# MICROLOCAL METHODS IN TOMOGRAPHY AND ELASTICITY

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# ABSTRACT

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This thesis compiles my work on three projects.

The first project studies the cancellation of singularities in the inversion of two X-ray type transforms in the presence of conjugate points. In the first part of this project, we study the integral transform over a general family of broken rays in $\mathbb{R}^2$. One example of the broken rays is the family of rays reflected from a curved boundary once. There is a natural notion of conjugate points for broken rays. If there are conjugate points, we show that the singularities conormal to the broken rays cannot be recovered from local data and therefore artifacts arise in the reconstruction. As for global data, more singularities might be recoverable. We apply these conclusions to two examples, the V-line transform and the parallel ray transform. In each example, a detailed discussion of the local and global recovery of singularities is given and we perform numerical experiments to illustrate the results. This part is based on the paper [1]. In the second part of this project, we extend the result of cancellation of singularities in the presence of conjugate points to the integral transform over a generic family of smooth curves. This part is based on the draft [2].

The second project studies the recovery of singularities for the weighted cone transform $I_\kappa$ of distributions with compact support in a domain $M$ of $\mathbb{R}^3$, over cone surfaces whose vertexes are located on a smooth surface away from $M$ and opening angles are limited to an open interval of $(0, \pi/2)$. This transform models data are obtained by a Compton camera with attenuation and a realistic angle of view. We show that when the weight function has compact support and satisfies certain nonvanishing assumptions, the normal operator $I_\kappa^* I_\kappa$ is an elliptic $\Psi$DO at accessible

singularities. Then the accessible singularities are stably recoverable from local data. We prove a microlocal stability estimate for $I_\kappa$. Moreover, we show the same analysis can be applied to the cone transform with vertexes of cones restricted on a smooth curve and fixed opening angles. This chapter is based on the work [3].

The third project studies the phenomenon of Rayleigh waves and Stoneley waves in the isotropic elastic wave equation of variable coefficients with a curved boundary. Most recently in [4], the authors describe the microlocal behavior of solutions to the transmission problems in isotropic elasticity with variable coefficients and curved interfaces. Surface waves are briefly mentioned there as possible solutions of evanescent type which propagate on the boundary. In this project, we construct the microlocal solutions of Rayleigh waves and Stoneley waves, describe their microlocal behaviors, and compute the direction of their polarizations. Essentially, the existence of these two kind of waves come from the nonempty kernel of the Dirichlet-to-Neumann map (DN map) on the boundary. Inspired by the diagonalization of the Neumann operator for the case of constant coefficients in [5], we diagonalize the DN map microlocally up to smoothing operators by a symbol construction in [6]. This gives us a system of one hyperbolic equation and two elliptic equations on the boundary. Then the solution to this system applied by a ΨDO of order zero serves as the Dirichlet boundary condition of the elastic system, and the Rayleigh wave can be constructed basically by using the parametrix of elliptic systems, as it is in the elliptic region. The wave front set and microlocal polarization can be derived during the procedure and they explain the propagation of Rayleigh waves and the retrograde elliptical particle motion. The part of Stoneley waves can be analyzed in a similar way with a more complicated system on the boundary and a similar result holds.

# 1. INTRODUCTION TO MICROLOCAL ANALYSIS

In this chapter, we summarize important results for Fourier integral operators and most of them are presented without proofs. For a more complete treatment, see [7–10]. Before that, I would like to introduce the applications of microlocal analysis in tomography first.

An important inverse problem arising in medical imaging and biomedical research is to reconstruct the unknown density function of the medium from its integral transform. The microlocal singularities of the density function (also called the "features") provides enough information about the shape of organs or boundary of layers, especially when in some cases the exact reconstruction is impossible due to limited data. An example of this is the local tomography, which studies what part of the wave front set could be obtained from the integral transform in a stable way.

From the perspective of microlocal analysis, these integral transforms are usually Fourier integral operators (FIOs) as defined in Section 1.3. On the one hand, the microlocal singularities i.e. the wave front set of the unknown densities and the data, can be related by the canonical relations of the integral transform $I$ as an FIO. On the other hand, when we apply the adjoint to the data as the first attempt of reconstruction, the calculus of FIOs tell us the properties of the normal operator $I^*I$. When the Bolker condition is satisfied, i.e. $\pi_X$ in (1.9) is an injective immersion, the normal operator is basically a pseudodifferential operator ($\Psi$DO), which admits a parametrix if it is elliptic. This is the case of the transform in Chapter 4. Otherwise artifacts may arise and the microlocal stability is violated, for example, the transforms in Chapter 2 and 3.

