HARMOMIC MAPS INTO TEICHMÜLLER SPACES AND SUPERRIGIDITY OF MAPPING CLASS GROUPS

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ABSTRACT

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This thesis contains two parts.

In the first part of the present work, we will study the harmonic maps onto Teichmüller space. We will formulate a general Bochner type formula for harmonic maps into Teichmüller space. We will also prove the existence theorem of equivariant harmonic maps from a symmetric space with finite volume into its Weil-Petersson completion $\overline{\mathcal{T}}$, by deforming an almost finite energy map in the sense of [1] into a finite energy map.

In the second part of the work, we discuss the superrigidity of mapping class group. We will provide a geometric proof of both the high rank and the rank one superrigidity of mapping class groups due to Farb-Masur [2] and Yeung [3].

1. INTRODUCTION

Given a smooth map $u : M \to N$ between two complete Riemannian manifolds, it is called harmonic if it is a critial point of energy functional defined by

$$E(u) = \int_M \|du\|^2 dV \tag{1.1}$$

An important milestone of the research on harmonic mappings is [4] by Eells and Sampson in 1964, where they established an existence theorem of harmonic maps from a compact Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, via heat equation method. In 1983, Schoen and Uhlenbeck [5] developed the regularity theory of harmonic maps.

The theory of harmonic maps to singular spaces is a generalization of the theory of harmonic maps between Riemmanian manifolds. It was originated by Gromov and Schoen in the seminal paper [6], In their remarkable work [6], the authors proposed a theory of harmonic mappings into buildings, as well as important applications to p-adic supperrigidity for certain discrete groups. Then it was subsequently extended for harmonic maps to maps into more general NPC (non positively curved) space by Korevaar and Schoen ([7] and [8]), where they defined the Korevaar-Schoen energy functional and asserted existence of harmonic maps and their general regularity.asserted and [9]).

In the first part of the present work, we will study the harmonic maps into Teichmüller space. Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ is the space of Riemann surface structures up to isotopy, and the mapping class group $Mod(S)$ is the group of isotopy classes of orientation-preserving homeomorphisms of S. Teichmüller space is non positively curved when equipped with the Weil-Petersson metric. However, the existence of harmonic maps into a Teichmüller space is not guaranteed due to the fact that the Weil-Petersson metric is not complete. We have proved the following existence theorem for equivariant harmonic maps from a symmetric space into its metric completion $\overline{\mathcal{T}}$ with respect to Weil-Petersson metric.

Theorem 1.0.1 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is neither complex nor real hyperbolic space, and Γ be a lattice in G. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a ρ-equivariant Lipschitz harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

The proof of the theorem follows from Theorem 2.1.3 and Corollary 2.1.5 in [8] under the assumption that there exists a finite energy ρ-equivariant map from $\widetilde{M}$ to $\overline{\mathcal{T}}$, which can be constructed directly when Γ is cocompact. We start with the almost finite energy retraction from $\widetilde{M}/\Gamma$ to the central tile of a tiling of the Borel-Serre compactification of M(cf. [1], theorem 6.1) and construct an almost finite energy ρ-equivariant map (defined in section 6.3) from $\widetilde{M}$ to $\overline{\mathcal{T}}$ by composition. The key point is to prove

Theorem 1.0.2 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type, Γ be a lattice in G, and $\rho : \Gamma \to Mod(S)$ be a homomorphism. If there is an almost finite energy ρ-equivalriant map from $\widetilde{M}$ to $\overline{\mathcal{T}}$, then there exits a finite energy ρ-equivariant map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

In the second part of the work, we discuss the superrigidity of mapping class groups. The history behind the problem of rigidity of Kähler manifolds was initiated by Calabi and Vesentini [10] in 1960. The authors implies that compact locally symmetric spaces of dimension at least 2 do not admit any nontrivial infinitesimal holomorphic deformation. In 1968, Mostow [11] celebrated the strong rigidity theorem that two compact ball quotients of complex dimension at least 2 with isomorphic fundamental groups are isometric and thus biholomorphic or conjugate biholomorphic. At that time, the results of Eells-Sampson [4] on existence of harmonic map is already known and people had been hoping to use the approach of harmonic maps to derive results in rigidity. This was not successful until the work of Siu [12], which proves a strong rigidity result for Kähler manifolds with curvature which is sufficiently negative, in particular if the curvature is strongly negative in the sense of Siu. This reproved the result of Mostow in the case of locally Hermitian symmetric spaces. The technique of harmonic maps has been generalized to study geometric superrigidity, cf. [13].

In another direction, Teichmuller space is equipped with a natural invariant metric, the Weil-Petersson metric. It is well-known that the Weil-Petersson metric is Kahler and is negatively curved, in particular, strongly negatively curved in the sense of Siu, from the work of Ahlfors, Wolpert, Schumacher and others, (cf. [14]) Hence the result of Siu applied once a harmonic map is shown to exist. The difficulty here is that the Weil-Petersson metric is incomplete and hence the result of Eells-Sampson did not applied.

Nevertheless, there is the following superrigidity type of results concerning lattices and mapping class groups.

Theorem 1.0.3 (Superrigidity of mapping class group) *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is not real or complex hyperbolic. Let Γ be a lattice in G. Then any homomorphism ρ from Γ to the mapping class group $Mod(S)$ has finite image.*

The above theorem is proved via different methods in both high rank and rank-one cases by Farb-Masur [2] and Yeung [3]. In our work, we give a uniform proof by using harmonic map approach and the Bochner formula which is modified from the one in [11]. These are the main ingredients of my proof:

Theorem 1.0.4 (Bochner type formula) *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type other than real or complex hyperbolic spaces, and Γ be a lattice in G. Assume that $u : M = \widetilde{M}/\Gamma \to \overline{\mathcal{T}}$ is a harmonic map, then the following Bochner type formula holds:*

$$\int_{R(u)} Q^{ijkl} \nabla_i u_l^\alpha \nabla_j u_k^\beta h_{\alpha\beta} = \frac{1}{2} \int_{R(u)} Q^{ijkl} R^{\mathcal{T}}_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{1.2}$$

where $\mathcal{R}(u)$ is the regular set of u, defined

$$\mathcal{R}(u) = \{x \in \Omega \mid \exists r > 0 \text{ such that } B_r(u(x)) \subset \mathcal{T}' \text{ for a stratum } \mathcal{T}' \text{ of } \overline{\mathcal{T}}\}$$

on which the map u is smooth.

h and $R^{\mathcal{T}}$ are the metric and curvature tensors on $\mathcal{T}$ with respect to the Weil-Petersson metric.

$Q = Q^{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ is a 4-tensor on $\widetilde{M}$ satisfying the following conditions

1. *$Q(X,Y,Z,W) = -Q(Y,X,Z,W) = Q(Z,W,X,Y)$.*

2. *Q is parallel, i.e. $\nabla Q = 0$.*

3. *Q is compatible with curvatures on M, i.e.*

$$\langle Q(.,.,.,X), R^M(.,.,.,Y)\rangle + \langle Q(.,.,.,Y), R^M(.,.,.,X)\rangle = 0$$

Using the above Bochner formula combined with a regularity theorem of harmonic maps in [15], we can prove the following theorem, which is the most important step in proving the theorem of supperrigidity of mapping class group.

Theorem 1.0.5 *Let f be a ρ-equivariant harmonic map from $\widetilde{M} = G/K$ to $\overline{\mathcal{T}}$, then f is totally geodesic on its regular set.*

2. SUMMARY

In Chapter 3, we will recall some notions for later discussion. In section 3.1, we will recall the definition of Teichmüller space and its Weil-Petersson metric completion. We will also recall the definition of mapping class group. In section 3.2, we will recall the definition definition of a nonpositively curved (NPC for short) space, which generalizes the definition of smooth manifold with negative curvature, and show its relationship with Teichmüller space. In section 3.3, we will introduce the harmonic map into singular spaces and some related notations. We will also recall the existence theorem in [8].

Theorem 2.0.1 ([8],Theorem 2.1.3 & **Remark 2.1.5)** .

