μ_l using the first n observations on the system l. It is valid since the, according to Condition A2, systems are (conditional) independent with each other. For cases that $\tau_l < n$, we set $\hat{\mu}_{l,n} = \hat{\mu}_{l,\tau_l}$.

In the following, we show the almost sure convergences of expression 2.104 and 2.105, respectively.

First, let us consider the expression 2.104. For any $l \neq \{i,j\}$, let us denote

$$\kappa_l = \sum_{m=1}^{\min\{n,\tau_l\}} [\log g_l(X_{l,m}|\hat{\mu}_{l,n}) - \log g_l(X_{l,m}|\hat{\mu}_{l,m-1})] \tag{2.106}$$

$$= \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\mu}_{l,n})}{g_l(X_{l,m}|\mu_l^\star)} - \log \frac{g_l(X_{l,m}|\hat{\mu}_{l,m-1})}{g_l(X_{l,m}|\mu_l^\star)}] \tag{2.107}$$

Denote the Kullback Leibler divergence for each system as

$$\rho_l(\mu_l^\star, \mu_l) = \mathbb{E}_{g_l(\cdot|\mu_l^\star)}[\log \frac{g_l(\cdot|\mu_l^\star)}{g_l(\cdot|\mu_l)}] \tag{2.108}$$

Therefore, we can expand κ_l as

$$\kappa_l = \sum_{m=1}^{\min\{n,\tau_l\}} \Big[[\log \frac{g_l(X_{l,m}|\hat{\mu}_{l,n})}{g_l(X_{l,m}|\mu_l^\star)} + \rho_l(\mu_l^\star, \hat{\mu}_{l,n})] - \rho_l(\mu^\star, \hat{\mu}_{l,n}) \tag{2.109}$$

$$- [\log \frac{g_l(X_{l,m}|\hat{\mu}_{l,m-1})}{g_l(X_{l,m}|\mu_l^\star)} + \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1})] + \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1}) \Big] \tag{2.110}$$

Let

$$\kappa_{l_1} = \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\mu}_{l,n})}{g_l(X_{l,m}|\mu_l^\star)} + \rho_l(\mu_l^\star, \hat{\mu}_{l,n})] \tag{2.111}$$

$$\kappa_{l_2} = \sum_{m=1}^{\min\{n,\tau_l\}} \rho_l(\mu_l^\star, \hat{\mu}_{l,n}) \tag{2.112}$$

$$\kappa_{l_3} = \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\mu}_{l,m-1})}{g_l(X_{l,m}|\mu_l^\star)} + \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1})] \tag{2.113}$$

$$\kappa_{l_4} = \sum_{m=1}^{\min\{n,\tau_l\}} \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1}) \tag{2.114}$$

Since $\mathbb{E}_{g_l(\cdot|\mu_l^\star)}[\log \frac{g_l(\cdot|\mu_l)}{g_l(\cdot|\mu_l^\star)} + \rho_l(\mu_l^\star, \mu_l)] = 0$ for any μ_l; thus, κ_{l_1} and κ_{l_3} are zero-sum independent incremental martingales. By Condition A4 that $g_l(\cdot|\mu_l) > 0$ for all μ_l, we have $|\log \frac{g_l(X_l|\mu_l)}{g_l(X_l|\mu_l^\star)}| < \infty$ uniformly for all X_l and μ_l. By Lemma 2.6.1, we have

$$\mathbb{E}_{g_l(\cdot|\mu_l^\star)}[|\kappa_{l_1}|] < constant \cdot n^{1/2} \tag{2.115}$$

$$\mathbb{E}_{g_l(\cdot|\mu_l^\star)}[|\kappa_{l_3}|] < constant \cdot n^{1/2} \tag{2.116}$$

Thus, $\frac{\kappa_{l_1}}{n}$ and $\frac{\kappa_{l_3}}{n}$ converges almost surely to 0.

By Condition A6, $\mathbb{E}_{g_l(\cdot|\mu_l^\star)}[\sum_{m=1}^{\infty} \rho_l(\mu_l^\star, \hat{\mu}_{l,m})] < \infty$, we have $\frac{\kappa_{l_2}}{n}$ and $\frac{\kappa_{l_4}}{n}$ converges almost surely to 0, as well.

Thus, $\frac{\kappa_l}{n} = \frac{1}{n}[\kappa_{l_1} - \kappa_{l_2} - \kappa_{l_3} + \kappa_{l_4}]$ converges almost surely to 0.

Next, we consider the expression 2.105. Let us denote the joint density of systems i and j as

$$f_{ij}(X_i, X_j|\boldsymbol{\mu}_{ij}) = g_i(X_i|\mu_i) \cdot g_j(X_j|\mu_j) \tag{2.117}$$

where $\boldsymbol{\mu}_{ij} = [\mu_i, \mu_j]$.

Denote the Kullback Leibler divergence as

$$\rho_{ij}(\boldsymbol{\mu}_{ij}^{\star}, \boldsymbol{\mu}_{ij}) = \mathbb{E}_{f_{i,j}(\cdot|\boldsymbol{\mu}_{ij}^{\star})}[\log \frac{f_{ij}(\cdot|\boldsymbol{\mu}_{ij}^{\star})}{f_{ij}(\cdot|\boldsymbol{\mu}_{ij})}] \tag{2.118}$$

where $\boldsymbol{\mu}_{ij}^{\star} = [\mu_i^{\star}, \mu_j^{\star}]$ is the true mean vector of systems i and j.

Denote the expression 2.105 as

$$\kappa_{ij} = \sup_{\mu_i > \mu_j} \sum_{m=1}^{n} \sum_{l \in \{i,j\}} \log g_l(X_{l,m}|\mu_l) - \sum_{m=1}^{n} \sum_{l \in \{i,j\}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) \tag{2.119}$$

$$= \sup_{\mu_i > \mu_j} \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}) - \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\mu}}_{ij,m-1}) \tag{2.120}$$

where $\hat{\boldsymbol{\mu}}_{ij,m-1}$ is the MLE of $\boldsymbol{\mu}_{ij}$ using the first $m-1$ observations on the system i and j.

κ_{ij} can be expanded as

$$\kappa_{ij} = \sup_{\boldsymbol{\mu}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}) - \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\mu}}_{ij,m-1}) \tag{2.121}$$

$$= \sup_{\boldsymbol{\mu}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})} - \sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\mu}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})} \tag{2.122}$$

$$= \sup_{\boldsymbol{\mu}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})}] + \sum_{m=1}^{n} \rho_{ij}(\boldsymbol{\mu}_{ij}^{\star}, \hat{\boldsymbol{\mu}}_{ij,m-1}) \tag{2.123}$$

$$- \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\mu}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})} + \rho_{ij}(\boldsymbol{\mu}_{ij}^{\star}, \hat{\boldsymbol{\mu}}_{ij,m-1})] \tag{2.124}$$

Denote

$$\kappa_{ij_1} = \sup_{\boldsymbol{\mu}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})}] \tag{2.125}$$

$$\kappa_{ij_2} = \sum_{m=1}^{n} \rho_{ij}(\boldsymbol{\mu}_{ij}^{\star}, \hat{\boldsymbol{\mu}}_{ij,m-1}) \tag{2.126}$$

$$\kappa_{ij_3} = \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\mu}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij}^{\star})} + \rho_{ij}(\boldsymbol{\mu}_{ij}^{\star}, \hat{\boldsymbol{\mu}}_{ij,m-1})] \tag{2.127}$$

By the same logic for the almost sure convergences of κ_{j_3}/n and κ_{j_4}/n, it is easy to verify that κ_{ij_2}/n and κ_{ij_3}/n converges almost surely to 0.

Given Condition A3 that the parameter space is compact, Condition A4 that $g_i(X_i|\mu_i)$ is continuous and $g_i(X_i|\mu_i) > 0$ for all X_i and μ_i, we can apply the Uniform Law of Large Number to show that, almost surely,

$$\sup_{\boldsymbol{\mu}_{ij}\in\{\mu_i>\mu_j\}} \left| \frac{1}{n}\sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}^{\star}_{ij})} + \rho_{ij}(\boldsymbol{\mu}^{\star}_{ij}, \boldsymbol{\mu}_{ij}) \right| \to 0 \quad (2.128)$$

$$\frac{\kappa_{ij_1}}{n} = \sup_{\boldsymbol{\mu}_{ij}\in\{\mu_i>\mu_j\}} \frac{1}{n}\sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\mu}^{\star}_{ij})} \to - \inf_{\boldsymbol{\mu}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\mu}^{\star}_{ij}, \boldsymbol{\mu}_{ij}) \quad (2.129)$$

Therefore, $\frac{\kappa_{ij}}{n} = \frac{1}{n}[\kappa_{ij_1} + \kappa_{ij_2} - \kappa_{ij_3}] \to - \inf_{\boldsymbol{\mu}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\mu}^{\star}_{ij}, \boldsymbol{\mu}_{ij})$ almost surely. Given the fact that $- \inf_{\boldsymbol{\mu}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\mu}^{\star}_{ij}, \boldsymbol{\mu}_{ij}) = - \inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\mu}^{\star}, \boldsymbol{\mu})$, we have

$$\frac{\lambda^{(2)}_{i,j,n}}{n} = \frac{1}{n}[\sum_{l\in\Omega/\{i,j\}} \kappa_l + \kappa_{ij}] \to - \inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\mu}^{\star}, \boldsymbol{\mu}) \quad (2.130)$$

Thus, we conclude the proof.

■

Theorem 2.6.3 *Under the Conditions A1-A6, there exists a number N and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that*

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^{\star})}[(\Lambda^{(2)}_{i,j,n})^s] \leq constant \cdot e^{-\epsilon n} \quad (2.131)$$

for all $n > N$.

Proof Consider the log of the GLR statistic $\lambda^{(2)}_{i,j,n}$

$$\lambda^{(2)}_{i,j,n} \triangleq \log \Lambda^{(2)}_{i,j,n} = \sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\mu}) - \log \pi_n \quad (2.132)$$

By Theorem 2.6.2, we have $\frac{\lambda^{(2)}_{i,j,n}}{n} \to - \inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\mu}^{\star}, \boldsymbol{\mu})$. By Condition A5, $\rho_i(\mu^{\star}_i, \mu_i) > 0$ for any $\mu \neq \mu^{\star}$ and $i \in \Omega$; thus there exists a constant $c_1 > 0$ such that $\inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\mu}^{\star}, \boldsymbol{\mu}) = \inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \sum_{l=1}^{K} \rho_l(\mu^{\star}_l, \mu_l) \geq c_1$ for any $\boldsymbol{\mu}^{\star} \notin \mathbb{U}_{i,j}$.

Let $\hat{\boldsymbol{\mu}}_n = [\hat{\mu}_{1,n}, ..., \hat{\mu}_{K,n}]$ and $\tilde{\boldsymbol{\mu}}_n = [\tilde{\mu}_{1,n}, ..., \tilde{\mu}_{K,n}] = \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\mu})$ be the unconstrained and constrained MLEs of $\boldsymbol{\mu}^\star$ at the nth round, respectively. Then denote $\lambda^{(2)}_{i,j,n}$ as the summation of the sequence $\{h_{m,n},\ m = 1, ..., n\}$

$$\lambda^{(2)}_{i,j,n} = \sum_{m=1}^{n} \left[\sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\tilde{\mu}_{l,n}) - \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\hat{\mu}_{l,m-1}) \right] = \sum_{m=1}^{n} h_{m,n} \tag{2.133}$$

It can be verified that $\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h_{m,n}] = \sum_{l\in\mathcal{S}_{cand,m}} \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1}) - \rho_l(\mu_l^\star, \tilde{\mu}_{l,n})$.

By Condition A6, $\mathbb{E}_{g_l(\cdot|\mu_l^\star)}[\sum_{m=1}^{\infty} \rho_l(\mu_l^\star, \hat{\mu}_{l,m})] < \infty$, and the fact that $\rho_l(\mu_l^\star, \hat{\mu}_{l,m})$ are bounded and non-negative, we have the almost sure convergence of $\rho_l(\mu_l^\star, \hat{\mu}_{l,m}) \to 0$ as $m, n \to \infty$. Therefore, there exists a number N such that $\rho_l(\mu_l^\star, \hat{\mu}_{l,m}) \leq \frac{c_1}{2K}$ for all $m \geq N$ and l.

Thus, for any n and m with $N < m < n$, we have

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h_{m,n}] = \sum_{l\in\mathcal{S}_{cand,m}} \rho_l(\mu_l^\star, \hat{\mu}_{l,m-1}) - \rho_l(\mu_l^\star, \tilde{\mu}_{l,n}) \tag{2.134}$$

$$\leq \frac{c_1}{2} - \inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} \sum_{l=1}^{K} \rho_l(\mu_l^\star, \mu_l) \leq -\frac{c_1}{2} \tag{2.135}$$

The first inequality holds since $\tilde{\mu}_n \in \mathbb{U}_{i,j}$.

Consider the function

$$\gamma(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[e^{s\cdot h_{m,n}}] \tag{2.136}$$

To find an upper bound of $\gamma(s)$, we firstly show that $\gamma(s)$ is *twice differentiable* and *bounded*.

To verity that $\gamma(s)$ is twice differentiable with respect to s within $[0, S]$, it is sufficient to show $\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h^2_{m,n} e^{s\cdot h_{m,n}}] < \infty$ for $s \in [0, S]$.

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h^2_{m,n} e^{s\cdot h_{m,n}}] = \int h^2_{m,n} e^{s\cdot h_{m,n}} f(\cdot|\boldsymbol{\mu}^\star) d\boldsymbol{X} \leq \left[\int h^4_{m,n} f(\cdot|\boldsymbol{\mu}^\star) d\boldsymbol{X} \right]^{1/2} \left[\int e^{2\cdot s\cdot h_{m,n}} f(\cdot|\boldsymbol{\mu}^\star) d\boldsymbol{X} \right]^{1/2} \tag{2.137}$$

Given Conditions A3, $f_{\boldsymbol{\mu}}(\boldsymbol{X})$ is continuous and $f_{\boldsymbol{\mu}}(\boldsymbol{X}) > 0$ for all $\boldsymbol{X}$ and $\boldsymbol{\mu} \in \mathbb{U}$, thus, the first term $\int h^4_{m,n} f_{\boldsymbol{\mu}^\star}(\boldsymbol{x}) d\boldsymbol{x}$ is bounded above; the second term is less than 1 for any $s \in [0, 1/2]$ by the definition of $h_{m,n}$. Therefore, there exists a constant c_2 such that

$$\gamma''(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h_{m,n}^2 e^{s\cdot h_{m,n}}] \leq c_2 < \infty \tag{2.138}$$

The first order derivative of $\gamma(s)$ is $\gamma'(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[h_{m,n} e^{s\cdot h_{m,n}}]$ with $\gamma'(0) \leq -\frac{c_1}{2}$.

Therefore, $\gamma(s) < 1 - \frac{c_1}{2}s + \frac{c_2}{2}s^2$ for all $s \in [0, 1/2]$. Thus ,there exists a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$ such that $\gamma(s) \leq e^{-\epsilon} < 1$.

Thus, there exists a large number N, such that, for any $n > N$,

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda_{i,j,n}^{(2)})^s] = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[e^{s\sum_{m=1}^{n}\cdot h_{m,n}}] \leq \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[e^{s\sum_{m=1}^{N}\cdot h_{m,n}}] \cdot e^{-\epsilon(n-N)} \leq c \cdot e^{-\epsilon\cdot n} \tag{2.139}$$

where c is a constant.

Thus, we conclude the proof.

■

Next, we give similar results of aforementioned theorems under the Conditions B1 to B6.

Theorem 2.6.4 *Under the Conditions B1-B6 and the settings of Procedure 1 and Procedure 2, assuming that the systems i and j, with $\mu_i^\star < \mu_j^\star$, are exempt from the elimination step, then $\frac{\lambda_{i,j,n}^{(2)}}{n}$ converges almost surely to $-\inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta})$.*

Remark 2.6.2 *The proof follows the exactly same logic as in the proof of Theorem 2.6.2, except that the parameter set is expanded from $\boldsymbol{\mu}$ to $\boldsymbol{\theta}$.*

Proof In the following proof, for the sake of brevity, we use τ_l to represent the stopping time of system l in both Procedure 1 and Procedure 2.

For any n, the log GLR can be written as

$$\lambda^{(2)}_{i,j,n} \triangleq \log \Lambda^{(2)}_{i,j,n} = \sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\theta}) - \log \pi_n \tag{2.140}$$

$$= \sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\theta_l) - \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) \tag{2.141}$$

$$= \sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \Bigg[\sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}/\{i,j\}} \log g_l(X_{l,m}|\theta_l) - \sum_{m=1}^{n} \sum_{l\in\mathcal{S}_{cand,m}/\{i,j\}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) + \tag{2.142}$$

$$\sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\theta_l) - \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) \Bigg] \tag{2.143}$$

$$= \sum_{l\in\tilde{\Omega}/\{i,j\}} \Bigg[\sum_{m=1}^{\min\{n,\tau_l\}} [\log g_l(X_{l,m}|\hat{\theta}_{l,n}) - \log g_l(X_{l,m}|\hat{\theta}_{l,m-1})] \Bigg] + \tag{2.144}$$

$$\Bigg[\sup_{\mu_i>\mu_j} \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\theta_l) - \sum_{m=1}^{n} \sum_{l\in\{i,j\}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) \Bigg] \tag{2.145}$$

where $\hat{\theta}_{l,n}$ represents the unconstrained MLE of θ_l using the first n observations on the system l. It is valid since the, according to Condition B2, systems are (conditional) independent with each other. For cases that $\tau_l < n$, we set $\hat{\theta}_{l,n} = \hat{\theta}_{l,\tau_l}$.

In the following, we show the almost sure convergences of expression 2.144 and 2.145, respectively.

First, let us consider the expression 2.144. For any $l \neq \{i,j\}$, let us denote

$$\kappa_l = \sum_{m=1}^{\min\{n,\tau_l\}} [\log g_l(X_{l,m}|\hat{\theta}_{l,n}) - \log g_l(X_{l,m}|\hat{\theta}_{l,m-1})] \tag{2.146}$$

$$= \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\theta}_{l,n})}{g_l(X_{l,m}|\theta^\star_l)} - \log \frac{g_l(X_{l,m}|\hat{\theta}_{l,m-1})}{g_l(X_{l,m}|\theta^\star_l)}] \tag{2.147}$$

Denote the Kullback Leibler divergence for each system as

$$\rho_l(\theta^\star_l, \theta_l) = \mathbb{E}_{g_l(X_{l,m}|\theta^\star_l)}[\log \frac{g_l(X_{l,m}|\theta^\star_l)}{g_l(X_{l,m}|\theta_l)}] \tag{2.148}$$

Therefore, we can expand κ_l as

$$\kappa_l = \sum_{m=1}^{\min\{n,\tau_l\}} \Big[[\log \frac{g_l(X_{l,m}|\hat{\theta}_{l,n})}{g_l(X_{l,m}|\theta_l^\star)} + \rho_l(\theta_l^\star, \hat{\theta}_{l,n})] - \rho_l(\theta^\star, \hat{\theta}_{l,n}) \tag{2.149}$$

$$- [\log \frac{g_l(X_{l,m}|\hat{\theta}_{l,m-1})}{g_l(X_{l,m}|\theta_l^\star)} + \rho_l(\theta_l^\star, \hat{\theta}_{l,m-1})] + \rho_l(\theta_l^\star, \hat{\theta}_{l,m-1}) \Big] \tag{2.150}$$

Let

$$\kappa_{l_1} = \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\theta}_{l,n})}{g_l(X_{l,m}|\theta_l^\star)} + \rho_l(\theta_l^\star, \hat{\theta}_{l,n})] \tag{2.151}$$

$$\kappa_{l_2} = \sum_{m=1}^{\min\{n,\tau_l\}} \rho_l(\theta_l^\star, \hat{\theta}_{l,n}) \tag{2.152}$$

$$\kappa_{l_3} = \sum_{m=1}^{\min\{n,\tau_l\}} [\log \frac{g_l(X_{l,m}|\hat{\theta}_{l,m-1})}{g_l(X_{l,m}|\theta_l^\star)} + \rho_l(\theta_l^\star, \hat{\theta}_{l,m-1})] \tag{2.153}$$

$$\kappa_{l_4} = \sum_{m=1}^{\min\{n,\tau_l\}} \rho_l(\theta_l^\star, \hat{\theta}_{l,m-1}) \tag{2.154}$$

Since $\mathbb{E}_{g_l(\cdot|\theta_l^\star)}[\log \frac{g_l(\cdot|\theta_l)}{g_l(\cdot|\theta_l^\star)} + \rho_l(\theta_l^\star, \theta_l)] = 0$ for any θ_l; thus, κ_{l_1} and κ_{l_3} are zero-sum independent incremental martingales. By Condition B4 that $g_l(\cdot|\theta_l) > 0$ for all θ_l, we have $|\log \frac{g_l(X_{l,m}|\theta_l)}{g_l(X_{l,m}|\theta_l^\star)}| < \infty$ uniformly for all X_l and θ_l. By Lemma 2.6.1, we have

$$\mathbb{E}_{g_l(\cdot|\theta_l^\star)}[|\kappa_{l_1}|] < constant \cdot n^{1/2} \tag{2.155}$$

$$\mathbb{E}_{g_l(\cdot|\theta_l^\star)}[|\kappa_{l_3}|] < constant \cdot n^{1/2} \tag{2.156}$$

Thus, $\frac{\kappa_{l_1}}{n}$ and $\frac{\kappa_{l_3}}{n}$ converges almost surely to 0.

By Condition B6, $\mathbb{E}_{g_l(\cdot|\theta_l^\star)}[\sum_{m=1}^{\infty} \rho_l(\theta_l^\star, \hat{\theta}_{l,m})] < \infty$, we have $\frac{\kappa_{l_2}}{n}$ and $\frac{\kappa_{l_4}}{n}$ converges almost surely to 0, as well.

Thus, $\frac{\kappa_l}{n} = \frac{1}{n}[\kappa_{l_1} - \kappa_{l_2} - \kappa_{l_3} + \kappa_{l_4}]$ converges almost surely to 0.

Next, we consider the expression 2.145. Let us represent the joint density of systems i and j as

$$f_{ij}(X_i, X_j|\boldsymbol{\theta}_{ij}) = g_i(X_i|\theta_i) \cdot g_j(X_j|\theta_j) \tag{2.157}$$

where $\boldsymbol{\theta}_{ij} = [\theta_i, \theta_j] = [[\mu_i, \sigma_i^2], [\mu_j, \sigma_j^2]]$.

Denote the Kullback Leibler divergence as

$$\rho_{ij}(\boldsymbol{\theta}^{\star}_{ij}, \boldsymbol{\theta}_{ij}) = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^{\star}_{ij})}[\log \frac{f_{ij}(\cdot|\boldsymbol{\theta}^{\star}_{ij})}{f_{ij}(\cdot|\boldsymbol{\theta}_{ij})}] \tag{2.158}$$

where $\boldsymbol{\theta}^{\star}_{ij} = [\theta^{\star}_i, \theta^{\star}_j]$ is the true parameters of systems i and j.

Denote the expression 2.145 as

$$\kappa_{ij} = \sup_{\mu_i > \mu_j} \sum_{m=1}^{n} \sum_{l \in \{i,j\}} \log g_l(X_{l,m}|\theta_l) - \sum_{m=1}^{n} \sum_{l \in \{i,j\}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) \tag{2.159}$$

$$= \sup_{\mu_i > \mu_j} \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij}) - \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\theta}}_{ij,m-1}) \tag{2.160}$$

where $\hat{\boldsymbol{\theta}}_{ij,m-1}$ is the MLE of $\boldsymbol{\theta}_{ij}$ using the first $m-1$ observations on the system i and j.

κ_{ij} can be expanded as

$$\kappa_{ij} = \sup_{\boldsymbol{\theta}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij}) - \sum_{m=1}^{n} \log f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\theta}}_{ij,m-1}) \tag{2.161}$$

$$= \sup_{\boldsymbol{\theta}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})} - \sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\theta}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})} \tag{2.162}$$

$$= \sup_{\boldsymbol{\theta}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})}] + \sum_{m=1}^{n} \rho_{ij}(\boldsymbol{\theta}^{\star}_{ij}, \hat{\boldsymbol{\theta}}_{ij,m-1}) \tag{2.163}$$

$$- \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\theta}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})} + \rho_{ij}(\boldsymbol{\theta}^{\star}_{ij}, \hat{\boldsymbol{\theta}}_{ij,m-1})] \tag{2.164}$$

Denote

$$\kappa_{ij_1} = \sup_{\boldsymbol{\theta}_{ij} \in \{\mu_i > \mu_j\}} \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})}] \tag{2.165}$$

$$\kappa_{ij_2} = \sum_{m=1}^{n} \rho_{ij}(\boldsymbol{\theta}^{\star}_{ij}, \hat{\boldsymbol{\theta}}_{ij,m-1}) \tag{2.166}$$

$$\kappa_{ij_3} = \sum_{m=1}^{n} [\log \frac{f_{ij}(X_{i,m}, X_{j,m}|\hat{\boldsymbol{\theta}}_{ij,m-1})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^{\star}_{ij})} + \rho_{ij}(\boldsymbol{\theta}^{\star}_{ij}, \hat{\boldsymbol{\theta}}_{ij,m-1})] \tag{2.167}$$

By the same logic for the almost sure convergences of κ_{j_3}/n and κ_{j_4}/n, it is easy to verify that κ_{ij_2}/n and κ_{ij_3}/n converges almost surely to 0.

Given Condition B3 that the parameter space is compact, Condition B4 that $g_i(X_i|\theta_i)$ is continuous and $g_i(X_i|\theta_i) > 0$ for all X_i and θ_i, we can apply the Uniform Law of Large Number to show that, almost surely,

$$\sup_{\boldsymbol{\theta}_{ij}\in\{\mu_i>\mu_j\}} \left| \frac{1}{n}\sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^\star_{ij})} + \rho_{ij}(\boldsymbol{\theta}^\star_{ij}, \boldsymbol{\theta}_{ij}) \right| \to 0 \tag{2.168}$$

$$\frac{\kappa_{ij_1}}{n} = \sup_{\boldsymbol{\theta}_{ij}\in\{\mu_i>\mu_j\}} \frac{1}{n}\sum_{m=1}^{n} \log \frac{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}_{ij})}{f_{ij}(X_{i,m}, X_{j,m}|\boldsymbol{\theta}^\star_{ij})} \to - \inf_{\boldsymbol{\theta}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\theta}^\star_{ij}, \boldsymbol{\theta}_{ij}) \tag{2.169}$$

Therefore, $\frac{\kappa_{ij}}{n} = \frac{1}{n}[\kappa_{ij_1} + \kappa_{ij_2} - \kappa_{ij_3}] \to -\inf_{\boldsymbol{\theta}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\theta}^\star_{ij}, \boldsymbol{\theta}_{ij})$ almost surely. Given the fact that $-\inf_{\boldsymbol{\theta}_{ij}\in\{\mu_i>\mu_j\}} \rho_{ij}(\boldsymbol{\theta}^\star_{ij}, \boldsymbol{\theta}_{ij}) = -\inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta})$, we have

$$\frac{\lambda^{(2)}_{i,j,n}}{n} = \frac{1}{n}[\sum_{l\in\Omega/\{i,j\}} \kappa_l + \kappa_{ij}] \to - \inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta}) \tag{2.170}$$

Thus, we conclude the proof.

