SOME CONNECTIONS BETWEEN COMPLEX DYNAMICS AND STATISTICAL MECHANICS

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To my parents Ann & Alan, my sister Ivy, and my wife Jessica.

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ABSTRACT

Ph.D., Purdue University, August 2020. Some Connections Between Complex Dynamics and Statistical Mechanics. Major Professor: Roland K.W. Roeder.

Associated to any finite simple graph Γ is the *chromatic polynomial* $\mathcal{P}_\Gamma(q)$ whose complex zeros are called the *chromatic zeros* of Γ. A hierarchical lattice is a sequence of finite simple graphs $\{\Gamma_n\}_{n=0}^\infty$ built recursively using a substitution rule expressed in terms of a generating graph. For each n, let μ_n denote the probability measure that assigns a Dirac measure to each chromatic zero of Γ_n. Under a mild hypothesis on the generating graph, we prove that the sequence μ_n converges to some measure μ as n tends to infinity. We call μ the *limiting measure of chromatic zeros* associated to $\{\Gamma_n\}_{n=0}^\infty$. In the case of the Diamond Hierarchical Lattice we prove that the support of μ has Hausdorff dimension two.

The main techniques used come from holomorphic dynamics and more specifically the theories of activity/bifurcation currents and arithmetic dynamics. We prove a new equidistribution theorem that can be used to relate the chromatic zeros of a hierarchical lattice to the activity current of a particular marked point. We expect that this equidistribution theorem will have several other applications, and describe one such example in statistical mechanics about the Lee-Yang-Fisher zeros for the Cayley Tree.

1. CHROMATIC ZEROS ON HIERARCHICAL LATTICES AND EQUIDISTRIBUTION ON PARAMETER SPACE

1.1 Introduction

Motivated by a concrete problem from combinatorics and mathematical physics, we will prove a general theorem about the equidistribution of certain parameter values for algebraic families of rational maps. We will begin with the motivating problem about chromatic zeros (Section 1.1.1) and then present the general equidistribution theorem (Section 1.1.2).

We expect the equidistribution theorem to have other applications. In Section 2 we will describe one such instance.

1.1.1 Chromatic Zeros on Hierarchical Lattices

Let Γ be a finite simple graph. The *chromatic polynomial* $\mathcal{P}_\Gamma(q)$ counts the number of ways to color the vertices of Γ with q colors so that no two adjacent vertices have the same color. It is straightforward to check that the chromatic polynomial is monic, has integer coefficients, and has degree equal to the number of vertices of Γ. The chromatic polynomial was introduced in 1912 by G.D. Birkhoff in an attempt to solve the Four Color Problem [1, 2]. Although the Four Color Theorem was proved later by different means, chromatic polynomials and their zeros have become a central part of combinatorics (For example, a search on Mathscinet yields 333 papers having the words "chromatic polynomial" in the title). For a comprehensive discussion of chromatic polynomials we refer the reader to the book [3].

A further motivation for study of the chromatic polynomials comes from statistical physics because of the connection between the chromatic polynomial and the partition function of the antiferromagnetic Potts Model; see, for example, [4–6] and [7, p.323-325].

We will call a sequence of finite simple graphs $\Gamma_n = (V_n, E_n)$, where the number of vertices $|V_n| \to \infty$, a "lattice". The standard example is the $\mathbb{Z}^d$ lattice where, for each $n \geq 0$, one defines Γ_n to be the graph whose vertices consist of the integer points in $[-n, n]^d$ and whose edges connect vertices at distance one in $\mathbb{R}^d$. For a given lattice, $\{\Gamma_n\}_{n=1}^{\infty}$, we are interested in whether the sequence of measures

$$\mu_n := \frac{1}{|V_n|} \sum_{\substack{q \in \mathbb{C} \\ \mathcal{P}_{\Gamma_n}(q)=0}} \delta_q \tag{1.1}$$

has a limit μ, and in describing its limit if it has one. Here, δ_q is the Dirac Measure which, by definition, assigns measure 1 to a set containing q and measure 0 otherwise. (In (1.1) zeros of $\mathcal{P}_{\Gamma_n}(q)$ are counted with multiplicity.) If μ exists, we call it the *limiting measure of chromatic zeros* for the lattice $\{\Gamma_n\}_{n=1}^{\infty}$.

This problem has received considerable interest from the physics community especially through the work of Shrock with and collaborators Biggs, Chang, and Tsai (see [8–12] for a sample) and Sokal with collaborators Jackson, Procacci, Salas and others (see [13–15] for a sample). Indeed, one of the main motivations of these papers is understanding the possible ground states (temperature $T = 0$) for the thermodynamic limit of the Potts Model, as well as the phase transitions between them. Most of these papers consider sequences of $m \times n$ grid graphs with $m \leq 30$ fixed and $n \to \infty$. This allows the authors to use transfer matrices and the Beraha-Kahane-Weiss Theorem [16] to rigorously deduce (for fixed m) properties of the limiting measure of chromatic zeros. The zeros typically accumulate to some real-algebraic curves in $\mathbb{C}$ whose complexity increases as m does; see [8, Figures 1 and 2] and [13, Figures 21 and 22] as examples. Indeed, this behavior was first observed in the 1972 work of Biggs-Damerell-Sands [17] and then, more extensively, in the 1997 work of Shrock-Tsai [18]. Beyond these cases with m fixed, numerical techniques are used in [14] to make conjectures about the limiting behavior of the zeros as $m \to \infty$, i.e. for the $\mathbb{Z}^2$ lattice.

To the best of our knowledge, it is an open and very difficult question whether there is a limiting measure of chromatic zeros for the $\mathbb{Z}^2$ lattice. If such a measure does exist, rigorously determining its properties also seems quite challenging. For this reason, we will consider the limiting measure of chromatic zeros for *hierarchical lattices*. They are constructed as follows: start with a finite simple graph $\Gamma \equiv \Gamma_1$ as the *generating graph*, with two vertices labeled a and b, such that Γ is symmetric over a and b. For each $n > 1$, Γ_n retains the two marked vertices a and b from Γ, and we inductively obtain Γ_{n+1} by replacing each edge of Γ with Γ_n, using Γ_n's marked vertices as if they were endpoints of that edge. A key example to keep in mind is the Diamond Hierarchical Lattice (DHL) shown in Figure 1.1. In fact, one can interpret the DHL as an anisotropic version of the $\mathbb{Z}^2$ lattice; see [19, Appendix E.4] for more details.

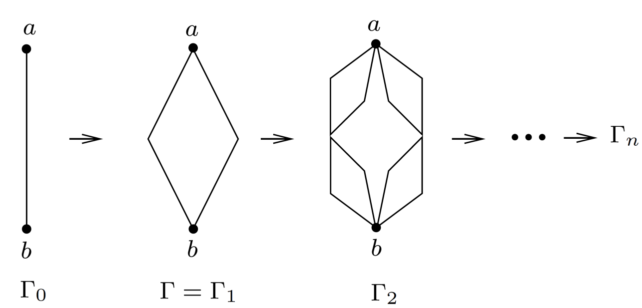


Fig. 1.1. Constructing the Diamond Hierarchical Lattice (DHL).

Several other possible generating graphs are shown in Figure 1.2, including a generalization of the DHL called the k-fold DHL. The k-fold DHL, Triangle, and Split Diamond are 2-connected, while the others are not.

Statistical physics on hierarchical lattices dates back to the work of Berker and Ostlund [20], followed by Griffiths and Kaufman [21], Derrida, De Seze, and Itzykson [22], Bleher and Žalys [23–25], and Bleher and Lyubich [26].

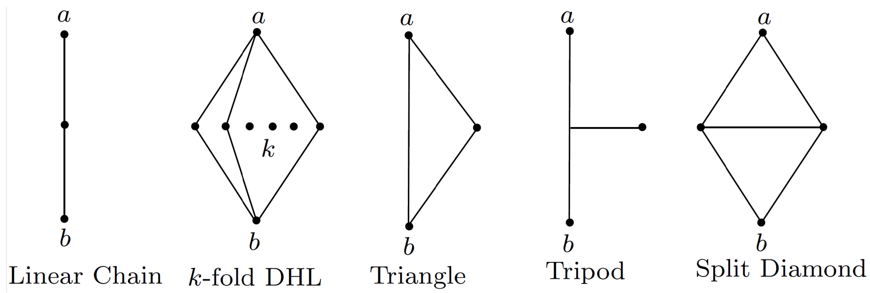


Fig. 1.2. Several possible generating graphs.

A graph Γ is called 2-*connected* if there is no vertex whose removal disconnects the graph. Our main results about the limiting measure of chromatic zeros are:

Theorem A. *Let* $\{\Gamma_n\}_{n=1}^{\infty}$ *be a hierarchical lattice whose generating graph* $\Gamma \equiv \Gamma_1$ *is* 2*-connected. Then its limiting measure* μ *of chromatic zeros exists.*

Theorem B. *Let* μ *be the limiting measure of chromatic zeros for the* k*-fold DHL and suppose* $k \geq 2$*. Then,* $\text{supp}(\mu)$ *has Hausdorff dimension* 2.

As shown in Figure 1.3, the support of the limiting measure of chromatic zeros for the DHL equals the union of boundaries of the black, blue, and white sets. Let $r_q(y)$ be the renormalization mapping for the DHL, given in (1.4). Points in white correspond to parameter values q for which $r_q^n(0) \to 1$, points in blue correspond to parameter values q for which $r_q^n(0) \to \infty$, and points in black correspond to parameter values for which $r_q^n(0)$ does neither. The region depicted on the left is approximately $-2 \leq \text{Re}(q) \leq 4$ and $-3 \leq \text{Im}(q) \leq 3$. The region on the right is a zoomed in view of the region shown in the red box on the left. See Section 1.7 for an explanation of the appearance of "baby Mandelbrot sets", as on the right. Their appearance will imply Theorem B. Figures 1.3 and 1.4 were made using the Fractalstream software [27].

Meanwhile, in Figure 1.4, the support of the limiting measure of chromatic zeros for the hierarchical lattice generated by the split diamond (see Figure 1.2) equals the union of boundaries of the black, blue, and white sets. Let $r_q(y)$ denote the renormalization mapping generated by the split diamond, given in (1.30). The coloring

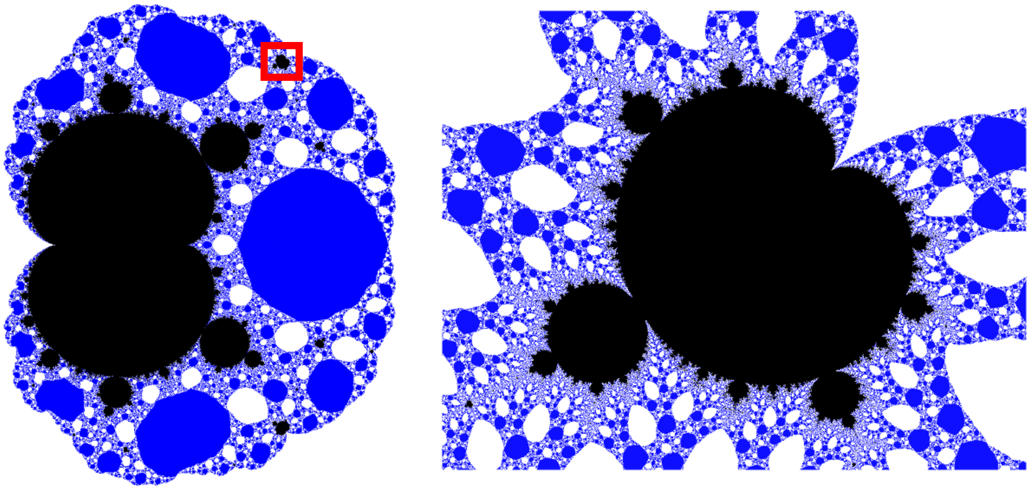


Fig. 1.3. The support of the limiting measure of chromatic zeros for the DHL.

scheme is the same as in Figure 1.3, but using this different mapping. The region depicted is approximately $-1 \leq \mathrm{Re}(q) \leq 5$ and $-3 \leq \mathrm{Im}(q) \leq 3$.

The technique for proving Theorems A and B comes from the connection between the antiferromagnetic Potts model in statistical physics and the chromatic polynomial; see, for example, [4–6] and [7, p.323-325]. For any graph Γ, one has:

$$\mathcal{P}_\Gamma(q) = Z_\Gamma(q, 0), \tag{1.2}$$

where $Z_\Gamma(q, y)$ is the partition function (1.18) for the antiferromagnetic Potts model with q states and "temperature" y. It is a polynomial in both q and y by the Fortuin-Kasteleyn [28] representation (1.19). See Section 1.5 for more details.