1. *Assume $M = \widetilde{M}/\Gamma$ is compact and ρ is a proper action of $\Gamma = \pi_1(M)$. Assume X is an NPC space. There exists an ρ equivariant, Lipschitz harmonic map $u : \widetilde{M} \to X$.*

2. *Assume $M = \widetilde{M}/\Gamma$ is complete, and ρ is a proper action of $\Gamma = \pi_1(M)$ on X. Assume X is an NPC space. If there exists a finite energy ρ-equivariant map $u_0 : \widetilde{M} \to X$, then there exists an ρ-equivariant, Lipschitz harmonic map $u : \widetilde{M} \to X$.*

In Chapter 4, we will discuss the regularity theory of harmonic maps in Teichmüller space. In section 4.1, we will give the model space of $\overline{\mathcal{T}}$ and the definition of regular set and singular set. In section 4.2, we recall the regularity threorem in [14] that the Hausdorff dimension of the singular set of a harmonic map into $\overline{\mathcal{T}}$ is at most $n - 2$.

Theorem 2.0.2 ([14], Theorem 1) *Let $\mathcal{T}$ be Teichmüller space of an oriented surface S of genus g and p marked points such that $k = 3g - 3 + p > 0$ and $\overline{\mathcal{T}}$*

be its metric completion with respect to the Weil-Petersson metric. If $u : \Omega \to \overline{\mathcal{T}}$ a harmonic map from an n-dimensional Lipschitz Riemannian domain, then the Haustorff dimension of singular set

$$dim_{\mathcal{H}}(\mathcal{S}(u)) \leq n-2$$

Theorem 2.0.3 ([14], Theorem 2) *Let $u : \Omega \to \overline{\mathcal{T}}$ be a harmonic map from an n-dimensional Lipschitz domain. For any compact subdomain Ω_1 of Ω, there exists a sequence of smooth function $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $S(u) \cap \bar{\Omega}_1$, $0 \leq \psi_i \leq 1$ and $\psi_i \to 1$ for all $x \in \Omega_1 \backslash S(u)$ such that*

$$\lim_{i\to\infty} \int_\Omega |\nabla\nabla u||\nabla\psi_i| d\mu = 0$$

In Chapter 5, we formulate a general Bochner type formula for harmonic maps into Teichmüller space, following the method of [13], which could be used to prove the supperrigidity theorem later.

Theorem 2.0.4 (Bochner type formula) *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type and Γ be a lattice in G. Assume that $u : M = \widetilde{M}/\Gamma \to \overline{\mathcal{T}}$ is a harmonic map, then the following Bochner formula holds:*

$$\int_{R(u)} Q^{ijkl} \nabla_i u_l^\alpha \nabla_j u_k^\beta h_{\alpha\beta} = \frac{1}{2} \int_{R(u)} Q^{ijkl} R^N_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{2.1}$$

In Chapter 6, we will discuss the equivariant harmonic maps into Teichmüller space. In section 6.1, we will recall the notation of equivariant harmonic maps. In section 6.2, we will prove a Poincaré inequality for a non-compact locally symmetric space.

Lemma 2.0.5 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type, Γ be a lattice in G, and $M = \widetilde{M}/\Gamma$. Let $f \in C_0^\infty(M) \cap H^2(M) \cap H^1(M)$. Then*

$$\int_M |f - \hat{f}|^2 dV_g \leq c \int_M |\nabla f|^2 dV \tag{2.2}$$

for some constant $c > 0$. where $\hat{f}$ is the average of f on M defined by

$$\hat{f} = \frac{1}{Vol(M)} \int_M |f| dV_g$$

In section 6.3, we will construct finite energy maps from non-compact locally symmetric spaces to $\overline{\mathcal{T}}$ starting from a retraction of [1]. In section 6.4, we will prove the existence theorem of equivariant harmonic maps.

Theorem 2.0.6 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is neither complex nor real hyperbolic spaces, and Γ be a lattice in G. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a ρ-equivariant Lipschitz harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

Remark: ρ is sufficiently large if $\rho(\Gamma) \subset Mod(S)$ contains two independent pseudo-Anosov's. More details can be found in section 6.4.

In Chapter 7, we will provide a geometric proof of both the high rank and the rank one superrigidity of mapping class groups due to Farb-Masur [2] and Yeung [3], by using the equivariant harmonic maps constructed in previous chapter.

Theorem 2.0.7 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is neither complex nor real hyperbolic spaces. Let Γ be a lattice in G. Then any homomorphism ρ from Γ to the mapping class group $Mod(S)$ has finite image.*

3. PRELIMINARIES

In this chapter, we recall some notions for later discussion.

3.1 Teichmüller space and mapping class group

Let S be punctured Riemann surface of finite type (g, p), where g denotes the genus of compact surface $\overline{S}$ and p denotes the number of punctures. We assume that $k = 3g - 3 + p > 0$.

Definition 3.1.1 (marked Riemann surface) *A marked Riemann surface (X, f) is a punctured Riemann surface X together with an orientation-preserving homemorphism $f : S \to X$.*

Two marked surfaces $(X, f) \sim (Y, g)$ are equivalent if $g \circ f^{-1} : X \to Y$ is isotopic to an isomorphism.

Definition 3.1.2 (Teichmüler Space) *We define the Teichmüler Space $\mathcal{T} = \mathcal{T}(S)$ of S by*

$$\mathcal{T} = \{(X, f)\}/\sim$$

where (X, f) is a marked Riemann surface of S and the equivalence is given by isotopy to an isometry.

The Teichmüller space $\mathcal{T}$ is equipped with a natural holomorphic local coordinate (cf. [16]).

Definition 3.1.3 (mapping class group) *The mapping class group is defined as the quotient*

$$Mod(S) = Homeo^{+}(S)/Homeo_0(S)$$

where $Homeo^+(S)$ denotes the group of orientation-preserving homeomorphisms of S. $Homeo_0(S)$ denotes the subgroup of homeomorphisms homotopic to identity.

A mapping class $[h]$ acts on Teichmüler space as following:

$$[(X, f)] \to [(X, f \circ h^{-1})]$$

The action of $Mod(S)$ on $\mathcal{T}$ is by biholomorphic maps. The moduli space of Riemann surfaces is

$$\mathcal{M} = \mathcal{T}/Mod(S)$$

Let (X, f) be a marked Riemmann surface

Definition 3.1.4 (quadratic differential) *A quadratic differential on X is a holomorphic section of $\kappa \otimes \kappa$ where κ is the canonical line bundle on X.*

Locally, a quadratic differential has the form $q(z)dz^2$, where $q(z)$ is holomorphic.

Denote

$$QD(X) = \{\text{integrable quadratic differentials on } X\}$$

Then a theorem in [16] shows that the holomorphic cotangent space of $\mathcal{T}$ at the marked surface (X, f) is isomorphic to $QD(X)$, whose dimension is equal to $3g-3+p$.

There are several kinds of metrics defined on Teichmüller spaces. Teichmüller introduced the Teichmüller metric, which is a complete metric. William Thurston introduced another metric on Teichmüller spaces. In our work, we focus on Teichmüller spaces eqquiped with the Weil-Petersson metric, which was introduced by André Weil using the Petersson inner product on forms on a Riemann surface as follow:

Definition 3.1.5 (Weil-Petersson metric) *For $q_1, q_2 \in QD(X)$, the Petersson inner product defined on the cotangent space of $\mathcal{T}$ is defined by*

$$h(q_1, q_2) = \int_X \frac{q_1(z)\overline{q_2(z)}}{\rho(z)} dz d\bar{z}$$

The dual metric is called the Weil-Petersson (WP) metric.

The following properties of The Weil-Petersson metric are used in the rest of the thesis:

Proposition 3.1.1 .

1. *The WP metric is a Kähler metric.*

2. *The WP metric is incomplete, but any two points in Teichmüller space can be joined by a Weil-Petersson geodesic.*

3. *The WP metric has negative holomorphic sectional curvatures, scalar curvatures, and Ricci curvatures.*

Proof The proof is given in [17] and [18]. ■

Denote the metric completion of $\mathcal{T}$ by $\overline{\mathcal{T}}$ with respect to the Weil-Petersson metric. $\overline{\mathcal{T}}$ is not a locally compact space. The action of $Mod(S)$ on $\mathcal{T}$ extends to an action by homeomorphisms on $\overline{\mathcal{T}}$.