■

Theorem 2.6.5 *Under the Conditions B1-B6 and the settings of Procedure 1 and Procedure 2, assuming that the systems i and j, with $\mu_i^\star < \mu_j^\star$, are exempt from the elimination step, then there exists a number N and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that*

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda^{(2)}_{i,j,n})^s] \leq constant \cdot e^{-\epsilon n} \tag{2.171}$$

for all $n > N$.

Remark 2.6.3 *The proof follows the exactly same logic as in the proof of Theorem 2.6.2, except that the parameter set is expanded from $\boldsymbol{\mu}$ to $\boldsymbol{\theta}$.*

Proof Consider the log of the GLR statistic $\lambda^{(2)}_{i,j,n}$ as

$$\lambda^{(2)}_{i,j,n} \triangleq \log \Lambda^{(2)}_{i,j,n} = \sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\theta}) - \log \pi_n \tag{2.172}$$

By Theorem 2.6.4, we have $\frac{\lambda^{(2)}_{i,j,n}}{n} \to -\inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta})$. By Condition B5, $\rho_i(\theta_i^\star, \theta_i) > 0$ for any $\theta \neq \theta^\star$ and $i \in \Omega$, so there exists a constant $c_1 > 0$ such that $\inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta}) = \inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \sum_{l=1}^{K} \rho_l(\theta_l^\star, \theta_l) \geq c_1$.

Let $\hat{\boldsymbol{\theta}}_n = [\hat{\theta}_{1,n}, ..., \hat{\theta}_{K,n}]$ and $\tilde{\boldsymbol{\theta}}_n = [\tilde{\theta}_{1,n}, ..., \tilde{\theta}_{K,n}] = \arg\sup_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \log L_n(\boldsymbol{\theta})$ be the unconstrained and constrained MLEs of $\boldsymbol{\theta}^\star$ at the nth round, respectively. $\lambda^{(2)}_{i,j,n}$ can be stated as a summation of a sequence $\{h_{m,n},\ m = 1, ..., n\}$ such that

$$\lambda^{(2)}_{i,j,n} = \sum_{m=1}^{n} \left[\sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\tilde{\theta}_{l,n}) - \sum_{l\in\mathcal{S}_{cand,m}} \log g_l(X_{l,m}|\hat{\theta}_{l,m-1}) \right] = \sum_{m=1}^{n} h_{m,n} \tag{2.173}$$

It can be verified that $\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[h_{m,n}] = \sum_{l\in\mathcal{S}_{cand,m}} \rho_l(\theta^\star_l, \hat{\theta}_{l,m-1}) - \rho_l(\theta^\star_l, \tilde{\theta}_{l,n})$.

By Condition B6, $\mathbb{E}_{g_l(\cdot|\theta^\star_l)}[\sum_{m=1}^{\infty} \rho_l(\theta^\star_l, \hat{\theta}_{l,m})] < \infty$, and the fact that $\rho_l(\theta^\star_l, \hat{\theta}_{l,m})$ are bounded and non-negative, we have the almost sure convergence of $\rho_l(\theta^\star_l, \hat{\theta}_{l,m}) \to 0$ as $m, n \to \infty$. Therefore, there exists a number N such that $\rho_l(\theta^\star_l, \hat{\theta}_{l,m}) \leq \frac{c_1}{2K}$ for all $m \geq N$ and l.

Thus, for any n and m with $N < m < n$, we have

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[h_{m,n}] = \sum_{l\in\mathcal{S}_{cand,m}} \rho_l(\theta^\star_l, \hat{\theta}_{l,m-1}) - \rho_l(\theta^\star_l, \tilde{\theta}_{l,n}) \tag{2.174}$$

$$\leq \frac{c_1}{2} - \inf_{\boldsymbol{\theta}\in\mathbb{U}_{i,j}} \sum_{l=1}^{K} \rho_l(\theta^\star_l, \theta_l) \leq -\frac{c_1}{2} \tag{2.175}$$

The first inequality holds since $\tilde{\theta}_n \in \mathbb{U}_{i,j}$.

Consider the function

$$\gamma(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[e^{s\cdot h_{m,n}}] \tag{2.176}$$

To find an upper bound of $\gamma(s)$, we firstly show that $\gamma(s)$ is *twice differentiable* and *bounded*.

To verity that $\gamma(s)$ is twice differentiable with respect to s within $[0, S]$, it is sufficient to show $\mathbb{E}_{\boldsymbol{\theta}^\star}[h^2_{m,n} e^{s\cdot h_{m,n}}] < \infty$ for $s \in [0, S]$. Let us denote $f_{\boldsymbol{\theta}}(\cdot) \triangleq \prod_{i=1}^{K} g_i(X_i|\theta_i)$.

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[h^2_{m,n} e^{s\cdot h_{m,n}}] = \int h^2_{m,n} e^{s\cdot h_{m,n}} f(\boldsymbol{X}|\boldsymbol{\theta}^\star) d\boldsymbol{X} \tag{2.177}$$

$$\leq \left[\int h^4_{m,n} f(\boldsymbol{X}|\boldsymbol{\theta}^\star) d\boldsymbol{X} \right]^{1/2} \left[\int e^{2\cdot s\cdot h_{m,n}} f(\boldsymbol{X}|\boldsymbol{\theta}^\star) d\boldsymbol{X} \right]^{1/2} \tag{2.178}$$

Given Conditions B3, $f(\cdot|\boldsymbol{\theta})$ is continuous and positive for all $\boldsymbol{\theta} \in \mathbb{U}$, thus, the first term $\int h^4_{m,n} f_{\boldsymbol{\theta}^\star}(\boldsymbol{x}) d\boldsymbol{x}$ is bounded above; the second term is less than 1 for any $s \in [0, 1/2]$ by the definition of $h_{m,n}$. Therefore, there exists a constant c_2 such that

$$\gamma''(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[h_{m,n}^2 e^{s \cdot h_{m,n}}] \leq c_2 < \infty \tag{2.179}$$

The first order directive of $\gamma(s)$ is $\gamma'(s) = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[h_{m,n} e^{s \cdot h_{m,n}}]$ with $\gamma'(0) \leq -\frac{c_1}{2}$.

Therefore, $\gamma(s) < 1 - \frac{c_1}{2}s + \frac{c_2}{2}s^2$ for all $s \in [0, 1/2]$. Thus ,there exists a pair of $(s^\star, \epsilon)$ with $s^\star \in [0, 1/2]$ and $\epsilon > 0$ such that $\gamma(s^\star) \leq e^{-\epsilon} < 1$.

Thus, there exists a large number N, such that, for any $n > N$,

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[(\Lambda_{i,j,n}^{(2)})^{s^\star}] = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[e^{s^\star \sum_{m=1}^{n} \cdot h_{m,n}}] \leq \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[e^{s^\star \sum_{m=1}^{N} \cdot h_{m,n}}] \cdot e^{-\epsilon(n-N)} \leq c \cdot e^{-\epsilon \cdot n} \tag{2.180}$$

where c is a constant.

Thus, we conclude the proof.

■

2.6.2 Proof of Theorem 2.2.3

Proof Under the normality assumption with unequal known variance σ^2, $L_n(\boldsymbol{\mu})$ and $g_j(X_{j,m}|\mu_j)$ are defined by

$$L_n(\boldsymbol{\mu}) = \prod_{m=1}^{n} \prod_{j \in \mathcal{S}_{cand,m}} g_j(X_{j,m}|\mu_j) \tag{2.181}$$

$$g_j(X_{j,m}|\mu_j) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\{-\frac{(X_{j,m} - \mu_j)^2}{2\sigma_i^2}\} \tag{2.182}$$

where $\mathcal{S}_{cand,n}$ contains systems that have not been eliminated by the nth iteration.

Given the definition of $\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i}$, we have

$$\hat{\boldsymbol{\mu}}_n^{\mathbb{U}_i} = \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_n(\boldsymbol{\mu}) \tag{2.183}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \prod_{m=1}^{n} \prod_{j\in\mathcal{S}_{cand,m}} g_j(X_{j,m}|\mu_j) \tag{2.184}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \prod_{m=1}^{n} \prod_{j\in\mathcal{S}_{cand,m}} \exp\{-\frac{(X_{j,m}-\mu_j)^2}{2\sigma_i^2}\} \tag{2.185}$$

$$= \arg\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} \sum_{m=1}^{n} \sum_{j\in\mathcal{S}_{cand,m}} \big(-\frac{(X_{j,m}-\mu_j)^2}{2\sigma_i^2}\big) \tag{2.186}$$

This optimization problem is solvable by using Lagrangian method. Denote $\boldsymbol{\lambda} = [\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ...\lambda_K]$ as the Lagrangian multipliers. We have the following KKT conditions.

$$\frac{1}{\sigma_i^2}(\mu_i - \hat{\mu}_{i,n}) - \sum_{j\neq i}\lambda_j = 0 \tag{2.187}$$

$$\frac{1}{\sigma_i^2}(\mu_j - \hat{\mu}_{j,n}) + \lambda_j = 0 \tag{2.188}$$

$$\lambda_j \cdot (\mu_j - \mu_i + \delta) = 0, \ \ \forall j \neq i \tag{2.189}$$

$$\mu_i - \mu_j - \delta \geq 0, \ \ \forall j \neq i \tag{2.190}$$

$$\lambda_j \geq 0 \tag{2.191}$$

It is straightforward to verify that Equation 2.24 satisfies the above conditions.

■

2.6.3 Proof of Theorem 2.2.4

Proof We will prove that $P(\text{Procedure } 1/2 \text{ terminates after } N \text{ iterations}) \to 0$ as $N \to \infty$.

The proposed procedures terminate when the cardinality of $\mathcal{S}_{cand,N+1}$ is 1. Thus, we can write the probability that the procedure terminates after N iterations as

$$P(\text{Procedure } 1/2 \text{ terminates after } N \text{ iterations}) \tag{2.192}$$

$$= P(|\mathcal{S}_{cand,N+1}| \geq 2) \tag{2.193}$$

$$= P(\cup_{i,j\in\Omega}[(i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})]) \tag{2.194}$$

$$\leq \sum_{i,j\in\Omega} P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.195}$$

To prove that $P(\text{Procedure } 1/2 \text{ terminates after } N \text{ iterations}) \to 0$ as $N \to \infty$, it is sufficient to show that $P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \to 0$ for any $i, j \in \Omega$.

Without loss of generality, let us assume $\mu_i^\star < \mu_j^\star$, i.e., system j is better than system i.

By Bayes rule,

$$P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) = P(j \in \mathcal{S}_{cand,N+1})) \times P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) \tag{2.196}$$

For Procedure 1, $P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1})$ is bounded above by

$$P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) = P(\min_{n=1,\ldots,N} \Lambda_{i,n}^{(1)} > \alpha | j \in \mathcal{S}_{cand,N+1}) \leq P(\Lambda_{i,N}^{(1)} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.197}$$

The last inequality holds since $\mathbb{U}_i = \{\boldsymbol{\mu} | \mu_i > \mu_l \forall l \neq i\} \subset \mathbb{U}_{i,j} = \{\boldsymbol{\mu} | \mu_i > \mu_j\}$ and thus

$$\Lambda_{i,N}^{(1)} = \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_i} L_N(\boldsymbol{\mu}|\cdot)}{\pi_N} \leq \frac{\sup_{\boldsymbol{\mu}\in\mathbb{U}_{i,j}} L_N(\boldsymbol{\mu}|\cdot)}{\pi_N} = \Lambda_{i,j,N}^{(2)} \tag{2.198}$$

For Procedure 2, $P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1})$ is bounded above by

$$P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) = P(\min_{n=1,\ldots,N} \Lambda_{i,\cdot,n}^{(2)} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.199}$$

$$\leq P(\Lambda_N^{i,\cdot} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.200}$$

$$\leq P(\Lambda_N^{i,j} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.201}$$

The last inequality holds since, by the definition of $\Lambda_N^{i,\cdot}$, $\Lambda_N^{i,\cdot} = \min_{l\neq i} \Lambda_N^{i,l} \leq \Lambda_{i,j,N}^{(2)}$.

Summarizing the above two inequalities, we have

$$P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.202}$$

$$= P(j \in \mathcal{S}_{cand,N+1})) \times P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) \tag{2.203}$$

$$\leq P(j \in \mathcal{S}_{cand,N+1})) \times P(\Lambda^{(2)}_{i,j,N} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.204}$$

$$\leq P(1_{\Lambda^{(2)}_{i,j,N} > \alpha} \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.205}$$

$$\leq P(\Lambda^{(2)}_{i,j,N} > \alpha) \tag{2.206}$$

Thus, to prove that both procedures terminate in finite time is necessary to show that $P(\Lambda^{i,j}_N > \alpha) \to 0$.

Apply Chebyshev inequality to $P(\Lambda^{(2)}_{i,j,N} > \alpha)$

$$P(\Lambda^{(2)}_{i,j,N} > \alpha) \leq \frac{\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda^{(2)}_{i,j,N})^s]}{\alpha^s}$$

for any real number $s > 0$.

By Theorem 2.6.3, there exists a number n and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda^{(2)}_{i,j,N})^s] \leq constant \cdot e^{-\epsilon N} \tag{2.207}$$

for all $N > n$.

Therefore, we have $P(\Lambda^{(2)}_{i,j,N} > \alpha) \to 0$ as $N \to \infty$. Thus, we conclude the proof.

■

2.6.4 Proof of Theorem 2.2.6

Proof Recall that $\tau^{(1)}_i = \min\{n | \Lambda^{(1)}_{i,n} < \alpha\}$ and $\tau^{(2)}_i = \min\{n | \Lambda^{(2)}_{i,\cdot,n} < \alpha\}$. And by definition, $\Lambda^{(1)}_{i,n} < \Lambda^{(2)}_{i,\cdot,n}$ for all n and i. Thus, we have

$$P(\tau^{(1)}_i > N) = P(\min_{n=1,\ldots,N} \Lambda^{(1)}_{i,n} > \alpha) \leq P(\min_{n=1,\ldots,N} \Lambda^{(2)}_{i,\cdot,n} > \alpha) = P(\tau^{(2)}_i > N), \forall N > 0 \tag{2.208}$$

Therefore, $\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[\tau^{(1)}_i] \leq \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}[\tau^{(2)}_i]$.

Recall that we defined $\Lambda^{(2)}_{i,\cdot,n}$ as $\Lambda^{(2)}_{i,\cdot,n} = \min_{j\in\mathcal{S}_{cand,n}} \Lambda^{(2)}_{i,j,n}$, therefore, we have $\Lambda^{(2)}_{i,\cdot,n} \leq \Lambda^{(2)}_{i,1,n}$. Thus, $\tau^{(2)}_i = \min\{n|\Lambda^{(2)}_{i,\cdot,n} < \alpha\} \leq \min\{n|\Lambda^{(2)}_{i,1,n} < \alpha\}$. Therefore, it is sufficient to derive asymptotic properties of $\tau^{(2)}_i$ using $\min\{n|\Lambda^{(2)}_{i,1,n} < \alpha\}$. Another reason that we want to use $\Lambda^{(2)}_{i,1,n}$ is that, by Theorem 2.2.5, as $\alpha \to 0$, system 1 is not eliminated w.p.1.

Let us denote $A = \log \alpha^{-1}$. Assume a large constant $M > 0$ and split the expected elimination time

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\tau^{(2)}_i / A\right] = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\tau^{(2)}_i / A \cdot 1_{\{\tau^{(2)}_i / A \leq M\}}\right] + \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\tau^{(2)}_i / A \cdot 1_{\{\tau^{(2)}_i / A > M\}}\right] \tag{2.209}$$

To prove the asymptotic limit of $\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\tau^{(2)}_i / A\right]$, we will show that (1) the first term is approximated by $\frac{1}{\inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,1}} \rho(\boldsymbol{\mu}^\star,\boldsymbol{\mu})}$ as $\alpha \to 0$ (or, equivalently, $A \to \infty$) and (2) the second term approaches to $o(1)$.

To begin with, we examine the first term. Consider the limiting process of the log of the GLR statistic $\lambda^{(2)}_{i,1,n} = \log \Lambda^{(2)}_{i,1,n}$.

By Theorem 2.6.2, $\lambda^{(2)}_{i,1,n}/n \to -\inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,1}} \rho(\boldsymbol{\mu}^\star, \boldsymbol{\mu})$ almost surely. Given that $\frac{\tau^{(2)}_i}{A} \leq M$, there exists $m < M$ such that $\lfloor A \cdot m \rfloor = \tau^{(2)}_i$. Thus, the first term in the Equation 2.209 converges to a limit

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\frac{\tau^{(2)}_i}{A} \cdot 1_{\{\tau^{(2)}_i / A \leq M\}}\right] \approx \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\frac{A \cdot m}{-\lambda^{(2)}_{i,1,\lfloor A\cdot m\rfloor}}\right] \to \frac{1}{\inf_{\boldsymbol{\mu}\in\mathbb{U}_{i,1}} \rho(\boldsymbol{\mu}^\star,\boldsymbol{\mu})} \text{ as } A \to \infty \tag{2.210}$$

Next we verify the second term in Equation 2.209 is $o(1)$.

Recall that

$$P(\tau^{(2)}_i > n) \leq P(\Lambda^{(2)}_{i,1,n} > \alpha) = \mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}\left[\frac{(\Lambda^{(2)}_{i,1,n})^s}{(\Lambda^{(2)}_{i,1,n})^s} 1_{\Lambda^{(2)}_{i,1,n} > \alpha}\right] \leq \alpha^{-s}\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda^{(2)}_{i,1,n})^s] \tag{2.211}$$

for any real number $s > 0$.

By Theorem 2.6.3, there exist a number N, a positive constant c, and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda^{(2)}_{i,1,n})^s] \leq c \cdot e^{-\epsilon\cdot n} \tag{2.212}$$

for all $n > N$. Hence,

$$P(\tau_i^{(2)} > n) \leq c \cdot e^{s \cdot A - \epsilon \cdot n} \tag{2.213}$$

For a sufficient large M, there exists an ϵ' with $0 < \epsilon' = \epsilon - \frac{s}{M} < \epsilon - \frac{s}{n}$ for all $n > M$; then we have

$$P(\tau_i^{(2)} > A \cdot n) \leq c \cdot e^{s \cdot A - \epsilon \cdot A \cdot n} = c \cdot e^{-(\epsilon - \frac{s}{n}) A \cdot n} \leq c \cdot e^{-\epsilon' \cdot A \cdot n} \text{ for } n > M \tag{2.214}$$

It states that, with a sufficient large M, the tail probability of $\tau_i^{(2)} > A \cdot n$ decays exponentially for $n > M$. Thus, we have

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[\tau_i^{(2)}/A \cdot 1_{\{\tau_i^{(2)}/A \geq M\}}] = \int_M^\infty n \cdot P(\frac{\tau_i^{(2)}}{A} > n) dn \leq \int_M^\infty n \cdot c \cdot e^{-\epsilon' \cdot A \cdot n} dn \to o(1) \tag{2.215}$$

as $A \to \infty$.

Combining Equations 2.210 and 2.215 together, we conclude the proof.

■

2.6.5 Proof of Theorem 2.3.1

Remark 2.6.4 *The proof for Procedure 1 can be easily verified since $\Lambda_{i,n}^{(1)} \leq \Lambda_{i,\cdot,n}^{(2)} \leq \Lambda_{i,j,n}^{(2)}$ for all $j \neq i$. If Procedure 2 terminates in finite time, then Procedure 1 should terminate in finite time.*

Proof We will prove that $P(\text{Procedure 2 terminates after } N \text{ iterations}) \to 0$ as $N \to \infty$.

The procedure terminates when the cardinality of $\mathcal{S}_{cand,N+1}$ is 1. Thus, we can write the probability that the procedure terminates after time N as

$$P(\text{Procedure 2 terminates after } N \text{ iterations}) \tag{2.216}$$

$$= P(|\mathcal{S}_{cand,N+1}| \geq 2) \tag{2.217}$$

$$= P(\cup_{i,j \in \Omega}[(i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})]) \tag{2.218}$$

$$\leq \sum_{i,j \in \Omega} P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.219}$$

To prove that $P(\text{Procedure 2 terminates after } N \text{ iterations}) \to 0$ as $N \to \infty$, it is sufficient to show that $P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \to 0$ for any $i, j \in \Omega$.

Without loss of generality, let us assume $\mu_i^\star < \mu_j^\star$, i.e., system j is better than system i.

By Bayes rule,

$$P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) = P(j \in \mathcal{S}_{cand,N+1})) \times P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) \tag{2.220}$$

$P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1})$ is bounded above by

$$P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) = P(\min_{n=1,\dots,N} \Lambda_{i,\cdot,n}^{(2)} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.221}$$

$$\leq P(\Lambda_N^{i,\cdot} > \alpha | j \in \mathcal{S}_{cand,N+1}) \leq P(\Lambda_N^{i,j} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.222}$$

The last inequality holds since, by definition of $\Lambda_N^{i,\cdot}$, $\Lambda_N^{i,\cdot} = \min_{l \neq i} \Lambda_N^{i,l} \leq \Lambda_{i,j,N}^{(2)}$.

Thus,

$$P((i \in \mathcal{S}_{cand,N+1}) \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.223}$$

$$= P(j \in \mathcal{S}_{cand,N+1})) \times P(i \in \mathcal{S}_{cand,N+1} | j \in \mathcal{S}_{cand,N+1}) \tag{2.224}$$

$$\leq P(j \in \mathcal{S}_{cand,N+1})) \times P(\Lambda_{i,j,N}^{(2)} > \alpha | j \in \mathcal{S}_{cand,N+1}) \tag{2.225}$$

$$\leq P(1_{\Lambda_{i,j,N}^{(2)} > \alpha} \cap (j \in \mathcal{S}_{cand,N+1})) \tag{2.226}$$

$$\leq P(\Lambda_{i,j,N}^{(2)} > \alpha) \tag{2.227}$$

Apply Chebyshev inequality to $P(\Lambda_{i,j,N}^{(2)} > \alpha)$

$$P(\Lambda_{i,j,N}^{(2)} > \alpha) \leq \frac{\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda_{i,j,N}^{(2)})^s]}{\alpha^s}$$

for any real number $s > 0$.

By Theorem 2.6.5, there exists a number n and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that

$$\mathbb{E}_{f(\cdot|\boldsymbol{\mu}^\star)}[(\Lambda_{i,j,N}^{(2)})^s] \leq constant \cdot e^{-\epsilon N} \tag{2.228}$$

for all $N > n$.

Therefore, we have $P(\Lambda^{(2)}_{i,j,N} > \alpha) \to 0$ as $N \to \infty$. Thus, we conclude the proof.

■

2.6.6 Proof of Theorem 2.3.4

Proof The second half of the proof uses the Theorem 5.4.3 in [29] on MASPRT asymptotic optimality.

Recall that we defined $\Lambda^{(2)}_{i,\cdot,n}$ as $\Lambda^{(2)}_{i,\cdot,n} = \min_{j\in\mathcal{S}_{cand,n}} \Lambda^{(2)}_{i,j,n}$, therefore, we have $\Lambda^{(2)}_{i,\cdot,n} \leq \Lambda^{(2)}_{i,1,n}$ and $\tau^{(2)}_i = \min\{n|\Lambda^{(2)}_{i,\cdot,n} < \alpha\} \leq \min\{n|\Lambda^{(2)}_{i,1,n} < \alpha\}$. Thus, it is sufficient to derive asymptotic properties of $\tau^{(2)}_i$ using $\min\{n|\Lambda^{(2)}_{i,1,n} < \alpha\}$. Another reason that we want to use $\Lambda^{(2)}_{i,1,n}$ is that, by Theorem 2.3.3, as $\alpha \to 0$, system 1 is not eliminated w.p.1.

Let us denote $A = \log \alpha^{-1}$. Assume a large constant $M > 0$ and split the expected elimination time

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\tau^{(2)}_i / A\right] = \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\tau^{(2)}_i / A \cdot 1_{\{\tau^{(2)}_i / A \leq M\}}\right] + \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\tau^{(2)}_i / A \cdot 1_{\{\tau^{(2)}_i / A > M\}}\right] \tag{2.229}$$

To prove the asymptotic limit of $\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\tau^{(2)}_i / A\right]$, we will show that (1) the first term is approximated by $\frac{1}{\inf_{\boldsymbol{\theta}\in\Theta_{i,1}} \rho(\boldsymbol{\theta}^\star,\boldsymbol{\theta})}$ as $\alpha \to 0$ (or, equivalently, $A \to \infty$) and (2) the second term approaches to $o(1)$.

To begin with, we examine the first term. Consider the limiting process of the log of the GLR statistic $\lambda^{(2)}_{i,1,n} = \log \Lambda^{(2)}_{i,1,n}$.

By Theorem 2.6.4, $\lambda^{(2)}_{i,1,n}/n \to -\inf_{\boldsymbol{\theta}\in\Theta_{i,1}} \rho(\boldsymbol{\theta}^\star, \boldsymbol{\theta})$ almost surely. Given that $\frac{\tau^{(2)}_i}{A} \leq M$, there exists $m < M$ such that $\lfloor A \cdot m \rfloor = \tau^{(2)}_i$. Thus, the first term in the Equation 2.229 converges to a limit

$$\mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\frac{\tau^{(2)}_i}{A} \cdot 1_{\{\tau^{(2)}_i / A \leq M\}}\right] \approx \mathbb{E}_{f(\cdot|\boldsymbol{\theta}^\star)}\left[\frac{A \cdot m}{-\lambda^{(2)}_{i,1,\lfloor A\cdot m\rfloor}}\right] \to \frac{1}{\inf_{\boldsymbol{\theta}\in\Theta_{i,1}} \rho(\boldsymbol{\theta}^\star,\boldsymbol{\theta})} \text{ as } A \to \infty \tag{2.230}$$

Next we verify the second term in Equation 2.229 is $o(1)$.