Remark 1.1.1. *The computer plots from Figures 1.3 and 1.4 lead to several further questions. Moreover, the techniques developed in this paper can also be used to study the q-plane zeros of the partition function for the Potts Model on hierarchical lattices for fixed values of $y \neq 0$. These will be explored in the upcoming work of Chang, Shrock, and the second author of the present paper [29].*

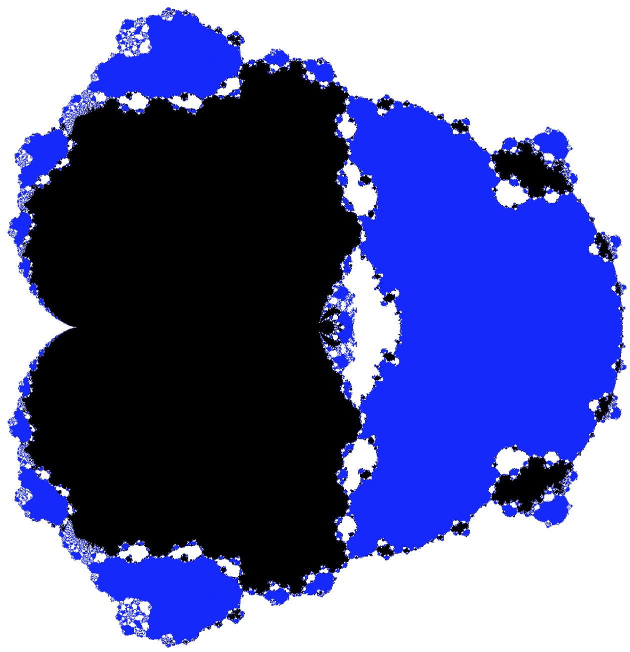


Fig. 1.4. The support of the limiting measure of chromatic zeros for the split diamond

Given a hierarchical lattice $\{\Gamma_n\}_{n=1}^{\infty}$ generated by $\Gamma = (V, E)$, let us write the partition functions $Z_n(q, y) \equiv Z_{\Gamma_n}(q, y)$ for each $n \in \mathbb{N}$. The zero locus of $Z_n(q, y)$ is a (potentially reducible) algebraic curve in $\mathbb{C}^2$. However, we will consider it as a divisor by assigning positive integer multiplicities to each irreducible component according to the order at which $Z_\Gamma(q, y)$ vanishes on that component. This divisor will be denoted by

$$\mathcal{S}_n := (Z_n(q, y) = 0),$$

where, in general, we denote the zero divisor of a polynomial $p(x, y)$ by $(p(x, y) = 0)$. Since Γ_0 is a single edge with its two endpoints, we have

$$\mathcal{S}_0 = (q(y + q - 1) = 0) = (q = 0) + (y + q - 1 = 0).$$

If Γ is 2-connected, there is a Migdal-Kadanoff renormalization procedure that takes Γ and produces a rational map

$$R \equiv R_\Gamma : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{C} \times \mathbb{P}^1 \qquad \text{be given by} \qquad R(q,y) = (q, r_q(y)),$$

with the property that

$$\mathcal{S}_{n+1} = R^* \mathcal{S}_n \qquad \text{for } n \geq 0. \tag{1.3}$$

Here, $\mathbb{P}^1$ denotes Riemann Sphere, R^* denotes the pullback of divisors, and the rational map $r_q : \mathbb{P}^1 \to \mathbb{P}^1$ has degree $|E|$ (However, the degree can drop below $|E|$ for finitely many values of q). Informally, one can think of the pullback on divisors R^* as being like the set-theoretic preimage, but designed to keep track of multiplicities. The reader should keep in mind the case of the DHL for which

$$r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2. \tag{1.4}$$

It will be derived in Section 1.5.

Because $R(q,y) = (q, r_q(y))$ is a skew product over the identity, for each $n \geq 0$ we have

$$\mathcal{S}_n \; = \; (R^n)^* \big((q=0) + (y+q-1=0) \big) \; = \; (q=0) + (R^n)^*(y+q-1=0).$$

The chromatic polynomial of a connected graph Γ has a simple zero at $q = 0$, which we can ignore when discussing the limiting measure of chromatic zeros. It corresponds to the divisor $(q = 0)$ above. Therefore, using (1.2), all of the chromatic zeros for Γ_n (other than $q = 0$) are given by

$$\tilde{\mathcal{C}}_n = (R^n)^*(y+q-1=0) \cap (y=0), \tag{1.5}$$

where each intersection point is assigned its Bezout multiplicity.

Since we want to normalize and then take limits as n tends to infinity, we re-write (1.5) in terms of currents (see [30, 31] for background). We find

$$\tilde{\mu}_n := \frac{1}{|V_n|}[\tilde{\mathcal{C}}_n] = (\pi_1)_* \left(\frac{1}{|V_n|}(R^n)^*[y+q-1=0] \;\wedge\; [y=0] \right), \tag{1.6}$$

where square brackets denote the current of integration over a divisor and $\wedge$ denotes the wedge product of currents. Since $[y=0]$ is the current of integration over a horizontal line, the wedge product is just the horizontal slice of $\frac{1}{|V_n|}(R^n)^*[y+q-1=0]$ at height $y=0$. Since the wedge product results in a measure on $\mathbb{C}\times\mathbb{P}^1$, we compose with the projection $(\pi_1)_*$ to obtain a measure on $\mathbb{C}$. (In the previous two paragraphs we have used tildes on $\tilde{\mathcal{C}}_n$ and $\tilde{\mu}_n$ to denote that we have dropped the simple zero at $q=0$.)

It is relatively standard to see that

$$\frac{1}{|E_n|}(R^n)^*[y+q-1=0]\to T,$$

where T is the *fiber-wise Green current* for the family of rational maps $r_q(y)$. In Proposition 1.5.4 we'll see that $\alpha:=\lim_{n\to\infty}\frac{|E_n|}{|V_n|}$ exists so that

$$T_n:=\frac{1}{|V_n|}(R^n)^*[y+q-1=0]\to\alpha T.$$

However:

First Main Technical Issue: $T_n\to\alpha T$ does not necessarily imply convergence of the slices $T_n\wedge[y=0]\to\alpha T\wedge[y=0]$.

This issue will be handled using the notion of activity currents which were introduced by DeMarco in [32, 33] to study bifurcations in families of rational maps (they are sometimes called bifurcation currents). Since then, they have been studied by Berteloot, DeMarco, Dujardin, Favre, Gauthier, Okuyama and many others. We refer the reader to the surveys by Berteloot [34] and Dujardin [35] for further details.

We can re-write (1.6) as

$$\tilde{\mu}_n:=\frac{1}{|V_n|}[(r_q^n\circ a)(q)=b(q)],$$

where $a,b:\mathbb{C}\to\mathbb{P}^1$ are the two marked points

$$a(q)=0\qquad\text{and}\qquad b(q)=1-q.$$

(Special care must be taken at the finitely many parameters q for which we have the drop of degree $\deg_y(r_q(y)) < |E|$. It is the Second Main Technical Issue for proving Theorem A and it will be be explained in the next subsection.)

Meanwhile, the *activity current* of the marked point a is defined by

$$T_a := \lim_{n\to\infty} \frac{1}{|E_n|}(r_q^n \circ a)^* \widehat{\omega},$$

where $\widehat{\omega}$ is the fiberwise Fubini-Study $(1,1)$ form on $\mathbb{C} \times \mathbb{P}^1$. Therefore, proving Theorem A reduces to proving the convergence

$$\tilde{\mu}_n = \frac{1}{|V_n|}[(r_q^n \circ a)(q) = b(q)] \to \alpha T_a. \tag{1.7}$$

It will be a consequence of Theorems C and C' that are presented in the next subsection.

1.1.2 Equidistribution in Parameter Space

Let V be a connected projective algebraic manifold. An *algebraic family of rational maps of degree d* is a rational mapping

$$f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$$

such that, there exists an algebraic hypersurface $V_{\deg} \subset V$ (possibly reducible) with the property that for each $\lambda \in V \setminus V_{\deg}$ the mapping

$$f_\lambda : \mathbb{P}^1 \to \mathbb{P}^1 \quad \text{defined by} \quad f_\lambda(z) = f(\lambda, z)$$

is a rational map of degree d. A *marked point* is a rational map $a : V \dashrightarrow \mathbb{P}^1$. (We will denote the indeterminacy locus of a by $I(a)$. It is a proper subvariety of codimension at least two.)

Our result will depend heavily on a theorem from arithmetic dynamics due to Silverman [36, Theorem E] and this will require us to assume that the manifold V, the family f, and the marked points a and b are defined over the algebraic numbers $\overline{\mathbb{Q}}$. In other words, every polynomial in the definitions of these objects has coefficients in $\overline{\mathbb{Q}}$.

Convention. *Throughout the paper, an algebraic family of rational maps*

$$f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$$

defined over $\overline{\mathbb{Q}}$ will mean that both V and f are defined over $\overline{\mathbb{Q}}$.

Theorem C. *Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ defined over $\overline{\mathbb{Q}}$ and let $a, b : V \dashrightarrow \mathbb{P}^1$ be two marked points defined over $\overline{\mathbb{Q}}$. Extending V_{deg}, if necessary, we can suppose $I(a) \cup I(b) \subset V_{\mathrm{deg}}$.*

Suppose that:

(i) *There is no iterate n satisfying $f_\lambda^n a(\lambda) \equiv b(\lambda)$.*

(ii) *The marked point $b(\lambda)$ is not persistently exceptional for f_λ.*

Then we have the following convergence of currents on $V \setminus V_{\mathrm{deg}}$

$$\frac{1}{d^n} \left[(f_\lambda^n \circ a)(\lambda) = b(\lambda) \right] \to T_a, \tag{1.8}$$

where T_a is the activity current of the marked point $a(\lambda)$.

The precise definition of activity current will be given in Section 1.2.

The following version of Theorem C holds on all of V, without removing V_{deg}, an essential feature for our application to Theorem A.

Theorem C'. *Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$ defined over $\overline{\mathbb{Q}}$ and let $a, b : V \dashrightarrow \mathbb{P}^1$ be two marked points defined over $\overline{\mathbb{Q}}$. Suppose that*

(i) *There is no iterate n satisfying $f_\lambda^n a(\lambda) \equiv b(\lambda)$.*

(ii) *The marked point $b(\lambda)$ is not persistently exceptional for f_λ.*

Consider the rational map

$$F : V \times \mathbb{P}^1 \dashrightarrow V \times \mathbb{P}^1 \quad \text{defined by} \quad F(\lambda, z) = (\lambda, f(\lambda, z)).$$

Then the following sequence of currents on V

$$(\pi_1)_* \left(\frac{1}{d^n} (F^n)^* \left[z = b(\lambda) \right] \ \wedge \ \left[z = a(\lambda) \right] \right) \tag{1.9}$$

converges and the limit equals T_a *when restricted to* $V \setminus V_{\mathrm{deg}}$*. Here,* $\pi_1 : V \times \mathbb{P}^1 \to \mathbb{P}^1$ *is the projection onto the first coordinate* $\pi_1(\lambda, z) = \lambda$*.*

Remark 1.1.2. *We have phrased Theorems C and C' in their natural level of generality. However, in most applications that we have in mind (in particular to the chromatic zeros), one can use* $V = \mathbb{P}^m$ *and define everything in the usual affine coordinates* $\mathbb{C}^m \subset \mathbb{P}^m$ *in the following ways:*

(i) $f(\lambda, z) = \frac{P(\lambda,z)}{Q(\lambda,z)}$ *with* $P, Q \in \overline{\mathbb{Q}}[\lambda, z]$ *and having no common factors of positive degree in* $\overline{\mathbb{Q}}[\lambda, z]$*, and*

(ii) $a(\lambda) = \frac{R(\lambda)}{S(\lambda)}$ *with* $R, S \in \overline{\mathbb{Q}}[\lambda]$ *and having no common factors of positive degree in* $\overline{\mathbb{Q}}[\lambda]$ *(and similarly for* $b(\lambda)$*).*

The reader can keep in mind the simple case of the renormalization mapping for the DHL (1.4) in which case everything is defined over $\mathbb{Q} \subset \overline{\mathbb{Q}}$*. Here* $V = \mathbb{P}^1$*,*

(i) $r(q, y) = \left(\frac{y^2+q-1}{2y+q-2} \right)^2$*,*

(ii) $a(q) \equiv 0$*, and* $b(q) = 1 - q$*.*

The degree of this family is $d = 4$ *and* $V_{\mathrm{deg}} = \{0, \infty\}$ *because the degree of* $r_q(y)$ *drops when* $q = 0$ *and* $q = \infty$ *but at no other values of* q*.*

The proofs of Theorem C and C' will closely follow the strategy that Dujardin-Favre use in [37, Theorem 4.2]. However:

Second Main Technical Issue: The proof of [37, Theorem 4.2] requires a technical "Hypothesis (H)" that is not satisfied for the Migdal-Kadanoff renormalization mapping (1.4) for the DHL (and presumably not satisfied for many other hierarchical lattices). Indeed, $q = 0 \in V_{\mathrm{deg}}$ for this mapping and there are active parameters accumulating to $q = 0$. One sees this in Figure 1.3 where $q = 0$ is the "main cusp" on the left side of the black region.

Our assumption that the family and the marked points are defined over $\overline{\mathbb{Q}}$ allows us to avoid Hypothesis (H). Note that, using quite different techniques, Okuyama has proved in [38, Theorem 1] a version of [37, Theorem 4.2] without Hypothesis (H). His proof requires the marked point to be critical, but does not require working over $\overline{\mathbb{Q}}$.

Once Theorem C is proved, one can extend the convergence (1.8) across V_{deg} by an application of the compactness theorem for families of plurisubharmonic functions [39, Theorem 4.1.9], thus proving Theorem C'. Note that a similar statement to Theorem C' is found in the work of Gauthier-Vigny [40, Corollary 3.1]. The proof there also uses such compactness to extend a given convergence across various "bad" parameters that are analogous to our V_{deg}.

1.1.3 Recent Works on Interplay Between Holomorphic Dynamics and Statistical Mechanics

The present work lies in the context of several recent papers where holomorphic dynamics has played a role in studying problems from statistical physics. We describe a sample of them here.