### 1.1 Distribution densities on manifolds

Let $X$ be a $n$-dimensional smooth manifold. To invariantly define Fourier integral operators and the principal symbols on $X$, we introduce the concept of disribition densities. Densities can be regarded as the sections of a line bundle over manifolds as we will see in the following. Recall the definition of vector bundles in [11]. A **vector bundle** of rank $k$ over $X$ is a topological space $E$ together with a subjective continuous map $\pi : E \to M$, called the bundle projection, satisfying the following properties for each $p \in X$

(1) the fiber $E_p = \pi^{-1}(p)$ is endowed with the structure of a $k$-dimensional real vector space.

(2) there exists a neighborhood $U$ of $p$ in $X$ and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, such that the diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

and for each $q \in U$, the restriction $\Phi$ to $E_q$ is a linear isomorphism from $E_q$ to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$. Such $\Psi$ is called a local trivialization of $E$ over $U$.

**Example 1.** (1) A vector bundle of rank 1 is called a **line bundle**. The product $X \times \mathbb{R}$ is a trivial line bundle.

(2) The tangent bundle $TX$ and cotangent bundle $T^*X$ are vector bundles of rank $n$.

Let $V$ be an $n$-dimensional vector space. A **density** $\rho$ on $V$ of order $s$ is a map

$$\rho : \underbrace{V \times \ldots \times V}_{\text{n copies}} \to \mathbb{R}$$

such that for any linear map $T : V \to V$, we have $\rho(Tv_1, \ldots, Tv_n) = |\det T|^s \rho(v_1, \ldots, v_n)$. We denote the space of all densities on $V$ of order $s$ by $\Omega_s(V)$. On a smooth manifold

$X$, we can define $\Omega_s(T_xX)$, the space of densities on the tangent space $T_xX$ at a point $x \in X$. Notice $\Omega_s(T_xX)$ can be naturally regarded as the fibers of a smooth line bundle $\Omega_s(X)$ over $X$. In this way, a density on $M$ of order $s$ is a section of $\Omega_s(X)$. We denote the space of smooth densities on $X$ of order $s$ by $C^\infty(X, \Omega_s)$.

Densities pull back under smooth maps in a similar way as differential forms. Moreover, let $(U_i, \kappa_i)$ be two coordinate charts on $X$, for $i = 1, 2$. Each $\kappa_i$ gives us a diffeomorphism from $U_i$ to $\mathbb{R}^n$ and defines a density $\rho_{\kappa_i} = (\kappa_i^*)^{-1}(\rho)$ over $\mathbb{R}^n$. If $U_1 \cap U_2$ is nonempty, then we have

$$\rho_{\kappa_2} = (\rho_{\kappa_1} \circ (\kappa_1 \circ \kappa_2^{-1})) \cdot |\det D(\kappa_1 \circ \kappa_2^{-1})|^s.$$

This implies that for a density $\rho$ of order one with compact support, we can define a coordinate invariant integral $\int \rho \, \mathrm{d}x$ over $X$. For $\rho \in C^\infty(X, \Omega_s)$ and $\tau \in C^\infty(X, \Omega_{1-s})$, there is a continuous bilinear form

$$(\rho, \tau) \equiv \int \rho \cdot \tau \, \mathrm{d}x,$$

if one of them has compact support. Furthermore, we can define $\mathcal{D}'(X, \Omega_s)$, the **distribution densities** of order $s$, as the dual space of $C^\infty(X, \Omega_{1-s})$. In particular, when $s = \frac{1}{2}$, we have the half distribution densities $\mathcal{D}'(X, \Omega_{1/2})$ as the dual of the half smooth densities $C^\infty(X, \Omega_{1/2})$.