Recall that

$$P(\tau_i^{(2)} > n) \leq P(\Lambda_{i,1,n}^{(2)} > \alpha) = \mathbb{E}_{f(\cdot|\theta^\star)}\left[\frac{(\Lambda_{i,1,n}^{(2)})^s}{(\Lambda_{i,1,n}^{(2)})^s} 1_{\Lambda_{i,1,n}^{(2)} > \alpha}\right] \leq \alpha^{-s}\mathbb{E}_{f(\cdot|\theta^\star)}[(\Lambda_{i,1,n}^{(2)})^s] \tag{2.231}$$

for any real number $s > 0$.

By Theorem 2.6.5, there exist a number N, a positive constant c, and a pair of (s, ϵ) with $s \in [0, 1/2]$ and $\epsilon > 0$, such that

$$\mathbb{E}_{f(\cdot|\theta^\star)}[(\Lambda_{i,1,n}^{(2)})^s] \leq c \cdot e^{-\epsilon \cdot n} \tag{2.232}$$

for all $n > N$. Hence,

$$P(\tau_i^{(2)} > n) \leq c \cdot e^{s \cdot A - \epsilon \cdot n} \tag{2.233}$$

For a sufficient large M, there exists an ϵ' with $0 < \epsilon' = \epsilon - \frac{s}{M} < \epsilon - \frac{s}{n}$ for all $n > M$; then we have

$$P(\tau_i^{(2)} > A \cdot n) \leq c \cdot e^{s \cdot A - \epsilon \cdot A \cdot n} = c \cdot e^{-(\epsilon - \frac{s}{n}) A \cdot n} \leq c \cdot e^{-\epsilon' \cdot A \cdot n} \text{ for } n > M \tag{2.234}$$

It states that, with a sufficient large M, the tail probability of $\tau_i^{(2)} > A \cdot n$ decays exponentially for $n > M$. Thus, we have

$$\mathbb{E}_{f(\cdot|\theta^\star)}[\tau_i^{(2)}/A \cdot 1_{\{\tau_i^{(2)}/A \geq M\}}] = \int_M^\infty n \cdot P(\frac{\tau_i^{(2)}}{A} > n) dn \leq \int_M^\infty n \cdot c \cdot e^{-\epsilon' \cdot A \cdot n} dn \to o(1) \tag{2.235}$$

as $A \to \infty$.

Combining Equations 2.230 and 2.235 together, we conclude the proof.

■

3. BONFERRONI-BASED PROCEDURES FOR MULTI-OBJECTIVE RANKING AND SELECTION

3.1 Introduction

The Ranking and Selection (R&S) problems are to select the best system among a finite set of alternatives. In terms of single-objective R&S problem, the best system refers to the system with the largest expected performance. In the past few decades, there is a substantial body of literature on single-objective R&S procedures. See [7–9] for thorough overviews on single-objective R&S procedures.

A less studied R&S problem is in the presence of multiple performance measures. We call this problem the multi-objective Ranking and Selection (MOR&S). Existing procedures fall into two categories: (a) the fixed-budget approaches that efficiently allocate a finite computing budget among alternatives in order to optimize an objective, i.e., the posterior probability of correct selection (*PCS*); and (b) the fixed-precision approaches that attain a target precision by using a random total number of simulation budget.

The fixed-budget approaches include the multi-objective optimal computing budget allocation (MOCBA) [31, 32], which proposes a computing allocation procedure for MOR&S under the assumption of normally distributed outputs and independent objectives; optimal sampling laws proposed by [33] that allocates sampling budget accounting for dependence between the objective vector estimates for a single system in the context of bi-objective; myopic strategy, M-MOBA [34, 35], that targets to minimize the difference in hyper-volume between the observed means of the perceived Pareto front and the Oracle Pareto front; SCORE for multi-objective Ranking and Selection proposed in [36, 37] that allocates simulation bud-

get among suboptimal systems according to their corresponding squared standardized optimality gaps.

As to the researches on the fixed-precision procedures, Batur and Choobineh [38] present a fixed-precision procedure designed for bi-objective R&S where the objectives are the mean and variance of the system performances. [39] provides three general fixed-precision MOR&S procedures based on an indifference-zone formulation, where the indifference zone on one objective is the smallest difference in system performances that a decision-maker cares to detect. Interesting readers may refer to [40] for an overview of the existing MOR&S literature.

Same as other researches on multi-objective optimization, we consider the "best" systems characterized by the *Pareto front*. Loosely speaking, systems within the Pareto front are better than any other systems in at least one objective. In the context of multi-objectives, there might be more than one "best" systems since comparisons can be made along many objectives. In this thesis, we aim to design procedures that achieve a target *PCS*, namely, the probability of correctly selecting the exact *Pareto front*.

Procedures that control *PCS* for the single-objective R&S problems are well-studied, i.e., KN procedure [13], KN++ procedure [14], BIZ procedure [16], and IZ-free procedure [17, 18]. These procedures and many others conduct pairwise-comparison between a system and the perceived best system in order to test whether a system is inferior. This strategy fails in the context of multi-objective since there may be more than one best system in the Pareto front.

To attain a target *PCS* for MOR&S problems, we directly test whether a system is in the Pareto front or not. We call such a test the *candidacy test* and group the *candidacy tests* together to form our procedures. For a *candidacy test*, we approach it using the generalized sequential probability ratio test (GSPRT) procedure [27, 41]. In essence, the GSPRT procedure conducts sequential sampling, and calculates the generalized likelihood ratio (GLR) statistics. Moreover, it constructs the threshold on the GLR statistics in order to control the level of error probabilities. For imple-

mentation, we give the closed-form solution of the GLR statistic under normality assumption. The proposed procedures have the following properties,

1. **An indifference-zone-free formulation**. The new formulation eliminates the necessity of specifying the indifference-zone parameter.

2. **A GLR-based procedure for MOR&S problems**. We propose to test directly on whether an alternative is the best, i.e., in the Pareto front. We call such a test the *candidacy test* and approach it using the generalized sequential probability ratio test (GSPRT).

The rest of this chapter is organized as follows. In Section 3.2, we introduce the problem statement on a Multi-Objective Ranking and Selection Problem under the normality assumption. Section 3.3 presents the algorithms and properties of the proposed procedures. Section 3.4 gives numerical evaluations.

3.2 The Problem Statement

Multi-Objective Ranking and Selection (MOR&S) problems are the natural extensions of single-objective Ranking and Selection problems. In this section, we consider a MOR&S problem with K independent candidate systems. Let $\Omega = \{1, ..., K\}$ be the set of all candidate systems. In the context of multi-objective, each system can be measured in D objectives. Let us denote the vector of the true objectives of the system i as $\boldsymbol{\mu}_i^\star = [\mu_{i1}^\star, \mu_{i2}^\star, ..., \mu_{iD}^\star]'$, where $\boldsymbol{\mu}_i^\star$ is an unknown vector. We assume that only samples from normally distributed random vector $X_{i,n} \sim N(\boldsymbol{\mu}_i^\star, \Sigma_i)$ are available. In this section, we assume that the covariance matrix, $\{\Sigma_i, i = 1, ..., K\}$, is known beforehand. We denote $f_i(\cdot|\boldsymbol{\mu}_i)$ as the density function of the system i with respect to the parameter $\boldsymbol{\mu}_i$.

For the sake of brevity, let us denote $\boldsymbol{\theta}^\star = [\boldsymbol{\mu}_1^\star, ..., \boldsymbol{\mu}_K^\star]$ as the matrix of true mean vectors across all systems. Assume Θ is the parameter space of all possible configurations of $\boldsymbol{\theta}^\star$, i.e., $\boldsymbol{\theta}^\star \in \Theta \subset \mathbb{R}^{D \times K}$. Throughout the paper, we use notations

with superscript $\star$ to denote the true parameters, i.e., $\boldsymbol{\mu}_i^\star$, and notations without superscript to denote the dummy parameters and decision variables, i.e., $\boldsymbol{\mu}_i$.

The R&S problems are to identify the "best" system(s). In the context of multi-objective, we consider the "best" systems as the systems that are *not-dominated* by any others.

Definition 3.2.1 *The system i is **dominated** by the system j, denoted by $\boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star$, if $\mu_{id}^\star \leq \mu_{jd}^\star$ for all objectives $d = 1, 2, ..., D$.*

*The system i is **not dominated** by the system j, denoted by $\boldsymbol{\mu}_i^\star \npreceq \boldsymbol{\mu}_j^\star$, if $\mu_{id}^\star > \mu_{jd}^\star$ for at least one objective.*

*The system i is **Pareto optimal** if it is not dominated by all other systems; and the set of all Pareto optimal system is called the **Pareto front**. Denote the **Pareto front** as $\mathcal{S}_{pf} \triangleq \{i | \forall j \in \Omega, \boldsymbol{\mu}_i^\star \npreceq \boldsymbol{\mu}_j^\star\}$ and the **non-Pareto-front** as $\mathcal{S}_{npf} \triangleq \{i | \exists j \in \Omega, \boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star\}$.*

Remark 3.2.1 *In the context of multi-objective optimization, the definition of dominated above is also called the weakly dominated, as opposed to the concept of strongly dominated that requires $\mu_{id}^\star < \mu_{jd}^\star$ for all objectives. This distinction, in stochastic optimization, however, is impractical as a sharp distinction is not possible due to sampling error. Therefore, we do not distinguish these two concepts.*

The objective of the proposed procedure is to correctly identify the Pareto front, $\mathcal{S}_{pf}$ with a guaranteed *probability of correct selection*. Let us denote $\mathcal{S}_{epf}$ as the empirical Pareto front selected by the procedure; then, *PCS* can be put as $PCS = P(\mathcal{S}_{epf} = \mathcal{S}_{pf}) \geq 1 - \alpha$, where α is the pre-specified error probability.

To deliver the *PCS* guarantee, we introduce the following two types of hypothesis tests, namely, *pairwise test* and *candidacy test*.

Definition 3.2.2 *The **pairwise test** between the system i and the system j determines whether the system i is dominated by the system j.*

$$H_0 : \boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star \text{ and } H_1 : \boldsymbol{\mu}_i^\star \npreceq \boldsymbol{\mu}_j^\star \tag{3.1}$$

*The **candidacy test** of the system i determines whether the system i is in the Pareto front or not.*

$$H_0 : \ i \notin \mathcal{S}_{pf} \text{ and } H_1 : \ i \in \mathcal{S}_{pf} \tag{3.2}$$

The **pairwise test** defined above differs from pairwise comparison in single-objective R&S procedures (i.e. KN, KN++, and IZ-free) in two aspects. First, those procedures only compare the system with the practical best system; whereas the pairwise test in our procedure considers comparisons between all pairs of systems. Second, pairwise comparisons in those procedures are reflexive: $\mu_i^\star > \mu_j^\star$ is equivalent to $\mu_j^\star < \mu_i^\star$; whereas $\boldsymbol{\mu}_i^\star \not\preceq \boldsymbol{\mu}_j^\star$ doesn't imply $\boldsymbol{\mu}_j^\star \preceq \boldsymbol{\mu}_i^\star$ in our procedures.

Throughout this section, we assume the following conditions hold:

C1 Systems have different mean values, i.e., $\mu_{i,d}^\star \neq \mu_{j,d}^\star$ for $i \neq j$;

C2 Observations of systems are independent with each other, i.e., $\{X_{i,n}\}$ and $\{X_{j,n}\}$ are independent for all $i \neq j$;

C3 The parameter space Θ is compact;

C4 $f_i(\cdot|\boldsymbol{\mu}_i)$ is the Normal density function with $\boldsymbol{\mu}_i$ as the mean vector;

C5 $\rho_i(\boldsymbol{\mu}_i, \boldsymbol{\mu}_i') > 0$ for all $\boldsymbol{\mu}_i \neq \boldsymbol{\mu}_i'$, where $\rho_i(\boldsymbol{\mu}_i, \boldsymbol{\mu}_i')$ is the Kullback-Leibler divergence from $f(\cdot|\boldsymbol{\mu}_i)$ to $f(\cdot|\boldsymbol{\mu}_i')$:

$$\rho_i(\boldsymbol{\mu}_i, \boldsymbol{\mu}_i') = \mathbb{E}_{f_i(\cdot|\boldsymbol{\mu}_i)}\{\log f_i(X|\boldsymbol{\mu}_i) - \log f_i(X|\boldsymbol{\mu}_i')\} \tag{3.3}$$

where $\mathbb{E}_{f_i(\cdot|\boldsymbol{\mu}_i)}$ represents taking the expectation with respect to $f_i(\cdot|\boldsymbol{\mu}_i)$;

C6 $\sum_{m=1}^{\infty} \mathbb{E}_{f_i(\cdot|\boldsymbol{\mu}_i)}[\rho_i(\boldsymbol{\mu}_i, \hat{\boldsymbol{\mu}}_{i,m})] < \infty$ where $\hat{\boldsymbol{\mu}}_{i,m}$ is the MLE estimator of $\boldsymbol{\mu}_i$ based on the first m observations of the system i.

For the sake of simplicity, we denote the Kullback-Leibler divergence from $\boldsymbol{\theta} = [\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_K]$ to $\boldsymbol{\theta}' = [\boldsymbol{\mu}_1', ..., \boldsymbol{\mu}_K']$ as

$$\rho(\boldsymbol{\theta}, \boldsymbol{\theta}') = \mathbb{E}_{f(\cdot|\boldsymbol{\theta})}[\sum_{i=1}^{K} \log \frac{f_i(X_i|\boldsymbol{\mu}_i)}{f_i(X_i|\boldsymbol{\mu}_i')}] \tag{3.4}$$

where $\mathbb{E}_{f(\cdot|\boldsymbol{\theta})}$ represents taking the expectation with respect to $f(\cdot|\boldsymbol{\theta}) \triangleq \prod_{i=1}^{K} f_i(X_i|\boldsymbol{\mu}_i)$. Under the Conditions C2, C5, and C6, it is easy to verify that $\rho(\boldsymbol{\theta}, \boldsymbol{\theta}') > 0$ and $\sum_{m=1}^{\infty} \mathbb{E}_{f(\cdot|\boldsymbol{\theta})}[\rho(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}_m)] < \infty$, where $\hat{\boldsymbol{\theta}}_m$ is the MLE of $\boldsymbol{\theta}$ using the first m observations of all systems.

Condition C1 is set to avoid the situations when two or more systems tie with each other. Condition C2 requires no dependencies among alternatives so that the early eliminations of inferior systems do not compromise the validity of the proposed procedure. It may be replaced by the less-restricted conditional-independent condition to allow the usage of the common random number. Condition C3 and C4 assume that the observations are from multivariate normal densities and the parameter space of the joint density function is compact. Condition C5 ensures the validity of the likelihood ratio method. Note that Condition C5 can be easily deducted from Conditions C3 and C4. Condition C6 states that the MLE $\hat{\boldsymbol{\mu}}_m$ converges to the true parameter $\boldsymbol{\mu}$ fast enough so that, for a sufficient large enough n, $\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_n) \approx 0$. It is worth noting that Conditions C5 and C6 are automatically fulfilled by many distributions, i.e., Gaussian, Poisson, Bernoulli, etc, where $\rho(\boldsymbol{\mu}, \hat{\boldsymbol{\mu}}_m) \leq C|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_m|^2$ [29].

3.3 The Procedures

In this section, we propose three procedures for Multi-Objective R&S problems and provide theoretical results that the first two procedures attain the target *PCS* asymptotically.

3.3.1 Procedure 3

Procedure 3 is illustrated in Algorithm 3. Over the course of Procedure 3, it records three set of systems: (1) the *undetermined pool*, $\mathcal{S}_{pool,n}$, that contains systems have not been determined as either Pareto optimal or non-Pareto optimal ; (2) the *empirical Pareto front*, $\mathcal{S}_{epf,n}$, that contains systems are identified as Pareto optimal;

and (3) the *empirical non-Pareto front*, $\mathcal{S}_{enpf,n}$. Procedure 3 starts with considering all systems are within the *undetermined set* $\mathcal{S}_{pool,n}$. It sequentially eliminates systems from $\mathcal{S}_{pool,n}$ and puts them into either $\mathcal{S}_{epf,n}$ or $\mathcal{S}_{enpf,n}$. When terminates, systems within the *empirical Pareto front* are returned as the "best" system.

Note that the mis-classification may happen since Procedure 3 is stochastic. To control the probability of the mis-classification at the level of $\alpha = 1 - PCS$, the procedure applies the generalized sequential probability test (GSPRT) and uses the generalized likelihood ratio (GLR) as the testing statistic. This test requires two testing boundaries: the *elimination boundary* at $A = \frac{\alpha}{K}$ and the *acceptance boundary* at $B = \frac{K}{\alpha}$. As the name suggests, systems with the GLR statistics above the acceptance boundary are accepted as *Pareto optimal*; whereas systems with the GLR statistics below the elimination boundary are considered as *non-Pareto optimal*.

The procedure is an iterative method. In the nth iteration, the procedure samples each system in Ω. Given the previous samples as well as the newly obtained samples, the procedure calculates the GLR statistics for systems remained in $\mathcal{S}_{pool,n}$ and compares them against the testing boundaries. The procedure sequentially eliminates systems from $\mathcal{S}_{pool,n}$ by comparing their corresponding GLR statistics with the testing boundaries. It terminates when there is no system left in $\mathcal{S}_{pool,n}$. When terminates, those systems with the GLR statistics that are above the acceptance boundary are identified as the Pareto front; whereas other systems are the non-Pareto front.

Figure 3.1 illustrates the behavior of the proposed procedure. In Figure 3.1, we consider an example with $K = 10$ systems. In this example, the observations of the alternative i are from a normal distribution $N(\boldsymbol{\mu}_i, \begin{bmatrix} 4, & 0 \\ 0, & 4 \end{bmatrix})$. The configurations of $\{\boldsymbol{\mu}_i\}$ are plotted in the upper left of Figure 3.1. The Pareto front contains system 3, 9, and 10. In Figure 3.1, initially, the undetermined set $\mathcal{S}_{pool,1}$ contains all alternatives. At each iteration, the procedure requests one more sample for each system in the set Ω and calculate the GLR statistics. The procedure sequentially elim-

Algorithm 3 Procedure 3 for Multi-Objective Ranking and Selection

Inputs:

$PCS = 1 - \alpha$ ▷ Desired PCS level

$\Omega = \{1, ..., K\}$ ▷ Pool of candidate systems

$f_i(X|\boldsymbol{\mu}_i)$ ▷ Density functions under $\boldsymbol{\mu}_i$

Procedure:

set $n = n_0$;

set $A = \frac{\alpha}{K}$ and $B = \frac{K}{\alpha}$; ▷ A and B are the GSPRT test boundaries

set $\mathcal{S}_{pool,n+1} = \Omega$, $\mathcal{S}_{enpf,n+1} = \emptyset$ and $\mathcal{S}_{epf,n+1} = \Omega$;

generate n_0 samples for each system in Ω;

repeat

  set $n = n + 1$;

  generate 1 more sample for each system in Ω;

  set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n}$, $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n}$;

  for $i \in \mathcal{S}_{pool,n+1}$ **do**

    calculate the GLR statistic $\Lambda^{(3)}_{i,\cdot,n}$;

    if $\Lambda^{(3)}_{i,\cdot,n} < A$ **then**

      set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n+1} \cup \{i\}$, and $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n+1}/\{i\}$;

    end if

    if $\Lambda^{(3)}_{i,\cdot,n} > B$ **then**

      set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$;

    end if

  end for

until $|\mathcal{S}_{pool,n+1}| = 0$ or $|\mathcal{S}_{epf,n+1}| = 1$

return $\mathcal{S}_{epf} = \mathcal{S}_{epf,n+1}$

inates the inferior candidates when their corresponding GLR numbers cross the elimination boundary. For instance, in the plot, the system 6 is eliminated since at

$n = 2$ when it hits the elimination boundary. The procedure keeps sampling all the systems and calculating the GLR numbers until the termination criteria is met.

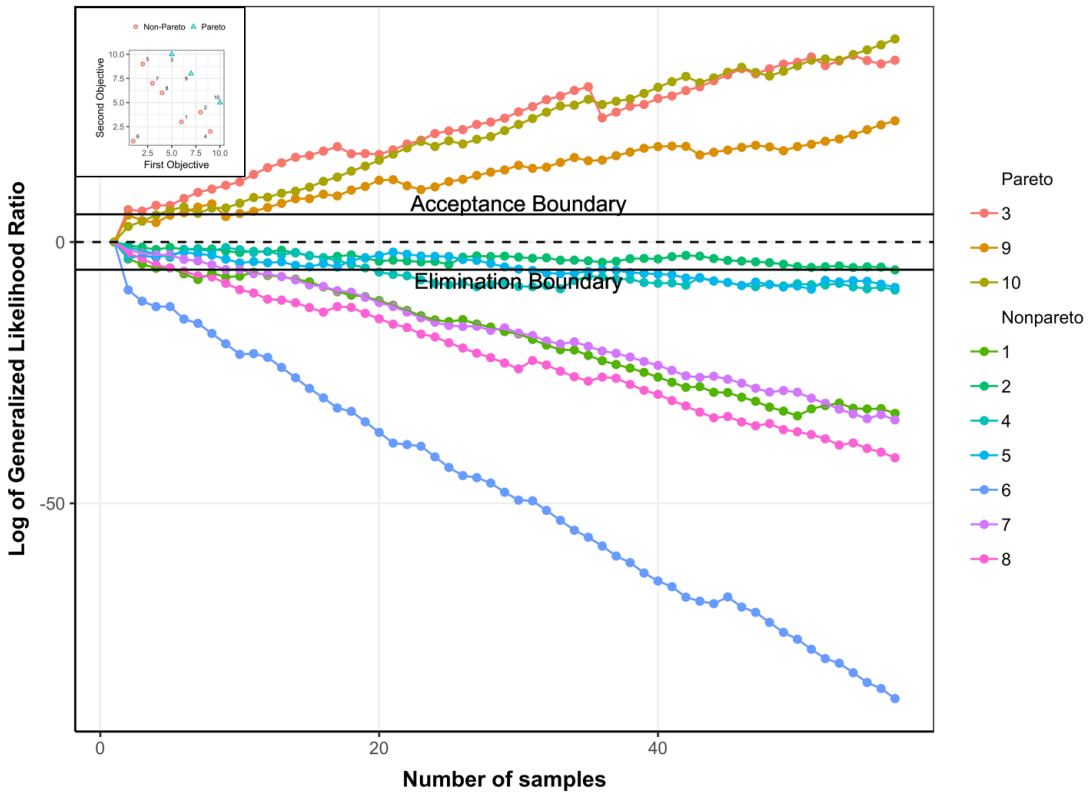


Figure 3.1.: Illustration of the proposed procedures with $K = 10$ systems. The log of the acceptance boundary is at $\log 10/\alpha$, and the log of the elimination boundary is at $\log \alpha/10$. The procedure follows the log of the GLR statistic $\log \Lambda_{i,n}^{(4)}$ and screens out alternatives when their GLR numbers hit the elimination boundary. The procedure terminates when the corresponding GLR statistics are either above the acceptance boundary or below the elimination boundary. Those systems with the GLR statistics that are above the acceptance boundary are accepted as the Pareto front.

The GLR statistic used in Procedure 3, $\Lambda^{(3)}_{i,\cdot,n}$, is defined upon the GLR statistic of the pairwise tests. Recall that the pairwise test between the system i and the system j determines whether the system i is dominated by the system j or not, i.e.,

$$H_0 : \boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star \text{ and } H_1 : \boldsymbol{\mu}_i^\star \npreceq \boldsymbol{\mu}_j^\star$$

Let $\Lambda^{(3)}_{i,j,n}$ be the GLR statistic of the pairwise test between the system i and the system j:

$$\Lambda^{(3)}_{i,j,n} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\npreceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.5}$$

where $\boldsymbol{\theta} = [\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_K]$ are the decision variables, $\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \npreceq \boldsymbol{\mu}_j\}$ corresponds to the feasible set of $\boldsymbol{\theta}$ that the system i is not dominated by the system j, $\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}$ the set of $\boldsymbol{\theta}$ that the system i is dominated by the system j, and $L_n(\boldsymbol{\theta})$ is the likelihood functions of all observations up to the nth iterations under the parameter $\boldsymbol{\theta}$.

$$L_n(\boldsymbol{\theta}) = L_n(\boldsymbol{\theta}|\text{all observations up to the } n\text{th round}) \tag{3.6}$$

$$= \prod_{m=1}^{n} \prod_{j\in\Omega} f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j) \tag{3.7}$$

The GLR statistic used in Procedure 3 is defined as

$$\Lambda^{(3)}_{i,\cdot,n} \triangleq \min_{j\neq i} \Lambda^{(3)}_{i,j,n} \tag{3.8}$$

The procedure uses $\Lambda^{(3)}_{i,\cdot,n}$ to test whether the system i is in Pareto front or not. Intuitively, $\Lambda^{(3)}_{i,j,n}$ measures how likely that the system i is not dominated by the system j. The higher the $\Lambda^{(3)}_{i,j,n}$, the more likely that the system i is not dominated by the system j. Since the scenario that the system i is in Pareto front implies that the system i is not dominated by any other systems. In other words, we are confident that the system i is in Pareto front if all of the pairwise tests between the system i and other systems are in favor of the decision that the system i is not dominated. Under this intuition, we are motivated to use the $\Lambda^{(3)}_{i,\cdot,n}$ as the GLR statistic. Section 3.3.5 provides a rigorous proof on $\Lambda^{(3)}_{i,\cdot,n}$ as a valid testing statistic.

3.3.2 Procedure 4

Procedure 4 is introduced to analyze the validity of Procedure 3. It is exactly the same as Procedure 3 except with the different GLR statistics. In this procedure, the GLR statistic is defined as

$$\Lambda_{i,n}^{(4)} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta})} \tag{3.9}$$

where $\boldsymbol{\theta} = [\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_K]$ are the decision variables.

The numerator of $\Lambda_{i,n}^{(4)}$ is the maximized likelihood that any other systems within $\mathcal{S}_{epf,n}$ do not dominate the system i; the denominator is the maximized likelihood that the system i is dominated by at least one system within Ω.

The closed-form expression of $\Lambda_{i,n}^{(4)}$ is difficult, if still possible, to obtain since the optimization problem is constrained by all possible scenarios that the system i is (not-)dominated by another system. However, it is of theoretical importance as it bounded the GLR statistic $\Lambda_{i,\cdot,n}^{(3)}$ so that the error rates in Procedure 3 can be analyzed.

The following theorem states the relation between $\Lambda_{i,n}^{(4)}$ and $\Lambda_{i,\cdot,n}^{(3)}$.