One can interpret a rooted Cayley Tree as a type of hierarchical lattice, and this allows one to apply a renormalization theory that is similar to the Migdal-Kadanoff version used in this paper, in order to study statistical physics on such trees. This led to holomorphic dynamics playing an important role in proof of the Sokal Conjecture by Peters and Regts [41] and also in their work on the location of Lee-Yang zeros for bounded degree graphs [42]. The same renormalization theory was also recently used in combination with techniques from dynamical systems by He, Ji, and the authors of the present paper to characterize the limiting measure of Lee-Yang zeros for the Cayley Tree [43].

Meanwhile, holomorphic dynamics has been used by Bleher, Lyubich, and the second author of the present paper to characterize the limiting measure of Lee-Yang zeros for the DHL [19] and also to describe the limit behavior of the Lee-Yang-Fisher zeros for the DHL [44].

1.1.4 Structure of the Paper

In Section 1.2 we present the background on activity currents and describe the Dujardin-Favre classification of the passive locus, that will play an important role in the proofs of Theorems C and C'. Theorems C and C' are proved in Section 1.3 and Section 1.4.

We return to the problem of chromatic zeros in Section 1.5 by providing background on their connection with the Potts Model from statistical physics. We also set up the renormalization mapping $r_q(y)$ associated to any hierarchical lattice having 2-connected generating graph. We prove Theorem A in Section 1.6 by verifying the hypotheses of Theorem C'.

For the k-fold DHL with $k \geq 2$, one can check that the critical points $y = \pm\sqrt{1-q}$ satisfy $r_q(\pm\sqrt{1-q}) \equiv 0 \equiv a(q)$. Therefore, a result of McMullen [45, Theorem 1.1] gives that $\mathrm{supp}(T_a)$ has Hausdorff dimension 2. This is explained in Section 1.7, where we prove Theorem B.

In Section 1.8 we discuss the chromatic zeros associated with the hierarchical lattices generated by each of the graphs shown in Figure 1.2. We conclude this dissertation in Section 2 with a second application of Theorem C in statistical mechanics: the limiting current of Lee–Yang–Fisher zeros for the Cayley Tree.

1.2 Basics in Activity Currents

1.2.1 Activity Current for Holomorphic Families

Let Λ be a connected complex manifold. A *holomorphic family of rational maps of degree* $d \geq 2$ is a holomorphic map $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$ such that$f_\lambda := f(\lambda, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map of degree d for every $\lambda \in \Lambda$. In this context, a marked point is just a holomorphic map $a : \Lambda \to \mathbb{P}^1$. Remark that if one starts with an algebraic family of rational maps $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ one can delete V_{deg} to obtain a holomorphic family $f : (V \setminus V_{\text{deg}}) \times \mathbb{P}^1 \to \mathbb{P}^1$.

The marked point $a : \Lambda \to \mathbb{P}^1$ is called *passive* at $\lambda_0 \in \Lambda$ if the family $\{f_\lambda^n a(\lambda)\}$ is normal in some neighborhood of λ_0, otherwise a is said to be *active* at λ_0. The set of active parameters is called the *active locus*. The active locus naturally supports an invariant current T_a which we will describe now. (Classically, one usually chooses a to be a marked critical point, but it is not necessary.)

Associated to a holomorphic family f_λ of rational maps with degree $d \geq 2$ is a skew product mapping

$$F : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1 \quad \text{given by} \quad F(\lambda, z) = (\lambda, f_\lambda(z)). \tag{1.10}$$

Let ω be the Fubini-Study $(1,1)$ form on $\mathbb{P}^1$ and let $\widehat{\omega} = \pi_2^* \omega$, where $\pi_2(\lambda, z) = z$ is the projection onto the second coordinate. The next proposition and corollary are standard results in complex dynamics, see for example [37, Proposition 3.1].

Proposition 1.2.1. *The sequence of closed positive* $(1,1)$ *currents,* $d^{-n}(F^n)^* \widehat{\omega}$*, converges to a closed positive* $(1,1)$ *current* $\widehat{T}$.

Let v_n, v_∞ be the local potentials of $d^{-n}(f^n)^* \widehat{\omega}$ and $\widehat{T}$ respectively. In the proof of Proposition 1.2.1 one sees that that $v_n \to v_\infty$ locally uniformly, which implies the following corollary:

Corollary 1.2.2. *For any marked point* $a : \Lambda \to \mathbb{P}^1$*, we have the following convergence of intersection of currents:*

$$\frac{1}{d^n} (f^n)^* \widehat{\omega} \wedge [z = a(\lambda)] \to \widehat{T} \wedge [z = a(\lambda)]\,. \tag{1.11}$$

Let $\Gamma := \{(\lambda, a(\lambda)\} \subset \Lambda \times \mathbb{P}^1$. Since $\pi_1 : \Gamma \to \Lambda$ is an isomorphism, one defines the *activity current of the pair* (f, a) by

$$T_a := (\pi_1)_* \left(\widehat{T} \wedge [z = a(\lambda)] \right). \tag{1.12}$$

The next result can be found in [37, Theorem 3.2], in which it was assumed that $a(\lambda)$ is a marked critical point. However, one can check that its proof does not rely on the fact that the marked point is critical.

Theorem 1.2.1. *The support of the activity current T_a coincides with the active locus of a.*

1.2.2 Activity Current for Algebraic Families

Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree d. The construction from the previous subsection defines the activity current T_a for the corresponding holomorphic family $f : (V \setminus V_{\text{deg}}) \times \mathbb{P}^1 \to \mathbb{P}^1$. We will now show that there is a natural extension of T_a through the hypersurface V_{deg}.

As the construction is local, without loss of generality we can suppose V is an open subset of $\mathbb{C}^m$. We can choose a lift $\tilde{f} : V \times \mathbb{C}^2 \to \mathbb{C}^2$ which is holomorphic and so that for each $\lambda \in V$,

$$\tilde{f}_\lambda(z, w) := \tilde{f}(\lambda, z, w) = (P_\lambda(z, w), Q_\lambda(z, w)),$$

where P_λ, Q_λ are homogeneous polynomials. Moreover, P_λ, Q_λ have degree $d' \leq d$, with equality iff $\lambda \in V \setminus V_{\text{deg}}$. Similarly, the marked point $a : V \dashrightarrow \mathbb{P}^1$ can be lifted to a holomorphic map

$$\tilde{a} : V \to \mathbb{C}^2.$$

Let $G_n : V \to [-\infty, \infty)$ be the PSH function defined by

$$G_n(\lambda) := \frac{1}{d^n} \log ||(\tilde{f}_\lambda^n \circ \tilde{a})(\lambda)||.$$

Proposition 1.2.3. *Suppose $V \subset \mathbb{C}^m$ is open. The pointwise limit*

$$G(\lambda) := \lim_{n\to\infty} G_n(\lambda)$$

exists and is PSH in V. When restricted to $V \setminus V_{\deg}$, the current $\tilde{T}_a := dd^c G$ is identically equal to the activity current T_a.

Remark 1.2.4. *On $V \setminus V_{\deg}$ the functions $G_n(\lambda)$ correspond to local potentials for the currents on the left side of (1.11) from Corollary 1.2.2 and the limiting function $G(\lambda)$ corresponds to the local potential for the current on the right side of (1.11). In particular, on $V \setminus V_{\deg}$ the continuous functions $G_n(\lambda)$ converge locally uniformly to the continuous function $G(\lambda)$ and the latter is not equal to $-\infty$ anywhere. See the proof of [37, Proposition 3.1] for details.*

Proof. Fix a parameter $\lambda \in V$ so that $f_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map with degree $d' \leq d$. Since its lift $\tilde{f}_\lambda$ is defined up to a multiplicative constant, we may assume that in the unit sphere $S \subset \mathbb{C}^2$ we have $\sup_S |\tilde{f}_\lambda| = 1$. By the homogeneity of $\tilde{f}_\lambda$,

$$||\tilde{f}_\lambda(z,w)|| \leq ||(z,w)||^{d'}, \text{ which implies } ||\tilde{f}_\lambda^{n+1}(z,w)|| \leq ||\tilde{f}_\lambda^{n}(z,w)||^{d'},$$

so the maps G_n satisfy

$$\begin{aligned} G_{n+1}(\lambda) &= \frac{1}{d^{n+1}} \log ||(\tilde{f}_\lambda^{n+1} \circ \tilde{a})(\lambda)|| \\ &\leq \frac{1}{d^{n+1}} \log ||(\tilde{f}_\lambda^{n} \circ \tilde{a})(\lambda)||^{d'} \leq \frac{1}{d^{n}} \log ||(\tilde{f}_\lambda^{n} \circ \tilde{a})(\lambda)|| = G_n(\lambda), \end{aligned}$$

which implies that $\{G_n\}_{n=1}^{\infty}$ is a decreasing sequence of PSH functions, so it either converges to a PSH limit function G or to $-\infty$ identically. The latter is impossible, by Remark 1.2.4. □

Proposition 1.2.5. *Suppose $V \subset \mathbb{C}^m$ is open. The sequence of PSH functions G_n converges to G in $L^1_{\mathrm{loc}}(V)$. Equivalently, the sequence of currents $dd^c G_n$ converges to $\tilde{T}_a = dd^c G$.*

Proof. Assume on the contrary that $G_n \not\to G$ in $L^1_{\mathrm{loc}}(V)$. Then the compactness theorem for PSH functions [39, Theorem 4.1.9] implies that there is a subsequence

G_{n_k} and some PSH function $G' \neq G$ in $L^1_{\text{loc}}(V)$ such that $G_{n_k} \to G'$ in $L^1_{\text{loc}}(V)$. Then there is a set $\Omega \subset V$ of positive measure in which $G'(\lambda) \neq G(\lambda)$ for all λ. In particular, since V_{deg} is a hypersurface and hence has zero measure, we can find a compact set $K \subset \Omega \setminus V_{\text{deg}}$ in which $G_{n_k} \to G'$. However, by Corollary 1.2.2, $G_n \to G$ uniformly in K, which is a contradiction. □

1.2.3 Classification of the Passivity Locus

Dujardin-Favre Classification of Passivity Locus [37, Theorem 4].
Let $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic family and let $a(\lambda)$ be a marked point. Assume $U \subset \Lambda$ is a connected open subset where $a(\lambda)$ is passive. Then exactly one of the following cases holds:

(i) *$a(\lambda)$ is never preperiodic in U. In this case the closure of the orbit of $a(\lambda)$ can be followed by a holomorphic motion.*

(ii) *$a(\lambda)$ is persistently preperiodic in U.*

(iii) *There exists a persistently attracting (possibly superattracting) cycle attracting $a(\lambda)$ throughout U and there is a closed subvariety $U' \subsetneq U$ such that the set of parameters $\lambda \in U \setminus U'$ for which $a(\lambda)$ is preperiodic is a proper closed subvariety in $U \setminus U'$.*

(iv) *There exists a persistently irrationally neutral periodic point such that $a(\lambda)$ lies in the interior of its linearization domain throughout U and the set of parameters $\lambda \in U$ for which $a(\lambda)$ is preperiodic is a proper closed subvariety in U.*

This classification is stated in [37] with the marked point being critical. However, the heart of the proof is a separate theorem (Theorem 1.1 in [37]) whose statement does not require the marked point to be critical. Meanwhile, the remainder of the proof is short and it is easy to check that the marked point need not be critical.

However, one should note that the statement in [37] claims that in Case (iii) the set λ such that $a(\lambda)$ is preperiodic is a closed subvariety of U itself, without first removing a proper closed subvariety U'. Unfortunately, that is not true, but fortunately that particular claim is not used anywhere later in their paper.

Consider the following holomorphic family of polynomial mappings

$$f_\lambda(z) = z(z-\lambda)(z-1/2)$$

where $\lambda \in \mathbb{C}$. The critical points of f_λ are

$$c_\pm(\lambda) = \frac{(1+2\lambda) \pm \sqrt{4\lambda^2 - 2\lambda + 1}}{6},$$

which vary holomorphically in a neighborhood $\mathbb{D}_r(0)$ of $\lambda = 0$, for some $r > 0$. Notice that $c_-(0) = 0$ and $c_+(0) = \frac{1}{3}$. Consider the marked critical point $c(\lambda) := c_+(\lambda)$. One can check that

1. There exists $0 < \epsilon < r$ such that $\lambda \in \mathbb{D}_\epsilon(0)$ implies that $f_\lambda^n(c(\lambda)) \to 0$ with $|f_\lambda^n(c(\lambda))| < 1/2$ for all $n \geq 0$, and

2. There exists an infinite sequence $\{\lambda_k\}_{k=1}^\infty$ in $\mathbb{D}_\epsilon(0) \setminus \{0\}$ with $\lambda_k \to 0$ such that for each k there is an iterate n_k with $f_{\lambda_k}^{n_k}(c(\lambda_k)) = 0$.

Therefore, the set of preperiodic parameters $\lambda \in \mathbb{D}_\epsilon(0)$ is not a closed subvariety, but they are in $\mathbb{D}_\epsilon(0) \setminus \{0\}$.

Proof of the claim about preperiodic parameters in Case (iii):
By taking a suitable iterate, let us suppose that $a(\lambda)$ is attracted to an attracting fixed point $p(\lambda)$. For each $\lambda \in U$ let $m(\lambda)$ denote the local multiplicity of $p(\lambda)$ for f_λ.