3.2 NPC space

In the classical theory of harmonic maps, Eells and Sampson shows that there exists a unique harmonic maps from a compact Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. In the theory of harmonic maps into singular spaces, one assume the target space have similar characteristic features in geometry compared with non-positive sectional curvature in Riemannian manifold. The following definition of metric space is a natrual generalization of the Riemannian manifold with non-positive sectional curvature.

Definition 3.2.1 (NPC space, [7]) *A complete metric space (X, d) (possibly infinite dimensional) is said to be non-positively curved (NPC) if the following two conditions are satified:*

- *(X, d) is a length space, that is, for any two points P, Q in X, the distance $d(P, Q)$ is realized as the length of a rectifiable curve connecting P to Q. (We call such distance-realizing curves geodesics.)*

- *For any three points P, Q, R in X and choices of geodesics γ_{PQ} (of length r), γ_{QR} (of length p), and γ_{RP} (of length q) connecting the respective points, the following comparison property is to hold: For any $0 < \lambda < 1$, write Q_λ for the point on γ_{QP} which is a fraction λ of the distance from Q to R. That is,*

  $$d(Q_\lambda, Q) = \lambda p, d(Q_\lambda, R) = (1-\lambda)p$$

  On the (possibly degenerate) Euclidean triangle of side lengths p, q, r and opposite vertices $\bar{P}$, $\bar{Q}$,$\bar{R}$, there is a corresponding point

  $$Q_\lambda = \bar{Q} + \lambda(\bar{R} - \bar{Q})$$

  The NPC hypothesis is that the metric distance $d(P, Q_\lambda)$ (from Q_λ to the opposite vertex P) is bounded above by the Euclidean distance $|\bar{P} - \bar{Q}_\lambda|$.

  This inequality can be written precisely as

  $$d^2(P, Q_\lambda) \leq (1-\lambda)d^2(P, Q) + \lambda d^2(P, R) - \lambda(1-\lambda)d^2(Q, R)$$

  .

Proposition 3.2.1 *$\overline{\mathcal{T}}$ is a NPC space.*

Proof By Proposition 3.1.1, Teichmüller space $\mathcal{T}$ is a length space. So is it metric completion.

Again, by Proposition 3.1.1, the sectional curvature Teichmüller space $\mathcal{T}$ is negative. hence the distance induced by the metric satisfies the comparison property of NPC space. It is obvious that the distance function d is continuous, it follows that the metric completion of $\mathcal{T}$ also satisfies the comparison property.

Hence it follows that its Weil-Petersson metric completion $\overline{\mathcal{T}}$ is an NPC space. ∎

3.3 Harmonic maps into singular spaces

The theory of harmonic maps to singular spaces is a generalization of the theory of harmonic maps between Riemmanian manifolds. It was originated by Gromov and Schoen in the seminal paper [6], In their remarkable work [6], the authors proposed a theory of harmonic mappings into buildings, as well as important applications to p-adic supperrigidity for certain discrete groups. Then it was subsequently extended for harmonic maps to maps into more general NPC (non positively curved) space by Korevaar and Schoen ([7] and [8]), where they defined the Korevaar-Schoen energy functional and asserted existence of harmonic maps and their general regularity.asserted and [9]).

In this section, we will first briefly recall the definition of harmonic maps into singular spaces in [7] and the recall the existence theorem of equivariant harmonic maps in [8].

Let (Ω, μ) be a Riemannian manifold, and $X = (X, d)$ a metric space. Consider a map $u : \Omega \to X$.

Recall that the harmonic map between Riemannian manifold is a critical point of the energy functional. In general metric spaces, we are going to first define a similar energy functional, which is called Korevaar-Schoen energy functional.

Assume $1 < p < \infty$. First we can define the space $L^p(\Omega, X)$ by assuming the integration of the p-th power of the distance function over Ω is finite. i.e. fixing $Q \in \Omega$,

$$L^p(\Omega, X) = \{u : \Omega \to X \mid \int_\Omega d^p(u(x), Q) d\mu(x) < \infty\}$$

To define the (p-th) Korevaar-Schoen energy of a function $u \in L^p(\Omega, X)$, we first fix a real number $\epsilon > 0$ and the unit vectors $V \in S^{n-1}$, and let

$$e_\epsilon(x) = \int_{S^{n-1}} (\frac{d(u(x), u(exp(x, \epsilon V))}{\epsilon})^p d\sigma(V)$$

Let ν be any Borel measure on the interval $(0, 2)$ satisfying

$$\nu \geq 0, \nu(0, 2) = 1, \int_0^2 \lambda^p d\nu(\lambda) < \infty$$

Define the approximate energy density function ${}_\nu e_\epsilon(x)$ by averaging the spherical averages $e_\epsilon(x)$ on $[0,2]$ with respect to the measure ν:

$$_\nu e_\epsilon(x) = \int_0^2 e_{\lambda\epsilon}(x) d\nu(\lambda)$$

For $u \in L^p(\Omega, X)$ and $f \in C_c(\Omega)$, define the functional

$$_\nu E_\epsilon(f) = \int_\Omega f(x)_\nu e_\epsilon(x) d\mu(x)$$

We write $u \in W^{1,p}(\Omega, X)$ if

$$\sup_{f \in C_c(\Omega), 0 \le f \le 1} (\limsup_{\epsilon \to 0})_\nu E_\epsilon(f) =_\nu E < \infty$$

Theorem 3.3.1 ([7], theorem 1.5.1) *Let $1 < p < \infty$, $u \in L^p(\Omega, X)$ have finite energy ${}_\nu E$ with respect to some measure ν. Then it has finite energy with respect to all such ν, and each measure ${}_\nu e_\epsilon(x) d\mu(x)$ converges weakly to the same energy density measure de, having total mass ${}_\nu E$.*

One can give a formal definition of the norm of the directional derivative of u in the direction V at x by

$$|u_*(V)| = \lim_{\epsilon \to 0} (\frac{d(u(x), u(exp(x, \epsilon V))}{\epsilon})$$

Remark: The limit exists according to [7].

Theorem 3.3.2 ([7],Theorem 1.10) *Let (Ω, g) be a Riemannian domain, and let $1 < p < \infty$. Let $u \in W^{1,p}(\Omega, X)$. Then the energy density measure de is absolutely continuous with respect to Lebesgue measure, i.e. there exists $|\nabla u|_p(x) \in L^1(\Omega, R)$ s.t.*

$$de = |\nabla u|_p(x) d\mu(x)$$

More precisely,

$$|\nabla u|_p(x) = \int_{S^{n-1}} |u_*(V)|^p d\sigma(V)$$

For $p = 2$, one can define the energy of a map u by

$$E(u) = \int_\Omega |\nabla u|^2(x) d\mu(x) = \frac{1}{\omega_n} \int_\Omega |\nabla u|_2(x) d\mu(x)$$

Remark: If X is a Riemannian manifold with metric h, then $|\nabla u|^2(x)$ is the square of the norm the usual gradient of u, which is

$$|\nabla u|^2(x) = \mu^{ij} h_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\beta}{\partial x_j}$$

In this case, Korevaar-Schoen energy functional $E(u)$, is the usual energy functional.

Definition 3.3.1 (harmonic maps) *A map $u \in W^{1,p}(\Omega, X)$ is said to be harmonic if it is a critical point of the energy functional $E(u)$ amongst all finite energy maps with the same boundary value on every bounded Lipschitz subdomain $\Omega' \subset \Omega$.*

By the definition above, we can extend the theory of harmonic mappings from Riemannian manifolds into NPC metric spaces.

Theorem 3.3.3 ([8],Theorem 2.1.3 & Remark 2.1.5) .