Theorem 3.3.1 *Under the Conditions C1 to C6, and the definitions of* $\Lambda_{i,n}^{(4)}$ *as in 3.9 and* $\Lambda_{i,\cdot,n}^{(3)}$ *as in 3.8, we have*

1. $\Lambda_{i,\cdot,n}^{(3)} > 1$ *if* $\Lambda_{i,n}^{(4)} > 1$
2. $\Lambda_{i,\cdot,n}^{(3)} < 1$ *if* $\Lambda_{i,n}^{(4)} < 1$,
3. *and* $|\log \Lambda_{i,n}^{(4)}| \geq |\log \Lambda_{i,\cdot,n}^{(3)}|$.

Remark 3.3.1 $\Lambda_{i,\cdot,n}^{(3)} > 1$ *if* $\Lambda_{i,n}^{(4)} > 1$ *and* $\Lambda_{i,\cdot,n}^{(3)} < 1$ *if* $\Lambda_{i,n}^{(4)} < 1$ *means that* $\Lambda_{i,n}^{(4)}$ *and* $\Lambda_{i,\cdot,n}^{(3)}$ *are in favor of the same decision.* $|\log \Lambda_{i,n}^{(4)}| \geq |\log \Lambda_{i,\cdot,n}^{(3)}|$ *indicates* $\Lambda_{i,n}^{(4)}$ *overshoots the decision boundaries (the elimination boundary and the acceptance boundary) when*

Algorithm 4 Procedure 4 for Multi-Objective Ranking and Selection

Inputs:

1: $PCS = 1 - \alpha$ ▷ Desired PCS level

2: $\Omega = \{1, ..., K\}$ ▷ Pool of candidate systems

3: $f_i(X|\boldsymbol{\mu}_i)$ ▷ Density functions under $\boldsymbol{\mu}_i$

Procedure:

1: set $n = n_0$;

2: set $A = \frac{\alpha}{K}$ and $B = \frac{K}{\alpha}$; ▷ A and B are the GSPRT test boundaries

3: set $\mathcal{S}_{pool,n+1} = \Omega$, $\mathcal{S}_{enpf,n+1} = \emptyset$ and $\mathcal{S}_{epf,n+1} = \emptyset$;

4: generate n_0 samples for each system in Ω;

5: **repeat**

6: set $n = n + 1$;

7: generate 1 more sample for each system in Ω;

8: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n}$, $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n}$;

9: **for** $i \in \mathcal{S}_{pool,n+1}$ **do**

10: calculate the GLR statistic $\Lambda^{(4)}_{i,n}$;

11: **if** $\Lambda^{(4)}_{i,n} < A$ **then**

12: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n+1} \cup \{i\}$, and $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n+1}/\{i\}$;

13: **end if**

14: **if** $\Lambda^{(4)}_{i,n} > B$ **then**

15: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$;

16: **end if**

17: **end for**

18: **until** $|\mathcal{S}_{pool,n+1}| = 0$ or $|\mathcal{S}_{epf,n+1}| = 1$

19: **return** $\mathcal{S}_{pool} = \mathcal{S}_{epf,n+1}$

$\Lambda^{(3)}_{i,\cdot,n}$ crosses the boundaries. Therefore, $\Lambda^{(3)}_{i,\cdot,n}$ is a less powerful GLR statistic than $\Lambda^{(4)}_{i,n}$. Thus,

$$\mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(3)}_{i,\cdot,\tau} \leq B) \leq \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \leq B) \text{ if } B > 1 \tag{3.10}$$

$$\mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(3)}_{i,\cdot,\tau} \geq A) \leq \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \geq A) \text{ if } A < 1 \tag{3.11}$$

for any $\boldsymbol{\theta}$.

3.3.3 Procedure 5: a heuristic procedure

We propose Procedure 5 as a modified version of Procedure 3. In Procedure 3, all systems are sampled in each iteration; even for systems that are already classified as *non-Pareto optimal*; whereas procedure 5 stops sampling systems when they are identified as *non-Pareto optimal*. An intuitive justification for such a modification is that a system is *Pareto optimal* if and only if it is not dominated by any other *Pareto optimal* systems, and non-Pareto optimal systems are not necessary to identify other non-Pareto optimal systems.

Procedure 5 differs from Procedure 3 in two places. Firstly, Procedure 5 samples only system within $\mathcal{S}_{epf,n+1}$ and $\mathcal{S}_{pool,n+1}$ in the nth iteration; secondly, the GLR statistic in Procedure 5 is $\Lambda^{(5)}_{i,\cdot,n}$, which is defined as

$$\Lambda^{(5)}_{i,\cdot,n} \triangleq \min_{j\neq i, j\in\mathcal{S}_{epf,n+1}\cup\mathcal{S}_{pool,n+1}} \Lambda^{(3)}_{i,j,n} \tag{3.12}$$

Remark 3.3.2 *For the deterministic multi-objective optimization problem, it is solid that the identified non-Pareto optimal systems are unnecessary for identifying other non-Pareto optimal systems. However, given the stochastic nature of MOR&S problem, the probability that we mis-classify the Pareto optimal system as non-Pareto optimal system is nonzero. And the events of mis-classifications, together, form the error probability of the entire procedure. To analyze the error probability of the proposed procedures, we desire the events of mis-classifications are independent. This property, however, does not stand if the future tests are dependent on the previous decisions. (* $\Lambda^{(5)}_{i,\cdot,n}$ *in Procedure 5 depends on* $\mathcal{S}_{epf,n+1}$, $\mathcal{S}_{enpf,n+1}$, $\mathcal{S}_{pool,n+1}$*).*

The behavior of the proposed procedure is illustrated in Figure 3.2. We use the same example as in Figure 3.1. We consider a bi-objective ranking and selection problem with $K = 10$ systems. Samples of the system i are from a normal distribu-

Algorithm 5 Procedure 5 for Multi-Objective Ranking and Selection

Inputs:

1: $PCS = 1 - \alpha$ ▷ Desired PCS level

2: $\Omega = \{1, ..., K\}$ ▷ Pool of candidate systems

3: $f_i(X|\boldsymbol{\mu}_i)$ ▷ Density functions under $\boldsymbol{\mu}_i$

Procedure:

1: set $n = n_0$;

2: set $A = \frac{\alpha}{K}$ and $B = \frac{K}{\alpha}$; ▷ A and B are the GSPRT test boundaries

3: set $\mathcal{S}_{pool,n+1} = \Omega$, $\mathcal{S}_{enpf,n+1} = \emptyset$ and $\mathcal{S}_{epf,n+1} = \Omega$;

4: generate n_0 samples for each system in Ω;

5: **repeat**

6: set $n = n + 1$;

7: generate 1 more sample for each system in $\mathcal{S}_{pool,n+1}$ or $\mathcal{S}_{epf,n}$;

8: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n}$, $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n}$;

9: **for** $i \in \mathcal{S}_{pool,n+1}$ **do**

10: calculate the GLR statistic $\Lambda^{(5)}_{i,\cdot,n}$;

11: **if** $\Lambda^{(5)}_{i,\cdot,n} < A$ **then**

12: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$, $\mathcal{S}_{enpf,n+1} = \mathcal{S}_{enpf,n+1} \cup \{i\}$, and $\mathcal{S}_{epf,n+1} = \mathcal{S}_{epf,n+1}/\{i\}$;

13: **end if**

14: **if** $\Lambda^{(5)}_{i,\cdot,n} > B$ **then**

15: set $\mathcal{S}_{pool,n+1} = \mathcal{S}_{pool,n+1}/\{i\}$;

16: **end if**

17: **end for**

18: **until** $|\mathcal{S}_{pool,n+1}| = 0$ or $|\mathcal{S}_{epf,n+1}| = 1$

19: **return** $\mathcal{S}_{epf} = \mathcal{S}_{epf,n+1}$

tion $N(\boldsymbol{\mu}_i, \begin{bmatrix} 4, & 0 \\ 0, & 4 \end{bmatrix})$. The configurations of $\boldsymbol{\mu}_i$ are plotted in the upper left of Figure

3.2. The Pareto front contains system 3, 9, and 10. In the beginning, the undetermined set $\mathcal{S}_{pool,0}$ contains all alternatives. At each iteration, the procedure requests one more sample for each system in the set $\mathcal{S}_{pool,n}$ and $\mathcal{S}_{epf,n}$, and calculates and plots the GLR numbers. The inferior candidates are sequentially eliminated when their corresponding GLR numbers cross the elimination boundary. The procedure keeps sampling the remaining systems and calculating the GLR numbers until no system is left in $\mathcal{S}_{pool,n}$.

The differences between Figure 3.1 and Figure 3.2 are remarkable. First, Procedure 5 uses much less number of samples since inferior systems are stopped sampling as soon as they touch the elimination boundary; second, Procedure 5 terminates earlier than Procedure 3. This relates to the choice of the GLR statistics: $\Lambda^{(3)}_{i,\cdot,n}$ is more conservative than $\Lambda^{(5)}_{i,\cdot,n}$ as $\Lambda^{(3)}_{i,\cdot,n}$ considers all systems, whereas $\Lambda^{(5)}_{i,\cdot,n}$ only considers a subset of systems.

3.3.4 The Implementation

In this subsection, we implement $\Lambda^{(3)}_{i,j,n}$ under the assumption that observations are from multi-normal distribution with known variance.

Let us denote $\{X_{i,1}, ..., X_{i,n}\}$ as the i.i.d. random observations obtained from the alternative i. Assume $\{X_{i,n}\}$ are from multivariate normal distribution $N(\boldsymbol{\mu}_i^\star, \Sigma_i)$, where $\boldsymbol{\mu}_i^\star$ is the unknown mean and Σ_i is the known covariance matrix. The direct approach to calculate $\Lambda^{(3)}_{i,j,n}$ is to consider $\boldsymbol{\theta}$ as the decision variable in the following formula

$$\Lambda^{(3)}_{i,j,n} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.13}$$

where $L_n(\boldsymbol{\theta})$ is the likelihood function under $\boldsymbol{\theta}$,

$$L_n(\boldsymbol{\theta}) = \prod_{m=1}^{n} \prod_{l\in\Omega} f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l) \tag{3.14}$$

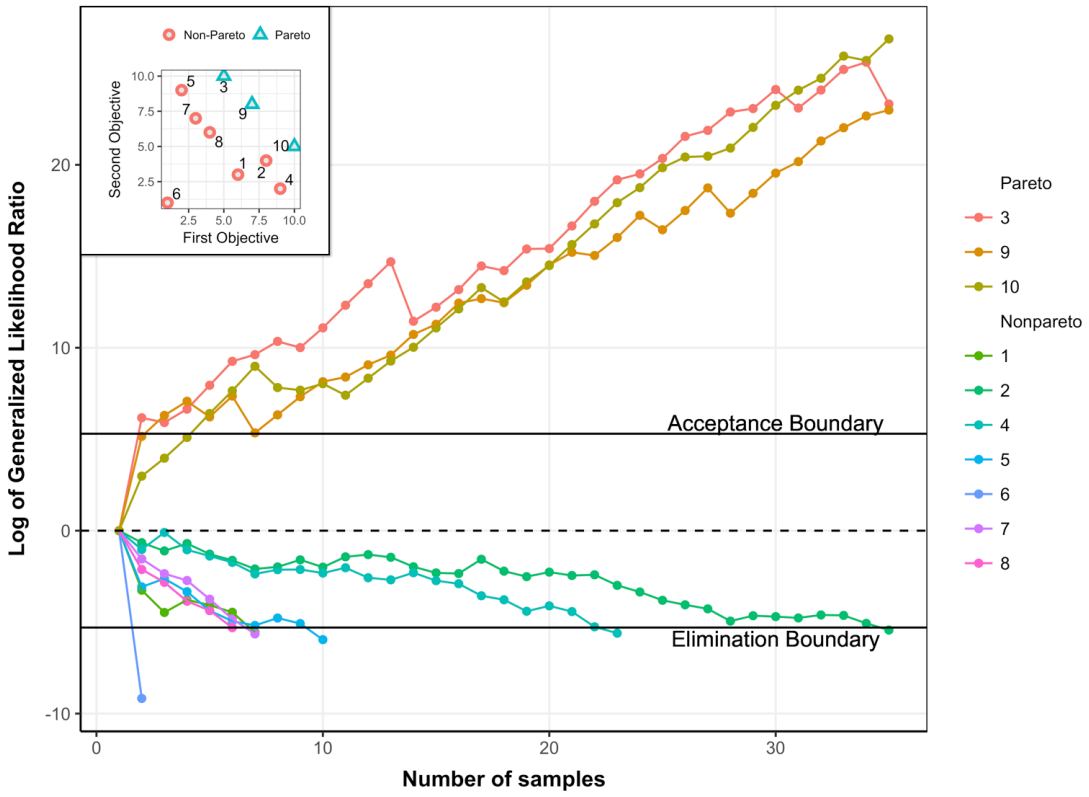


Figure 3.2.: Illustration of the proposed procedures with $K = 10$ alternatives. The log of the acceptance boundary is at $\log 10/\alpha$ and the log of the elimination boundary is at $\log \alpha/10$. The procedure follows the log of the GLR statistic $\log \Lambda_n^i$ and screens out alternatives when their GLR numbers hit the elimination boundary. The procedure terminates when there is no system left between the acceptance boundary and elimination boundary. The systems that are above the acceptance boundary are reported as the Pareto front.

Given the fact that the feasible regions, $\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}$ and $\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}$, only restrict $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_j$, it is easy to verify that

$$\Lambda_{i,j,n}^{(3)} = \frac{\sup_{[\boldsymbol{\mu}_i,\boldsymbol{\mu}_j]\in\{\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} \prod_{m=1}^{n} f_i(X_{i,m}|\boldsymbol{\mu}_i) f_j(X_{j,m}|\boldsymbol{\mu}_j)}{\sup_{[\boldsymbol{\mu}_i,\boldsymbol{\mu}_j]\in\{\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} \prod_{m=1}^{n} f_i(X_{i,m}|\boldsymbol{\mu}_i) f_j(X_{j,m}|\boldsymbol{\mu}_j)} \tag{3.15}$$

$$= \frac{\sup_{[\boldsymbol{\mu}_i,\boldsymbol{\mu}_j]\in\{\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} \prod_{m=1}^{n} \exp\{-\frac{(X_{i,m}-\boldsymbol{\mu}_i)'\Sigma_i^{-1}(X_{i,m}-\boldsymbol{\mu}_i)}{2} - \frac{(X_{j,m}-\boldsymbol{\mu}_j)'\Sigma_j^{-1}(X_{j,m}-\boldsymbol{\mu}_j)}{2}\}}{\sup_{[\boldsymbol{\mu}_i,\boldsymbol{\mu}_j]\in\{\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} \prod_{m=1}^{n} \exp\{-\frac{(X_{i,m}-\boldsymbol{\mu}_i)'\Sigma_i^{-1}(X_{i,m}-\boldsymbol{\mu}_i)}{2} - \frac{(X_{j,m}-\boldsymbol{\mu}_j)'\Sigma_j^{-1}(X_{j,m}-\boldsymbol{\mu}_j)}{2}\}} \tag{3.16}$$

$\Lambda^{(3)}_{i,j,n}$, in Equation 3.16, is solvable since the density kernel is convex. However, in general cases, it is still difficult to find the analytical solution. To speed up the calculation, we are motivated to find an approximated solution. Let us denote $Z_{ij,m} = X_{i,m} - X_{j,m}$ as the differences between $\{X_{i,m}\}$ and $\{X_{j,m}\}$. Since $\{X_{i,m}, m = 1, ..., n\}$ and $\{X_{j,m}, m = 1, ..., n\}$ are normally distributed, it is evident that $\{Z_{ij,m}\}$ are also of Normal distribution, i.e, $Z_{ij,m} \sim N(\boldsymbol{\mu}^{\star}_{ij} \triangleq \boldsymbol{\mu}^{\star}_{i} - \boldsymbol{\mu}^{\star}_{j}, \Sigma_{ij} \triangleq \Sigma_i + \Sigma_j)$. Moreover, let us denote $f_{ij}(Z|\boldsymbol{\mu}_{ij})$ as the density function of Z_{ij} under the parameter $\boldsymbol{\mu}_{ij}$. The intuition behind this approximation is that, the test using $\{X_{i,m}\}$ and $\{X_{j,m}\}$,

$$H_0 : \boldsymbol{\mu}^{\star}_{i} \preceq \boldsymbol{\mu}^{\star}_{j} \text{ and } H_1 : \boldsymbol{\mu}^{\star}_{i} \npreceq \boldsymbol{\mu}^{\star}_{j} \tag{3.17}$$

is equivalent to the test using $\{Z_{ij,m}\}$

$$H_0 : \boldsymbol{\mu}^{\star}_{ij} \preceq \mathbf{0} \text{ and } H_1 : \boldsymbol{\mu}^{\star}_{ij} \npreceq \mathbf{0} \tag{3.18}$$

Thus, we approximate $\Lambda^{(3)}_{i,j,n}$ by calculating the likelihood of $\{Z_{ij,m}\}$, instead of the likelihood of $\{X_{i,m}\}$ and $\{X_{j,m}\}$.

$$\Lambda^{(3)}_{i,j,n} \approx \frac{\sup_{\boldsymbol{\mu}_{ij} \in \{\boldsymbol{\mu}_{ij} \npreceq \mathbf{0}\}} \prod_{m=1}^{n} f_{ij}(Z_{ij,m}|\boldsymbol{\mu}_{ij})}{\sup_{\boldsymbol{\mu}_{ij} \in \{\boldsymbol{\mu}_{ij} \preceq \mathbf{0}\}} \prod_{m=1}^{n} f_{ij}(Z_{ij,m}|\boldsymbol{\mu}_{ij})} \tag{3.19}$$

$$= \frac{\sup_{\boldsymbol{\mu}_{ij} \in \{\boldsymbol{\mu}_{ij} \npreceq \mathbf{0}\}} \prod_{m=1}^{n} \exp\{-\frac{(Z_{ij,m}-\boldsymbol{\mu}_{ij})' \Sigma_{ij}^{-1} (Z_{ij,m}-\boldsymbol{\mu}_{ij})}{2}\}}{\sup_{\boldsymbol{\mu}_{ij} \in \{\boldsymbol{\mu}_{ij} \preceq \mathbf{0}\}} \prod_{m=1}^{n} \exp\{-\frac{(Z_{ij,m}-\boldsymbol{\mu}_{ij})' \Sigma_{ij}^{-1} (Z_{ij,m}-\boldsymbol{\mu}_{ij})}{2}\}} \tag{3.20}$$

The approximation reduces the number of decision variables and thus significantly saves the computation.

3.3.5 The Validity of Procedure 3 and 4

In this section, we show that the proposed procedure is asymptotically valid to deliver the *PCS* guarantee. This section contains two parts. In the first part, we present asymptotic error probabilities of the GSPRT method. In the second part, we show the asymptotic validities for the proposed procedures.

The Asymptotic Error Probabilities of GSPRT

Let us denote Θ as the parameter space of the decision variable $\boldsymbol{\theta}$ and consider a composite hypothesis test of the form

$$H_0 \ : \boldsymbol{\theta}^\star \in \Theta_0 \ \text{ and } H_1 \ : \boldsymbol{\theta}^\star \in \Theta_1 \tag{3.21}$$

where Θ_0 and Θ_1 are compact and disjoint subsets of Θ.

For such a test, the corresponding generalized likelihood ratio (GLR) statistics is defined as,

$$\Lambda_n = \frac{\sup_{\boldsymbol{\theta}\in\Theta_1} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\Theta_0} L_n(\boldsymbol{\theta})} \tag{3.22}$$

where $L_n(\boldsymbol{\theta})$ is the likelihood function with observations collected up to the nth iteration.

Essentially, the GLR statistic "self-tunes" to information about the true parameter, $\boldsymbol{\theta}^\star$, over the course of the test. To determine whether the null hypothesis or the alternative hypothesis is supported, the GSPRT stops sampling after the nth observations if the GLR crosses one of the two boundaries B or A. The null hypothesis is rejected if $\Lambda_n \geq B$ and is accepted if $\Lambda_n \leq A$. For a given pair of A and B, denote the stopping time as

$$\tau = \inf\{n : \Lambda_n \geq B \text{ or } \Lambda_n \leq A\} \tag{3.23}$$

The error probabilities at the stopping time τ is asymptotically approximated in the following theorem.

Theorem 3.3.2 *Under the Conditions C1-C6, consider the composite null hypothesis against composite alternative hypothesis given as in (3.21). Let τ be the first time that the GLR statistic crosses one of the boundaries, A and B with $A < 1 < B$, i.e.,*

$$\tau = \inf\{n : \Lambda_n \geq B \text{ or } \Lambda_n \leq A\} \tag{3.24}$$

The maximal error probability admits following approximations

$$\sup_{\boldsymbol{\theta}\in\Theta_1} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda_\tau \leq B) \cong B^{-1} \tag{3.25}$$

$$\sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda_\tau \geq A) \cong A \tag{3.26}$$

as $A \to 0$ and $B \to \infty$, where $\mathbf{P}_{\boldsymbol{\theta}}$ represents the probability measure for a given $\boldsymbol{\theta}$.

Proof The result can be easily verified using Theorem 2.1 in [27] which presents this result under a set of less restricted conditions. Similar results can also be found in [42] and [29]. ■

Remark 3.3.3 $\sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda_\tau \geq A) \cong A$ *is equivalent to* $\lim_{A\to 0} \frac{\sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda_\tau \geq A)}{A} = 1$.

Remark 3.3.4 *In the classic SPRT method, the thresholds $A = \frac{\alpha_2}{1-\alpha_1}$ and $B = \frac{1-\alpha_2}{\alpha_1}$, where α_1 and α_2 are the pre-specified type I and type II error probabilities, have the same decay rate as in 3.25.*

The Asymptotic Validity

With above theorems, we are ready to present the statistical validity of Procedure 3. Please refer to the appendix for the proof of the following theorem.

Theorem 3.3.3 *Under the Conditions C1 to C6, Procedure 3 terminates in finite time with probability 1 and selects the Pareto front with probability at least $1-\alpha$ in asymptotically sense, i.e.,*

$$\limsup_{\alpha\to 0^+} \frac{1 - P(CS)}{\alpha} \geq 1 \tag{3.27}$$

The above theorem states that Procedure 3 selects the entire Pareto front at the termination with a statistical guarantee. Nonetheless, if the procedure stops before the termination criteria, the candidate pool $\mathcal{S}_{epf,n}$ contains the Pareto front with a guaranteed probability. The result is stated as the following Theorem.

Theorem 3.3.4 *Under the Conditions C1 to C6, if Procedure 3 is stopped at iteration n before termination, then the candidate set $\mathcal{S}_{epf,n+1}$ contains the Pareto front $\mathcal{S}_{pf}$ with probability at least $1-\alpha$, in asymptotically sense, i.e.,*

$$\limsup_{\alpha \to 0^+} \frac{1 - P(\mathcal{S}_{pf} \in \mathcal{S}_{epf,n+1})}{\alpha} \geq 1 \tag{3.28}$$

3.4 Simulation Studies

In this section, we conduct numerical experiments to test the performance of the proposed procedures.

3.4.1 Study 1: MOR&S under different number of objectives

The first numerical study considers MOR&S problems with 20 or 50 systems and 2 to 5 objectives. The configurations of true mean vectors $\Theta^\star$ are randomly generated and reported in Appendix. We use the independent and identical variance configuration $\Sigma = I_K$. We apply Procedure 3 and Procedure 5 ,and conduct 1,000 macroreplications for each scenario. The estimated *PCS* and average sample size are reported in Table 3.1.

Table 3.1.: Procedure 3 for MOR&S to select the *Pareto front* with the target $PCS = 0.95$. (In each cell, the first line is the estimated *PCS*, the second line is the average sample size with 95% confidence interval over 1,000 macroreplications.)

K	Sequential Procedure ($\Sigma = I_K$)			
	$D = 2$	$D = 3$	$D = 4$	$D = 5$
20	1.000	1.000	1.000	1.000
	547.0 ± 5.7	643.8 ± 8.3	865.2 ± 9.3	1075.4 ± 10.5
50	1.000	0.999	1.000	1.000
	1064.3 ± 12.5	1456.2 ± 20.4	2532.5 ± 35.1	3139.2 ± 41.1

Table 3.2.: Procedure 5 for MOR&S to select the Pareto front with $PCS = 0.95$ and three variance structures. (In each cell, the first line is the estimated PCS, the second line is the average sample size with 95% confidence interval over 1,000 macroreplications.)

K	Sequential Procedure ($\Sigma = I_K$)			
	$D = 2$	$D = 3$	$D = 4$	$D = 5$
20	1.000 266.0 ± 4.0	0.994 376.8 ± 4.6	1.000 478.4 ± 6.4	0.996 494.4 ± 8.0
50	1.000 636.6 ± 6.4	0.996 1061.5 ± 12.4	0.999 1812.3 ± 23.6	1.000 1939.2 ± 30.9

3.4.2 Study 2: MOR&S with various covariance configuration

We consider bi-objective R&S problems with 20, 50, 100 candidates. True means are generated randomly and depicted in Figure 3.3. We consider three variance configurations: (1) Independent, $\Sigma_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, (2) Positive-Correlated, $\Sigma_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, (3) Negative-Correlated, $\Sigma_3 = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$. We conduct 1,000 independent macroreplications for each scenario and report the estimated PCS and the average sample size in Table 3.3. The results indicate the proposed procedure is valid for various types of covariance structure.

Table 3.3.: Procedure 5 for Bi-Objective R&S to select the Pareto front from K normal variables with $PCS = 0.95$ and three variance structures. (In each cell, the first line is the estimated PCS, the second line is the average sample size with 95% confidence interval over 1,000 macroreplications.)