Let $m_0 := \min\{m(\lambda) \, : \, \lambda \in U\}$ and let

$$U' := \{\lambda \in U \, : \, m(\lambda) > m_0\}.$$

Suppose $\lambda_0 \in U \setminus U'$ and choose a neighborhood W of λ_0 such that $\overline{W} \Subset U \setminus U'$. Then, there exists $\epsilon > 0$ such that:

(i) $f_\lambda(\mathbb{D}_\epsilon(p(\lambda)) \Subset \mathbb{D}_\epsilon(p(\lambda))$, and

(ii) for each $\lambda \in W$ and each $z \in \mathbb{D}_\epsilon(p(\lambda)) \setminus \{p(\lambda)\}$ we have that $f_\lambda(z) \neq p(\lambda)$, i.e. $p(\lambda)$ is the only preimage of $p(\lambda)$ under f_λ within $\mathbb{D}_\epsilon(p(\lambda))$.

Since $\overline{W}$ is compact and $f_\lambda^n(a(\lambda)) \to p(\lambda)$ for all $\lambda \in \overline{W}$ there exists $k > 0$ such that for all $\lambda \in W$ we have that $f_\lambda^k(a(\lambda)) \subset \mathbb{D}_\epsilon(p(\lambda))$. Then, using (ii) above, the set of preperiodic parameters in W is

$$\{\lambda \in W \,:\, f_\lambda^n(a(\lambda)) = p(\lambda) \text{ for some } 0 \leq n \leq k\},$$

which is a closed subvariety of W.

$\square$

1.3 Proof of Theorem C

Our proof of Theorem C will closely follow the strategy that Dujardin-Favre use to prove Theorem 4.2 in [37] and we will assume some of the basic results from their proof.

Let $f : V \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ be an algebraic family of rational maps of degree d defined over $\overline{\mathbb{Q}}$. Let $a, b : V \dashrightarrow \mathbb{P}^1$ be marked points and assume, without loss of generality, that the indeterminacy $I(a) \cup I(b) \subset V_{\text{deg}}$. Let T_a be the activity current of $a(\lambda)$ and suppose that all hypotheses of Theorem C are satisfied.

Proposition 1.3.1. *The following convergence of currents*

$$\frac{1}{d^n}\left[(f_\lambda^n \circ a)(\lambda) = b(\lambda)\right] \to T_a \tag{1.13}$$

holds in $V \setminus V_{\text{deg}}$ *if and only if there is a dense set of parameters* $\lambda \in V_{\text{good}} \subset V \setminus V_{\text{deg}}$ *such that*

$$h_n(\lambda) := \frac{1}{d^n} \log \operatorname{dist}_{\mathbb{P}^1}\left(f_\lambda^n a(\lambda), b(\lambda)\right) \to 0, \tag{1.14}$$

where $\operatorname{dist}_{\mathbb{P}^1}$ *denotes the chordal distance on* $\mathbb{P}^1$.

Proof. A direct adaptation of the first four paragraphs of the proof of Theorem 4.2 in [37] shows that (1.13) holds if and only if $h_n \to 0$ in $L^1_{\text{loc}}(V \setminus V_{\text{deg}})$. If $h_n \not\to 0$ in $L^1_{\text{loc}}(V \setminus V_{\text{deg}})$, then, as in paragraph seven of the same proof, one uses Hartogs' Lemma [39, Theorem 4.1.9(b)] to find an open set $U \subset V \setminus V_{\text{deg}}$ and a subsequence n_k such that $h_{n_k}(\lambda) \to h(\lambda) < 0$ for all $\lambda \in U$. □

The proof of Theorem C will then follow immediately from the following:

Proposition 1.3.2. *There is a dense set of parameters* $\lambda \in V_{\text{good}} \subset V \setminus V_{\text{deg}}$ *such that (1.14) holds.*

This is almost an immediate consequence of the following beautiful theorem:

Silverman's Theorem E [36]. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$ defined over a number field K. Let $A, B \in \mathbb{P}^1(\overline{K})$ and assume that B is not exceptional for ϕ and that A is not preperiodic for ϕ. Then*

$$\lim_{n\to\infty} \frac{\delta(\phi^n A, B)}{d^n} = 0, \tag{1.15}$$

where $\delta(P,Q) = 2 - \log \mathrm{dist}_{\mathbb{P}^1}(P,Q)$ is the logarithmic distance function[1].

Remark that (1.15) holds if and only if $\lim_{n\to\infty} \frac{1}{d^n} \log \mathrm{dist}_{\mathbb{P}^1}(\phi^n A, B) = 0$.

If we want to use Silverman's Theorem E directly, we need that there is a dense set of parameters $V_\infty \subset V \setminus V_{\mathrm{deg}}$ such that the marked point $a(\lambda)$ has infinite orbit under f_λ for all $\lambda \in V_\infty$. We do not know if this true at this level of generality, even if the active locus is non-empty.

1.3.1 Proof of Proposition 1.3.2

We will need the following result:

Algebraic Points Are Dense. *Let $V \subset \mathbb{P}^n$ be a projective algebraic manifold that is defined over $\overline{\mathbb{Q}}$. Then, the set of points $a \in V$ that can be represented by homogeneous coordinates in $\overline{\mathbb{Q}}$ form a dense subset of V (in the complex topology). I.e. $V(\overline{\mathbb{Q}})$ is dense in V.*

We could not find this statement in the literature, but it can be proved by induction on $\dim(V)$. The base of the induction, when $\dim(V) = 0$, plays an important role in the theory of Kleinian Groups, see for example [46, Lemma 3.1.5].

Proof of Proposition 1.3.2. We will consider the active and passive loci separately. Let λ_0 be an active parameter, and $W \subset V$ be any open neighborhood containing λ_0. We will find a parameter $\lambda_1 \in W$ at which (1.14) holds. We will do this by showing that there exists a parameter $\lambda_1 \in W$ such that iterates of $a(\lambda_1)$ under f_{λ_1}

[1]In [36, Theorem E] a different logarithmic distance function was used. However, as mentioned in Section 3 of the referenced paper, the result still holds if we use $2 - \mathrm{dist}_{\mathbb{P}^1}(\cdot,\cdot)$ instead.

will eventually land on a repelling cycle disjoint from $b(\lambda_1)$. This will immediately imply (1.14) at λ_1.

Pick three distinct points in a repelling cycle of f_{λ_0} which is disjoint from $b(\lambda_0)$. By reducing W to a smaller neighborhood of λ_0 if necessary, we can ensure that the repelling cycle moves holomorphically as λ varies over W, and that $b(\lambda)$ is disjoint from the cycle for every $\lambda \in W$. Since the family $\{f_\lambda^n a(\lambda)\}_{n=1}^\infty$ is not normal in W, it cannot avoid all three points.

We now suppose λ_0 is in the passive locus for a and let U be the connected component of the passive locus containing λ_0. Then, the Dujardin-Favre classification gives four possible behaviors for $f_\lambda^n(a(\lambda))$ in U.

In Cases (i),(iii), and (iv) the classification gives a (possibly empty) closed subvariety $U' \subsetneq U$ such that the set of parameters for which $a(\lambda)$ is preperiodic is contained in a proper closed subvariety $U_1 \subset (U \setminus U')$. Moreover, the hypothesis that marked point $b(\lambda)$ is not persistently exceptional gives that there is another proper closed subvariety $U_2 \subset U$ such that $b(\lambda)$ is not exceptional for $\lambda \in U \setminus U_2$. Then, $U \setminus (U' \cup U_1 \cup U_2)$ is an open dense subset of U. Since $V(\overline{\mathbb{Q}})$ is dense in V, see the beginning of this subsection, arbitrarily close to λ_0 is a point $\lambda_1 \in U \setminus (U' \cup U_1 \cup U_2)$ with coordinates in $\overline{\mathbb{Q}}$. Since there are only finitely many coefficients to consider, we can find a number field K so that $f_{\lambda_1} \in K(z)$ and the points $a(\lambda_1), b(\lambda_1) \in \mathbb{P}^1(K)$. Since $\lambda_1 \in U \setminus (U' \cup U_1 \cup U_2)$, the point $a(\lambda_1)$ has infinite orbit under f_{λ_1}, and the point $b(\lambda_1)$ is not exceptional for f_{λ_1}. Hence Silverman's Theorem E implies that (1.14) holds for the parameter λ_1.

Finally suppose we are in Case (ii), so that the marked point $a(\lambda)$ is persistently preperiodic. By assumption, there is no iterate n that satisfies $f_\lambda^n(a(\lambda)) \equiv b(\lambda)$, so there is a proper closed subvariety $U_1 \subset U$ such that for all $\lambda \in U \setminus U_1$ and for all $n \geq 0$, we have $f_\lambda^n(a(\lambda)) \neq b(\lambda)$. It follows that for each $\lambda \in U \setminus U_1$, the quantities $\mathrm{dist}_{\mathbb{P}^1}(f_\lambda^n(a(\lambda)), b(\lambda))$ are uniformly bounded in $n \geq 0$, which implies (1.14) for all $\lambda \in U \setminus U_1$. $\square$

1.3.2 Arithmetic Proof of Proposition 1.3.2 Under Additional Hypotheses

The additional hypotheses are:

(iii) The parameter space is $\mathbb{P}^1$.

(iv) The marked point a is not passive on all of $\mathbb{P}^1 \setminus V_{\mathrm{deg}}$.

For applications in chromatic zeros our parameter space is $\mathbb{P}^1$ so that Hypothesis (iii) will automatically hold (in fact, we typically think of it as $\mathbb{C} \subset \mathbb{P}^1$). Meanwhile, for the renormalization mappings associated with many hierarchical lattices one can check Hypothesis (iv) directly, but it does not hold for all such mappings (e.g. when the generating graph is a triangle, as discussed in Section 1.8.3).

The proof will not depend on the Dujardin-Favre classification of the passive locus but instead requires technical results from arithmetic dynamics. Proposition 1.3.2 will follow from Silverman's Theorem E and the next statement (choosing K to be dense in $\mathbb{C}$), which is due to Laura DeMarco and Niki Myrto Mavraki.

Proposition 1.3.3. (DeMarco-Mavraki) *Suppose the hypotheses in Theorem C and additionally hypotheses (iii) and (iv) above. Then, for any number field K there are at most finitely many parameters $\lambda \in \mathbb{P}^1(K) \setminus V_{\mathrm{deg}}$ such that the marked point $a(\lambda)$ is preperiodic under f_λ.*

We will need the following two results, which depend on having a one-dimensional parameter space. Denote the logarithmic absolute Weil height on $\overline{\mathbb{Q}}$ by $h : \overline{\mathbb{Q}} \to \mathbb{R}$. For a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over K and a point $P \in \mathbb{P}^1(K)$, we denote the canonical height function associated to ϕ by $\widehat{h}_\phi(P)$. For more background on these definitions, see [47].

Call-Silverman Specialization [48, Theorem 4.1].
Let (f, a) be a one-dimensional algebraic family of rational maps of degree $d \geq 2$ with

a marked point a, both defined over a number field K. Then, for any sequence of parameters $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{P}^1(K) \setminus V_{\text{deg}}$ such that $h(\lambda_n) \to \infty$, we have

$$\lim_{n\to\infty} \frac{\widehat{h}_{f_{\lambda_n}}(a(\lambda_n))}{h(\lambda_n)} = \widehat{h}_f(a),$$

where $\widehat{h}_f(a)$ is the canonical height associated to the pair (f, a).

The canonical height of the pair $\widehat{h}_f(a)$ was introduced in [48].

The pair (f, a) is *isotrivial* if there exists a branched covering $W \to \mathbb{P}^1 \setminus V_{\text{deg}}$ and a family of holomorphically varying Möbius transformations M_λ such that $M_\lambda \circ f_\lambda \circ M_\lambda^{-1}$ is independent of $\lambda \in W$ and also $M_\lambda \circ a$ is a constant function of $\lambda \in W$.

Theorem 1.3.1. (DeMarco [49, Theorem 1.4]) *Suppose $f : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ is a non-isotrivial one-dimensional algebraic family of rational maps. Let $\widehat{h}_f : \mathbb{P}^1(\bar{k}) \to \mathbb{R}$ be a canonical height of f, defined over the function field $k = \mathbb{C}(\mathbb{P}^1)$. For each $a \in \mathbb{P}^1(\bar{k})$, the following are equivalent:*

(1) *The marked point a is passive in all of $\mathbb{P}^1 \setminus V_{\text{deg}}$;*

(2) $\widehat{h}_f(a) = 0$*;*

(3) (f, a) *is preperiodic.*

Moreover, the set

$$\{a \in \mathbb{P}^1(k) : a \textit{ is passive in all of } \mathbb{P}^1 \setminus V_{\text{deg}}\}$$

is finite.

Proof of Proposition 1.3.3. Assume on the contrary that there is a sequence of distinct parameters $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{P}^1(K) \setminus V_{\text{deg}}$ such that $a(\lambda_n)$ is preperiodic for f_{λ_n}. It follows from Northcott property [47, Theorem 3.7] that the parameters λ_n satisfies $h(\lambda_n) \to \infty$. Meanwhile, since $a(\lambda_n)$ is preperiodic for f_{λ_n}, we have $\widehat{h}_{f_{\lambda_n}}(a(\lambda_n)) = 0$. Then Call-Silverman Specialization implies $\widehat{h}_f(a) = 0$, so by Theorem 1.3.1 the marked point a must be passive in all of $\mathbb{P}^1 \setminus V_{\text{deg}}$, which contradicts hypothesis (iv). □

1.4 Proof of Theorem C'

The following statement about convergence of sequences of PSH functions is probably standard, but we will include a proof because we cannot find an appropriate reference.