1. *Assume $M = \widetilde{M}/\Gamma$ is compact and ρ is a proper action of $\Gamma = \pi_1(M)$ on X. Assume X is an NPC space. There exists a ρ-equivariant, Lipschitz harmonic map $u : \widetilde{M} \to X$.*

2. *Assume $M = \widetilde{M}/\Gamma$ is complete, and ρ is a proper action of $\Gamma = \pi_1(M)$ on X. Assume X is an NPC space. If there exists a finite energy ρ-equivariant map $u_0 : \widetilde{M} \to X$, then there exists a ρ-equivariant, Lipschitz harmonic map $u : \widetilde{M} \to X$.*

The following definition is from [6] will be used in the prove of the regularity theorem of harmonic maps later.

$$E_{x_0}(\sigma) = \int_{B_\sigma(x_0)} |\nabla u|^2 d\mu$$

$$I_{x_0}(\sigma) = \int_{\partial B_\sigma(x_0)} d^2(u, u(x_0)) d\Sigma$$

By Section 1.2 of [6], there exists a constant $c > 0$ depending only on the metric on Ω such that

$$\sigma \to e^{c\sigma^2} \frac{\sigma E_{x_0}(\sigma)}{I_{x_0}(\sigma)}$$

is non-decreasing for any $x_0 \in \Omega$. One can define the order of a map u at x_0 by

$$Ord(x_0) = \lim_{\sigma \to 0} e^{c\sigma^2} \frac{\sigma E_{x_0}(\sigma)}{I_{x_0}(\sigma)}$$

4. REGULARITY THEOREMS FOR HARMONIC MAPS IN TEICHMÜLLER SPACE

4.1 model space of $\overline{\mathcal{T}}$

Let $\mathcal{T}$ be the Teichmüller space of punctured Riemann surface of finite type (g,p) such that $k = 3g-3+p > 0$ and $\overline{\mathcal{T}}$ be the Weil-Petersson completion of $\mathcal{T}$. The complex dimension of $\mathcal{T}$ is $k = 3g-3+p$. The space $\overline{\mathcal{T}}$ is a stratified space, i.e.

$$\overline{\mathcal{T}} = \bigcup \mathcal{T}'$$

where $\mathcal{T}' = \mathcal{T}$ or $\mathcal{T}'$ is a lower dimensional Teichmüller space, where all the strata are totally geodesic with respect to the Weil-Petersson distance. ([16]).

Definition 4.1.1 (regular set & singular set) *Given a map (not necessarily harmonic) $u : \Omega \to \overline{\mathcal{T}}$, we define its regular set and singular set as*

$$\mathcal{R}(u) = \{x \in \Omega \mid \exists r > 0 \textit{ such that } u(B_r(x)) \subset \mathcal{T}' \textit{ for a stratum } \mathcal{T}' \textit{ of } \overline{\mathcal{T}}\}$$

$$\mathcal{S}(u) = \Omega \backslash \mathcal{R}(u)$$

A point in $\mathcal{R}(u)$ is called a regular point and a point in $\mathcal{S}(u)$ is called a singular point.

If $u : \Omega \to \overline{\mathcal{T}}$ is harmonic, x is a regular point of u, then the usual regularity theory for harmonic maps implies that u is C^∞ in a neighborhood of x.

We can decompose the singular set $\mathcal{S}(u)$ of a harmonic map u as a disjoint union of sets

$$\mathcal{S}(u) = \bigcup_{j=0}^{k} \mathcal{S}_j(u)$$

where

$$\mathcal{S}_j(u) = \{x \in \mathcal{S}(u) \mid \sharp u(x) = j\}$$

4.2 Regularity theorems

The following regularity theorems are used in constructing the Bochner formula.

Theorem 4.2.1 ([14], Theorem 1) *Let $\mathcal{T}$ be the Teichmüller space of an oriented surface S of genus g and p marked points such that $k = 3g - 3 + p > 0$and $\overline{\mathcal{T}}$ be its metric completion with respect to the Weil-Petersson metric. If $u :\to \overline{\mathcal{T}}$ is a harmonic map from an n-dimensional Lipschitz Riemannian domain, then the Haustorff dimension of the singular set*

$$dim_{\mathcal{H}}(\mathcal{S}(u)) \leq n - 2$$

Theorem 4.2.2 ([14], Theorem 2) *Let $u : \Omega \to \overline{\mathcal{T}}$ be a harmonic map from an n-dimensional Lipschitz domain. For any compact subdomain Ω_1 of Ω, there exists a sequence of smooth function $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $S(u) \cap \bar{\Omega}_1$, $0 \leq \psi_i \leq 1$ and $\psi_i \to 1$ for all $x \in \Omega_1 \backslash S(u)$ such that*

$$\lim_{i\to\infty} \int_{\Omega} |\nabla\nabla u||\nabla\psi_i| d\mu = 0$$

Remark:

1. $\nabla\nabla u$ denotes the Hessian of u. It is well-defined on $\mathcal{R}(u)$. $|\nabla\nabla u||\nabla\psi_i|$ is well-defined on Ω since $\nabla\psi_i = 0$ near singular points.

Corollary 4.2.2.1 *Let $u : \Omega \to \overline{\mathcal{T}}$ be a harmonic map defined as above. Then there exists a stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$ such that $u(\mathcal{R}(u)) \subset \mathcal{T}'$.*

Proof Let $\mathcal{T}'$ be the stratum such that

$$\mathcal{T}' \cap u(\mathcal{R}(u)) \neq \emptyset$$

We are going to prove that $u(\mathcal{R}(u)) \subset \mathcal{T}'$. By theorem 4.2.1, the singular set is of Hausdorff codimension 2, so the regular set $\mathcal{R}(u)$ is connected. It suffices to prove that $u^{-1}(\mathcal{T}') \cap \mathcal{R}(u)$ is a nonempty open subset of $\mathcal{R}(u)$.

$u^{-1}(\mathcal{T}') \cap \mathcal{R}(u)$ is nonempty since $\mathcal{T}' \cap u(\mathcal{R}(u)) \neq \emptyset$. For any $x \in u^{-1}(\mathcal{T}') \cap \mathcal{R}(u)$, by the definition of $\mathcal{R}(u)$, there exists $r > 0$ such that $u(B_r(x)) \subset \mathcal{T}'$, which implies that $B_r(x) \subset u^{-1}(\mathcal{T}') \cap \mathcal{R}(u)$.

■

5. BOCHNER TYPE FORMULA

In this section, we formulate a general Bochner type formula for harmonic maps into Teichmüller space, which we will use prove the supperrigidity theorem later.

5.1 A Bochner type formula

Let Q be a covariant 4-tensor on $\widetilde{M}$ satisfying the following conditions:

1. $Q(X,Y,Z,W) = -Q(Y,X,Z,W) = Q(Z,W,X,Y)$.

2. Q is parallel, i.e. $\nabla Q = 0$.

3. $\langle Q(.,.,.,X), R^M(.,.,.,Y)\rangle + \langle Q(.,.,.,Y), R^M(.,.,.,X)\rangle = 0$.

Theorem 5.1.1 (Bochner type formula) *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type other than real or complex hyperbolic spaces, and Γ be a lattice of G with finite volume quotient, i.e. $M = \widetilde{M}/\Gamma$ is of finite volume. Assume that $u : M \to \overline{\mathcal{T}}$ is a harmonic map, then the following Bochner formula holds:*

$$\int_{R(u)} Q^{ijkl}\nabla_i u_l^\alpha \nabla_j u_k^\beta h_{\alpha\beta} = \frac{1}{2}\int_{R(u)} Q^{ijkl} R^N_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{5.1}$$

Proof We follow the proof of [13]. Since M is non-compact, we need to make sure that the boundary term is trivial. So we use a cut-off function η supported on geodesic ball of radius $2R$ from a fixed point $x_0 \in \widetilde{M}/\Gamma$,

$$\Omega_{2R} = \{x \in \widetilde{M}/\Gamma \mid d(x,x_0) < 2R\}$$

which has the following property:

$\eta \equiv 1$ on Ω_R, $\eta \equiv 0$ on Ω_{2R}^c, and $|d\eta| < \frac{4}{R}$.