K	Sequential Procedure ($D = 2$)		
	$\Sigma_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$	$\Sigma_2 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$	$\Sigma_3 = \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$
20	1.000 283.3 ± 3.6	1.000 381.5 ± 6.1	1.000 397.9 ± 5.9
50	1.000 368.7 ± 2.7	1.000 422.6 ± 3.9	1.000 423.6 ± 4.5
100	1.000 816.6 ± 4.6	0.999 998.8 ± 8.2	1.000 993.5 ± 7.9

These two studies test the validity of the proposed procedure from two important aspects: (1) covariance structure, (2) number of objectives. From the above results, we have two interesting findings. Firstly, the independent covariance struc-

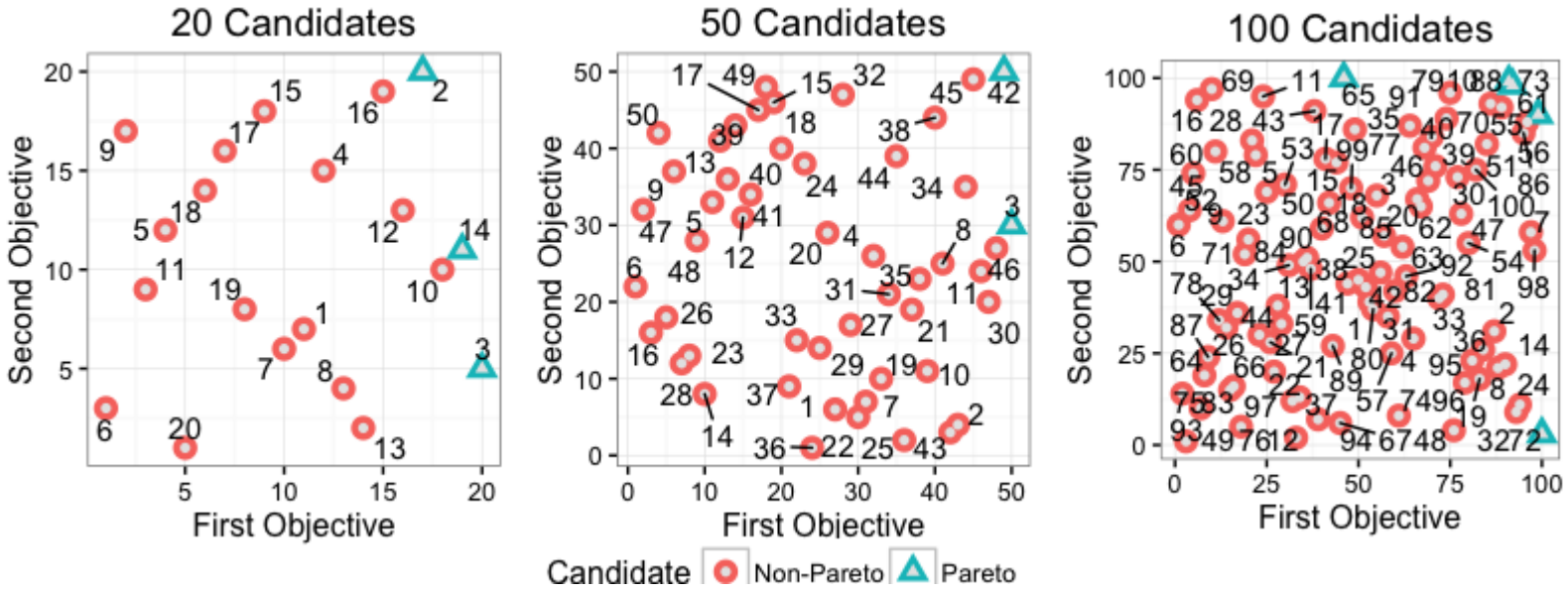


Figure 3.3.: True means of 2-Dimensional candidates. Values along each objective are generated by randomly shuffling $[1, 2..., K]$. For $K = 20$, the Pareto front are $\mathcal{S}_{pf} = \{x_2, x_3, x_{14}\}$; for $K = 50$, $\mathcal{S}_{pf} = \{x_3, x_{42}\}$; and for $K = 100$, $\mathcal{S}_{pf} = \{x_{61}, x_{65}, x_{72}, x_{73}, x_{88}\}$.

ture uses the fewer samples than the dependent covariance structure (Note that the proposed procedure does not assume a particular covariance structure). It makes sense since dependent samples provide redundant due to correlation and thus, to achieve the same amount of information, the procedure needs less independent samples. Secondly, the sample size increases as the number of objectives increases.

3.5 Conclusion

In this chapter, we propose three sequential testing procedures for MOR&S with a guaranteed PCS. The new procedures are adapted from the Generalized Sequential Probability Ratio tests and solve the multiple-testing problem using Bonferroni Correction over the number of systems. It is worth noting that the procedures disregard the number of objectives and achieves the same level of efficiency as the KN family procedures.

3.6 Appendix: Technical Proofs

3.6.1 Proof of Theorem 3.3.1

Proof Recall that

$$\Lambda_{i,n}^{(4)} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta})}$$

$$\Lambda_{i,\cdot,n}^{(3)} = \min_{j\neq i} \Lambda_{i,j,n}^{(3)} = \min_{j\neq i} \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}$$

where $\boldsymbol{\theta}$ is the matrix of decision variables.

Let us denote $\hat{\boldsymbol{\theta}}_n = [\hat{\boldsymbol{\mu}}_{1,n}, ..., \hat{\boldsymbol{\mu}}_{K,n}]$ as the MLE of $\boldsymbol{\theta}$ at the nth iteration. Clearly, $\hat{\boldsymbol{\theta}}_n$ maximize $L_n(\boldsymbol{\theta})$.

To prove the theorem, we consider two scenarios:

Scenario 1: $\hat{\boldsymbol{\mu}}_{i,n} \not\preceq \hat{\boldsymbol{\mu}}_{l,n}$ for all $l \neq i$.

This scenario implies that the empirical evaluations of all systems left in $\mathcal{S}_{epf,n}$ show that any other systems do not dominate system i.

Under this scenario, it is easy to verify that

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta}) = L_n(\hat{\boldsymbol{\theta}}_n) \tag{3.29}$$

since $\hat{\boldsymbol{\theta}}_n \in \{\boldsymbol{\theta}|\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j, \forall j \neq i\}$ and $\hat{\boldsymbol{\theta}}_n$ maximizes the $L_n(\boldsymbol{\theta})$.

Similarly, for the numerator in $\Lambda^{(3)}_{i,\cdot,n}$, we also have, for any j,

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta}) = L_n(\hat{\boldsymbol{\theta}}_n) \tag{3.30}$$

The denominator in $\Lambda^{(4)}_{i,n}$ can be written as

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta}) = \max_{j\neq i} \sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L(\boldsymbol{\theta}) \tag{3.31}$$

Note that the RHS is equal to the denominator in $\Lambda^{(3)}_{i,\cdot,n}$.

Put these together, we have

$$\Lambda^{(3)}_{i,\cdot,n} = \min_{j\neq i} \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.32}$$

$$= \min_{j\neq i} \frac{L_n(\hat{\boldsymbol{\theta}}_n)}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.33}$$

$$= \frac{L_n(\hat{\boldsymbol{\theta}}_n)}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta})} = \Lambda^{(4)}_{i,n} \tag{3.34}$$

Moreover, $\Lambda^{(4)}_{i,n} \geq 1$ since $L_n(\hat{\boldsymbol{\theta}}_n) > L(\boldsymbol{\theta})$ for all $\boldsymbol{\theta} \neq \hat{\boldsymbol{\theta}}_n$.

Scenario 2: $\exists l \neq i$ such that $\hat{\boldsymbol{\mu}}_{i,n} \preceq \hat{\boldsymbol{\mu}}_{l,n}$

This scenario represents that the empirical evaluations show that system i is dominated by some systems within the set Ω.

The numerator of $\Lambda^{(4)}_{i,n}$ can be written as

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta}) \leq \sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta}), \forall j \neq i \tag{3.35}$$

the above inequality holds for any $j \neq i$. Therefore,

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta}) \leq \min_{j\neq i} \sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta}) \tag{3.36}$$

As to the denominator, we have

$$\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta}) = \max_{j\neq i} \sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L(\boldsymbol{\theta}) \tag{3.37}$$

Thus,

$$\Lambda_{i,\cdot,n}^{(3)} = \min_{j\neq i} \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.38}$$

$$\geq \frac{\min_{j\neq i}\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\max_{j\neq i}\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.39}$$

$$\geq \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j,\forall j\neq i\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j,\exists j\neq i\}} L_n(\boldsymbol{\theta})} = \Lambda_{i,n}^{(4)} \tag{3.40}$$

Moreover, under the scenario that there exists a system l such that $\hat{\boldsymbol{\mu}}_{i,n} \preceq \hat{\boldsymbol{\mu}}_{l,n}$, $\Lambda_{i,l,n}^{(3)} < 1$. We have, $\Lambda_{i,\cdot,n}^{(3)} = \min_{j\in\mathcal{S}_{epf,n}} \Lambda_{i,j,n}^{(3)} \leq \Lambda_{i,l,n}^{(3)} < 1$.

To sum it up, in Scenario 1, $\Lambda_{i,n}^{(4)} = \Lambda_{i,\cdot,n}^{(3)} > 1$; in Scenario 2, $0 < \Lambda_{i,n}^{(4)} \leq \Lambda_{i,\cdot,n}^{(3)} < 1$. In conclusion, we have $|\log \Lambda_{i,n}^{(4)}| \geq |\log \Lambda_{i,\cdot,n}^{(3)}|$.

■

3.6.2 Proof of Theorem 3.3.3

Proof There are two parts of the theorem. The first part is that the procedure terminates in finite time; the second part is that it is asymptotically valid to deliver the *PCS* guarantee.

Part 1: the procedure terminates in finite time w.p.1

To begin with, we show that the procedure terminates in finite time with probability 1. The proposed procedure sequentially removes systems from the set $\mathcal{S}_{pool,n+1}$. Let us denote $\tau_i = \min\{n|\Lambda_{i,\cdot,n}^{(3)} > B \text{ or } \Lambda_{i,\cdot,n}^{(3)} < A\}$ with $A = \frac{\alpha}{K}$ and $B = \frac{K}{\alpha}$, as the first time that the system i crosses A or B. The procedure terminates at time N when $\mathcal{S}_{pool,N+1}$ is empty or the cardinality of $\mathcal{S}_{epf,N+1}$ is 1. Thus, we have the following inequality

$$P(\text{Procedure 3 does not terminate before } N) \tag{3.41}$$

$$= P(1_{|\mathcal{S}_{pool,N+1}|\geq 1} \cap 1_{|\mathcal{S}_{epf,N+1}|\geq 2}) \tag{3.42}$$

$$\leq P(1_{|\mathcal{S}_{pool,N+1}|\geq 1} \big| |\mathcal{S}_{epf,N+1}| \geq 2) = P(\cup_{i\in\Omega} 1_{i\in\mathcal{S}_{pool,N+1}} \big| |\mathcal{S}_{epf,N+1}| \geq 2) \tag{3.43}$$

$$\leq \sum_{i\in\Omega} P(i \in \mathcal{S}_{pool,N+1} \big| |\mathcal{S}_{epf,N+1}| \geq 2) = \sum_{i\in\Omega} P(\tau_i > N \big| |\mathcal{S}_{epf,N+1}| \geq 2) \tag{3.44}$$

For the sake for brevity, in the following proofs, we denote $P_c(\cdot)$ as $P(\cdot \big| |\mathcal{S}_{epf,N+1}| \geq 2)$. To prove that the procedure terminates in finite time with probability 1, it is sufficient to show that $\lim_{N\to\infty} P_c(\tau_i > N) \to 0$ for all i.

To demonstrate, we consider two scenarios.

Scenario 1. there exists a system j such that $\boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star$.

In this scenario, there exists a system j that dominates the system i. Moreover, both systems i and j survive after N iterations.

By definition, $\Lambda_{i,\cdot,N}^{(3)} \triangleq \min_{l\in\Omega/i} \Lambda_{i,l,N}^{(3)}$, where

$$\Lambda_{i,j,N}^{(3)} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})} \tag{3.45}$$

where $L_N(\boldsymbol{\theta})$ is the likelihood of all observations up to the N iterations,

$$L_N(\boldsymbol{\theta}) = L_N(\boldsymbol{\theta}|\text{all observations up to the } n\text{th round}) \tag{3.46}$$

$$= \prod_{m=1}^{N} \prod_{j\in\Omega} f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j) \tag{3.47}$$

Therefore,

$$P_c(\tau_i > N) \leq P_c(A < \Lambda_{i,\cdot,N}^{(3)} < B) \leq P_c(\Lambda_{i,\cdot,N}^{(3)} > A) \leq P_c(\Lambda_{i,j,N}^{(3)} > A) \tag{3.48}$$

The last inequality holds since, by definition, $\Lambda_{i,\cdot,N}^{(3)} = \min_{l\in\Omega/i} \Lambda_{i,l,N}^{(3)} \leq \Lambda_{i,j,N}^{(3)}$.

By Chebyshev inequality,

$$P_c(\Lambda_{i,j,N}^{(3)} > A) \leq \frac{\mathbb{E}[(\Lambda_{i,j,N}^{(3)})^s]}{(A)^s}, \ \forall s > 0 \tag{3.49}$$

Under the Conditions C1 to C5, the maximized likelihood is solvable. Thus, there exists $\tilde{\boldsymbol{\theta}}_N = \arg\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})$, with $\tilde{\boldsymbol{\theta}}_N = [\tilde{\boldsymbol{\mu}}_{1,N}, ..., \tilde{\boldsymbol{\mu}}_{K,N}]$. We have

$$\Lambda^{(3)}_{i,j,N} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\not\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_n(\boldsymbol{\theta})} \tag{3.50}$$

$$= \frac{L_N(\tilde{\boldsymbol{\theta}}_N)}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i\preceq\boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})} \leq \frac{L_N(\tilde{\boldsymbol{\theta}}_N)}{L_N(\boldsymbol{\theta}^\star)} \tag{3.51}$$

The last inequality holds since, Scenario 1 asserts that $\boldsymbol{\mu}_i^\star \preceq \boldsymbol{\mu}_j^\star$, the true parameters $\boldsymbol{\theta}^\star$ are within the parameter space $\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}$.

$$\mathbb{E}[(\Lambda^{(3)}_{i,j,N})^s] \leq \mathbb{E}\left[(\frac{L_N(\tilde{\boldsymbol{\theta}}_N)}{L_N(\boldsymbol{\theta}^\star)})^s\right] \tag{3.52}$$

$$=\mathbb{E}\left[\prod_{m=1}^{N}\prod_{l\in\Omega}(\frac{f_l(\boldsymbol{X}_{l,m}|\tilde{\boldsymbol{\mu}}_{l,N})}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \tag{3.53}$$

$$=\prod_{m=1}^{N}\prod_{l\in\Omega}\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\tilde{\boldsymbol{\mu}}_{l,N})}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \tag{3.54}$$

$$=\prod_{m=1}^{N}\prod_{l\in\{i,j\}}\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\tilde{\boldsymbol{\mu}}_{l,N})}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \times \prod_{m=1}^{N}\prod_{l\in\Omega/\{i,j\}}\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\tilde{\boldsymbol{\mu}}_{l,N})}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \tag{3.55}$$

The equality in 3.54 holds because $\{X_{l,m}\}$ are independent for all l and m. We next show that Equation 3.55 is strictly less than 1 and approaches to 0 as $N \to \infty$.

For any $0 < s < 1$ and l, by Jensen's inequality, we have

$$\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l)}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \leq \mathbb{E}\left[\frac{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l)}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)}\right]^s = \left[\int \frac{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l)}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)} f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star) d\boldsymbol{X}_{l,m}\right]^s = 1 \tag{3.56}$$

Moreover, for any $\boldsymbol{\mu}_l \neq \boldsymbol{\mu}_l^\star$, $\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l)}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] < \mathbb{E}\left[\frac{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l)}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)}\right]^s$.

Thus, by replacing the first half of Equation 3.55 with 1, we have

$$\mathbb{E}[(\Lambda^{(3)}_{i,j,N})^s] \leq \prod_{m=1}^{N}\prod_{l\in\Omega/\{i,j\}}\mathbb{E}\left[(\frac{f_l(\boldsymbol{X}_{l,m}|\tilde{\boldsymbol{\mu}}_{l,N})}{f_l(\boldsymbol{X}_{l,m}|\boldsymbol{\mu}_l^\star)})^s\right] \tag{3.57}$$

$$= \mathbb{E}\left[(\frac{f_i(\boldsymbol{X}_i|\tilde{\boldsymbol{\mu}}_{i,N})}{f_i(\boldsymbol{X}_i|\boldsymbol{\mu}_i^\star)}\frac{f_j(\boldsymbol{X}_j|\tilde{\boldsymbol{\mu}}_{j,N})}{f_j(\boldsymbol{X}_j|\boldsymbol{\mu}_j^\star)})^s\right]^N \tag{3.58}$$

Since $(\boldsymbol{\mu}_i^\star, \boldsymbol{\mu}_j^\star) \notin \{(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j) | \boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}$, we have $\mathbb{E}\left[\left(\frac{f_i(\boldsymbol{X}_{i,m}|\boldsymbol{\mu}_i)}{f_i(\boldsymbol{X}_{i,m}|\boldsymbol{\mu}_i^\star)} \frac{f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j)}{f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j^\star)} \right)^s \right] < 1$ for any $(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \in \{(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j) | \boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}$. Thus, there exists a positive number ϵ such that

$$\sup_{(\boldsymbol{\mu}_i, \boldsymbol{\mu}_j) \in \{\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}} \mathbb{E}\left[\left(\frac{f_i(\boldsymbol{X}_{i,m}|\boldsymbol{\mu}_i)}{f_i(\boldsymbol{X}_{i,m}|\boldsymbol{\mu}_i^\star)} \frac{f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j)}{f_j(\boldsymbol{X}_{j,m}|\boldsymbol{\mu}_j^\star)} \right)^s \right] \leq 1 - \epsilon < 1$$

. Furthermore, $\mathbb{E}[(\Lambda_{i,j,N}^{(3)})^s] \leq (1-\epsilon)^N \to 0$ as $N \to \infty$.

In summary, under this scenario, we have $P_c(\tau_i > N) \leq P_c(\Lambda_{i,j,N}^{(3)} > A_i) \leq \frac{\mathbb{E}[(\Lambda_{i,j,N}^{(3)})^s]}{(A)^s} \to 0$ as $N \to \infty$.

Scenario 2: $\boldsymbol{\mu}_i^\star \not\preceq \boldsymbol{\mu}_j^\star$ for all $j \in \Omega / \{i\}$

This scenario considers the situations that system i is not eliminated from $\mathcal{S}_{pool,N+1}$ given that any other systems do not dominate it in $\mathcal{S}_{epf,N+1}$

$$P_c(\tau_i > N) \leq P_c(A < \Lambda_{i,\cdot,N}^{(3)} < B) \leq P_c(\Lambda_{i,\cdot,N}^{(3)} < B) \leq P_c(\min_{j \in \Omega} \Lambda_{i,j,N}^{(3)} < B) \leq \sum_{j \in \Omega} P_c(\Lambda_{i,j,N}^{(3)} < B) \tag{3.59}$$

We need to show that $P_c(\Lambda_{i,j,N}^{(3)} < B) \to 0$ as $N \to \infty$.

By Chebyshev inequality,

$$P_c(\Lambda_{i,j,N}^{(3)} < B) = P_c\left(\frac{1}{\Lambda_{i,j,N}^{(3)}} > \frac{1}{B}\right) \leq \frac{\mathbb{E}[(\Lambda_{i,j,N}^{(3)})^{-s}]}{(B)^{-s}}, \ \forall s > 0 \tag{3.60}$$

Under the Conditions C1 to C5, the maximized likelihood is solvable. Thus, there exists $\tilde{\boldsymbol{\theta}}'_N = \arg\sup_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | \boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}, \boldsymbol{\theta}} L_N(\boldsymbol{\theta})$, with $\tilde{\boldsymbol{\theta}}'_N = [\tilde{\boldsymbol{\mu}}'_{1,N}, ..., \tilde{\boldsymbol{\mu}}'_{K,N}]$. We have

$$\Lambda_{i,j,N}^{(3)} = \frac{\sup_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | \boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | \boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j\}} L_N(\boldsymbol{\theta})} \tag{3.61}$$

$$= \frac{\sup_{\boldsymbol{\theta} \in \{\boldsymbol{\theta} | \boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}, \boldsymbol{\theta}} L_N(\boldsymbol{\theta})}{L_N(\tilde{\boldsymbol{\theta}}'_N)} \geq \frac{L_N(\boldsymbol{\theta}^\star)}{L_N(\tilde{\boldsymbol{\theta}}'_N)} \tag{3.62}$$

The last inequality holds since, by assumption that $\boldsymbol{\mu}_i^\star \not\preceq \boldsymbol{\mu}_j^\star$, the true parameters $\boldsymbol{\theta}^\star$ are within the parameter space defined by $\{\boldsymbol{\theta} | \boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j\}$.

Using the same logic as the proof in Scenario 1, it is evident that $\mathbb{E}[(\Lambda^{(3)}_{i,j,N})^{-s}] \to 0$ as $N \to \infty$. Thus, we have $P_c(\tau_i > N) \leq \sum_{j\in\Omega} P_c(\Lambda^{(3)}_{i,j,N} < B) \leq \sum_{j\in\Omega} \frac{\mathbb{E}[(\Lambda^{(3)}_{i,j,N})^{-s}]}{(B)^{-s}} \to 0$ as $N \to \infty$.

Part 2: the procedure delivers the PCS guarantee

Recall that the PCS for MOR&S is defined by

$$PCS = P(\mathcal{S}_{epf} = \mathcal{S}_{pf})$$

meaning the selected Pareto front, $\mathcal{S}_{epf}$, should match the true Pareto front, $\mathcal{S}_{pf}$.

The proposed procedure uses the Bonferroni correction to deliver the PCS guarantee.

$$\begin{aligned}
PCS &= P\{\mathcal{S}_{epf} = \mathcal{S}_{pf}\} \\
&= 1 - P\{\mathcal{S}_{epf} \neq \mathcal{S}_{pf}\} \\
&= 1 - P\{[\cup_{i\in\mathcal{S}_{pf}} i \notin \mathcal{S}_{epf}] \cup [\cup_{i\notin\mathcal{S}_{pf}} i \in \mathcal{S}_{epf}]\} \\
&\geq 1 - \sum_{i\in\mathcal{S}_{pf}} P(i \notin \mathcal{S}_{epf}) - \sum_{i\notin\mathcal{S}_{pf}} P(x_i \in \mathcal{S}_{epf}) \\
&= 1 - \sum_{i\in\Omega} P(\text{system i is classified incorrectly})
\end{aligned}$$

Next, we need to show that $P(\text{system i is classified incorrectly})$ is controlled at approximately α/K level. Recall the candidacy test:

$$H_{0,i} : i \notin \mathcal{S}_{pf} \text{ and } H_{1,i} : i \in \mathcal{S}_{pf}$$

$\Lambda^{(4)}_{i,n}$ is defined as

$$\Lambda^{(4)}_{i,n} = \frac{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \not\preceq \boldsymbol{\mu}_j, \forall j\in\Omega/i\}} L_n(\boldsymbol{\theta})}{\sup_{\boldsymbol{\theta}\in\{\boldsymbol{\theta}|\boldsymbol{\mu}_i \preceq \boldsymbol{\mu}_j, \exists j\in\Omega/i\}} L_n(\boldsymbol{\theta})} \tag{3.63}$$

By Theorem 3.3.2, by setting $A = \frac{\alpha}{K}$ and $B = \frac{K}{\alpha}$, we have the error rate is approximately at $\frac{\alpha}{K}$ since

$$\sup_{\boldsymbol{\theta}\in\Theta_1} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \leq B) \cong B^{-1} = \frac{\alpha}{K} \tag{3.64}$$

$$\sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \geq A) \cong A = \frac{\alpha}{K} \tag{3.65}$$

as $\alpha \to 0$.

By Theorem 3.3.1, we have

$$\sup_{\boldsymbol{\theta}\in\Theta_1} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(3)}_{i,\cdot,\tau} \leq B) \leq \sup_{\boldsymbol{\theta}\in\Theta_1} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \leq B) \cong B^{-1} = \frac{\alpha}{K} \tag{3.66}$$

$$\sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(3)}_{i,\cdot,\tau} \geq A) \leq \sup_{\boldsymbol{\theta}\in\Theta_0} \mathbf{P}_{\boldsymbol{\theta}}(\Lambda^{(4)}_{i,\tau} \geq A) \cong A = \frac{\alpha}{K} \tag{3.67}$$

as $\alpha \to 0$.

Thus, the error probability of each test on the alternative i is $\frac{\alpha}{K}$. Therefore, we conclude that the *PCS* is bounded below by $1 - \sum \frac{\alpha}{K} = 1 - \alpha$.

■

4. SEQUENTIAL PROCEDURES FOR MULTI-OBJECTIVE FACTOR SCREENING

4.1 Introduction

Simulation models for complex systems typically involve hundreds to thousands of factors. According to Pareto or sparsity-of-effects principle, in many cases, only a few factors among many are responsible for most of the response variation [43]. Factor screening experiments are designed to identify a small subset of important factors efficiently so that the later effort can be focused on them, therefore significantly save the overall experimental effort.

There has been considerable research in this area. Procedures including one-factor-at-a-time designs [44], edge designs [45], and the Trocine screening procedure [46] are designed for stochastic simulation experiments. However, these procedures are based on homogeneous variance assumption and, more importantly, do not consider the error control. Later, procedures including Controlled Sequential Bifurcation [47], Two-stage Controlled Fractional Factorial Screening (TCFF) [47], Controlled Sequential Factorial Design [48], and hybrid method of CSB and CSFD [49] were proposed to address both Type I and Type II error controls, and relax the homogeneous variance requirement. The interested reader should refer to [50] for a thorough review.

Most of the past literature as discussed above focused on the single response model. In practice, however, there are usually multiple responses of interest that can be observed simultaneously in one experiment. For example, risk and return are two responses of a portfolio and finding factors that significantly contribute to both responses is of great importance. The primary challenge for multiple re-

sponse factor screening is to achieve desired error control with efficiency. Note that the multiple responses add another layer of complexity to the error control. The definitions of important and unimportant factors for single response experiments, for example, are not applicable anymore. Lee et al. [31] studied how to allocate computation resources efficiently to identify "Pareto Set" of designs for a multiple objective Ranking and Selection Problem. The concept of "Pareto Set" inspires our definition of important/unimportant factors in this paper. In this research, we want to identify all factors that are important to at least one response. In other words, a factor is considered as unimportant if it is unimportant for all responses. Multiple responses factor screening problem has been firstly studied by Shi et al. [51]. They proposed Multiple Sequential Bifurcation (MSB) to identify important factors. MSB extends CSB error control for each response and adopts sequential bifurcation to classify a group or an individual factor as important or unimportant. However, MSB is conservative since it controls error rate and power across responses through Bonferroni procedures. It also fails to incorporate covariance information among responses.

Most of these researches involve multiple inferences, which is defined as performing more than one statistical inference procedure on the same data set. However, none of these researches so far, to the best knowledge of the authors, perform multiple inferences without adjusting the Type I error rate accordingly. According to the American Statistical Association, "Running multiple tests on the same data set at the same stage of analysis increases the chance of obtaining at least one wrong result. Selecting the one "significant" result from a multiplicity of parallel tests poses a grave risk of an incorrect conclusion. Failure to disclose the full extent of tests and their results in such a case would be highly misleading." (Professionalism Guideline 8 [52]).