Proposition 1.4.1. *Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of PSH functions in an open connected set $U \subseteq \mathbb{C}^m$ which is uniformly bounded above in compact sets. Suppose there is a PSH function ϕ in U such that $\phi_n \to \phi$ in $L^1_{\mathrm{loc}}(U \setminus X)$, where $X \subset U$ is an analytic hypersurface. Then $\phi_n \to \phi$ in $L^1_{\mathrm{loc}}(U)$.*

Proof. Assume by contradiction that $\phi_n \not\to \phi$ in $L^1_{\mathrm{loc}}(U)$. Then there is an $\epsilon > 0$, a compact set K with positive Lebesgue measure, and a subsequence ϕ_{n_k} such that

$$||\phi_{n_k} - \phi||_{L^1(K)} > \epsilon \quad \text{for all } k.$$

Note that since $\phi_n \to \phi$ in $L^1_{\mathrm{loc}}(U \setminus X)$, the compact set K must intersect X. By the hypotheses, the sequence ϕ_{n_k} satisfies the conditions for the compactness theorem for PSH functions [39, Theorem 4.1.9], so we can find a further subsequence (which we still denote by ϕ_{n_k}), and a PSH function $\tilde{\phi}$ in U such that

$$\phi_{n_k} \to \tilde{\phi} \quad \text{in} \quad L^1_{\mathrm{loc}}(U).$$

In particular $\phi_{n_k} \to \tilde{\phi}$ in $L^1(K)$, which implies $\tilde{\phi} \neq \phi$ in $L^1(K)$, so there exist $\delta > 0$ and a compact subset $K' \subset K$ with positive Lebesgue measure such that $|\tilde{\phi}(z) - \phi(z)| > \delta$ for all $z \in K'$. Let X_ϵ be the ϵ-neighborhood of X in U, and let $X'_\epsilon := X_\epsilon \cap K'$. Choose $\epsilon > 0$ which satisfies $\mathrm{Leb}(K') = 2\mathrm{Leb}(X'_\epsilon)$. It follows that

$$\int_{K' \setminus X'_\epsilon} |\tilde{\phi} - \phi| \, d\mathrm{Leb} > \delta \cdot \mathrm{Leb}(K' \setminus X'_\epsilon) = \frac{\delta}{2}\mathrm{Leb}(K') > 0. \tag{1.16}$$

Meanwhile, since $K' \setminus X'_\epsilon$ is a compact subset of U disjoint from X, we must have $\tilde{\phi} = \phi$ in $L^1(K' \setminus X'_\epsilon)$, which contradicts (1.16). $\square$

Proof of Theorem C'. This is a local statement, so we can suppose without loss of generality that V is an open subset of $\mathbb{C}^m$. Choose a lift $\tilde{f} : V \times \mathbb{C}^2 \to \mathbb{C}^2$ and denote the iterates of each $\tilde{f}_\lambda : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\tilde{f}^n_\lambda(z, w) = \left(P^{(n)}_\lambda(z, w), Q^{(n)}_\lambda(z, w)\right).$$

Choose lifts $\tilde{a}, \tilde{b} : V \to \mathbb{C}^2$ and denote their coordinates by $\tilde{a}(\lambda) = (a_1(\lambda), a_2(\lambda))$ and $\tilde{b}(\lambda) = (b_1(\lambda), b_2(\lambda))$.

Recall from Section 1.3 that

$$h_n(\lambda) := \frac{1}{d^n} \log \mathrm{dist}_{\mathbb{P}^1} \left(f^n_\lambda a(\lambda), b(\lambda)\right). \tag{1.17}$$

Although (1.17) is only defined on $V \setminus V_{\deg}$, interpreting h_n in terms of the lifts allows its extension to all of V:

$$h_n(\lambda) := \frac{1}{d^n} \log |P^{(n)}_\lambda(\tilde{a}(\lambda))b_2(\lambda) - Q^{(n)}_\lambda(\tilde{a}(\lambda))b_1(\lambda)|^2 \\ - \frac{1}{d^n} \log ||(\tilde{f}^n_\lambda \circ \tilde{a})(\lambda)||^2 - \frac{1}{d^n} \log ||\tilde{b}(\lambda)||^2.$$

Note that the last term satisfies

$$\frac{1}{d^n} \log ||\tilde{b}(\lambda)||^2 \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(V) \quad \text{as } n \to \infty,$$

and by Proposition 1.2.5,

$$\frac{1}{d^n} \log ||(\tilde{f}^n_\lambda \circ \tilde{a})(\lambda)||^2 \to 2G \quad \text{in} \quad L^1_{\mathrm{loc}}(V) \quad \text{as } n \to \infty,$$

where G is the local potential for T_a. Moreover, by Proposition 1.3.1, $h_n \to 0$ in $L^1_{\mathrm{loc}}(V \setminus V_{\deg})$. Therefore we can conclude that

$$\frac{1}{d^n} \log |P^{(n)}_\lambda(\tilde{a}(\lambda))b_2(\lambda) - Q^{(n)}_\lambda(\tilde{a}(\lambda))b_1(\lambda)|^2 \longrightarrow 2G \quad \text{in} \quad L^1_{\mathrm{loc}}(V \setminus V_{\deg}).$$

Since G and the sequence $\frac{1}{d^n} \log |P^{(n)}_\lambda(\tilde{a}(\lambda))b_2(\lambda) - Q^{(n)}_\lambda(\tilde{a}(\lambda))b_1(\lambda)|$ are PSH functions in V, it follows from Proposition 1.4.1 that

$$\frac{1}{d^n} \log |P^{(n)}_\lambda(\tilde{a}(\lambda))b_2(\lambda) - Q^{(n)}_\lambda(\tilde{a}(\lambda))b_1(\lambda)| \longrightarrow G \quad \text{in} \quad L^1_{\mathrm{loc}}(V),$$

The PSH functions on the left hand side are local potentials for the currents expressed in (1.9) and G is a local potential for T_a on $V \setminus V_{\deg}$. $\square$

1.5 The Potts Model, Chromatic Zeros, and Migdal Kadanoff Renormalization

We first give a brief account of the antiferromagnetic Potts model on a graph Γ and its connection with the chromatic zeros of $\mathcal{P}_\Gamma$. Suitable references include [4–6], [7, p.323-325], and references therein. We then describe the Migdal-Kadanoff Renormalization procedure that produces a rational function $r_q(y)$ relating the zeros for the Potts Model on one level of a hierarchical lattice to the zeros for the next level. The remainder of the section is devoted to proving properties of the renormalization mappings $r_q(y)$.

1.5.1 Basic Setup

Fix a graph $\Gamma = (V, E)$ and fix an integer $q \geq 2$. A *spin configuration* of the graph Γ is a map

$$\sigma : V \to \{1, 2, ..., q\}.$$

Fix the coupling constant $J < 0$. The energy $H_\Gamma(\sigma)$ associated with a configuration σ on Γ is defined as

$$H_\Gamma(\sigma) = -J \sum_{\{v_i,v_j\}\in E} \delta(\sigma(v_i), \sigma(v_j)) = -J\mathcal{E}(\sigma),$$

where $\delta(a, b) = 1$ if $a = b$ and 0 otherwise, and $\mathcal{E}(\sigma)$ is the number of edges whose endpoints are assigned the same spin under σ. Remark that since $J < 0$ it is energetically favorable to have different spins at the endpoints of each edge, if possible. This means that we are in the *antiferromagnetic* regime.

The Boltzmann distribution assigns a configuration σ on Γ probability proportional to the weight

$$W_\Gamma(\sigma) = \exp\left(-\frac{H_\Gamma(\sigma)}{\mathrm{T}}\right) = \exp\left(\frac{J\mathcal{E}(\sigma)}{\mathrm{T}}\right),$$

where $\mathrm{T} > 0$ is the temperature of the system (We set the Boltzmann constant $k_B = 1$). The probability $\Pr(\sigma)$ of σ occurring is therefore

$$\Pr(\sigma) = W_\Gamma(\sigma)/Z_\Gamma \quad \text{where} \quad Z_\Gamma := \sum_\sigma W_\Gamma(\sigma), \tag{1.18}$$

and the sum is over all possible spin configurations. Some intuition for this distribution can be gained by considering the following two extreme cases: when T is near zero, configurations with minimum energy are strongly favored. Meanwhile for high temperature, all configurations occur with nearly equal probability.

Let us introduce the temperature-like variable $y := e^{J/\mathrm{T}}$, so that $W_\Gamma(\sigma) = y^{\mathcal{E}(\sigma)}$. All the quantities above implicitly depend on q, y, and the graph Γ. The normalizing factor $Z_\Gamma(q, y)$ is called the *partition function* and given by

$$Z_\Gamma(q, y) := \sum_\sigma y^{\mathcal{E}(\sigma)}.$$

It turns out that $Z_\Gamma(q, y)$ is actually a polynomial in *both* q and y. To see this it will be helpful to express the partition function in terms of (q, v) where $v = y - 1$. For any subset of the edge set $A \subseteq E$ is a subgraph (V, A). We have

$$Z_\Gamma(q, v) = \sum_\sigma \prod_{(i,j)\in E} [1 + v\delta(\sigma_i, \sigma_j)] = \sum_{A\subseteq E} q^{k(A)} v^{|A|}. \tag{1.19}$$

where $k(A)$ is the number of connected components of (V, A), including isolated vertices. This is called the *Fortuin-Kasteleyn [28] representation of* $Z_\Gamma(q, v)$; see, for example, [6, Section 2.2]. (We will only express Z_Γ in terms of v instead of y in this paragraph and in Subsection 1.5.2.)

As discussed in the introduction, we will describe the zeros of $Z_\Gamma(q, y)$ as a divisor denoted

$$\mathcal{S} := (Z_\Gamma(q, y) = 0).$$

Remark that in the next subsection we will see that if Γ is 2-connected, then we have that $Z_\Gamma(q, y) = q\tilde{Z}_\Gamma(q, y)$ with $\tilde{Z}_\Gamma(q, y)$ irreducible, implying $\mathcal{S}$ is a *reduced divisor*, i.e.

all multiplicities are one. Therefore, if Γ is 2-connected there is no harm in thinking of $\mathcal{S}$ as a (reducible) algebraic curve.

To establish the connection between the chromatic polynomial $\mathcal{P}_\Gamma(q)$ and the partition function $Z_\Gamma(q,y)$ of the Potts model note that

$$\mathcal{P}_\Gamma(q) = \sum_{\substack{\sigma \text{ such that} \\ \mathcal{E}(\sigma)=0}} 1 = Z_\Gamma(q,0).$$

Therefore, the chromatic zeros are given by the intersection:

$$\mathcal{C} := \mathcal{S} \cap (y=0),$$

where Bezout intersection multiplicities and multiplicities of the divisor $\mathcal{S}$ are taken into account.

1.5.2 Irreducibility of $\tilde{Z}_\Gamma(q,y)$ for 2-Connected Γ

It follows from (1.19) that we can always factor $Z_\Gamma(q,v) = q\widetilde{Z}_\Gamma(q,v)$ in the polynomial ring $\mathbb{C}[q,v]$. The goal of this subsection is to prove:

Proposition 1.5.1. *If Γ is 2-connected, then $\widetilde{Z}_\Gamma(q,v)$ is irreducible in $\mathbb{C}[q,v]$. (The same holds in the (q,y) variables.)*

We will prove this proposition using the well-known relationship between $\widetilde{Z}_\Gamma(q,y)$ and the *Tutte Polynomial* of Γ. It is defined as

$$\mathcal{T}_\Gamma(x,y) = \sum_{A \subset E} (x-1)^{k(A)-1}(y-1)^{|A|+k(A)-|V|}, \tag{1.20}$$

where $k(A)$ has the same interpretation as in (1.19). The variables (x,y) in the Tutte Polynomial are related[2] to the variables (q,v) in the Partition Function (1.19) by:

$$x = 1 + (q/v) \qquad \text{and} \qquad y = v+1.$$

[2]Although the variable y appears in Equation (1.18) for the partition function and also in Equation (1.20) for the Tutte Polynomial, there is no conflict of notation because both satisfy $y = v+1$.

Comparing (1.20) with (1.19) we see the following relationship [6, Section 2.5] between $\mathcal{T}(x,y)$ and $Z_\Gamma(q,v)$:

$$\mathcal{T}_\Gamma(x,y) = (x-1)^{-1}(y-1)^{-|V|} Z_\Gamma((x-1)(y-1), y-1). \tag{1.21}$$

Proposition 1.5.1 will be a corollary to the following nice result by de Mier, Merino, and Noyi [50].

Irreducibility Of Tutte Polynomials. *If Γ is a connected matroid, in particular a 2-connected graph, then $\mathcal{T}_\Gamma(x,y)$ is irreducible in $\mathbb{C}[x,y]$.*

Lemma 1.5.2. *$Z_\Gamma(q,v)$ vanishes to order exactly $|V|$ at the origin.*

Proof. For any subgraph (V,A), it follows from a counting argument that we have the inequality $k(A)+|A| \geq |V|$. Moreover, for the subgraph (V,A_0) without any edges, the sum $k(A_0)+|A_0|$ is exactly $|V|$. Therefore the order of vanishing is exactly $|V|$ at the origin. $\square$

Proof of Proposition 1.5.1. By the Irreducibility of the Tutte Polynomial, it suffices to prove that if $\widetilde{Z}_\Gamma$ is reducible then so is $\mathcal{T}_\Gamma$. Suppose $\widetilde{Z}_\Gamma$ is reducible:

$$\widetilde{Z}_\Gamma = A_1 \cdot A_2 \cdot B,$$

where A_1, A_2 are non-constant irreducible factors, and B can potentially be a unit. Denote by C_i the zero set of A_i.

Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be the birational map $(x,y) \mapsto ((x-1)(y-1), y-1)$, so that by (1.21) we have

$$\mathcal{T}_\Gamma(x,y) = (y-1)^{-|V|+1}(\tilde{Z}_\Gamma \circ H).$$

Therefore, in order to prove that $\mathcal{T}_\Gamma$ is reducible it suffices to find at least two irreducible factors of $\tilde{Z}_\Gamma \circ H$ each of which is not equal to $y-1$.

For $i=1$ and 2, although $H^{-1}(C_i)$ can possibly contain the line

$$E := \{(x,y) \in \mathbb{C}^2 : y = 1\},$$

it cannot be the only irreducible component of $H^{-1}(C_i)$ because $H(E)$ is a single point $(0,0)$. From this observation we now have to consider two separate cases.

(i) If $A_1 \not\equiv A_2$, then the zero set of $\mathcal{T}_\Gamma$ contains at least two distinct irreducible components, neither of which is the line E.

(ii) If $A_1 \equiv A_2$, then the zero set of $\mathcal{T}_\Gamma$ contains an irreducible component of multiplicity at least two, which is not the line E.

In either case, we conclude that $\mathcal{T}_\Gamma$ is reducible.

□

1.5.3 Combinatorics of Hierarchical Lattices

Proposition 1.5.3. *Suppose $\{\Gamma_n\}_{n=1}^{\infty}$ is a hierarchical lattice that is generated by a 2-connected generated graph $\Gamma = (V, E)$. Then, Γ_n is 2-connected for each $n \geq 0$.*

Proof. The proof is by induction on n. Since Γ_0 is a single edge with two vertices at its endpoints it is 2-connected. Suppose now that Γ_n is 2-connected for some $n \geq 0$ to show that Γ_{n+1} is 2-connected. Recall that Γ_{n+1} is built by replacing each edge of the generating graph Γ with a copy of Γ_n using the marked vertices a and b as endpoints. The vertices of Γ_{n+1} fall into two classes:

1. The $|V|$ vertices of Γ_{n+1} that come from the vertices of Γ. Each of them is a marked vertex a or b from some copy of Γ_n, and

2. The remaining vertices.

If the removal of a vertex of Type (1) disconnects Γ_{n+1} then, since each Γ_n is 2-connected, this would imply that removal of the corresponding vertex of Γ disconnects Γ. This is impossible because Γ is 2-connected. Meanwhile, if removal of a vertex of Type (2) disconnects Γ_{n+1} then its removal will also disconnect the unique copy of Γ_n that the vertex is contained in. This contradicts the induction hypothesis. □

Proposition 1.5.4. *Let $\Gamma_n = (V_n, E_n)$ be a hierarchical lattice generated by generating graph $\Gamma = (V, E)$. Then $|V_n|$ and $|E_n|$ grow at the same exponential rate as $n \to \infty$.*

Proof. Observe that for any $n \geq 1$,

$$|V_{n+1}| \; = \; |V_n| + |E_n| \cdot (|V| - 2) \; = \; |V_n| + |E|^n \cdot (|V| - 2) \, .$$

It follows from induction that

$$|V_n| \; = \; |V| + (|V| - 2) \cdot \sum_{i=1}^{n-1} |E|^i \; = \; |V| + (|V| - 2) \cdot |E| \frac{|E|^{n-1} - 1}{|E| - 1},$$

which proves the assertion. ☐

1.5.4 Migdal-Kadanoff Renormalization for the DHL

Let $\{\Gamma_n = (V_n, E_n)\}_{n=0}^{\infty}$ be the Diamond Hierarchical Lattice (DHL). For each $n \geq 0$ the partition function $Z_n(q, y) \equiv Z_{\Gamma_n}(q, y)$ has zero divisor

$$\mathcal{S}_n := (Z_n(q, y) = 0).$$

Remark that Γ_0 is always a single edge with two vertices at its endpoints, so a simple calculation yields $Z_{\Gamma_n}(q, y) = q(y + q - 1)$ so that

$$\mathcal{S}_0 := (q(y + q - 1) = 0).$$

Associated to the hierarchical lattice $\{\Gamma_n\}_{n=0}^{\infty}$ is a Migdal-Kadanoff renormalization mapping that relates the zero divisor $\mathcal{S}_{n+1}$ to the zero divisor $\mathcal{S}_n$.

Proposition 1.5.5. *For the DHL we have that for each $n \geq 0$*

$$\mathcal{S}_n = (R^n)^*(\mathcal{S}_0)$$

where $R : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{C} \times \mathbb{P}^1$ is given by

$$R(q, y) = (q, r_q(y)) \, , \quad \textit{where} \quad r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2} \right)^2 . \tag{1.22}$$

As usual, the superscript $*$ denotes pullback of a divisor and we will denote points $y \in \mathbb{P}^1$ using the standard chart $\mathbb{C} \subset \mathbb{P}^1$.

The proof will be very similar to the derivation of the Migdal-Kadanoff renormalization transformation for the Ising Model on the DHL [19, Section 2.5] and it relies on the multiplicativity of the conditional partition functions which is proved in [19, Lemma 2.1], in the context of the Ising Model.

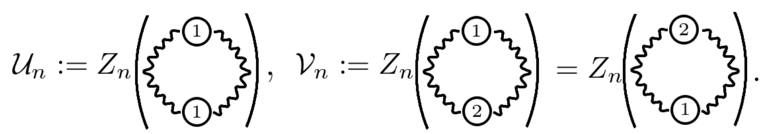
Proof. For each $n \geq 0$ consider the following conditional partition functions:

$$\mathcal{U}_n \equiv \mathcal{U}_n(q,y) := \sum_{\substack{\sigma \text{ such that} \\ \sigma(a)=\sigma(b)=1}} W(\sigma) \qquad \text{and} \qquad \mathcal{V}_n \equiv \mathcal{V}_n(q,y) := \sum_{\substack{\sigma \text{ such that} \\ \sigma(a)=1,\, \sigma(b)=2}} W(\sigma).$$

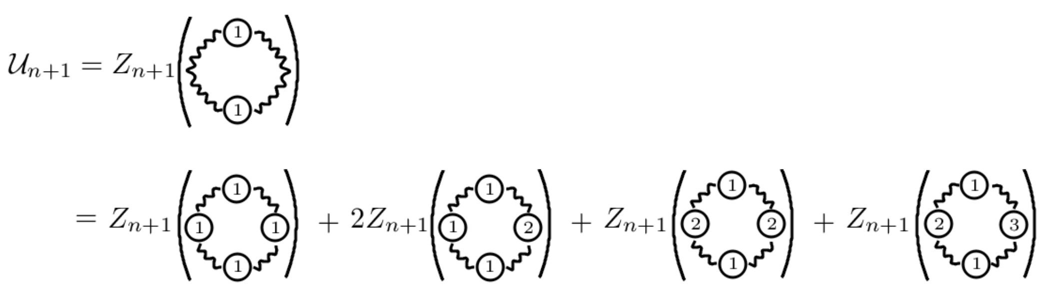
We claim for each $n \geq 0$ that

$$\mathcal{U}_{n+1} = \left(\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2\right)^2 \quad \text{and} \quad \mathcal{V}_{n+1} = \left(2\,\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2\right)^2. \tag{1.23}$$

To show this, it will be helpful to depict them graphically as follows:



The ones and twos in the figure denote the spins at the marked vertices a and b. Let us graphically illustrate the derivation of the first equation from (1.23):



$$= \quad \mathcal{U}_n^4 \quad + \quad 2(q-1)\mathcal{U}_n^2\,\mathcal{V}_n^2 \quad + \quad (q-1)\mathcal{V}_n^4 \quad + \quad (q-1)(q-2)\mathcal{V}_n^4.$$

The numbers one, two, and three in the second row of the figure above are meant to denote the boundary conditions imposed on each of the four copies of Γ_n that are glued together to form Γ_{n+1}. The third line is obtained from the second using multiplicativity of the conditional partition functions. (Once the spins at those four vertices are fixed, the conditional partition function is the same as that of a disjoint union of the four copies of Γ_n, each with its corresponding boundary conditions.) The expression for $\mathcal{V}_{n+1}$ in (1.23) can be obtained similarly.

In order to use an iteration on $\mathbb{P}^1$ instead of $\mathbb{C}^2$ it will be more convenient to iterate the ratio $y_n := \mathcal{U}_n/\mathcal{V}_n$, where $n \geq 0$. A simple calculation shows that $y_0 = y = e^{J/T}$. Using (1.23) we find that

$$y_{n+1} = \frac{\mathcal{U}_{n+1}}{\mathcal{V}_{n+1}} = \left(\frac{\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2}{2\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2}\right)^2 = \left(\frac{y_n^2 + q - 1}{2y_n + q - 2}\right)^2 = r_q(y_n).$$

Therefore, $(q_n, y_n) = R^n(q, y)$ where $q_n = q$ for all n.

Note that

$$Z_n(q, y) = q\,\mathcal{U}_n + q(q-1)\mathcal{V}_n. \tag{1.24}$$

Since the generating graph Γ is 2 connected Proposition 1.5.3 implies that Γ_n is 2-connected for each $n \geq 0$. Therefore, Proposition 1.5.1 gives that

$$\tilde{Z}_n(q, y) = \mathcal{U}_n + (q-1)\mathcal{V}_n$$

is irreducible, implying that $\mathcal{U}_n$ and $\mathcal{V}_n$ have no common factors of positive degree in q or y. Therefore,

$$\begin{aligned} \mathcal{S}_n &= (Z_n(q, y) = 0) \\ &= (q_n(\mathcal{U}_n + (q_n - 1)\mathcal{V}_n) = 0) \\ &= (q_n(y_n + q_n - 1) = 0) \;=\; (R^n)^*\mathcal{S}_0, \end{aligned} \tag{1.25}$$

where in the third equality we used that $\mathcal{U}_n$ and $\mathcal{V}_n$ have no common factors of positive degree. $\square$

The map $r_q(y)$ given in (1.22) is called the *Migdal-Kadanoff renormalization mapping* for the q-state Potts model on the DHL. Remark that this is an algebraic family of rational mappings of degree 4 defined over $\mathbb{Q}$. As a consequence of Proposition 1.5.5, the chromatic zeros for the DHL can be obtained dynamically:

$$\mathcal{C}_n = (R^n)^*(\mathcal{S}_0) \cap (y = 0) \tag{1.26}$$

and note that up to the simple zero at $q = 0$ we can use

$$\tilde{\mathcal{C}}_n = (R^n)^*(y + q - 1 = 0) \cap (y = 0). \tag{1.27}$$

When considering the limiting measure of chromatic zeros it suffices to consider $\tilde{\mathcal{C}}_n$.

1.5.5 Migdal-Kadanoff Renormalization for Arbitrary Hierarchical Lattices

Now suppose $\Gamma_n = (V_n, E_n)$ is the hierarchical lattice generated by an arbitrary generating graph $\Gamma = (V, E)$. It is clear that we can repeat the procedure in Proposition 1.5.5 to produce a renormalization mapping $r_q(y)$ associated to the generating graph Γ, which is a rational map in y on the Riemann sphere of degree at most $|E|$, parameterized by polynomials in q with integer coefficients.

However, it is possible that the generic degree of $r_q(y)$ is strictly smaller than $|E|$. One such example is the Tripod shown in Figure 1.2 for which we have

$$\begin{aligned} \mathcal{U}_{n+1} &= (\mathcal{U}_n + (q-1)\mathcal{V}_n)\left(\mathcal{U}_n^2 + (q-1)\mathcal{V}_n^2\right) \quad \text{and} \\ \mathcal{V}_{n+1} &= (\mathcal{U}_n + (q-1)\mathcal{V}_n)\left(2\mathcal{U}_n\mathcal{V}_n + (q-2)\mathcal{V}_n^2\right). \end{aligned}$$

The common factor of positive degree $(\mathcal{U}_n + (q-1)\mathcal{V}_n)$ is a consequence of the "horizontal" edge that is connected to the remainder of the generating graph Γ at a single vertex. When taking the ratios $y_n = \mathcal{U}_n/\mathcal{V}_n$ we lose track of these common factors resulting in the drop of generic degree:

$$R(q, y) = (q, r_q(y)) \quad \text{where} \quad r_q(y) = \frac{y^2 + q - 1}{2y + q - 2},$$

which has degree two even though Γ has three edges. This drop in generic degree results in $(R^n)^*\mathcal{S}_0 < (Z_n(q, y))$ for the hierarchical lattice generated by the Tripod.

This phenomenon can be avoided if the generating graph is 2-connected and the proof is exactly the same as for the DHL. We summarize:

Proposition 1.5.6. *Let $\{\Gamma_n\}_{n=0}^{\infty}$ be the hierarchical lattice generated by $\Gamma = (V, E)$. If Γ is 2-connected, then the associated renormalization mapping $R(q, y) = (q, r_q(y))$ has generic degree $|E|$ and satisfies*

$$\mathcal{S}_n = (R^n)^*(\mathcal{S}_0),$$

where $\mathcal{S}_n = (Z_n(q, y))$ and $\mathcal{S}_0 = (q(y + q - 1))$. Moreover, r_q is defined over $\mathbb{Q}$.