Let ψ be the smooth function in Theorem 4.2.2 with respect to $\bar{\Omega}_{2R}$. Let Q be a 4-tensor on $\widetilde{M}$ defined as above. Then on the regular set $R(u)$, we have

$$\begin{aligned}\eta\psi Q^{ijkl}u_l^\alpha\nabla_i\nabla_j u_k^\beta h_{\alpha\beta} &= \frac{1}{2}\eta\psi Q^{ijkl}u_l^\alpha[\nabla_i,\nabla_j]u_k^\beta h_{\alpha\beta}\\ &= \frac{1}{2}\eta\psi Q^{ijkl}(g^{ms}R^M_{ijkm}u_l^\alpha u_s^\beta h_{\alpha\beta} - R^N_{\alpha\beta\gamma\delta}u_l^\alpha u_j^\beta u_k^\gamma u_l^\delta)\end{aligned}$$

The first term equals zero since

$$Q^{ijkl}g^{ms}R^M_{ijkm}u_l^\alpha u_s^\beta = Q^{ijks}g^{ml}R^M_{ijks}u_l^\alpha u_s^\beta$$

and

$$Q^{ijkl}g^{ms}R^M_{ijkm} + Q^{ijks}g^{ml}R^M_{ijks} = 0$$

Take the integration on $R(u)$, we have

$$\int_{R(u)}\eta\psi Q^{ijkl}u_l^\alpha\nabla_i\nabla_j u_k^\beta h_{\alpha\beta} = \frac{1}{2}\int_{R(u)}\eta\psi Q^{ijkl}R^N_{\alpha\beta\gamma\delta}u_l^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{5.2}$$

Denote the neighborhood of $\Omega_{2R}\backslash S(u)$ where ψ vanishes by B. Integrate the L.H.S by part, recall that $\nabla Q = 0$, we have

$$\begin{aligned} L.H.S &= \int_{\Omega_{2R}\backslash B}\eta\psi Q^{ijkl}u_l^\alpha\nabla_i\nabla_j u_k^\beta h_{\alpha\beta}\\ &= -\int_{\Omega_{2R}\backslash B}\nabla_i(\eta\psi)Q^{ijkl}u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} - \int_{\Omega_{2R}\backslash B}\eta\psi Q^{ijkl}\nabla_i u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta}\\ &= -\int_{R(u)}\nabla_i(\eta\psi)Q^{ijkl}u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} - \int_{R(u)}\eta\psi Q^{ijkl}\nabla_i u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} \qquad (5.3)\end{aligned}$$

The first term of equation (5.3) is bounded by

$$\int_{R(u)}\nabla_i(\eta\psi)Q^{ijkl}u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} = \int_{R(u)}\psi\nabla_i\eta Q^{ijkl}u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} + \int_{R(u)}\eta\nabla_i\psi Q^{ijkl}u_l^\alpha\nabla_j u_k^\beta h_{\alpha\beta} \tag{5.4}$$

Since Q is bounded and u is Lipschitz, the bounded by

$$c\int_M(\frac{4}{R} + |\nabla\psi|)|\nabla\nabla u|$$

where c depends on the Lipschitz constant of u. By theorem 4.2.2 ([14], Theorem 2), it vanishes by taking $\psi \to 1$ and $R \to \infty$. ■

The two additional conditions in [13] are the following.

4. The quadratic form $(\xi^{ij}) \to Q_{ijkl}\xi^{il}\xi^{jk}$ is positive definite for all traceless symmetric 2-tensors (ξ^{ij}).

5. The inner product $\langle Q, T \rangle$ is nonpositive for any tensor T of curvature type with nonpositive Riemannian sectional curvature in the case of rank > 2 and with nonpositive complexified sectional curvature in the rank 1 case.

Remark: 1. If the 4-tensor satisfies additionally (4) and (5), the Bochner formula (5.1) gives by (4) a nonpositive right-hand side, which by (5) implies that ∇df vanishes identically on its regular set.

2. As is shown in [13], let M be a g1obally symmetric irreducible Riemannian manifold of noncompact type. Assume that either M is of rank at least two or M is the quaternionic hyperbolic space of quaternionic dimension at least two or the hyperbolic Cayley plane, such 4-tensor Q satisfying condition (1)-(5) exists. Furthermore, Q is parallel with respect to the Killing metric on the symmetric space satisfying various eigenvalue bounds as given in [13].

6. EQUIVARIANT HARMONIC MAPS INTO TEICHMÜLLER SPACE

In this chapter, we are going to prove the following existence theorem for equivariant harmonic maps from a symmetric space into the Weil-Petersson completion $\overline{\mathcal{T}}$.

Theorem 6.0.1 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is neither complex nor real hyperbolic space, and Γ be a discrete subgroup of G with finite volume quotient. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a ρ-equivariant Lipschitz harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

The proof of the theorem follows from Theorem 2.1.3 and Corollary 2.1.5 in [8] under the assumption that there exists a finite energy ρ-equivariant map from $\widetilde{M}$ to $\overline{\mathcal{T}}$, which can be constructed directly when Γ is cocompact. We start with the almost finite energy retraction from $\widetilde{M}/\Gamma$ to the central tile of a tiling of the Borel-Serre compactification of M(cf. [1], theorem 6.1) and construct an almost finite energy ρ-equivariant map (defined in section 6.3) from $\widetilde{M}$ to $\overline{\mathcal{T}}$ by composition.

6.1 Equivariant harmonic maps

Let (M, g) be a metrically complete Riemannian manifold, possibly with smooth compact boundary .Denote the fundamental group of M by Γ and the universal cover of M by $\widetilde{M}$. Let X be a metric space, and $\rho : \Gamma \to isom(X)$ a homomorphism. Such a ρ is also called representations of Γ. Let now ρ be a representation in isom($\overline{\mathcal{T}}$).

Definition 6.1.1 *A map $u : \widetilde{M} \to \overline{\mathcal{T}}$ is ρ-equivariant if*

$$u(\rho(\gamma)x) = \rho(\gamma)u(x)$$

for any $\gamma \in \Gamma$ and $x \in \widetilde{M}$.

For a ρ-equivariant map u the real-valued functions $d(u(x), u(y))$ are invariant under the action of Γ, then it follows that the Sobolev energy densities considered in Chapter 3 are ρ-invariant, so we may think of them as being defined on the quotient M.

An equivariant map u is said to have finite energy if its Korevaar-Schoen energy on M

$$E(u) = \int_\Omega |\nabla u|^2(x) d\mu(x)$$

is finite. An equivariant map u is said to be harmonic if its push-forward (still written as u) is a harmonic map on M.

6.2 Poincaré inequality for a non-compact locally symmetric space

We are going to use Poincaré inequality for a non-compact locally symmetric space of finite volume. The proof of such a statement does not seem to be available in the literature. Yeung explained to the author the following argument of Donnelly concerning the Poincare Lemma. It turns out that the same proof essentially works without assuming arithmeticity of Γ. The proof of the following proposition is based on an argument provided by Donnelly.

First, we proof the following lemma.

Lemma 6.2.1 *Suppose $h \in \mathcal{C}_o^\infty(R^+)$ and $\theta > 0$ satisfies $|\theta'/\theta| \geq c_1 > 0$. Then*

$$\int_0^\infty h^2 \theta dx \leq c \int_0^\infty (h')^2 \theta dx. \tag{6.1}$$

Proof From integration by parts,

$$2\int_0^\infty hh'\theta dx = -\int_0^\infty h^2\theta' dx = -\int_0^\infty h^2 \frac{\theta'}{\theta}\theta dx.$$

Hence our assumption implies that

$$c_1 \int_0^\infty h^2\theta dx \leq 2\left|\int_0^\infty h^2\frac{\theta'}{\theta}\theta dx\right| \leq 2\left(\int_0^\infty h^2\theta dx\right)^{1/2}\left(\int_0^\infty (h')^2\theta dx\right)^{1/2},$$

from which the lemma follows.

■

Theorem 6.2.2 *Let $f \in C_0^\infty(M)$. Then*

$$\int_M |f|^2 dV_g \leq c \int_M |\nabla f|^2 \tag{6.2}$$

for some constant $c > 0$.

Proof From the Decomposition Principle as shown in [19], we know that it suffices to prove that the essential spectrum of $\triangle$ on a neighborhood U of the the cusp

$$D = \overline{\widetilde{M}}/\Gamma - \widetilde{M}/\Gamma$$

is bounded away from 0, where $\overline{\widetilde{M}}$ is the Borel-Serre compactification of $\widetilde{M}$.