To account for multiple inferences, we target our proposed procedures to control the Family-Wise Error Rates (FWERs), instead of individual Type I error rate

and power for each factor. We consider two types of FWER, Family-Wise Error Rate I and Family-Wise Error Rate II, which are defined as follows:

Definition 4.1.1 *Let V denote the number of true null hypotheses being rejected and T the number of false null hypotheses being accepted. The Family-Wise Error Rate I (FWERI) is defined as the probability to reject at least one true null hypothesis; the Family-Wise Error Rate II (FWERII) is the probability of accepting at least one false null hypothesis.*

$$FWERI = Prob\,(V \geq 1) \tag{4.1}$$

$$FWERII = Prob\,(T \geq 1) \tag{4.2}$$

Note that FWERI and FWERII are independent of the number of individual tests. See more discussions on FWERI and FWERII in [53].

Most previous studies on multiple hypothesis testings are non-sequential. Such procedures, including Bonferroni, Sidak, and Holm's step-down methods, are designed to control FWERI. Due to the non-sequential nature, these procedures are unable to provide controls on FWERII. De and Baron first explored the sequential analogs of these procedures in [54]. They extended the Holm's step-down methods into likelihood ratio test; and proposed Holm's intersection Procedure (HIP) to control the FWERI and FWERII in the strong sense. HIP is rigorous, yet conservative. This paper proposes two stronger stopping criteria than HIP, Sum Intersection Procedure (SUMIP) and Sort Intersection Procedure (SORTIP).

Both SUMIP and SORTIP are Sequential Probability Ratio Test (SPRT) procedures. SPRT is a fully sequential likelihood-ratio testing procedure that varies with the specifics of hypotheses and assumed distributions. From Neyman-Pearson lemma, among all tests with the same significance level, likelihood ratio test has the most significant power. Moreover, the sequential nature of SPRT allows it to achieve the same power more efficiently than the fixed number of samples [21]. SPRT first chooses a pair of constants A and B with $0 < A < 1 < B < \infty$ as thresholds of likelihood ratio. After each observation, SPRT will calculate Λ, the

likelihood ratio of hypothesis H_1 against H_0, and choose H_0 if Λ is smaller than A, H_1 if Λ is higher than B, or take one more observation. The smaller the A, the smaller the value of Λ needed to accept the null hypothesis, and then the higher the power; and the higher the B, the larger the value of Λ needed to reject the null hypothesis and then the smaller the Type I error.

There are two phases in SUMIP and SORTIP. Phase 1 is to calculate Likelihood Ratio (LR) for each factor; phase 2 is to check whether LRs of all factors are sufficient to categorize factors into important and unimportant sets within permitted FWERI and FWERII. For Phase 1, we derive formula and algorithm to calculate the likelihood ratio for each factor. In multiple-response factor screening problem, individual test on each factor takes the forms of multivariate normal union-intersection tests. Literature provides two other test methods. The first method is to approximate the test as Hotelling t test or χ^2 test. These tests are substantially less powerful than likelihood ratio test (LRT) for testing multivariate normal mean vector $H_0: \mu = 0$ against $H_1: \mu \neq 0$ according to Neymann-Pearson Lemma [21]. Moreover, χ^2 or Hotelling t tests are bias for one-sided test [55]. Another choice is to decompose the testing problem into a collection of sub-problems and treat them independently. [56,57] discussed procedures in this fashion. However, their procedures neither fit sequential manner nor take advantage of covariance information. In this paper, we provide the exact likelihood ratio and prove that maximum likelihood estimator (MLE) is the solution of a quadratic programming problem. Also, our method is robust under various covariance structure as demonstrated in our numerical evaluation.

For Phase 2, we propose two efficient and powerful stopping criteria to control FWERI and FWERII. The complexity of such schemes lies in the number of possible hypotheses combinations. For the test on a single factor, there are only two possibilities H_0 and H_1. However, for familywise test that formed by I individual tests, the decision schemes should be able to make decisions among 2^I possibilities. The common practice is to apply Bonferroni method among these I tests, which is

intuitive but too conservative in large-scale case. In literature, the most efficient procedure is the Holm Intersection Procedure (HIP). Our procedures are more efficient than the HIP concerning using fewer samples. Numerical results show that our proposed methods achieve the same power as HIP with only approximately half of the sample size.

In our numerical evaluation, we use a factorial design to estimate factor coefficients. Thus, one sample is equivalent to one replicate of the factorial design. It is worth noting here that SUMIP and SORTIP can be used for any designs, as long as the estimated coefficient vector of each factor follows a multivariate normal distribution.

The rest of this paper is organized as follows: Section 4.2 states the problem details and assumptions. Section 4.3 presents the two proposed procedures. Numerical evaluate are given in Section 4.4. We conclude the paper with discussions of future research in Section 5.

4.2 Problem Statement

In this paper, we assume a first-order model for all responses,

$$Y_j = \beta_{0j} + \sum_{i=1}^{I} \beta_{ij} x_i + \epsilon_j,\ \epsilon_j \sim N(0, \sigma_j^2),\ j = 1, 2, ...J \tag{4.3}$$

which could also be written in matrix form as

$$Y = X'\beta + \epsilon \tag{4.4}$$

where $Y = \left[Y_j\right]_J$, $X = [x_i]_I$, $\epsilon = \left[\epsilon_j\right]_J$ and $\beta = \left[\beta_{ij}\right]_{I \times J}$. Assume Y_j is the output of the j^{th} response, and β_{ij} represents the main effect of the i^{th} factor on the j^{th} response. Quantitative inputs, x_i, are normalized such that they have only two levels which are arbitrarily denoted as -1 or 1. ϵ are the metamodel residuals with mean 0. Furthermore, we assume ϵ_j for $j = 1, 2, ..., J$ can be correlated and their scale may depend on the size of the output. In other words, we assume (1)

unknown covariance structure among different responses, and (2) heterogeneous variance condition.

In this paper, we define "important factor" in multiple responses setting. We use definitions from Lee et al. (2010) and Shi et al. (2014), where important factors are defined as factors which have an important effect on at least one response. Let Δ_{0j} be the threshold for important effect on the jth response, and Δ_{1j} the threshold for critical effect on the jth response.

Definition 4.2.1 *The ith factor β_i is **unimportant** if $|\beta_{ij}| \leq \Delta_{0j}$ for all $j = 1, 2, ..., J$, **important** if there exists at least one j such that $|\beta_{ij}| \geq \Delta_{0j}$, and **critical** if there exists at least one j such that $|\beta_{ij}| \geq \Delta_{1j}$. Specifically,*

$$\Theta_c = \left\{ \beta_i | \max_{j \in \{1,2,..,J\}} \left(|\beta_{ij}| - \Delta_{1j} \right) \geq 0,\ i \in \{1, 2, ..I\} \right\}$$
$$\Theta_{imp} = \left\{ \beta_i | \max_{j \in \{1,2,..,J\}} \left(|\beta_{ij}| - \Delta_{0j} \right) \geq 0,\ i \in \{1, 2, ..I\} \right\}$$
$$\Theta_0 = \left\{ \beta_i | \max_{j \in \{1,2,..,J\}} \left(|\beta_{ij}| - \Delta_{0j} \right) \leq 0,\ i \in \{1, 2, ..I\} \right\}$$

With this definition, our goal is, under proper error control, to screen out factors within Θ_0 and claim them unimportant, and to find out factors in Θ_c, which have significant impacts on responses and claim them important. For factors within the "indifference zone", which means factor is in Θ_{imp} but not in Θ_c, although they are important, it is neither possible nor our intention to control the power for these factors.

To distinguish, we call the single test for each factor the **elementwise test**, and the I elementwise tests form the **familywise test**. In this paper, elementwise test is defined as follows:

$$H_0^i :\ \beta_i \in \Theta_0 \quad H_1^i :\ \beta_i \in \Theta_{imp} \quad \text{for } i = 1, ..., I \tag{4.5}$$

This elementwise test can be shown in the Union-Intersection form,

$$H_0^i = \bigcap_{j=1}^{J} H_o^{ij} \quad H_1^i = \bigcup_{j=1}^{J} H_1^{ij}$$

where

$$H_0^{ij}: \beta_{ij} < \Delta_{0j} \quad H_1^{ij}: \beta_{ij} \geq \Delta_{0j} \text{ for } j = 1, ..., J \tag{4.6}$$

Familywise test includes I elementwise tests. As discussed earlier, we intend to control FWERI and FWERII, where FWERI is the probability of identifying at least one unimportant factor as important, and FWERII is the probability of classifying at least one critical factor as unimportant.

4.3 The Procedures

Procedures proposed in this paper contain two phases as depicted in 4.1. Phase 1 is to calculate Likelihood Ratio (LR) for each factor, whose effects follow multivariate normal distribution; phase 2 is to aggregate LRs of all factors and decide whether the LRs are sufficient to make statistic inference with control of FWERI and FWERII. This section will discuss these two phases and state the algorithms.

4.3.1 Phase 1: Elementwise Likelihood Ratio

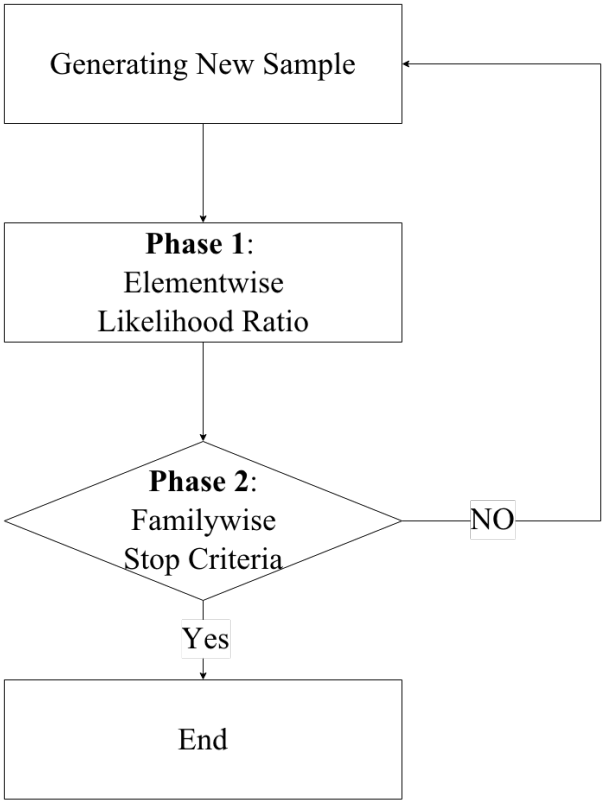
Recall the test for factor i,

$$H_0^i: \beta_i \in \Theta_0 \quad H_1^i: \beta_i \in \Theta_{imp} \tag{4.7}$$

From experiments, we have $\{Z_{i1}, Z_{i2}, ...Z_{iK}\}$ as K estimators of β_i, where Z_{ik} is of dimension $J \times 1$ and from distribution $N(\beta_i, \Sigma_i)$ with unknown mean β_i and covariance matrix Σ_i. If $K \geq J + 1$, then $S_i = \sum_{k=1}^{K} (Z_{ik} - \bar{Z}_i)(Z_{ik} - \bar{Z}_i)'$ is positive definite with probability one. In the scope of factor screening problem, these estimators can be derived via regression analysis.

Assuming $\{Z_{i1}, Z_{i2}, ...Z_{iK}\}$ are independent. $\bar{Z}_i$ and S_i are then sufficient statistics for β_i and Σ, and $\sqrt{K}\bar{Z}_i \sim N(\beta_i, \Sigma_i)$, $S_i \sim$ Wishart$(K - 1, \Sigma_i)$, and $\bar{Z}_i$ and S_i are independent. Based on these assumptions, the likelihood function and likelihood ratio for (4.7) are,

Figure 4.1.: Procedure flowchart



$$\Lambda_i\left(\Theta_0,\Theta_c\right)=\frac{L_1}{L_0}=\frac{\left(1+\left\|\bar{Z}_i-\beta_{mle}\left(\bar{Z}_i,S_i,\Theta_c\right)\right\|^2_{\frac{S_i}{K}}\right)^{-\frac{K}{2}}}{\left(1+\left\|\bar{Z}_i-\beta_{mle}\left(\bar{Z}_i,S_i,\Theta_0\right)\right\|^2_{\frac{S_i}{K}}\right)^{-\frac{K}{2}}} \tag{4.8}$$

where $\|Z-\beta\|^2_S = (\beta-Z)' S^{-1}(\beta-Z)$ and $\beta_{mle}(Z,S,\Theta)$ is the maximum likelihood estimator for β within the parameter space Θ,

$$\beta_{mle}(Z,S,\Theta)=\arg\min_{\beta\in\Theta}\|Z-\beta\|^2_S$$

See the appendix for the proof.

For $\Theta = \Theta_0$, $\beta_{mle}(Z, S, \Theta_0)$ is the solution of the following quadratic programming problem with convex constraints,

$$\begin{aligned} &\min (\beta - Z)' S^{-1} (\beta - Z) \\ &\text{s.t.} \left|\beta_j\right| \leq \Delta_{0j} \text{ for } j = 1, 2, ..., J \end{aligned} \tag{4.9}$$

For $\Theta = \Theta_c$, $\beta_{mle}(Z, S, \Theta_c)$ is the solution of the following problem,

$$\begin{aligned} &\min (\beta - \bar{Z})' S^{-1} (\beta - \bar{Z}) \\ &\text{s.t.} \left|\beta_j\right| \geq \Delta_{1j} \text{ for at least one } j,\ j = 1, 2, ..., J \end{aligned} \tag{4.10}$$

Note that this is no longer a convex quadratic programming problem. We prove that it can be solved as a convex problem as demonstrated in Theorem 4.3.1.

Theorem 4.3.1 *If $Z \in \Theta_c$, $\beta_{mle}(Z, S, \Theta_c) = Z$;*

If $Z \notin \Theta_c$, $\beta_{mle}(Z, S, \Theta_c)$ is the solution of the following quadratic programming problem,

$$\begin{aligned} &\min (\beta - Z)' S^{-1} (\beta - Z) \\ &s.t.\ \beta_{j^\star} = sign(Z_{j^\star}) \Delta_{1j^\star} \end{aligned} \tag{4.11}$$

where $j^\star = \arg\min_{j=1,...,J} \frac{\left(\Delta_{1j} - z_j\right)^2}{s_j}$ with $s_j > 0$ as the j^{th} diagonal element of S.

Thus, both $\beta_{mle}(Z, S, \Theta_0)$ and $\beta_{mle}(Z, S, \Theta_c)$ could be modeled as quadratic optimization problems with linear constraints and, thus, be solved easily.

4.3.2 Phase 2: Familywise Stopping Scheme

For the familywise test, we could choose whether to decide for elementwise tests simultaneously or separately, which correspond to two types of stopping rules. One is the intersection rule, which makes inferences simultaneously; the other one is the maximum rule, which treats I elementwise tests separately. The intersection rule is less conservative since it dynamically allocates the "allowed errors" according to the likelihood ratios, instead of assigning the same amount of errors to each factor [54].

Intersection Rule

Let A_i and B_i ($0 < A_i < 1 < B_i < \infty$) be the pair of thresholds for Likelihood Ratio Test of the i^{th} elementwise test. The intersection rule states that if all elementwise tests reach decision regions, which means that the likelihood ratio $\Lambda^i \notin (A_i, B_i)$ for all $i \in \{1, 2, ...I\}$, then stop sampling and choose H_{0i} if Λ^i is smaller than A_i or H_{1i} if Λ_i is greater than B_i; otherwise take one more observation for all elementwise tests. Therefore, the intersection rule stops at the first time when $\Lambda_n^i \notin (A_i, B_i)$ for all $i = 1, 2, 3, ...I$.

$$N_{int} = \inf \left\{ n : \bigcap_{i=1}^{I} \Lambda_n^i \notin (A_i, B_i) \right\}$$

Intersection rule is a proper stopping time since it stops in finite time with probability 1 (De and Baron, 2012). The following corollary reveals the connection between thresholds, A_i, and B_i, moreover, Type I error and Type II error for the intersection rule.

Theorem 4.3.2 *(De and Baron, 2012) Let τ be any stopping time satisfying*

$$Prob\left(\Lambda_\tau^i \in (A_i, B_i)\right) = 0 \; for \; A_i < 1, \; B_i > 1 \; for \; i \in \{1, 2, ..., I\}$$

with the decision rule of rejecting H_0^i if and only if $\Lambda_\tau^i \geq A_i$. For such a test

$$Prob\left(Type\ I\ on\ i^{th}\ test\right) = Prob\left(\Lambda_\tau^i > B_i | H_{0i}\ is\ true\right) \leq B_i^{-1}$$
$$Prob\left(Type\ II\ on\ i^{th}\ test\right) = Prob\left(\Lambda_\tau^i < A_i | H_{1i}\ is\ true\right) \leq A_i$$

Based on Theorem 4.3.2, we propose two schemes, Sum Intersection Scheme and Sort Intersection Scheme, to connect thresholds, A_i, and B_i, with Family Wise Error Rate.

Sum Intersection Scheme

We propose Sum Intersection Scheme with the intersection stopping time and boundaries as

$$N_{int} = \inf\left\{ n : \bigcap_{i=1}^{I} \Lambda_n^i \notin (A_i, B_i) \right\}, \sum_{i\in I_A} A_i \leq \gamma, \sum_{i\in I_B} B_i^{-1} \leq \alpha \tag{4.12}$$

where $I_A = \left\{ i : \Lambda_n^i \leq 1 \right\}$, $I_B = \left\{ i : \Lambda_n^i > 1 \right\}$ and α and γ are the predefined FWERI and FWERII, respectively. I_A and I_B are determined after each iteration and hence may change at each step.

Intuitively, this scheme applies Bonferroni procedure, and sets $B_i = \infty$ for factors to be claimed as unimportant (factors within I_A) and $A_i = 0$ for factors to be claimed as unimportant (factors within I_B); so we have FWERI$\leq \sum_{i=1}^{I} B_i^{-1} = \sum_{i\in I_B} B_i^{-1} \leq \alpha$. Same logic applies to FWERII. At N_{int}, factors within I_B are claimed important, while factors within I_A unimportant. We have following result regarding to Sum Intersection Scheme (See proof in appendix 4.7.3.

Theorem 4.3.3 *The Sum Intersection Scheme strongly controls both FWERI and FWERII. That is,*

$$FWERI = Prob\left(At\ least\ 1\ Type\ I\ error\ among\ I\ tests\right) \leq \alpha$$
$$FWERII = Prob\left(At\ least\ 1\ Type\ II\ error\ among\ I\ tests\right) \leq \gamma$$

Sort Intersection Scheme

We define Sort Intersection Scheme with stopping time and boundaries as,

$$N_{int} = \inf\left\{ n : \left(\bigcap_{i\in I_A} \Lambda_n^{(i)} \notin (A_i, 1) \right) \bigcap \left(\bigcap_{j\in I_B} \Lambda_n^{(j)} \notin (1, B_j) \right) \right\}, A_i = \frac{\gamma}{|I_A| - i + 1}, B_j = \frac{j}{\alpha} \tag{4.13}$$

where α and γ be the upper bounds of FWERI and FWERII, and $\Lambda_n^{(i)}$ is the ith smallest likelihood ratios. After each sampling, factors are sorted in increasing order and then separated into two groups according to their likelihood ratios, $I_A =$

$\{i : \Lambda^i \leq 1\}$ and $I_B = \{i : \Lambda^i > 1\}$, with cardinalities $|I_A|$ and $|I_B|$. For factors within I_A, it is certain that we would not commit Type I error. Thus, we only need to consider FWERI for factors within I_B. The same logic applies to factors within I_B.

Theorem 4.3.4 *The Sort Intersection Scheme strongly controls both FWERI and FWERII. That is,*

$$FWERI = Prob\,(At\ least\ 1\ Type\ I\ error\ among\ I\ tests) \leq \alpha$$
$$FWERII = Prob\,(At\ least\ 1\ Type\ II\ error\ among\ I\ tests) \leq \gamma$$

4.3.3 Procedures

We now apply the schemes mentioned above into screening procedures. It is worth mentioning that the hypothesis testing procedures can be applied in screening problem with any experimental designs as long as the estimator of the factor effect follows a multivariate normal distribution.

Sum Intersection Procedure

The first procedure, Sum Intersection Procedure (SUMIP for short), is a two-phase procedure.

Phase 1 calculates likelihood ratio for each elementwise test. Phase 2 applies Sum Intersection Scheme. This procedure calculates Likelihood Ratios (LRs) after sampling; then assigns factors into I_B if $LR > 1$ or I_A otherwise. If LRs in I_B and I_A satisfy (4.12), then we claim that factors within I_A are unimportant and I_B are important. If not, we collect one more set of samples. Notice that Y_k is the kth simulation result.

Algorithm 6 SUMIP Procedure

Step 0 Select a factorial design X for I factors, generate n_0 replications of observations. Let $n = n_0$.

Step 1

Likelihood Ratio Calculate Likelihood Ratio Λ_n^i for each factor $i \in \{1, 2, ..., I\}$ with

$$Z_k = \left(X'X\right)^{-1} X'Y_k,\ k = 1, 2, ...n_0$$
$$S_n = \sum_{k=1}^{n} \left(Z_k - \bar{Z}\right)\left(Z_k - \bar{Z}\right)'$$

Stopping Criteria Apply Sum Intersection Scheme

Divide factors into two set $I_A = \left\{i : \Lambda_n^i \leq 1\right\}$ and $I_B = \left\{i : \Lambda_n^i > 1\right\}$
If $\alpha \geq \sum_{i \in I_B} \left(\Lambda_n^i\right)^{-1}$ and $\gamma \geq \sum_{i \in I_A} \Lambda_n^i$, go to Step 2.
Otherwise, generate 1 more observation with X, and make $n = n + 1$; back to Step 1.

Step 2 Claim factors within I_B are important, and factors within I_A are unimportant.

Sort Intersection Procedure

This procedure, Sort Intersection Procedure (SORTIP), is based on Sort Intersection Scheme. It has the same structure as SUMIP except for stopping criteria. This procedure will run n_0 initial replications of the design at first, where $n_0 \geq J$; then split factors into two groups, $I_A = \left\{i : \Lambda^i \leq 1\right\}$ and $I_B = \left\{i : \Lambda^i > 1\right\}$. If likelihood ratios satisfy (4.13), then we claim factors within I_B as important and I_A as unimportant. If not, we conduct one more factorial design for all factors.

Algorithm 7 SORTIP Procedure

Step 0 Select a factorial design X for I factors, generate n_0 replications of observations. Let $n = n_0$.

Step 1

Stopping Criteria Calculate Likelihood Ratio Λ_n^i for each factor $i \in \{1, 2, ..., I\}$ with

$$Z_k = \left(X'X\right)^{-1} X'Y_k,\ k = 1, 2, ...n_0$$
$$S_n = \sum_{k=1}^{n} \left(Z_k - \bar{Z}\right)\left(Z_k - \bar{Z}\right)'$$

Stopping Criteria Apply Sort Intersection Scheme

Divide factors into two set $I_A = \left\{i : \Lambda_n^i \leq 1\right\}$ and $I_B = \left\{i : \Lambda_n^i > 1\right\}$ and sort $\Lambda_n^i, i \in I_A$ and $\Lambda_n^j, j \in I_B$ ascending.

If $\Lambda_n^{(i)} \notin (A_i, 1)$ where $A_i = \frac{\gamma}{I_A - i + 1}$, and $\Lambda_n^{(j)} \notin \left(1, B_j\right)$ where $B_j = \frac{j}{\alpha}$ for all $i \in I_A,\ j \in I_B$, then go step 2.

Otherwise, generate 1 more observation with X, and make $n = n + 1$; back to step 1.

Step 2 Claim factors within I_B are important, and factors within I_A are unimportant.

Mixed Intersection Procedure

We now turn from SUMIP and SORTIP to a procedure under parallel fashion.

Assume P workers and one master available in the parallel environment. The worker-master structure requires us to (a) Balance workload among workers, and (b) Reduce communications between workers and master.

While, both SUMIP and SORTIP are sequential procedures and not ready for the parallel environment. Recall that both SUMIP and SORTIP are a two-phase

procedure, whereas stage 1 is to calculate the likelihood ratio for each factor and stage 2 is to test if there is enough information to make a decision. For the first requirement, both SUMIP and SORTIP could divide factors into P groups (or close to) evenly. Since all factors will be sampled for the same times, the workload on each worker is well-balanced if the number of factors assigned to workers is the same. For the second requirement, neither SUMIP nor SORTIP could meet if the number of factors is enormous. If workers send all likelihood ratios information to master, communications would be the bottleneck.

To address this problem, we implement an additional step at worker level and send only two numbers α_p and γ_p, namely, the minimum amount of error levels to decide for a worker, back to master after each sampling stage and keep on sampling until master says not. At master level, master makes decision according to Bonferroni rules, go on sampling if $\sum_p \alpha_p \leq \alpha$ and $\sum_p \gamma_p \leq \gamma$, for $p \in \{1, 2, ..., P\}$.

α_{pn} and γ_{pn} are defined in as follow.

$$\alpha_{pn} = \max_{i=\{1:|I_B|\}} \left\{ \frac{i}{\Lambda_{p,n}^{(i+|I_A|)}} \right\}$$
$$\gamma_{pn} = \max_{i=\{1:|I_A|\}} \left\{ (|I_A - i + 1| \cdot \Lambda_{p,n}^{(i)} \right\}$$

where $\Lambda_{p,n}^{i}$ be the likelihood ratio for the ith factor on worker p, $I_A^p = \left\{ i : \Lambda_{p,n}^{i} \leq 1 \right\}$ and $I_B^p = \left\{ i : \Lambda_{p,n}^{i} > 1 \right\}$; then sort $\Lambda_{p,n}^{i}$ in ascending order.

For worker p, it is easy to verify that $\text{FWERI}_\text{p} \leq \alpha_{pn}$ and $\text{FWRII}_\text{p} \leq \gamma_{pn}$ according to 4.3.3. For master, let V be the number of unimportant factor classified as important among all factors and V_p the number among factors in worker p.

$$\begin{aligned}
\text{FWERI} =& Prob\left(V \geq 1\right) = Prob\left(\sum_p V_p \geq 1\right) \\
=& Prob\left(\bigcup_p \left(V_p \geq 1\right)\right) \leq \sum_p Prob\left(V_p \geq 1\right) \\
=& \sum_p \text{FWERI}_\text{p} \leq \sum_p \alpha_{pn}
\end{aligned}$$

The first "$\leq$" sign holds since Boole's inequality.