Several concrete examples are presented in Section 1.8.

1.6 Proof of Theorem A

Let $\{\Gamma_n\}_1^\infty$ be a hierarchical lattice, whose generating graph $\Gamma = (V, E)$ is 2-connected. Let us denote its Migdal-Kadanoff renormalization mapping by

$$R(q, y) = (q, r_q(y)).$$

Since Γ is 2-conneced, Proposition 1.5.6 implies that the chromatic zeros for Γ_n (omitting the simple zero at $q = 0$) are given by $\tilde{\mathcal{C}}_n = (R^n)^*(y + q - 1 = 0) \cap (y = 0)$. Therefore, in the language of currents,

$$\tilde{\mu}_n := \frac{1}{|V_n|} \sum_{\substack{q\in\mathbb{C}\setminus\{0\} \\ \mathcal{P}_{\Gamma_n}(q)=0}} \delta_q = (\pi_1)_* \left(\frac{1}{|V_n|}(R^n)^*[y + q - 1 = 0] \wedge [y = 0] \right),$$

where the zeros of $\mathcal{P}_{\Gamma_n}(q)$ are counted with multiplicities, as always. Since $\tilde{\mu}_n$ and μ_n (see (1.1)) differ by $1/|V_n|$ times a Dirac measure at $q = 0$, it suffices to prove that the sequence $\tilde{\mu}_n$ converges. Moreover, Proposition 1.5.4 allows us to replace the normalizing factor of $|V_n|$ with $|E_n|$. Therefore, it suffices to verify that $R = (q, r_q(y))$ and the marked points $a(q) = 0$ and $b(q) = 1-q$ satisfy the hypotheses of Theorem C'.

By Proposition 1.5.6, the algebraic family r_q is defined over $\mathbb{Q}$. Hypotheses (i) and (ii) on the marked points will be verified in Propositions 1.6.1 and 1.6.2 below.

Proposition 1.6.1. *There are no iterates $n \geq 0$ satisfying $r_q^n(0) \equiv 1 - q$.*

Proof. Away from the finitely many points in V_{deg}, the chromatic zeros of Γ_n are solutions in q to $r_q^n(0) = 1 - q$. If there is some iterate $n \geq 0$ such that

$$r_q^n(0) \equiv 1 - q,$$

this will imply that Γ_n has infinitely many chromatic zeros, which is impossible because $\deg(\mathcal{P}_{\Gamma_n}) = |V_n|$. $\square$

Proposition 1.6.2. *The marked point $b(q) = 1-q$ is not persistently exceptional for the maps r_q.*

Proof. Assume by contradiction that the marked point $b(q) = 1 - q$ is persistently exceptional. Taking the second iterate, we can suppose it is a fixed point. Then by (1.25), the pullback of the divisor $(y = 1 - q)$ by the map R^2 satisfies

$$(\tilde{Z}_2(q, y)) = \left(R^2\right)^* (y = 1 - q) = |E|^2(y = 1 - q),$$

which implies that the partition function, $\tilde{Z}_2(q, y) = (y+q-1)^{|E|^2}$, for Γ_2 is reducible. However, since the generating graph Γ is assumed to be 2-connected, Γ_2 is also 2-connected, so $\tilde{Z}_2(q, y)$ is irreducible by Proposition 1.5.1, which is a contradiction.

□

□(Theorem A)

1.7 Proof of Theorem B

We will use the following famous result:

Theorem 1.7.1 (McMullen [45]). *For any holomorphic family of rational maps over the unit disk Δ, the bifurcation locus $B(f) \subset \Delta$ is either empty or has Hausdorff dimension two.*

Although the above theorem states that the bifurcation locus, which is the union of the active loci of all the critical points, has Hausdorff dimension two (unless it is empty), one can check that the proof still applies to each individual marked critical point $c(\lambda)$, as long as it bifurcates. Indeed the proof of Theorem 1.7.1 consists of using activity of the marked point to construct a holomorphically-varying family of polynomial-like mappings, whose critical point is the marked one $c(\lambda)$. Associated to this family is the space of parameters λ for which the orbit of the critical point remains bounded (in the polynomial-like mapping). McMullen shows that this set is a quasiconformal image of the Mandelbrot set (or a higher degree generalization). The boundary of this "baby Mandelbrot set" has Hausdorff Dimension two [51], and, by definition, the marked point $c(\lambda)$ is active at such points.

Proof of Theorem B. Using an analogous proof to that of Proposition 1.5.5 one finds that the renormalization mapping for the k-fold DHL is

$$r_q(y) = \left(\frac{y^2 + q - 1}{2y + q - 2}\right)^k. \tag{1.28}$$

For this family of mappings we have $V_{\text{deg}} = \{0, \infty\}$. Since the generating graph is 2-connected Theorem A implies that the limiting measure of chromatic zeros exists for the k-fold DHL and the proof of Theorem A implies that on $\mathbb{C} \setminus V_{\text{deg}}$ it coincides with the activity measure for the marked point $a(q) \equiv 0$.

One can check that $c(q) := \sqrt{1-q}$ is a critical point for $r_q(y)$, which we can suppose is marked after replacing $\mathbb{C}$ with a branched cover. A direct calculation shows that $r_q(c(q)) \equiv 0 \equiv a(q)$. Therefore, the activity loci of marked point $a(q)$ (and hence of our limiting measure of μ of chromatic zeros) coincides with the activity locus for the critical point $c(q)$.

It remains to check that these are non-empty and not entirely contained in the set of parameters for which the degree of $r_q(y)$ drops. Drop in degree of $r_q(y)$ corresponds to values of q for which numerator and denominator of $r_q(y)$ have a common zero. One can check that this only happens when $q = 0$.

One can also check by direct calculation that $y = 1$ and $y = \infty$ are both persistently superattracting fixed points for $r_q(y)$. One has that $r_q(0)$ is a degree $k \geq 2$ rational function of q and that $r_0(0) = (1/2)^k$. Therefore, there is some parameter $q_1 \neq 0$ for which $r_{q_1}(0) = 1$. On some open neighborhood of this parameter one has $r_q^n(0) \to 1$. Meanwhile, one has $r_2(0) = \infty$ and so there is an open neighborhood of $q = 2$ on which $r_q^n(0) \to \infty$. This implies that the marked point $a(q)$ cannot be passive on the connected set $\mathbb{C} \setminus \{0\}$ by the identity theorem.

Theorem 1.7.1 and the paragraph following it then give that the activity locus of $c(q)$ has Hausdorff Dimension equal to two.

□

In the special case that $k = 2$, Laura DeMarco and Niki Myrto Mavraki observed the following:

Proposition 1.7.1. *Let $r_q(y)$ be the renormalization mapping for the 2-fold DHL given by (1.28) with $k = 2$. Then, $B(r_q) = \operatorname{supp}(T_a)$.*

Proof. The critical points of the map $r_q(y)$ are $y = 1, 1 - q, \infty, \frac{2-q}{2}, \pm\sqrt{1-q}$. Three of them behave similarly: $y = 1$ and $y = \infty$ are superattracting fixed points, while $y = \frac{2-q}{2}$ is just a preimage of ∞. Meanwhile, note that $\pm\sqrt{1-q}$ are both preimages of $y = 0$, so the bifurcation locus of the family is the union of the activity loci of the two marked points $y = 0$, $y = 1 - q$.

The map r_q commutes with

$$C_q(y) := \left(\frac{y + q - 1}{y - 1}\right)^2,$$

which satisfies $C_q(1-q) = 0$ and $C_q(0) = r_q(1-q) = (1-q)^2$. Therefore, the activity loci of $y = 0$ and $y = 1-q$ coincide, and it follows that the bifurcation locus of the family is equal to the activity locus of the non-critical marked point $y = 0$. $\square$

1.8 Examples

We conclude the paper with a discussion of the chromatic zeros associated with the hierarchical lattices generated by the graphs shown in Figure 1.2. We also provide a more detailed explanation of Figures 1.3 and 1.4.

1.8.1 Linear Chain

In this case, each graph Γ_n is a tree so that $\mathcal{P}_{\Gamma_n}(q) = q(q-1)^{|V_n|}$. See, for example, [52]. Therefore, the limiting measure of chromatic zeros for the linear chain is a Dirac measure at $q = 1$.

Meanwhile, even though the generating graph is not 2-connected, the statement of Proposition 1.5.6 still applies with

$$r_q(y) = \frac{y^2 + q - 1}{2y + q - 2},$$

which is the same formula as for the k-fold DHL, except with exponent $k = 1$. One can check that r_q has $y = 1 - q$ as a persistent exceptional point, so that Theorem C' does not apply. Indeed, the activity locus for marked point $a(q) \equiv 0$ is the round circle $|q - 1/2| = 1/2$ while for each $n \geq 0$ the sequence of wedge products (1.9) is just the Dirac measure at $q = 1$.

1.8.2 k-Fold DHL, Where $k \geq 2$

In the proofs of Theorems A and B we already saw that the limiting measure μ of chromatic zeros exists for this lattice and that outside of $V_{\text{deg}} = \{q = 0, \infty\}$ it coincides with the activity measure for the marked point $a(q) \equiv 0$. Here, we will explain the claim the activity locus, and hence $\text{supp}(\mu)$, is the boundary between any two of the colors (blue, black, and white) in Figure 1.3.

The Migdal-Kadanoff renormalization mapping is given by (1.28). One can check that this mapping has $y = 1$ and $y = \infty$ as persistent superattracting fixed points. In Figure 1.3, the set q for which $r_q^n(0) \to 1$ is shown in white (i.e. not colored) and the set of q for which $r_q^n(0) \to \infty$ is shown in blue. Each of these corresponds to passive behavior for the marked point $a(q) \equiv 0$. Meanwhile, if there is some neighborhood N of $q_0 \in \mathbb{C} \setminus V_{\rm deg}$ on which $r_q^n(0)$ does not have one of these two behaviors, then Montel's Theorem implies that $a(q)$ is also passive on N. Such points are colored black.

Conversely, if q_0 is on the boundary of two colors (blue, black, and white), then q_0 is an active parameter for the marked point $a(q)$. Indeed, if N is any neighborhood of q_0 then along any subsequence n_k we have that $r_q^{n_k}(0)$ will converge uniformly to 1 or ∞ the parts of N that are white or blue, respectively, and $r_q^{n_k}(0)$ will remain bounded away from 1 and ∞ on the black. Therefore, $r_q^n(0)$ cannot form a normal family on N.

1.8.3 Triangles

As the generating graph is 2-connected, Proposition 1.5.6 applies and one can compute that the Migdal-Kadanoff renormalization mapping is:

$$r_q(y) = y\left(\frac{y^2+q-1}{2y+q-2}\right). \tag{1.29}$$

It is the same as for the linear chain, but with an extra factor of y. Notice that for this family of mappings $V_{\rm deg} = \{q = 0, 2, \infty\}$. The proof of Theorem A applies and one concludes that on $\mathbb{C} \setminus V_{\rm deg}$ the limiting measure of chromatic zeros μ coincides with the activity measure of the marked point $a(q) \equiv 0$. However, a curious thing happens: for every iterate n we have $r_q^n(0) = 0$ so that the marked point a is globally passive on $\mathbb{C} \setminus V_{\rm deg}$. Therefore, μ is supported on $V_{\rm deg}$. This illustrates why it was important to use Theorem C' (instead of just Theorem C) when proving Theorem A. Working inductively with (1.29) one can directly prove that μ is the Dirac measure at $q = 2$.

1.8.4 Tripods

As explained in Section 1.5.5, the Migdal-Kadanoff renormalization mapping for the tripod coincides with that of the linear chain, due to a common factor appearing in the numerator and denominator. This drop in degree makes r_q not useful for studying the chromatic zeros on the hierarchical lattice generated by the tripod. However, since each of the graphs Γ_n in this hierarchical lattice is a tree, the limiting measure of chromatic zeros exists and is a Dirac measure at $q = 1$, by the same reasoning as for the linear chain.

1.8.5 Split Diamonds

The split diamond is 2-connected and Theorem A implies that there is a limiting measure of chromatic zeros μ for the associated lattice. One can check that the Migdal-Kadanoff renormalization mapping for this generating graph is

$$r_q(y) = \frac{y^5 + 2(q-1)y^2 + (q-1)y + (q-1)(q-2)}{2y^3 + 2y^2 + 5(q-2)y + (q-2)(q-3)}. \tag{1.30}$$

As for the k-fold DHL, one can check that r_q has $y = 1$ and $y = \infty$ as persistent superattracting fixed points. Therefore, one can use the the same coloring scheme as for the k-fold DHL to make computer images of the activity locus of $a(q) \equiv 0$, and hence of $\text{supp}(\mu)$; See Figure 1.4. With some explicit calculations, one can rigorously verify that each of the three behaviors (white, blue, and black) actually occurs for $q \notin V_{\text{deg}}$.

2. SECOND APPLICATION OF THEOREM C & C' - LEE-YANG-FISHER ZEROS FOR THE CAYLEY TREE

In this section we will describe another application of Theorem C & C' in statistical mechanics. Particular we show that the limiting current of Lee-Yang-Fisher zeros for the Cayley Tree exists. We expect that this result will be useful in studying thermodynamics quantities for this model, such as critical components.