The metric behavior of the Killing metric on M near a cusp is given by [20], Proposition 4.3 with volume form given by Corollary 4.4. It follows from the formula there that the volume form near the cusp is of the form $\theta dtdx$ with volume element θ satisfying

$$\frac{\theta'}{\theta} \geq c_1 \geq 0$$

where we may assume that $t \in [0, \infty)$. From the construction in [1] D is a disjoint union of D_i and U is a finite disjoint union of $V_i, i = 1, \cdots, N$ of neighborhoods of D_i. Apply Lemma 6.2.1, the energy near the cusp is of form

$$\int_{V_i} f^2 dV = \int_{D_i} \int_0^\infty f^2 \theta dtdx \leq c \int_{D_i} \int_0^\infty (\frac{\partial f}{\partial t})^2 \theta dtdx \leq c \int_{V_i} |\nabla f|^2 dV \tag{6.3}$$

Then it follows that

$$\begin{aligned} \int_U |\nabla f|^2 dV_g &= \int_{\cup_i^N V_i} |\nabla f|^2 dV_g = \sum_{i=1}^N \int_{V_i} |\nabla f|^2 dV_g \\ &\geq c_2 \sum_{i=1}^N \int_{V_i} |f|^2 dV_g = c_2 \int_{\cup_i V_i} u^2 dV_g \end{aligned} \tag{6.4}$$

■

Here is a slightly different version of Poincaré Inequality to be used later.

Theorem 6.2.3 *Let $f \in C_0^\infty(M) \cap H^2(M) \cap H^1(M)$. Let $\hat{f}$ be the average of f on M. Then*

$$\int_M |f-\hat{f}|^2 dV_g \leq c\int_M |\nabla f|^2 \tag{6.5}$$

for some constant $c > 0$.

Proof Fix a point $x_0 \in M$. Let $B_r(x_0)$ be the geodesic ball of radius r centered at x_0. Let η be the cut-off function supported on $B_{2r}(x_0)$ and is identically 1 on $B_r(x_0)$ with $|\nabla \eta| \leq \frac{2}{r}$. From our assumption,

$$f - \hat{f} \in L^2(M)$$

since M has finite volume. It the follows that

$$\begin{aligned}
\int_M |(f-\hat{f})\eta|^2 dV_g &\leq c\int_M |\nabla((f-\hat{f})\eta)|^d V_g \\
&\leq 2c\int_M |\nabla f|^2\eta^2 dV_g + 2c\int_M |f-\hat{f}|^2|\nabla\eta|^2 dV_g \\
&\leq 2c\int_M |\nabla f|^2 dV_g + \frac{4c}{r}\int_M |f-\hat{f}|^2 dV_g
\end{aligned}$$

Proposition follows by letting $r \to \infty$ ■

6.3 Construction of finite energy map

Assume that M is compact, we can conclude from theorem 3.3.3 that there exists a ρ-equivariant harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$. However, when M is only of finite volume, one has to construct a finite energy map first, which in general, is open.

In this section, we exhibit a method for constructing finite energy maps when M is a symmetric space. We start with the result of Saper [1] that the identity map of M can be deformed into a retraction onto a compact submanifold.

Definition 6.3.1 (almost finite energy [21]) *A ρ-equivariant map f is an almost finite energy map if there exists a function $\varphi \in C^\infty(M)$, satisfying the following conditions*

1. *φ is an exhaustion function on M and $\varphi \geq 1$.*

2. *$|\nabla\varphi| \leq C$.*

3. *$|\triangle\varphi| \leq C$.*

4. *$\int_M \varphi^{1+\alpha} dV_g < \infty$ where $0 < \alpha \leq \frac{1}{2}$.*

such that

$$\int_M |\nabla f|^2 \varphi^{-(1+\alpha)} dV_g < \infty \tag{6.6}$$

The following theorem from [1] is used to construction the almost finite energy map into $\overline{\mathcal{T}}$

Theorem 6.3.1 ([1], Theorem 7.3) *Let $r : \widetilde{M} \to \widetilde{M_0}$ be the ρ-invariant retraction onto the central tile of a tiling and let r' be the induced retraction on M. Assume G is almost Q-simple and that M is noncompact. Then r' has almost finite energy if and only if $G_C \neq SL(2,C)$. Furthermore, r' has finite energy of and only if G_C is not equal to $SL(2,C), SL(2,C) \times SL(2,C), SL(3,C)$, or a Q-split form of $SO(5,C)$.*

Lemma 6.3.2 *Let $\widetilde{M} = G/K$ be an irriducible symmetric space of noncompact type which admits a finite energy retraction onto its compact retract (Theorem 7.3 in [1]), Γ be a discrete subgroup of G with finite volume quotient. Let X be a NPC space, and $\rho : \Gamma \to iso(X)$ be a homomorphism. Then there exists a finite energy ρ-equivariant map from $\widetilde{M}$ to X.*

Proof By using the theorem 7.3 of [1], we can construct a retraction r from $\widetilde{M}$ onto the central tile of a tiling $\widetilde{M_0}$, whose induced retraction on M has finite energy. According to the Proposition 2.6.1 in [7], there also exists a ρ-equivariant local Lipschitz map v from $\widetilde{M_0}$ to X. Now construct

$$u = v \circ r$$

It is obvious that u is ρ-equivariant.

Since v is local Lipschitz and $\widetilde{M}_0/\Gamma$ is compact, v is Lipschitz.

$$E(u) = \int_M |\nabla u|^2(x) d\mu(x) \leq c^2 \int_M |dr|^2(x) d\mu(x) \tag{6.7}$$

i.e. u is of finite energy. ∎

Corollary 6.3.2.1 *Let $\widetilde{M} = G/K$ be an irriducible symmetric space of noncompact type which admits a finite energy retraction onto its compact retract (Theorem 7.3 [1]), and Γ be a discrete subgroup of G with finite volume quotient. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a ρ-equivariant Lipschitz harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

Proof Follows immediately from theorem 3.3.3. and the theorem 1.2 in [22], which shows that the homomorphism is proper if it is sufficiently large. ∎

Lemma 6.3.3 *If the induced retraction only has an almost finite energy, we can only construct a finite energy ρ-equivariant map from $\widetilde{M}$ to $\mathcal{T}$.*

Proof The proof is similar to Lemma 4.3.2. Let

$$u = v \circ r$$

Then

$$E(u) = \int_M |\nabla u|^2(x) \varphi^{-(1+\alpha)} d\mu(x) \leq c^2 \int_M |dr|^2(x) \varphi^{-(1+\alpha)} d\mu(x)$$

∎

Denote $(\widetilde{M}, g)$ be a symmetric space as studied before. Let $g_1 = \varphi^{-\frac{2(1+\alpha)}{n-2}} g$. Then the energy of u with respect to g_1 is

$$E_{g_1}(u) = \frac{1}{2} \int_M |\nabla u|^2_{g_1} dV_{g_1} = \frac{1}{2} \int_M |\nabla u|^2_g \varphi^{-(1+\alpha)} dV_g < \infty \tag{6.8}$$

Lemma 6.3.4 *Assume (M, g_1) defined as above. Then (M, g_1) is a complete Riemannian manifold.*

Proof Using Proposition 4.3 in [20] and $\varphi = \log(3+a)$ in [21], one can get the metric form of (M, g_1). Completeness is given by the fact that

$$\int |ds_{g_1}| = \int_0^\infty \frac{da}{(\log(3+a))^{(1+\alpha)/(n-2)}} = \infty \tag{6.9}$$

In this case, the exponent here is not greater than 3/4 since all examples involved has $n \geq 4$, actually 5. ∎

By Theorem 3.3.3, there exists a ρ-equivariant harmonic map u from $(\widetilde{M}, g_1)$ to $\bar{\mathcal{T}}$.