4.4 Simulation Studies

In this section, we compare our proposed procedures, SORTIP and SUMIP, with HIP proposed by De and Baron (2012) and MSB (Shi et.al 2014) to demonstrate the efficiency and validity of both SORTIP and SUMIP. MSB procedure controls Type I error rate and Type II error rate for the elementwise test. Thus, we compare the Likelihood ratio testing procedure used as Phase 1 in SUMIP and SORTIP with MSB. HIP differs from SUMIP and SORTIP in stopping criteria. We run all three procedures for factor screening in the second experiment to compare their performances and conclude both SUMIP and SORTIP outperform HIP.

Numerical simulations use common random numbers across different methods to compare these algorithms.

4.4.1 Study 1: Single-Factor Multiple-Responses Factor Screening

This experiment is to show the performance of the proposed Likelihood Ratio Test, which is phase 1 of SORTIP and SUMIP, with MSB. Consider a factor screening problem with a single factor and multiple responses. Parameters are set as follows:

In all cases, the presented results are the averages of 1000 independent macro replications. For this example, we take three covariance matrix forms to represent independent case, positive dependent case, and nested case, respectively. For nested case, there are both positive and negative correlations among responses.

$$\Sigma_1 = \begin{bmatrix} 4 & 0 & & 0 \\ 0 & 4 & & 0 \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & 4 \end{bmatrix} \quad \Sigma_2 = \begin{bmatrix} 4 & 1 & & 1 \\ 1 & 4 & & 1 \\ & & \ddots & \vdots \\ 1 & 1 & \cdots & 4 \end{bmatrix} \quad \Sigma_3 = \begin{bmatrix} 4 & (-1)^2 & & (-1)^J \\ (-1)^2 & 4 & & (-1)^{J+1} \\ & & \ddots & \vdots \\ (-1)^J & (-1)^{J+1} & \cdots & 4 \end{bmatrix}$$

Table 4.2 and Figure 4.2 present the average sample sizes required by our proposed likelihood ratio test and MSB. Table 4.2 shows the average samples and achieved errors for Likelihood Ratio Test and MSB when $J = 5$. Notice that we

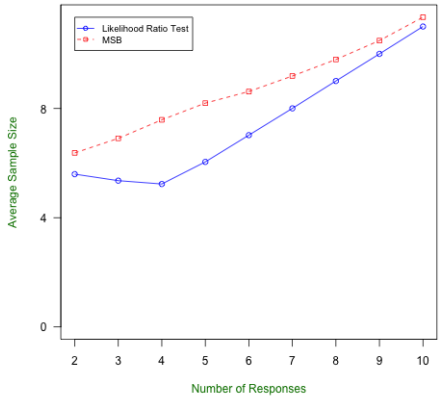
Table 4.1.: Study 1: Configurations of Simulation Experiment on a Single Factor Screening

Parameter	Value	Meaning
J	Specified below	Number of Responses
Δ_0	2	Threshold of Important Factors
Δ_1	4	Threshold of Critial Factors
α	0.05	Upperbound of Type I Error
γ	0.05	Upperbound of Type II Error
β	Specified below	Responses size
Σ	Specified below	Covariance matrix
n_0	6	Initial sample size

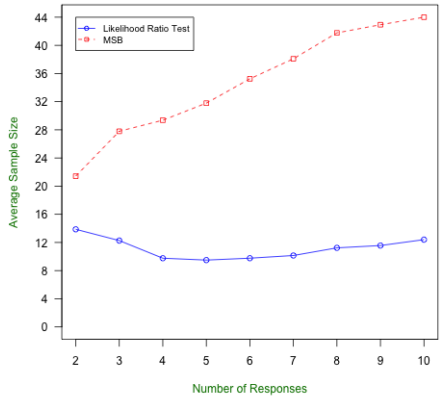
have many "-" in Table 4.2 since, for instance, if the factor is unimportant, we cannot commit Type II error. Figure 4.2 compares the performance of likelihood ratio test with MSB under different setting of J, the number of responses. We set J to take value from 2 to 10. Let us assume β take one of three following configurations

$$\beta_1' = (0, 0, ..., 0, 1) \qquad \beta_2' = (0, 0, ..., 0, 2) \qquad \beta_3' = (0, 0, ..., 0, 5)$$

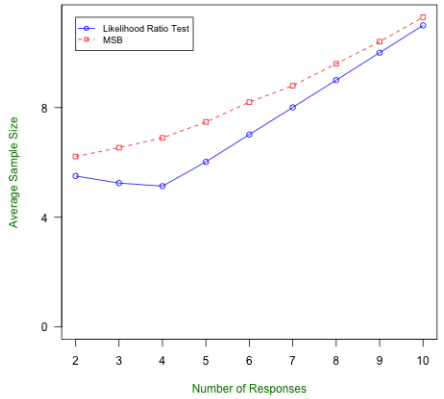
We can see the proposed likelihood ratio test are more efficient than MSB. The advantage of the likelihood ratio test over MSB is even more apparent when factors are within the indifferent zone. In the five response case, likelihood ratio test saves 30% simulation effort on average when compared with MSB. When the factor is important but not critical, likelihood ratio test needs only approximately $1/4$ of the computation effort of MSB. Figure 4.2 indicates that in some case, the gap between the likelihood ratio test and MSB is closing as the number of responses is increasing. However, the likelihood ratio test still performs universally better than MSB regarding computational effort. Moreover, for MSB, the computational effort highly depends on the coefficient of the factors and the average sample size varies



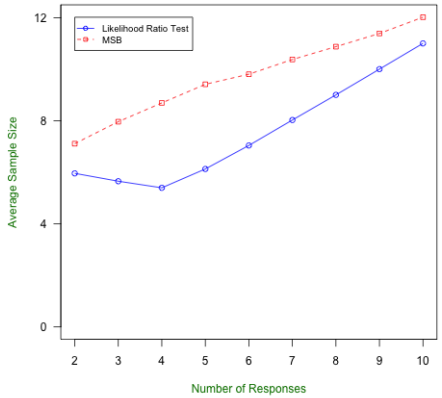
(a) $\beta = \beta_1'$ and $\Sigma = \Sigma_1$



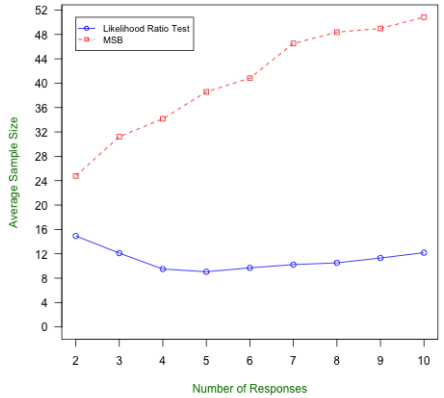
(b) $\beta = \beta_2'$ and $\Sigma = \Sigma_1$



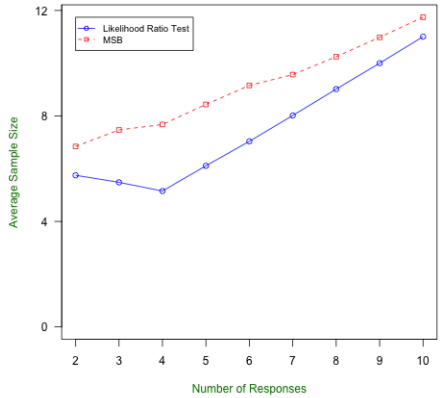
(c) $\beta = \beta_3'$ and $\Sigma = \Sigma_1$



(d) $\beta = \beta_1'$ and $\Sigma = \Sigma_2$



(e) $\beta = \beta_2'$ and $\Sigma = \Sigma_2$



(f) $\beta = \beta_3'$ and $\Sigma = \Sigma_2$

Figure 4.2.: Study 1: plots of the average sample size of a factor screening problem with a single factor and observations from $N(\beta, \Sigma)$. The x coordinate represents the number of objectives (dimensions); the y coordinate represents the average sample size. Numbers are summarized from 10,000 independent macroreplications of likelihood ratio test procedure we proposed and the MSB procedure.

Table 4.2.: Study 1: the average sample size (Avg. SSize) and the estimated error probability (Est. Errors) of a factor screening problem with $I = 1$ factor and observations from $N(\beta, \Sigma)$. Numbers are summarized from 10,000 independent macroreplications of likelihood ratio test procedure we proposed and the MSB procedure.

		Likelihood Ratio Test		MSB	
β	Covariance	Avg. SSize	Est. Errors	Avg. SSize	Est. Errors
$(0,0,0,0,1)$	Σ_1	6.154	0.1%	9.513	0.0%
	Σ_2	6.163	0.2%	9.456	0.0%
	Σ_3	6.100	0.9%	9.549	0.0%
$(1,1,1,1,1)$	Σ_1	6.536	2.2%	11.895	0.0%
	Σ_2	6.495	3.4%	11.598	0.1%
	Σ_3	6.418	3.3%	11.954	0.0%
$(0,0,0,0,3)$	Σ_1	9.818	-	36.606	-
	Σ_2	9.121	-	38.307	-
	Σ_3	9.214	-	39.753	-
$(3,3,3,3,3)$	Σ_1	6.350	-	22.766	-
	Σ_2	7.008	-	27.678	-
	Σ_3	6.174	-	22.097	-
$(0,0,0,0,5)$	Σ_1	6.141	0.0%	8.535	0.1%
	Σ_2	6.063	0.2%	8.278	0.0%
	Σ_3	6.073	0.3%	8.405	0.0%
$(5,5,5,5,5)$	Σ_1	6.000	0.0%	6.192	0.0%
	Σ_2	6.001	0.0%	6.296	0.0%
	Σ_3	6.000	0.0%	6.201	0.0%

from 10 to 40 when $j = 10$. As for the likelihood ratio test, the average sample size is between 10 to 12 when $j = 10$.

4.4.2 Study 2: Multiple-Factors Multiple-Responses Factor Screening

This experiment is to compare SUMIP and SORTIP with the HIP. Resolution III fractional factorial design is used in all three procedures. Consider a factor screening problem with five responses. Table 4.3 lists all parameters for this experiment.

Table 4.3.: Study 2: Configurations of Simulation Experiment on a multiple Factor Screening Problem

Parameter	Value	Meaning
I	50, 100, and 150	Number of Factors
J	4	Number of Responses
Δ_0	2	Threshold of Important Factors
Δ_1	4	Threshold of Critial Factors
α	0.05	Upperbound of FWERI
γ	0.1	Upperbound of FWERII
β	Specified below	Responses
Σ	Specified below	Covariance matrix
n_0	5	Initial sample size

We consider two scenarios for β_{ij}. Scenario one has 5% critical factors, 5% important but not critical factors, moreover, 90% unimportant factors. Scenario two has 10% critical factors, 10% important but not critical factors, and 80% unimportant factors. Critical factors' coefficients are randomly generated from uniform distribution on $(\Delta_1, 6)$, unimportant factors' coefficients are generated from uniform distribution $(0, \Delta_0)$, the important factors are generated from uniform distribution (Δ_0, Δ_1).

For the i^{th} factor, its covariance is one of the three cases,

$$\Sigma_i = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} + m_i \begin{bmatrix} \beta_{i1} & 0 & 0 & 0 \\ 0 & \beta_{i2} & 0 & 0 \\ 0 & 0 & \beta_{i3} & 0 \\ 0 & 0 & 0 & \beta_{i4} \end{bmatrix}$$

where $m_1 = 0$, $m_2 = 0.5$ and $m_2 = 2$.

We randomly generate two scenarios of β and Table 4.4 and Table 4.5 present the average sample sizes of 1,000 independent experiments required by SUMIP, SORTIP, and HIP. The sample size of each experiment is the product of two parts: (1) the number of factors in the Resolution III fractional factorial design, (2) the number of replications to conclude. For instance, if we have 50 factors and ten replications to conclude since the number of the design points in Resolution III fractional factorial design is 64, the sample size would be 64 times 10. In all cases in these two scenarios, both SUMIP and SORTIP dominate HIP regarding average sample size. SUMIP and SORTIP use approximately 50% to 60% sample size required in the HIP in most cases without the loss in power. Although SUMIP performs little worse than SORTIP, its stopping criteria is much simpler than both SORTIP's and HIP.

4.5 Case Study

The case study discussed in this section is a re-entrant bottleneck simulation model with attributes of the real-world wafer fabrication environment of a semiconductor facility production system. The significant characteristics of the simulation model include re-entrant process, unreliable machines, and batching machines. The simulation model is made up of 11 stations. In this study, we assume all the jobs follow the same routing through the system as in Figure **??**. The system contains 11 stations. For a particular production job, **??** shows a sequence of sta-

Table 4.4.: Study 2: the average sample size (Avg. SSize) and the estimated type I and type II FWER (FWERI, FWERII) of a factor screening problem with I factor and observations from $N(\beta, \Sigma)$. The configuration of β uses the first scenario, 5% factors are critical. Numbers are summarized from 10,000 independent macroreplications of HIP, SUMIP, and SORTIP.

		HIP			SUMIP			SORTIP		
I	Σ	Avg Samples	FWERI	FWERII	Avg Samples	FWERI	FWERII	Avg Samples	FWERI	FWERII
50	Σ_1	1038.208	0.0%	0.0%	520.512	0.0%	0.5%	474.816	0.0%	1.0%
	Σ_2	1089.216	0.0%	0.6%	481.984	0.0%	0.7%	470.848	0.0%	0.7%
	Σ_3	1179.264	0.0%	0.2%	535.808	0.0%	0.3%	535.920	0.0%	0.3%
100	Σ_1	2583.040	0.0%	0.0%	1229.560	0.0%	0.6%	1113.984	0.0%	0.6%
	Σ_2	2597.120	0.0%	0.0%	1136.896	0.0%	0.6%	1078.144	0.0%	0.6%
	Σ_3	2923.392	0.0%	0.0%	1298.048	0.0%	1.3%	1189.130	0.0%	1.3%
150	Σ_1	10965.504	0.0%	0.2%	5142.528	0.0%	0.2%	4904.960	0.0%	0.2%
	Σ_2	11048.960	0.0%	0.1%	5204.480	0.0%	0.5%	4874.240	0.0%	0.5%
	Σ_3	12914.416	0.0%	0.1%	6337.536	0.0%	0.2%	5766.144	0.0%	0.2%

Table 4.5.: Study 2: the average sample size (Avg. SSize) and the estimated type I and type II FWER (FWERI, FWERII) of a factor screening problem with I factor and observations from $N(\beta, \Sigma)$. The configuration of β uses the second scenario, 10% factors are critical. Numbers are summarized from 10,000 independent macroreplications of HIP, SUMIP, and SORTIP.

		HIP			SUMIP			SORTIP		
I	Σ	Avg Samples	FWERI	FWERII	Avg Samples	FWERI	FWERII	Avg Samples	FWERI	FWERII
50	Σ_1	696.448	0.0%	0.0%	401.920	0.0%	0.6%	390.208	0.0%	0.6%
	Σ_2	828.032	0.0%	0.1%	470.976	0.0%	0.7%	458.944	0.0%	0.7%
	Σ_3	1049.088	0.0%	0.0%	636.608	0.0%	0.1%	603.84	0.0%	0.2%
100	Σ_1	1752.576	0.0%	0.0%	993.024	0.0%	0.1%	923.136	0.0%	0.3%
	Σ_2	2418.176	0.0%	0.0%	1300.350	0.0%	0.6%	1243.008	0.0%	0.7%
	Σ_3	3097.600	0.0%	0.3%	1704.704	0.0%	0.9%	1623.424	0.0%	0.9%
150	Σ_1	7746.54	0.0%	0.0%	4401.664	0.0%	0.1%	4039.680	0.0%	0.2%
	Σ_2	11716.096	0.0%	0.0%	6277.12	0.0%	0.0%	6024.704	0.0%	0.0%
	Σ_3	15515.136	0.0%	0.1%	7777.792	0.0%	0.2%	7432.193	0.0%	0.2%

tions to process that job. Some stations are used more than once in the processing sequence.

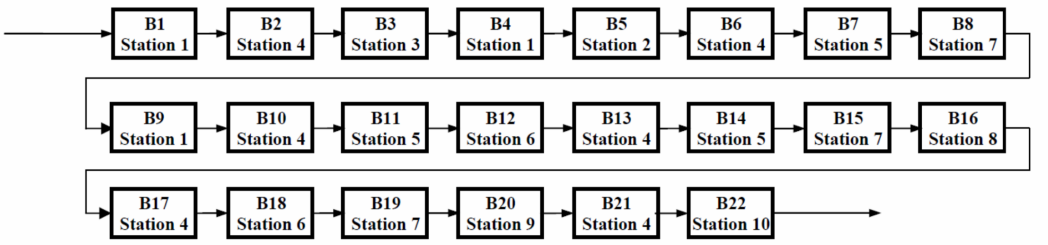


Figure 4.3.: Case Study: job processing sequence chart

One interesting question of this simulation model is to predict the **lead time** of a particular order, which is the time between receiving an order and the promised due date. More specifically, given the current status of the system, how to reliably quote a tight time for the newly arrived order. Literature suggests a **three parameter Gamma distribution** to model the lead time [58].

$$g(y;\gamma,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}[y-\gamma]^{\alpha-1}\exp\{-\frac{y-\gamma}{\beta}\} \tag{4.14}$$

To solve the problem by multiple-response factor screening, we model these three parameters (α, β, γ in Gamma distribution) as responses and system status as factors. Note that, our factor screening procedure does not intend to fit the distribution, but rather to select those influential/important factors that shape this distribution so that these critical factors can be more thoroughly studied in the later experiment as in [59].

Table 4.6 lists 41 inputs variable. To generate the responses, we run the simulation model for multiple times to get finishing time t_i, and then estimates three responses α, β, γ using

$$\hat{\mu} = \frac{1}{n}\sum_{i}^{n} t_i \tag{4.15}$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i}^{n} [t_i - \hat{\mu}]^2 \tag{4.16}$$

$$\hat{\gamma} = 2.2t_{(1)} - 1.2t_{(2)} \tag{4.17}$$

$$\hat{\alpha} = \frac{[\hat{\mu} - \hat{\gamma}]^2}{\hat{\sigma^2}} \tag{4.18}$$

$$\hat{\beta} = \frac{\hat{\sigma^2}}{\hat{\mu} - \hat{\gamma}} \tag{4.19}$$

To solve the problem by factor screening, we assume the functional form is of the following form

$$F(X,\theta) = \sum_{i=1}^{14} X_{A,i} + \sum_{i=1}^{14} X_{B,j}(1 + X_{C,i}) + \sum_{j=1} X_{D,j}(1 + X_{E,j}) + (1 - X_{D,j})(1 + X_{F,j}) + X_R \tag{4.20}$$

These factors are of different types: qualitative variables and continuous/discrete variables. Also, these factors are not independent, for example, if a station is down, then the downtime should be non-zero; if a station is up, then the uptime should be positive. Follow the 2-stage D-optimal design generation procedure in [59], we obtained a 84×41 screening design.

We conducted the SORTIP procedure for different importance importance threshold Δ_0 and Δ_1. Note that the bigger Δ_0 and Δ_1 are, the fewer factors are identified as important. Since factors are of different natures, it is wise to pick various sets of Δ_0 and Δ_1 for different factors. In our experiment, we set Δ_0 and Δ_1 so that a small number of factors in X_A, X_B, and X_C are selected as important. For WIP factors in X_A, when Δ_0 increases to $[0.25, 0.25, 0.25]$ and Δ_1 increases to $[0.5, 0.5, 0.5]$, the two most important WIP factors are selected, which are $X_{A,9}$ and $X_{A,13}$, both corresponding to the 4^{th} station.[1]. For busy/idle factors, when we set the Δ_0 as

[1]the 4^{th} station is the most-visited station and accumulates the most WIPs during simulation

Table 4.6.: Case Study: descriptions of factors

	Levels	Number	Description
X_A	$\star$	14	number of jobs at each buffers
X_B	2	10	the busy or idle status of server
X_C	3	10	the elapsed processing time of a busy server
X_D	2	2	the up or down status of unreliable server
X_E	3	2	the elapsed down time of a currently down server
X_F	3	2	the elapsed up time of a currently up server
X_R	3	1	arrival rate of the future order

X_A: is generated via uniform design algorithm by setting the overall WIP levels is at a low (15) or high (75) level.

X_C: it is related to X_b. If a station $X_{B,i}$ is idle, then its processing time $X_{proc,i}$ is 0; otherwise, $X_{proc,i}$ takes two values [ML_i, HL_i], where HL_i is the 95^{th} percentile of the distribution of the station's processing time and $ML_i = HL_i/2$.

X_E: down time. There are two stations subject to failure. When the station is up, then X_E is 0; otherwise, X_E takes value from [ML_i, HL_i], where HL_i is the 95^{th} percentile of the distribution of the station's down time and $ML_i = HL_i/2$.

X_F: up time. There are two stations subject to failure. When the station is down, then X_F is 0; otherwise, X_F takes value from [ML_i, HL_i], where HL_i is the 95^{th} percentile of the distribution of the station's up time and $ML_i = HL_i/2$.

X_R: arriving rate. The system is jammed when the rate is high. We take three levels of arriving rate, [0.5, 0.7, 0.9].

$[1, 100, 1$ and Δ_1 as $[2, 200, 2]$, then the 3^{rd} and 7^{th} station status are determined as important; when we decrease the threshold to include more factors, the 4^{th} factors is then selected as important. These results confirms [59] that the $[3, 4, 7]$ stations are bottleneck stations.

4.6 Conclusion

SUMIP and SORTIP are sequential multiple responses factor screening procedures that provide strong controls on FWERI and FWERII simultaneously. With the option of using efficient screening designs, SUMIP and SORTIP can handle large-scale problems efficiently. Numerical evaluation indicates the performances of SUMIP and SORTIP are robust and efficient across different simulation configurations.

Our future research will concentrate on developing sequential bifurcation or grouping procedure that controls the FWERI and FWERII. Besides, since previous research of simulation factor screening focused on controlling the error rate with economical designs, a related research topic would be the optimal computational budget allocations, which is how to allocate design budgets in order to minimize the error rates and maximize the power of the tests.

It worths mentioning that these procedures do not fit all factor screening problems. One drawback is that since our model only considers the main effects, factors with strong interactions may not be selected as important factors due to their corresponding main effect are not important.

4.7 Appendix: Technical Proofs

4.7.1 Proof of Equation 4.8

Proof The likelihood function of Z is

$$L = (2\pi)^{-\frac{KJ}{2}} |\Sigma|^{-\frac{K}{2}} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{K} (Z_k - \beta)' \Sigma^{-1} (Z_k - \beta) \right\}$$

While,

$$\begin{aligned}
&\sum_{k=1}^{K} (Z_k - \beta)' \Sigma^{-1} (Z_k - \beta) \\
=&\sum_{k=1}^{K} \text{tr}\left(\Sigma^{-1} (Z_k - \beta) (Z_k - \beta)' \right) \\
=&\text{tr} \sum_{k=1}^{K} \left(\Sigma^{-1} (Z_k - \beta) (Z_k - \beta)' \right) \\
=&\text{tr}\left(\Sigma^{-1} \sum_{k=1}^{K} (Z_k - \beta) (Z_k - \beta)' \right)
\end{aligned}$$

Then, the likelihood function is proportional to

$$\begin{aligned}
L &= (2\pi)^{-\frac{KJ}{2}} |\Sigma|^{-\frac{K}{2}} \exp\left\{ -\frac{1}{2} \sum_{k=1}^{K} (Z_k - \beta)' \Sigma^{-1} (Z_k - \beta) \right\} \\
&\propto |\Sigma|^{-\frac{K}{2}} \exp\left\{ -\frac{1}{2} \text{tr}\left(\Sigma^{-1} \sum_{k=1}^{K} (Z_k - \beta) (Z_k - \beta)' \right) \right\} \qquad (4.21)
\end{aligned}$$

Then, we need to find the maximum likelihood estimators of β and Σ.

For fixed β, let $D = \sum_{k=1}^{K} (Z_k - \beta) (Z_k - \beta)' = EE'$ and $H = E'\Sigma^{-1}E$. Then, D is a positive definite matrix with rank J.

Take log on the right-hand side of likelihood function L, then let

$$f(\Sigma) = -K \cdot \ln |\Sigma| - \text{tr}\left(\Sigma^{-1} D \right)$$

Since $\text{tr}\left(\Sigma^{-1}D\right) = \text{tr}\left(\Sigma^{-1}EE'\right) = \text{tr}(H)$ and $|\Sigma| = \left|EH^{-1}E'\right| = |E| \left|H^{-1}\right| |E'| = \frac{|D|}{|H|}$,

$$f(\Sigma) = -K \cdot \ln |D| + K \cdot \ln |H| - \text{tr}(H) \qquad (4.22)$$

Let $H = TT'$, where $T = \left[t_{ij}\right]_{J\times J}$ is lower triangular. Then 4.22 is

$$\begin{aligned} f(\Sigma) &= -K\cdot\ln|D| + K\cdot\ln|T|^2 - \operatorname{tr}\left(TT'\right) \\ &= -K\cdot\ln|D| + \sum_{j=1}^{J}\left(K\cdot\ln t_{jj}^2 - t_{jj}^2\right) - \sum_{i>j} t_{ij}^2 \end{aligned} \tag{4.23}$$

Maximum of 4.23 occurs at $t_{ii}^2 = K$ and $t_{ij} = 0,\ i \neq j$. That is $H = K\cdot I$. Then, $\Sigma = \frac{D}{K}$.

Since

$$\begin{aligned} &\sum_{k=1}^{K}(Z_k-\beta)(Z_k-\beta)' \\ =&\sum_{k=1}^{K}\left((Z_k-\bar{Z})(Z_k-\bar{Z})' + (\bar{Z}-\beta)(\bar{Z}-\beta)' + (Z_k-\bar{Z})(\bar{Z}-\beta)' + (\bar{Z}-\beta)(Z_k-\bar{Z})'\right) \\ =&\sum_{k=1}^{K}(Z_k-\bar{Z})(Z_k-\bar{Z})' + K(\bar{Z}-\beta)(\bar{Z}-\beta)' \\ =&S + K(\bar{Z}-\beta)(\bar{Z}-\beta)' \end{aligned}$$

The third and forth terms on the right hand side are 0 because $\sum(Z_k-\bar{Z}) = 0$. Then, L is maximized when $\Sigma = \frac{1}{K}S + (\bar{Z}-\beta)(\bar{Z}-\beta)'$. Thus,

$$\begin{aligned} \max_{\Sigma\succ 0} L &\propto \left|\frac{1}{K}S + (\bar{Z}-\beta)(\bar{Z}-\beta)'\right|^{-\frac{K}{2}} \\ &\propto |S|^{-\frac{K}{2}}\left(1 + K(\bar{Z}-\beta)'S^{-1}(\bar{Z}-\beta)\right)^{-\frac{K}{2}} \end{aligned}$$

Let $X'A^{-1}X = \|X\|_A^2$ and $\pi_S(Z,\ \Theta_i)$ denote the MLE of μ under the restriction $\beta\in\Theta_i$ with Σ unknown. Then,

$$L_i = \max_{\Sigma\succ 0,\ \mu\in\Theta_i} L \propto |S|^{-\frac{K}{2}}\left(1 + \|\bar{Z} - \pi_S(Z,\ \Theta_i)\|_{\frac{S}{K}}^2\right)^{-\frac{K}{2}} \tag{4.24}$$

Therefore, likelihood ratio Λ is

$$\Lambda = \frac{L_1}{L_0} = \frac{\left(1 + \|\bar{Z} - \pi_S(Z, \Theta_1)\|^2_{\frac{S}{K}}\right)^{-\frac{K}{2}}}{\left(1 + \|\bar{Z} - \pi_S(Z, \Theta_0)\|^2_{\frac{S}{K}}\right)^{-\frac{K}{2}}}$$

■

4.7.2 Proof of Theorem 4.3.1

Proof If $\bar{Z} \in \Theta_1$, $\pi_S(Z, \Theta_1) = \bar{Z}$ is clearly the optimal solution.