### 1.2 Symplectic geometry

A **symplectic vector space** is a vector space $V$ with a non-degenerate antisymmetric bilinear form $\sigma$, where the non-degeneracy means $\sigma(\gamma, \gamma') = 0$ for $\forall \gamma' \in V$ implies $\gamma = 0$. An example is $T^*\mathbb{R}^n$ equipped with $\sigma((x, \xi), (x', \xi')) = \langle x,' \xi \rangle - \langle x, \xi' \rangle$. Conversely, any finite dimensional symplectic vector space have even dimension and is symplectically isomorphic to $T^*\mathbb{R}^n$. If $V_1$ is a linear subspace of $V$, then we can define the compliment of $V_1$ under $\sigma$ as

$$V_1^\sigma = \{v \in V : \sigma(v, v') = 0, \forall v' \in V_1\}.$$

Then we say $V_1$ is isotropic (coisotropic) if $V_1 \subset V_1^\sigma$ ($V_1^\sigma \subset V_1$). If $V_1 = V_1^\sigma$, we say $V_1$ is **Lagrangian**. Notice a symplectic vector space $V$ mush have even dimension and its Lagrangian subspace must have dimension equal to $\frac{1}{2}\dim V$.

A **symplectic manifold** $S$ is a smooth manifold with a closed smooth two form $\sigma$, which defines a symplectic form $\sigma_x$ on the tangent space $T_xX$ for any $x \in X$. A submanifold $S_1$ is said to be isotropic (coisotropic, or **Lagrangian**) if this is true for $T_sS_1$ as a linear subspace of the $T_sS$, for each $s \in S_1$. A diffeomorphism between two symplectic manifolds that preserves the symplectic forms are called a symplectomorphism. Locally there is **symplectomorphism** between $S$ and $T^*\mathbb{R}^n$ and we can choose a so-called symplectic local coordinates $x, \xi$ such that

$$\{x_i, x_j\} = \{\xi_i, \xi_j\} = \{x_i, \xi_j\} - \delta_{ij} = 0, \quad \sigma = \sum \mathrm{d}\xi_j \wedge \mathrm{d}x_j,$$

where $\{\}$ is the Poisson bracket induced by the symplectic form. In particular, the cotangent bundle $T^*X$ is a symplectic manifold with $\dim T^*X = 2n$. There is a canonical one form $\omega$ satisfying $\sigma = \mathrm{d}\omega$. If $Y$ is a submanifold of $X$, then the conormal bundle $N^*Y = \{(y, \eta) \in T^*X : y \in Y, \eta|_{T_yY} = 0\}$ is a Lagrangian submanifold.

**Example 2.** Let $f_1, \ldots, f_k$ be smooth functions in an open set $U$ of $X$ and suppose their differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_k$ are linearly independent. Then $Y = \{x \in U; f_1(x) = \ldots = f_k(x) = 0\}$ is a submanifold of $X$ with dimension $n - m$. The conormal bundle is $N^*Y = \{(y, \sum_j \lambda_j \,\mathrm{d}f_j) : y \in Y, \lambda_j \in \mathbb{R}\}$.

The conormal bundle of a smooth submanifold is the most common type of Lagrangian submanifold we have when using microlocal analysis to deal with tomography problems.

Besides, the cotangent bundle $T^*X$ is a conic submanifold. Roughly speaking, by conic we mean we can define a free group action of $\mathbb{R}_+$ in its fibers. The $N^*Y$ is also a conic Lagrangian submanifold. One can show that a $n$-dimensional closed submanifold $\Lambda$ is a conic Lagrangian submanifold of $T^*X \setminus 0$ if and only if the canonical one form $\omega$ vanishes on $\Lambda$. Although a conic Lagrangian submanifold $\Lambda$ is not necessarily in form of $N^*Y$ with $Y$ as a smooth submanifold, yet it can be locally defined as this. More precisely, there exists homogeneous symplectic coordinates $x, \xi$ such that in a small conic neighborhood $\Lambda$ is defined by $x = 0$. Furthermore, $\Lambda$ can be locally parameterized by a non-degenerate phase function, of which the definition is in the following.

Let $\phi(x, \theta)$ be a smooth real valued function in an open conic set $\Gamma$ of $X \times (\mathbb{R}^N \setminus 0)$. We say $\phi$ is a **phase function** if it is a smooth real valued function homogeneous of degree one in $\theta$ and $\mathrm{d}\phi \neq 0$.