Lemma 6.3.5 *Assume that $\widetilde{M}, g_1$ are defined as before, f is a ρ-equivariant harmonic map from $(\widetilde{M}, g_1)$ to $\bar{\mathcal{T}}$. Then there exists a constant $c > 0$, s.t.*

$$\|\nabla\nabla u\|^2 \leq c\|\nabla u\|^2_{g_1} \tag{6.10}$$

Proof If M is irreducible, as shown in [13], we can define a 4-tensor Q such that (1)(2)(3) in section 5.1 is satisfied. Recall the Bochner formula in previus chapter

$$\int_M Q^{ijkl}\nabla_i u_l^\alpha \nabla_j u_k^\beta h_{\alpha\beta} = \frac{1}{2}\int_M Q^{ijkl} R^N_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta \tag{6.11}$$

The computations in [13] shows that for each symmetric space of rank at least two and each Q as chosen, $Q^{ijkl}\xi^{il}\xi^{jk}$ is positive define on symmetric traceless two tensor ξij. Decompose

$$\nabla\nabla f = (\nabla\nabla f)^0 + (\nabla\nabla f)^{tr}$$

into the direct sum of traceless part and

$$(\nabla\nabla f)^{tr} = \triangle_g f$$

then we have

$$\int_M Q^{ijkl} u_l^\alpha \nabla_i \nabla_j u_k^\beta h_{\alpha\beta} dV_g \leq c \int_M Q^{ijkl} R^N_{\gamma\delta\mu\beta} (\nabla_i u_l^\alpha)^{tr} (\nabla_j u_k^\beta)^{tr} h_{\alpha\beta} dV_g \tag{6.12}$$

Note that Q is a bounded tensor and is positive definite with eigenvalues bounded from below by a positive constant. Then we conclude that

$$\|\nabla\nabla f\|^2 \leq c\int_M |\triangle_g f|^2 \tag{6.13}$$

Since f is g_1-harmonic, we conclude that

$$\triangle_g f = (1+\alpha)\nabla^i \log\varphi \nabla_i f \tag{6.14}$$

where $\nabla^i \phi$ is bounded by the definition of almost finite energy map. Hence

$$\begin{aligned} \|\nabla\nabla f\|^2 &\leq c\int_M |(1+\alpha)\nabla^i \log\varphi\nabla_i f|^2 dV_g \\ &\leq c(1+\alpha)^2 \int_M \varphi^{-2}|\nabla\varphi|^2|\nabla f|^2 dV_g \end{aligned} \tag{6.15}$$

Since $\varphi \geq 1$ and $|\nabla\varphi| \leq C$, the last term of the above inequality is bounded above by

$$C_1 \int_M |\nabla f|^2 \varphi^{-(1+\alpha)} dV_g$$

which is finite. ■

Theorem 6.3.6 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is not real or complex hyperbolic, and Γ be a discrete subgroup of G with finite volume quotient. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a finite energy ρ-equivariant map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

Proof We can define a cut-off function $\eta : R \to R$ supported on $[-2, 2]$ such that $\equiv 1$ on $[-1, 1]$ and $|\eta'| \leq C_1$. Define $\eta_t : M \to R$ such that

$$\eta_t(x) = \eta(\frac{\varphi(x)}{t})$$

Since $|\nabla\varphi| \leq C$, we will have

$$|\nabla\eta_t| \leq \frac{C_2}{t}$$

Now let $\Omega_R = \{x \in M \mid \varphi(x) \leq R\}$. It follows from the Poincare Inequality that we have

$$\begin{aligned} \int_M |\eta_t\nabla f - \widehat{\eta_t\nabla f}|^2 dV &\leq \int_M |\nabla(\eta_t\nabla f)|^2 dV \\ &\leq \frac{C_2^2}{R^2}\int_{\Omega_{2R}-\Omega_R} |\nabla f|^2 V + \int_M |\nabla\nabla f|^2 dV_g \\ &\leq \frac{C_2^2}{R^\alpha}\int_{\Omega_{2R}-\Omega_R} |\nabla f|^2 \varphi^{-(1+\alpha)} V_g + \int_M |\nabla\nabla f|^2 dV \end{aligned} \tag{6.16}$$

where

$$\widehat{\eta_t \nabla f} = \frac{1}{Vol(M)} \int_M |\eta \nabla f| dV$$

Now from Cauchy-Schwarz Inequality,

$$\begin{aligned} \int_M |\eta_t \nabla f| dV &\leq (\int_M |\eta_t \nabla f|^2 \varphi^{-(1+\alpha)} dV)(\int_M \varphi^{1+\alpha} dV) \\ &\leq C \int_M |\nabla f|^2 \varphi^{-(1+\alpha)} dV \end{aligned} \quad (6.17)$$

Now letting $R \to \infty$, we conclude that

$$\|\nabla f\|^2 < C \quad (6.18)$$

■

6.4 Existence theorem

Let S be punctured Riemann surface of finite type (g, p), where g denotes the genus of compact surface $\overline{S}$ and p denotes the number of punctures. We assume that $k = 3g - 3 + p > 0$. A singular foliation of S is a foliation with 1-dimensional leaves except at isolated singular points.

Definition 6.4.1 (pseudo-Anosov, [23]) *An element $\gamma \in \Gamma$ is called pseudo-Anosov if there is $r > 1$ and transverse measured foliations F_+, F_- on S such that*

$$\gamma F_+ = r F_+$$

$$\gamma F_- = r^{-1} F_-$$

F_+ and F_- are called the stable and unstable foliations of γ, respectively.

Remark: We can scale the measure on a foliation F by a real number $r > 0$ to obtain a new measured foliation, which we denote by rF.

Definition 6.4.2 (sufficiently large, [23]) *A subgroup of $Mod(S)$ is sufficiently large if it contains two independent pseudo-Anosov's.*

A homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large if its image is sufficiently large.

Definition 6.4.3 (proper, [23]) *A finitely generated subgroup of $Mod(S)$ is called proper if there is a set of generators $\gamma_1, \ldots \gamma_k$ of the subgroup such that the sublevel sets of the displacement function*

$$\delta(\sigma) = max\{d_{WP}(\sigma, \gamma_i \sigma) : i = 1, \ldots, k\}$$

are bounded.

Theorem 6.4.1 ([23], Theorem 1.2) *Finitely generated subgroups of $Mod(S)$ is proper if it is sufficiently large.*

Theorem 6.4.2 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is not real or complex hyperbolic, and Γ be a discrete subgroup of G with finite volume quotient. If a homomorphism $\rho : \Gamma \to Mod(S)$ is sufficiently large, then there exists a ρ-equivariant Lipschitz harmonic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$.*

Proof Theorem 5.3.5 shows that there exists a finit energy ρ-equivariant map from $\widetilde{M}$ to $\overline{T}$. By theorem 3.3.3, one can construct a ρ-equivariant harmonic map from $\widetilde{M}$ to $\overline{T}$. ■

7. SUPPERRIGIDITY OF MAPPING CLASS GROUP

7.1 Introduction

In this chapter, we discuss the superrigidity of mapping class group.

The history behind the problem of rigidity of Kähler manifolds was initiated by Calabi and Vesentini [10] in 1960. The authors implies that compact locally symmetric spaces of dimension at least 2 do not admit any nontrivial infinitesimal holomorphic deformation. In 1968, Mostow [11] celebrated the strong rigidity theorem that two compact ball quotients of complex dimension at least 2 with isomorphic fundamental groups are isometric and thus biholomorphic or conjugate biholomorphic. At that time, the results of Eells-Sampson [4] on existence of harmonic map is already known and people had been hoping to use the approach of harmonic maps to derive results in rigidity. This was not successful until the work of Siu [12], which proves a strong rigidity result for Kähler manifolds with curvature which is sufficiently negative, in particular if the curvature is strongly negative in the sense of Siu. This reproved the result of Mostow in the case of locally Hermitian symmetric spaces. The technique of harmonic maps has been generalized to study geometric superrigidity, cf. [13].

In another direction, Teichmuller space is equipped with a natural invariant metric, the Weil-Petersson metric. It is well-known that the Weil-Petersson metric is Kahler and is negatively curved, in particular, strongly negatively curved in the sense of Siu, from the work of Ahlfors, Wolpert, Schumacher and others, (cf. [14]) Hence the result of Siu applied once a harmonic map is shown to exist. The difficulty here is that the Weil-Petersson metric is incomplete and hence the result of Eells-Sampson did not applied.

In this section, we provide a geometric proof of both the high rank and the rank one superrigidity of mapping class groups due to Farb-Masur [2] and Yeung [3].

Theorem 7.1.1 (Superrigidity of mapping class group) *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is not real or complex hyperbolic. Let Γ be a lattice in G. Then any homomorphism ρ from Γ to the mapping class group $Mod(S)$ has finite image.*

The sketch of the proof is the following. First we construct a ρ-equivariant harmonic map from $\widetilde{M}$ to $\bar{\mathcal{T}}$. Then following the works of [22], we claim that this map is constant. Finally, we proof that the image of ρ is finite by using the property of properly discontinuity of the mapping class group.