2.1 Introduction

Let us first recall the Ising Model on a graph $\Gamma = (V, E)$ with vertex set V and edge set E. Assign a "spin" to each vertex using a spin configuration $\sigma : V \to \{\pm 1\}$. The total energy of the configuration σ is given as

$$H(\sigma) = -J \cdot \sum_{\{v,w\} \in E} \sigma(v)\sigma(w) - h \cdot \sum_{v \in V} \sigma(v), \tag{2.1}$$

where $J > 0$ is the coupling constant that describes the interaction between neighboring spins, and h is the externally applied magnetic field.

Let $W(\sigma) := \exp(-H(\sigma)/T)$ for temperature T. The *partition function* Z is defined by

$$Z \equiv Z(\Gamma, h, T) := \sum_{\sigma} W(\sigma),$$

which is summed over all possible spin configurations σ. The partition function is a fundamental quantity in statistical mechanics as most aggregate thermodynamic quantities of a physical system can be derived from it.

Using the change of variables

$$z = \exp(-2h/T) \quad \text{(field-like variable) and}$$

$$t = \exp(-2J/T) \quad \text{(temperature-like variable),}$$

$Z(z,t)$ becomes a polynomial after multiplying by $\sqrt{z}^{|V|}\sqrt{t}^{|E|}$ to clear the denominators. For fixed $t \in [0,1]$, T. D. Lee and C. N. Yang [53] characterized these zeros, now known as *Lee–Yang zeros* in their famous theorem.

Lee–Yang Theorem. *For $t \in [0,1]$, the complex zeros in z of the partition function $Z(z,t)$ for the Ising model on any graph lie on the unit circle $\mathbb{T} = \{|z| = 1\}$.*

2.1.1 Limiting Measure of Lee–Yang Zeros μ_t

Let us loosely define a "lattice" to be a sequence of connected graphs $\Gamma_n = (V_n, E_n)$ of increasing size. The standard example is the $\mathbb{Z}^2$ lattice: for each $n \geq 0$, one defines Γ_n to be the graph whose vertices consist of the integer points in $[1,n] \times [1,n]$ and whose edges connect vertices at distance one in $\mathbb{R}^2$.

For each $n \geq 0$, denote by $Z_n(z,t)$ the partition function associated to Γ_n, and let $z_1(t), \ldots, z_{|V_n|}(t)$ be the Lee–Yang zeros at temperature $t \in [0,1]$. For classical lattices ($\mathbb{Z}^d$, etc), it is a consequence of the van-Hove Theorem [54] and the Lee–Yang Theorem that for each $t \in [0,1]$ the sequence of measures

$$\mu_{t,n} := \frac{1}{|V_n|} \sum_{i=1}^{|V_n|} \delta_{z_i(t)}$$

weakly converges to a limiting measure μ_t that is supported on the unit circle $\mathbb{T}$, called the *limiting measure of Lee–Yang zeros* for the lattice $\{\Gamma_n\}$.

A famous unsolved problem from statistical physics is to understand the limiting measures of Lee–Yang zeros μ_t for the $\mathbb{Z}^d$ ($d \geq 2$) lattice and how they depend on t. Besides the one-dimensional lattice $\mathbb{Z}^1$, there are very few lattices for which a global description of the limiting measure of Lee–Yang zeros has been rigorously proved, these include the Diamond Hierarchical Lattice [19] and the Cayley Tree [43].

2.1.2 Lee–Yang Zeros for the Cayley Tree

Let Γ_n^k denote the n-th-level rooted Cayley Tree with branching number k and $\widehat{\Gamma}_n^k$ the unrooted (full) Cayley Tree of level n with branching number k. An illustration for $k = 2$ is given in Figure 2.1.

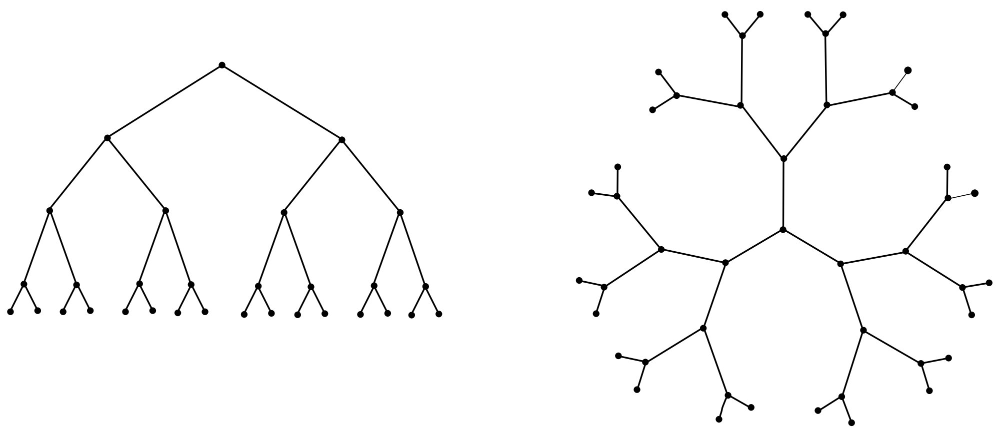


Fig. 2.1. Four levels of the rooted (left) and unrooted (right) Cayley tree with branching number $k = 2$.

We will denote the corresponding lattices by $\Gamma^k := \{\Gamma_n^k\}_{n=0}^\infty$ and $\widehat{\Gamma}^k := \{\widehat{\Gamma}_n^k\}_{n=0}^\infty$.

Lee-Yang zeros on the Cayley Tree (see Figure 2.1) were first studied by Müller-Hartmann and Zittartz in the 1970s [55, 56], also Barata–Marchetti [57], Barata-Goldbaum [58], and others. The hierarchical structure of the Cayley Tree allows the following Migdal-Kananoff renormalization procedure to locate these Lee-Yang zeros, which played a key role in the aforementioned papers:

Proposition 2.1.1. *For any $k \geq 2$, any $t \in [0, 1)$ and any $z \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ consider the following Blaschke Product:*

$$B_{z,t,k}(w) := z \left(\frac{w + t}{1 + wt} \right)^k . \tag{2.2}$$

The Lee–Yang zeros for the n-th rooted Cayley Tree with branching number $k \geq 2$ are solutions z to

$$B_{z,t,k}^n(z) = -1, \tag{2.3}$$

and the Lee–Yang zeros for the n-th full Cayley Tree with branching number $k \geq 2$ are solutions z to

$$B_{z,t,k+1} \circ B_{z,t,k}^{n-1}(z) = -1. \tag{2.4}$$

For the classical $\mathbb{Z}^d$ ($d \geq 2$) lattice, not much has been rigorously proved about the limiting measure of Lee–Yang zeros beside the case when $t > 0$ is sufficiently small. Nevertheless there is a conjectural description as detailed in [19, P.493-494]. Building on Proposition 2.1.1 and the works of the aforementioned papers, an almost global picture of the limiting measure of Lee–Yang zeros for the Cayley Tree was obtained in [43], using modern techniques from dynamical systems. The point was to determine to what extent the conjectural picture for the $\mathbb{Z}^d$ lattice is true for the Cayley Tree. We refer the reader to [43] for the main results and statements.

2.2 Lee–Yang–Fisher Zeros for the Cayley Tree

Recall that for any graph Γ, the partition function $Z_\Gamma(z,t)$ for the Ising model is a polynomial in two variables after clearing the denominators. In previous sections, one fixes the temperature $t \in [0,1]$ and consider the complex zeros of Z_Γ in the z-plane.

If we allow both z and t to vary, the zero locus of $Z_\Gamma(z,t)$ is a (potentially reducible) algebraic curve in $\mathbb{C}^2$. We will consider it as a divisor by assigning the multiplicity to each irreducible component according to its order of vanishing in $Z_\Gamma(z,t)$. This divisor is called the *Lee-Yang-Fisher (LYF) zeros* for Γ.

For simplicity, in the remainder of the section we will focus on the rooted Cayley Tree with a fixed branching number $k \geq 2$ (see Figure 2.1). Let $\mathcal{S}_n$ be LYF zeros for the n-th rooted Cayley Tree, and let d_n be the degree of $\mathcal{S}_n$. We are interested in whether the sequence $d_n^{-1}[\mathcal{S}_n]$ converges as $n \to \infty$ in weak topology of current. If such limit exists, it's called the *limiting current of LYF zeros* for the Cayley Tree. Similar convergence problems on the LYF zeros on the Diamond Hierarchical Lattice have been recently studied by Bleher-Lyubich-Roeder in [44]. Proposition 2.1.1 allows us to obtain $\mathcal{S}_n$ dynamically as follows:

By Proposition 2.1.1, the Lee-Yang-Fisher zeros for the n-th rooted Cayley Tree are the solutions z, t to

$$B^n_{z,t}(z) = -1,$$

where

$$B_{z,t}(w) = z\left(\frac{w+t}{1+wt}\right)^k.$$

Note that the degree of $B_{z,t}(w)$ is less than k if and only if $z = 0$ or $t = \pm 1$.

In order to apply Theorem C & C', the parameter space needs to be a projective variety, in our case we will use $\mathbb{P}^2$ as follows: Denote the coordinates in $V := \mathbb{P}^2$ by $[Z : T : U]$ such that $(z, t) = (Z/U, T/U)$. Then, $B_{z,t}$ can be extended as an algebraic family of rational maps (Section 1.1.2) $B_{[Z:T:U]}$ in $V = \mathbb{P}^2$, with

$$V_{\text{deg}} := \{Z = 0\} \cup \{T = U\} \cup \{T = -U\} \cup \{U = 0\} \subset \mathbb{P}^2.$$

In the affine coordinate $(z, t) = (Z/U, T/U)$ we recover the family $B_{z,t}$. We remark that

1. Points in V_{deg} are precisely the parameters for which the degree of $B_{z,t}$ degenerates to less than k.

2. Since $U = 0$ is the line at infinity with respect to the coordinates (z, t), the space $V \setminus V_{\text{deg}}$ is just $\mathbb{C}^2$ with three lines $\{z = 0\} \cup \{t = 1\} \cup \{t = -1\}$ removed.

Define the skew product $B : (V \setminus V_{\text{deg}}) \times \mathbb{P}^1 \to (V \setminus V_{\text{deg}}) \times \mathbb{P}^1$ by

$$B(z, t, w) := (z, t, B_{z,t}(w)),$$

and let $a(z, t) := z$ and $b(z, t) := -1$ be two marked points. Note that a, b can be extended as rational functions in $\mathbb{P}^2$ in a natural way. Then the current of integration over $\mathcal{S}_n$ can be expressed as

$$\begin{aligned}\frac{1}{d_n}[\mathcal{S}_n] &= \frac{1}{k^n}[B^n_{z,t} \circ a(z,t) = b(z,t)] \\ &= (\pi_1)_*\left(\frac{1}{k^n}(B^n_{z,t})^*[w = b(z,t)] \ \wedge [w = a(z,t)]\right),\end{aligned}$$

where π_1 is the projection onto the parameter space V.

Having this set up, the existence of the limiting current of LYF zeros for the rooted Cayley Tree is now a consequence of the equidistribution Theorems C'. Let us recall that T_a denotes the activity current of the marked point $a(z,t) = z$ as defined in Section 1.1.2.

Theorem D. *For any fixed branching number $k \geq 2$, the limiting current of Lee–Yang–Fisher zeros for the rooted Cayley Tree S exists. Moreover, when restricted to $V \setminus V_{\mathrm{deg}}$, the current S is equal to the activity current T_a of the marked point $a(z,t) = z$, i.e. the following weak convergence of currents holds in $V \setminus V_{\mathrm{deg}}$:*

$$\frac{1}{d_n}[\mathcal{S}_n] = \frac{1}{k^n}[B^n_{z,t}(z) = -1] \longrightarrow T_a.$$

Proof. It is enough to work in affine coordinate $U = 1$ and check that family of rational maps $\{B_{z,t}\}$ along with the marked points a, b in $V \setminus V_{\mathrm{deg}}$ satisfy the hypotheses of Theorem C & C'.

Clearly, the family of rational maps

$$B_{z,t} = z\left(\frac{w+t}{1+wt}\right)^k$$

and the two marked points $a(z,t) = z, b(z,t) = -1$ are all defined over $\mathbb{Q}$.

Suppose there is some integer $n \geq 1$ with $B^n_{z,t}(z) \equiv -1$, this implies that any $(z,t) \in V \setminus V_{\mathrm{deg}}$ is a zero of the partition function for the n-th rooted Cayley Tree, which is a non-constant polynomial in z and t, which is impossible.

Suppose $b(z,t) = -1$ is persistently exceptional. Then for all $(z,t) \in V \setminus V_{\mathrm{deg}}$, either $w = -1$ is a superattracting fixed point, or a point in a superattracting two cycle. However, for the parameter $(z,t) = (2,0) \notin V_{\mathrm{deg}}$, the rational map

$$B_{2,0}(w) = 2w^k$$

satisfies

$$B_{2,0}(-1) = 2(-1)^k, \quad B^2_{2,0}(-1) = 2^{k+1}(-1)^{k^2},$$

So that the point $w = -1$ cannot be a fixed point or in a two cycle. This shows that the marked point b cannot be persistently exceptional.

Since the hypotheses of Theorem C & C' hold, the theorem follows.

$\square$

Remark 2.2.1. *For any fixed branching number $k \geq 2$, the proof of Theorem D can be easily adjusted to show the existence of limiting current of LYF zeros for the unrooted Cayley Tree, but we will omit the details here.*

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