If $\bar{Z} \notin \Theta_1$, without loss of generality, assume elements of $\bar{Z}$ are nonnegative. It is easy to show that the optimal solution should lay on one of hyperplanes $\beta_j = \Delta_1$ for $j = 1, 2, 3, ...J$. Denote these hyperplanes as $P_{\beta_j = \Delta_1}$ respectively.

Let

$$S = \begin{bmatrix} \sqrt{s_1} & 0 & \cdots & 0 \\ 0 & \sqrt{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{s_J} \end{bmatrix} \begin{bmatrix} 1 & \frac{s_{12}}{\sqrt{s_1 s_2}} & \cdots & \frac{s_{1J}}{\sqrt{s_1 s_J}} \\ \frac{s_{21}}{\sqrt{s_1 s_2}} & 1 & & \\ \vdots & & \ddots & \\ \frac{s_{J1}}{\sqrt{s_1 s_J}} & & & 1 \end{bmatrix} \begin{bmatrix} \sqrt{s_1} & 0 & \cdots & 0 \\ 0 & \sqrt{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{s_J} \end{bmatrix}$$

Then,

$$(\beta - \bar{Z})' S^{-1} (\beta - \bar{Z}) = \eta' K \eta$$

where

$$\eta = \left(\begin{bmatrix} \sqrt{s_1} & 0 & \cdots & 0 \\ 0 & \sqrt{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{s_J} \end{bmatrix}^{-1} (\beta - \bar{Z}) \right)$$

$$K = \begin{bmatrix} 1 & \frac{s_{12}}{\sqrt{s_1 s_2}} & \cdots & \frac{s_{1J}}{\sqrt{s_1 s_J}} \\ \frac{s_{21}}{\sqrt{s_1 s_2}} & 1 & \cdots & \\ \vdots & \vdots & \ddots & \\ \frac{s_{J1}}{\sqrt{s_1 s_J}} & & & 1 \end{bmatrix}^{-1}$$

$(\beta - \bar{Z})' S^{-1} (\beta - \bar{Z})$ would be represented as a family of ellipse $\eta' K \eta$ with $\mathbf{1} = (1, 1, ...1)$ as its symmetric axis. Hyperplanes $P_{\beta_j = \Delta_j}$ under β is then equivalent to $P_{\eta_j = v_j}$ under η. Rotate hyperplanes $P_{\eta_j = v_j}$ around $\mathbf{1} = (1, 1, ...1)$ to get $P_{\eta_1 = v_j}$. Since hyperplanes $P_{\eta_1 = v_j}$ for $j = 1, .., J$ are all parallel with each other, clearly, the optimal solution lays on the plane who is closest to the origin.

Recall $j^* = \arg\min_{j=1,...,J} \frac{\left(\bar{z}_j - \Delta_1\right)^2}{s_j}$, so $P_{\eta_{j*} = v_{j*}}$ has the shortest distance to the origin. Thus, the optimal solution is on hyperplane $P_{\beta_{j*} = \Delta_1}$

■

4.7.3 Proof of Theorem 4.3.3

Proof From Theorem 2, $SPRT_i(A_i, B_i)$ with boundaries $A_i = \gamma_i$ and $B_i = \alpha_i^{-1}$ strongly controls FWERI and FWERII at levels α_i and γ_i respectively at $N_{int} = \inf\left\{n : \bigcap_{i=1}^{I} \Lambda_n^i \notin (A_i, B_i)\right\}$.

Then, at N_{int}, A_i and B_i satisfy

$$\alpha \geq \sum_{i=1,...,I} B_i^{-1}$$
$$\gamma \geq \sum_{i=1,...,I} A_i$$

And $\Lambda^i_{N_{int}}$ could be one of these two situations: $\Lambda^i_{N_{int}} \geq B_i$ and $\Lambda^i_{N_{int}} \leq A_i$.

Let $I_A = \left\{i : \Lambda^i \leq 1\right\}$ and $I_B = \left\{i : \Lambda^i > 1\right\}$. Then I_A is the index of factors that are accepted, and I_B The index of rejected factors. For factors within I_A, set $B_i = \infty$; for factors within I_B, set $A_i = 0$. Then we have

$$\alpha \geq \sum_{i=1,...,I} B_i^{-1} = \sum_{i \in I_B} B_i^{-1} \geq \sum_{i \in I_B} \left(\Lambda^i_{N_{int}}\right)^{-1} \tag{4.25}$$
$$\gamma \geq \sum_{i=1,...,I} A_i = \sum_{i \in I_A} A_i \geq \sum_{i \in I_A} \Lambda^i_{N_{int}} \tag{4.26}$$

Hence, if 4.25 and 4.26 are satisfied, FWERI and FWERII are at level α and γ. ■

4.7.4 Proof of Theorem 4.3.4

Proof We need to show that

$$N_{int} = \inf\left\{ n : \left(\bigcap_{i \in I_A} \Lambda_n^{(i)} \notin (A_i, 1) \right) \bigcap \left(\bigcap_{j \in I_B} \Lambda_n^{(j)} \notin (1, B_j) \right) \right\},\ A_i = \frac{\gamma}{|I_A| - i + 1},\ B_j = \frac{j}{\alpha} \tag{4.27}$$

controls FWERI and FWERII in the strong sense.

Recall that we will not reject H_0^i if $\Lambda^i_{N_{int}} \leq 1$. So, we seperate factors into two sets, $I_A = \{i : \Lambda^i \leq 1\}$ and $I_B = \{i : \Lambda^i > 1\}$, with cardinalities $|I_A|$ and $|I_B|$.

Let $I_{A0} = \{i \in I_A : H_0^i \text{ is true}\}$ be the set of true null hypotheses in I_A, and $I_{A1} = \{i \in I_A : H_1^i \text{ is true}\}$ be the set of true alternative hypotheses in I_A, $I_{B0} = \{i \in I_B : H_0^i \text{ is true}\}$ be the set of true null hypotheses in I_B, and $I_{B1} = \{i \in I_B : H_1^i \text{ is true}\}$ be the set of true alternative hypotheses in I_B, with cardinalities $|I_{A0}|$,$|I_{A1}|$,$|I_{B0}|$, and $|I_{B1}|$ respectively.

If there is no FWERI error, then $I_{B0} = \emptyset$ and $I_{B1} = I_B$; if there is no FWERII error, then $I_{A0} = I_A$ and $I_{A1} = \emptyset$.

Let

$$i_0 = \arg\max_{i \in I_{B0}} \Lambda^{(i)}$$

$$i_1 = \arg\min_{i \in I_{A1}} \Lambda^{(i)}$$

where i_0 denote the "first true null" in I_B and i_1 denote the "first true alternative" in I_A. If there are no such i_0 and i_1, then FWERI and FWERII are 0.

In other words, if $H_0^{(i)}$ denotes the null hypothesis that is tested by the likelihood ratio $\Lambda^{(i)}$ for $i = 1, 2, ..., I$, then i_0 is such that all $H_0^{(i)}$ are false for $i > i_0$ where $H_0^{(i_0)}$ is true and i_1 is such that all $H_0^{(i)}$ are true for $i < i_0$ where $H_0^{(i_0)}$ is false.

Thus, there are at least $(|I_B| - i_0)$ false hypotheses in I_B and $(i_1 - 1)$ true hypotheses in I_A, so that $|I_B| - i_0 \leq |I_{B1}|$ and $i_1 - 1 \leq |I_{A0}|$.

For FWERI, no Type I error can be made on hypothesis $H_0^{(i)}$ where $i \in I_A$ and hypothesis $H_0^{(i)}$ where $i \geq i_0$. If there is no such i_0 exists, then there is no Type I error at all.

$$\begin{aligned}
\text{FWERI} &= Prob\left(\Lambda_{N_{int}}^{(i_0)} \geq B_{i_0}\right) \leq Prob\left(\Lambda_{N_{int}}^{(i_0)} \geq B_{|I_B|-|I_{B1}|}\right) \\
&= Prob\left(\max_{i \in I_{B0}} \Lambda_{N_{int}}^{(i)} \geq B_{|I_B|-|I_{B1}|}\right) \\
&\leq \sum_{i \in I_{B0}} Prob\left(\Lambda_{N_{int}}^{(i)} \geq B_{|I_B|-|I_{B1}|}\right)
\end{aligned}$$

Recall that rejection boundaries are chosen as $B_i = \frac{i}{\alpha}$. Therefore, by theorem 4.3.2,

$$Prob\left(\Lambda_{N_{int}}^{(i)} \geq B_{|I_B|-|I_{B1}|}\right) \leq \frac{\alpha}{|I_B| - |I_{B1}|}$$

We have

$$\text{FWERI} \leq \sum_{i \in I_{B0}} \frac{\alpha}{|I_B| - |I_{B1}|} = \alpha$$

For FWERII, no Type II error can be made on hypothesis $H_0^{(i)}$ where $i \in I_B$ and hypothesis $H_0^{(i)}$ where $i \leq i_1$. If there is no such i_1 exists, then there is no Type II error at all.

$$\begin{aligned}
\text{FWERII} &= Prob\left(\Lambda_{N_{int}}^{(i_1)} \leq A_{i_0}\right) \leq Prob\left(\Lambda_{N_{int}}^{(i_1)} \leq A_{|I_{A0}|+1}\right) \\
&= Prob\left(\max_{i \in I_{B0}} \Lambda_{N_{int}}^{(i)} \leq A_{|I_{A0}|+1}\right) \\
&\leq \sum_{i \in I_{A1}} Prob\left(\Lambda_{N_{int}}^{(i)} \leq A_{|I_{A0}|+1}\right)
\end{aligned}$$

Recall that acceptance boundaries are chosen as $A_i = \frac{\gamma}{|I_A|-i+1}$. Therefore, by Theorem 4.3.2,

$$Prob\left(\Lambda_{N_{int}}^{(i)} \leq A_{|I_{A0}|+1}\right) \leq \frac{\gamma}{|I_A| - |I_{A0}|}$$

Therefore, we have

$$\text{FWERII} \leq \sum_{i \in I_{A1}} \frac{\gamma}{|I_A| - |I_{A0}|} = \gamma$$

■

4.8 Appendix: Holm Intersection Procedure

This procedure is based on HIS proposed by De and Baron (2012). In this paper, we call it as Holm Intersection Procedure (HIP). This procedure is for comparison. It is used to demonstrate the effectiveness and efficiency of our proposed procedures. This procedure will run n_0 initial replications at first, where $n_0 \geq J$. If likelihood ratios of all factors satisfy 4.28, then we claim factors with $\Lambda^i > B_i$ as important, while $\Lambda^i < A_i$ as unimportant. If not, we carry one more factorial design for all factors.

$$N_{int} = \inf \left\{ n : \bigcap_{i=1}^{I} \Lambda_n^{(i)} \notin (A_i, B_i) \right\}, \; A_i = \frac{\gamma}{I - i + 1}, \; B_i = \frac{i}{\alpha} \tag{4.28}$$

Algorithm 8 MIP Procedure

Step 1

Master

Assign I_p factors to worker p, where $p = \{1, 2, ..., P\}$, refer the factors on worker p as β_p.

Worker

Select a factorial design for I_p factors with design matrix X_p, generate n_0 replications of observations. Each observation of β_p is estimated as

$$Z_{p,n} = \left(X_p' X_p\right)^{-1} X_p' Y_{p,n},\ n = 1, 2, ...n_0$$
$$S_p = \sum_{n=1}^{n_0} \left(Z_{p,n} - \bar{Z}_p\right) \left(Z_{p,n} - \bar{Z}_p\right)'$$

Let $n = n_0$.

Step 2

Worker

1. For factors $\beta_{p,1},\ \beta_{p,2},\ \beta_{p,3}, ...\beta_{p,Ip}$, calculate their corresponding $\Lambda_{p,n}^i$ for $i \in \{1, 2, ..., I\}$. Divide factors into two set $I_A^p = \left\{i : \Lambda_{p,n}^i \leq 1\right\}$ and $I_B^p = \left\{i : \Lambda_{p,n}^i > 1\right\}$;
2. Sort $\Lambda_{p,n}^i, i \in I_A$ and $\Lambda_{p,n}^j, j \in I_B$ in ascending order, and calculate $\alpha_{p,n} = \max_{i=\{1:|I_B|\}} \left\{\frac{i}{\Lambda_{p,n}^{(i+|I_A|)}}\right\}$ and $\gamma_{p,n} = \max_{i=\{1:|I_A|\}} \left\{\left(|I_A - i + 1| \cdot \Lambda_{p,n}^{(i)}\right\}\right.$;
3. Send $\alpha_{p,n}$ and $\gamma_{p,n}$ back to master.

Master If $\sum_p \alpha_{p,n} \leq \alpha$ and $\sum_p \gamma_{p,n} \leq \gamma$, go step 3; otherwise, go step 2.

Step 3

Master Send a terminate instruction to all worker, Claim factors within $I_{p,B}$ are important, and factors within $I_{p,A}$ are not important for $p \in \{1, 2, ...P\}$.

Algorithm 9 HIP Procedure

Step 1 Select a factorial design for I factors with design matrix X, generate n_0 replications of observations. Each observation of β is estimated as

$$Z_n = \left(X'X\right)^{-1} X'Y_n,\ n = 1, 2, ...n_0$$
$$S = \sum_{n=1}^{n_0} \left(Z_n - \bar{Z}\right)\left(Z_n - \bar{Z}\right)'$$

Let $n = n_0$.

Step 2 DO

1. For factors $\beta_1,\ \beta_2,\ \beta_3, ...\beta_I$, calculate their corresponding Λ_n^i for $i \in \{1, 2, ..., I\}$.
2. Rank Λ_n^i according to its value. If $\Lambda_n^{(i)} \notin (A_i, B_i)$ where $A_i = \frac{\gamma}{I-i+1}$, $B_i = \frac{i}{\alpha}$ for all $i = 1, ..., I$, then go step 3. Otherwise, for I factors with design matrix X, generate 1 more observations, and make $n = n + 1$; back to step 2.

Step 3 Claim factors within I_B are important, and factors within I_A are not important.

REFERENCES

REFERENCES

[1] Raghu Pasupathy and Shane G. Henderson. Simopt: A library of simulation optimization problems. In *Proceedings of the 2011 Winter Simulation Conference (WSC)*, pages 4075–4085, December 2011.

[2] Michael C. Fu. Feature Article: Optimization for simulation: Theory vs. Practice. *INFORMS Journal on Computing*, 14(3):192–215, August 2002.

[3] Michael C. Fu, Jian-Qiang Hu, Chun-Hung Chen, and Xiaoping Xiong. Simulation allocation for Determining the Best Design in the Presence of Correlated Sampling. *INFORMS Journal on Computing*, 19(1):101–111, January 2007.

[4] Raghu Pasupathy and Soumyadip Ghosh. Simulation Optimization: A Concise Overview and Implementation Guide. In *Theory Driven by Influential Applications*, INFORMS Tutorials in Operations Research, pages 122–150. INFORMS, September 2013.

[5] Satyajith Amaran, Nikolaos V. Sahinidis, Bikram Sharda, and Scott J. Bury. Simulation optimization: a review of algorithms and applications. *Annals of Operations Research*, 240(1):351–380, May 2016.

[6] Chun-Hung Chen, Jianwu Lin, Enver Yucesan, and Stephen E. Chick. Simulation Budget Allocation for Further Enhancing the Efficiency of Ordinal Optimization. *Discrete Event Dynamic Systems*, 10(3):251–270, July 2000.

[7] Seong-Hee Kim and Barry L. Nelson. Chapter 17 Selecting the Best System. In Shane G. Henderson and Barry L. Nelson, editor, *Handbooks in Operations Research and Management Science*, volume 13 of *Simulation*, pages 501–534. Elsevier, 2006.

[8] Seong-Hee Kim and B. L. Nelson. Recent advances in ranking and selection. In *Proceedings of the 2007 Winter Simulation Conference*, pages 162–172, December 2007.

[9] Jurgen Branke, Stephen E. Chick, and Christian Schmidt. Selecting a selection procedure. *Management Science*, 53(12):1916–1932, 2007.

[10] Shanti Swarup Gupta. *On a Decision Rule for a Problem in Ranking Means*. University of North Carolina at Chapel Hill, 1956.

[11] Shanti S. Gupta. On Some Multiple Decision (Selection and Ranking) Rules. *Technometrics*, 7(2):225–245, May 1965.

[12] Edward Paulson. A Sequential Procedure for Selecting the Population with the Largest Mean from k Normal Populations. *The Annals of Mathematical Statistics*, 35(1):174–180, March 1964.

[13] Seong-Hee Kim and Barry L. Nelson. A fully sequential procedure for indifference-zone selection in simulation. *ACM Transactions on Modeling and Computer Simulation*, 11(3):251–273, July 2001.

[14] Seong-Hee Kim and Barry L. Nelson. On the Asymptotic Validity of Fully Sequential Selection Procedures for Steady-State Simulation. *Operations Research*, 54(3):475–488, June 2006.

[15] L. Jeff Hong and Barry L. Nelson. The tradeoff between sampling and switching: New sequential procedures for indifference-zone selection. *IIE Transactions*, 37(7):623–634, July 2005.

[16] Peter I. Frazier. A Fully Sequential Elimination Procedure for Indifference-Zone Ranking and Selection with Tight Bounds on Probability of Correct Selection. *Operations Research*, 62(4):926–942, June 2014.

[17] Weiwei Fan and L. Jeff Hong. A frequentist selection-of-the-best procedure without indifference zone. In *Proceedings of the 2014 Winter Simulation Conference*, pages 3737–3748, December 2014.

[18] Weiwei Fan, L. Jeff Hong, and Barry L. Nelson. Indifference-Zone-Free Selection of the Best. *Operations Research*, 64(6):1499–1514, October 2016.

[19] Sijia Ma and Shane G. Henderson. An efficient fully sequential selection procedure guaranteeing probably approximately correct selection. In *Proceedings of the 2017 Winter Simulation Conference*, pages 2225–2236, December 2017.

[20] Abraham Wald. Sequential Method of Sampling for Deciding between Two Courses of Action. *Journal of the American Statistical Association*, 40(231):277–306, September 1945.

[21] Abraham Wald and Jacob Wolfowitz. Optimum Character of the Sequential Probability Ratio Test. *The Annals of Mathematical Statistics*, 19(3):326–339, September 1948.

[22] Gary Lorden. 2-SPRT'S and The Modified Kiefer-Weiss Problem of Minimizing an Expected Sample Size. *The Annals of Statistics*, 4(2):281–291, March 1976.

[23] Tze Leung Lai. Power-One Tests Based on Sample Sums. *The Annals of Statistics*, 5(5):866–880, September 1977.

[24] Tze Leung Lai. Asymptotic optimality of invariant sequential probability ratio tests. *The Annals of Statistics*, pages 318–333, 1981.

[25] Igor V. Pavlov. Sequential Procedure of Testing Composite Hypotheses with Applications to the KieferWeiss Problem. *Theory of Probability & Its Applications*, 35(2):280–292, January 1991.

[26] Igor V. Pavlov. A Sequential Procedure for Testing Many Composite Hypotheses. *Theory of Probability & Its Applications*, 32(1):138–142, January 1988.

[27] Xiaoou Li, Jingchen Liu, and Zhiliang Ying. Generalized Sequential Probability Ratio Test for Separate Families of Hypotheses. *Sequential Analysis*, 33(4):539–563, October 2014.

[28] Herbert Robbins and David Siegmund. The Expected Sample Size of Some Tests of Power One. *The Annals of Statistics*, 2(3):415–436, May 1974.

[29] Alexander Tartakovsky, Igor Nikiforov, and Michele Basseville. *Sequential Analysis: Hypothesis Testing and Changepoint Detection*. CRC Press, August 2014.

[30] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Springer-Verlag, December 1999.

[31] Loo Hay Lee, Ek Peng Chew, and Suyan Teng. Finding the pareto set for multi-objective simulation models by minimization of expected opportunity cost. In *Proceedings of the 2007 Winter Simulation Conference*, pages 513–521, December 2007.

[32] Loo Hay Lee, Ek Peng Chew, Suyan Teng, and David Goldsman. Finding the non-dominated Pareto set for multi-objective simulation models. *IIE Transactions*, 42(9):656–674, June 2010.

[33] Guy Feldman, Susan R. Hunter, and Raghu Pasupathy. Multi-objective simulation optimization on finite sets: Optimal allocation via scalarization. In *Proceedings of the 2015 Winter Simulation Conference (WSC)*, pages 3610–3621, December 2015.

[34] Juergen. Branke and Wen Zhang. A new myopic sequential sampling algorithm for multi-objective problems. In *Proceedings of the 2015 Winter Simulation Conference*, pages 3589–3598, December 2015.

[35] Juergen Branke, Wen Zhang, and Yang Tao. Multiobjective Ranking and Selection Based on Hypervolume. In *Proceedings of the 2016 Winter Simulation Conference*, pages 859–870, December 2016.

[36] Guy Feldman and Susan R. Hunter. Score allocations for bi-objective ranking and selection. *ACM Trans. Model. Comput. Simul.*, 28(1):7:1–7:28, January 2018.

[37] Eric A. Applegate, Guy Feldman, Susan R. Hunter, and Raghu Pasupathy. Multi-objective ranking and selection: Optimal sampling laws and tractable approximations via SCORE. *Journal of Simulation*, 2019.

[38] Demet Batur and Fred Choobineh. Mean-variance based ranking and selection. In *Proceedings of the 2010 Winter Simulation Conference*, pages 1160–1166, Dec 2010.

[39] Judy S. Lee. *Advances in simulation: validity and efficiency*. PhD dissertation, Georgia Institute of Technology, 2014.

[40] Susan R. Hunter, Eric A. Applegate, Viplove Arora, Bryan Chong, Kyle Cooper, Oscar Rincn-Guevara, and Carolina Vivas-Valencia. An Introduction to Multiobjective Simulation Optimization. *ACM Transactions on Modeling and Computer Simulation*, 29(1):7:1–7:36, January 2019.

[41] Jack Kiefer and Lionel Weiss. Some Properties of Generalized Sequential Probability Ratio Tests. *The Annals of Mathematical Statistics*, 28(1):57–74, March 1957.

[42] Hock Peng Chan and Tze Leung Lai. Asymptotic approximations for error probabilities of sequential or fixed sample size tests in exponential families. *The Annals of Statistics*, 28(6):1638–1669, December 2000.

[43] Raymond H Myers, Christine M Anderson-Cook, and Douglas C Montgomery. *Response surface methodology: process and product optimization using designed experiments*. Wiley, Hoboken, NJ, 2009.

[44] Andrea. Saltelli, Stefano Tarantola, and Francesca Campolongo. Sensitivity Analysis as an Ingredient of Modeling. *Statistical Science*, 15(4):377–395, November 2000.

[45] Clemens Elster and Arnold Neumaier. Screening by conference designs. *Biometrika*, 82(3):589–602, September 1995.

[46] Linda Trocine and Linda C. Malone. Statistical tools for simulation design and analysis I: finding important independent variables through screening designs: a comparison of methods. In *Proceedings of the 2000 Winter Simulation conference*, pages 749–754, 2000.

[47] Hong Wan, Bruce E. Ankenman, and Barry L. Nelson. Controlled sequential bifurcation: A new factor-screening method for discrete-event simulation. *Operations Research*, 54(4):743–755, 2006.

[48] Hua Shen and Hong Wan. Controlled sequential factorial design for simulation factor screening. *European Journal of Operational Research*, 198(2):511–519, 2009.

[49] Hua Shen, Hong Wan, and Susan M. Sanchez. A hybrid method for simulation factor screening. *Naval Research Logistics*, 57(1):45–57, 2010.

[50] Jack PC Kleijnen, Susan M. Sanchez, Thomas W. Lucas, and Thomas M. Cioppa. State-of-the-art review: a users guide to the brave new world of designing simulation experiments. *INFORMS Journal on Computing*, 17(3):263–289, 2005.

[51] Wen Shi, Jack P. C. Kleijnen, and Zhixue Liu. Factor screening for simulation with multiple responses: Sequential bifurcation. *European Journal of Operational Research*, 237(1):136–147, August 2014.

[52] Shelley Hurwitz and John S. Gardenier. Ethical Guidelines for Statistical Practice: The First 60 Years and Beyond. *The American Statistician*, 66(2):99–103, May 2012.

[53] Mei-Ling Ting Lee. *Analysis of Microarray Gene Expression Data*. Springer US, 2004.

[54] Shyamal K. De and Michael Baron. Sequential Bonferroni Methods for Multiple Hypothesis Testing with Strong Control of Family-Wise Error Rates I and II. *Sequential Analysis*, 31(2):238–262, 2012.

[55] Michael D. Perlman and Lang Wu. Some Improved Tests for Multivariate One-Sided Hypotheses. *Metrika*, 64(1):23–39, August 2006.

[56] Peter Reitmeir and Gernot Wassmer. One-sided multiple endpoint testing in two-sample comparisons. *Communications in Statistics - Simulation and Computation*, 25(1):99–117, 1996.

[57] Brent R. Logan. A cone order monotone test for the one-sided multivariate testing problem. *Statistics & Probability Letters*, 63(3):315–323, July 2003.

[58] Feng Yang, Bruce E. Ankenman, and Barry L. Nelson. Estimating Cycle Time Percentile Curves for Manufacturing Systems via Simulation. *INFORMS Journal on Computing*, 20(4):628–643, July 2008.

[59] Minqi Li, Feng Yang, Hong Wan, and John Fowler. Simulation-based experimental design and statistical modeling for lead time quotation. *Journal of Manufacturing Systems*, 37:362–374, October 2015.