7.2 Totally geodesic

Lemma 7.2.1 *Let f be a ρ-equivariant harmonic map from $\widetilde{M} = G/K$ to $\bar{\mathcal{T}}$, then f is totally geodesic on its regular set.*

Proof We work on the quotient $\widetilde{M}/\Gamma$ and follow the proof of [MSY]. Since $\widetilde{M}/\Gamma$ is non-compact, we need to make sure that the boundary term is trivial. So we use a cut-off function η supported on geodesic ball of radius $2R$ from a fixed point $x_0 \in \widetilde{M}/\Gamma$,

$$\Omega_{2R} = \{x \in \widetilde{M}/\Gamma \mid d(x, x_0) < 2R\}$$

which has the following property:

$\eta \equiv 1$ on Ω_R, $\eta \equiv 0$ on Ω_{2R}^c, and $|d\eta| < \frac{4}{R}$. Let ψ be the smooth function in Theorem 2 of [DM] w.r.t. $\bar{\Omega}_{2R}$. Let Q be a 4-tensor on $\widetilde{M}/\Gamma$ satisfiying (i)-(v) in [MSY]. Then

$$\begin{aligned}\eta\psi Q^{ijkl} f_l^\alpha \nabla_i \nabla_j f_k^\beta h_{\alpha\beta} &= \frac{1}{2}\eta\psi Q^{ijkl} f_l^\alpha [\nabla_i, \nabla_j] f_k^\beta h_{\alpha\beta} \\ &= \frac{1}{2}\eta\psi Q^{ijkl}(g^{ms} R^M_{ijkm} f_l^\alpha f_s^\beta h_{\alpha\beta} - R^N_{\alpha\beta\gamma\delta} f_l^\alpha f_j^\beta f_k^\gamma f_l^\delta)\end{aligned}$$

The first term equals zero since

$$Q^{ijkl} g^{ms} R^M_{ijkm} f_l^\alpha f_s^\beta = Q^{ijks} g^{ml} R^M_{ijks} f_l^\alpha f_s^\beta$$

and

$$Q^{ijkl} g^{ms} R^M_{ijkm} + Q^{ijks} g^{ml} R^M_{ijks} = 0$$

Take the integration on $\widetilde{M}/\Gamma$, we have

$$\int_{\widetilde{M}/\Gamma} \eta\psi Q^{ijkl} f^\alpha_l \nabla_i \nabla_j f^\beta_k h_{\alpha\beta} = \frac{1}{2} \int_{\widetilde{M}/\Gamma} \eta\psi Q^{ijkl} R^N_{\alpha\beta\gamma\delta} f^\alpha_l f^\beta_j f^\gamma_k f^\delta_l \qquad (7.1)$$

Integrate the L.H.S by part, recall that $\nabla Q = 0$, we have

$$L.H.S = -\int_{\widetilde{M}/\Gamma} \nabla_i(\eta\psi) Q^{ijkl} f^\alpha_l \nabla_j f^\beta_k h_{\alpha\beta} - \int_{\widetilde{M}/\Gamma} \eta\psi Q^{ijkl} \nabla_i f^\alpha_l \nabla_j f^\beta_k h_{\alpha\beta} \qquad (7.2)$$

The first term is bounded by

$$c \int_{\widetilde{M}/\Gamma} (\frac{4}{R} + |\nabla\psi|)|\nabla\nabla f|$$

where c depends on the Lipschitz constant of f. By theorem 2 in [14], it vanishes by taking $\psi \to 1$ and $R \to \infty$. Thus we have

$$\int_M Q^{ijkl} \nabla_i f^\alpha_l \nabla_j f^\beta_k h_{\alpha\beta} = \frac{1}{2} \int_M Q^{ijkl} R^N_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_k f^\delta_l \qquad (7.3)$$

Since f is harmonic, df is traceless, by property (4)(5) of Q, L.H.S is nonnegative, while R.H.S is nonpositive. Hence

$$|\nabla df| = 0$$

■

Lemma 7.2.2 *Assume $u \to \overline{\mathcal{T}}$ is a totally geodesic on its regular set. Then u is totally geodesic on the entire $\widetilde{M}$.*

Proof It suffices to prove that u maps geodesics to geodesics. i.e. let

$$\gamma : [0, 1] \to \widetilde{M}$$

be a a constant speed parametrization of a geodesic, then we are going to prove that $u \circ \gamma$ is a constant speed parametrization of a geodesic.

Let $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$. We claim that there exists sequences $\{x_0^k\} \to x_0$ and $\{x_1^k\} \to x_1$, such that the geodesics γ_k joint x_0^k and x_1^k located in the regular set of u.

Let B be a hypersurface perpendicular to $\gamma'(1)$ at x_1 with local coordinate

$$\varphi : Br(0) \to B$$

Let γ_x be the unique geodesic joining x_0 and $x \in B$. (It is well-defined since $\widetilde{M}$ has negative sectional curvature.) One can define a map

$$\Phi : Br(0) \times [0,1] \to \widetilde{M}$$

by

$$\Phi(p,t) = \gamma_{\varphi(p)}(t)$$

Then for any $\epsilon > 0$,

$$\Phi\mid_{B_r(0)\times(\epsilon,1)}$$

is a diffeomorphism. Now since

$$dim_H(S(u))) \leq n-2$$

For any $\epsilon > 0$, there exists $p \in B_r(0)$, such that

$$\Psi(\{p\} \times (\epsilon, 1)) \cap S(u) = \emptyset$$

Especially, for $\epsilon = \frac{1}{k}$, we get a series p_k. Let

$$x_0^k = \Phi(p_k, \frac{1}{k})$$

$$x_1^k = \Phi(p_k, 1)$$

Then the image of $\gamma_k \subset \mathcal{R}(u)$.

Noticed that we have

$$\nabla u \equiv 0$$

in $\mathcal{R}(u)$. Thus $u \circ \gamma_k$ is a constant speed parametrization of a geodesic in $\mathcal{T}$. By the continuity of u, this then implies that $u \circ \gamma$ is also a geodesic, which proves the lemma. ■

7.3 Supperrigidity of mapping class group

Theorem 7.3.1 *Let $\widetilde{M} = G/K$ be an irreducible symmetric space of noncompact type which is not real or complex hyperbolic. Let Γ be a discrete subgroup of G with finite volume quotient. Then any homomorphism ρ from Γ to the mapping class group $Mod(S)$ has finite image.*

Proof First, we assume that ρ is sufficiently large.

By theorem 6.4.2, there exists a ρ-equivariant harmonic map u from $\widetilde{M}$ to $\overline{\mathcal{T}}$. By Corollary 4.2.2.1, the image of the regular set of u is located in a single stratum, that is $\mathcal{T} \subset \bar{\mathcal{T}}$ such that

$$u(\mathcal{R}(u)) \subset T$$

Now we will prove that u is actually a constant map. We prove by contradiction. We have already known that by lemma 7.2.1 that u is totally geodesic on the regular set $R(u)$. By Lemma 7.2.2, we can see that u is a totally geodesic map from $\widetilde{M}$ to $\overline{\mathcal{T}}$, which means that $\widetilde{M}$ can be regarded as a immersion submanifold of $\overline{\mathcal{T}}$.

But we already assume that either the rank of $\widetilde{M}$ is not less that 2 or not real or complex hyperbolic. This leads to a contradiction. This implies that u is a contant map on M, i.e. $\rho(\Gamma)$ fixes a point in Teichmüller space. Since the action of the mapping class group is properly discontinuous, this implies that $\rho(\Gamma)$ is finite.

Then we assume that ρ is not sufficiently large.

By ([24], Theorem 4.6) any irreducible subgroup of mapping class group that is not sufficiently large is either finite or virtually cyclic, i.e. there exists a cyclic subgroup Γ_2 of Γ such that the index$[\Gamma_2 : \Gamma]$ is finite. We assume that $\rho(\Gamma)$ is virtually cyclic. Then there exists a homeomorphism $\widetilde{\rho} : \Gamma_2 \to Z$ such that

$$h_1(M_2, Z) = dim(H_1(M_2, Z) \otimes R) > 0$$

where $M_2 = \widetilde{M}/\Gamma$.

Since M_2 is locally symmetric, Matsushima's Vanishing theorem implies that

$$h_1(M_2, Z) = 0$$

when rank of $M_2 \geq 2$. In the case of quaternionic and Cayley rank one cases, the vanishing follows from Kazhdan property (cf. [13]). This leads to a contradiction. $\blacksquare$

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