**Definition 1.** A phase function $\phi$ is called **clean** with excess $e$ if

$$C_\phi = \{(x, \theta) \in \Gamma; \phi'_\theta(x, \theta) = 0\}$$

is a smooth manifold with tangent plane defined by $\mathrm{d}\phi'_\theta(x, \theta) = 0$ and there are $N - e$ linearly independent differentials among $\mathrm{d}(\partial\phi/\partial\theta_j), j = 1, \ldots, N$.

**Definition 2.** A phase function $\phi$ is called **non-degenerate** if $e = 0$, i.e. the differentials $\mathrm{d}(\partial\phi/\partial\theta_j), j = 1, \ldots, N$ are linear independent.

If $\phi$ is non-degenerate we have $\dim C_\phi = \dim X$. If $\Lambda$ is a smooth conic submanifold, then it can be locally parameterized by a non-degenerate phase function $\phi$. Particularly, by choosing a proper local coordinates, the phase function has a unique form

$$\phi(x, \theta) = \langle x, \theta \rangle - H(\theta), \tag{1.1}$$

where $H(\theta)$ is smooth, homogeneous of degree one, and locally $\Lambda = \{(H'(\xi), \xi)\}$.

Besides this unique form (1.1), we have many choices of phase functions to parameterize $\Lambda$. One can increase or decrease the number of $\theta$ variables by performing change of variables, which implies that we can eliminate the excess of a clean phase. However, near fixed $\gamma_0 \in \Lambda$, the number of $\theta$ variables cannot be fewer than the dimension of $T_{\gamma_0}\Lambda \cap T_{x_0}X$, where $x_0$ is the projection of $\gamma_0$ onto $X$.

### 1.3 Lagrangian distributions

In the following we present Hörmander's definition of Lagrangian distribution sections in [12].

**Definition 3.** Let $X$ be a smooth manifold, $\Lambda \subset T^*X \setminus 0$ a smooth closed conic Lagrangian submanifold, and $E$ a smooth vector bundle over $X$. Then the space $I^m(X, \Lambda; E)$ of Lagrangian distribution sections of $E$, of order $m$ is defined as the set of all $u \in \mathcal{D}'(X, E)$ such that the iterated regularity condition holds

$$L_1 \dots L_N u \in {}^\infty H^{\text{loc}}_{(-m-n\backslash 4)}(X, E) \tag{1.2}$$

for all $N$ and all properly supported $L_j \in \Psi^1(X; E, E)$ with principal symbols $\sigma_p(L_j^0)$ vanishing on $\Lambda$.

Here ${}^\infty H^{\text{loc}}_{(s)}(X, E)$ is the Besov space, which can be roughly regarded as the space of elements whose Fourier transforms are in the modified Sobolev space $H^s$. This definition describes Lagrangian distribution sections of any smooth vector bundles. We can take $E$ as the line bundle $\Omega_s(X)$. In most cases, we consider the trivial line bundle $E = X \times \mathbb{R}$, which gives us the Lagrangian distributions in the usual sense. We will omit $E$ as $I^m(X, \Lambda)$ when it is the trivial line bundle in what follows.

**Example 3.** Let $Y$ be an arbitrary $C^\infty$ submanifold of $X$ and take the conormal bundle $N^*Y$ as the Lagrangian $\Lambda$. This gives us the **conormal distributions** $I^m(X, Y)$.

For a Lagrangian distribution $u \in I^m(X, \Lambda; E)$, its wave front set $\mathrm{WF}(u)$ is contained in the closed Lagrangian submanifold $\Lambda$. Indeed, if $(x_0, \xi^0)$ does not belong to $\Lambda$, then one find a small conic neighborhood of it away from $\Lambda$ such that $L_1, \dots, L_N$ is non-characteristic in this neighborhood. Then by (1.2), we have $u$ is regular up to arbitrary order if we choose $N$ large enough, which implies $u$ is smooth near $(x_0, \xi^0)$.

If $A$ is a ΨDO of order zero, then we have $Au \in I^m(X, \Lambda; E)$. Conversely, if for each $(x_0, \xi^0) \in T^*(X) \setminus 0$ one can find $A \in \Psi^0$ properly supported and nonvanishing at $(x_0, \xi^0)$, such that $Au \in I^m(X, \Lambda; E)$, then we have $u \in I^m(X, \Lambda; E)$.

This property allows us to describe a Lagrangian distribution $u$ by localizing it near some fixed $(x_0, \xi^0) \in \Lambda$. Moreover, in a small conic neighborhood $\Gamma'$ of $(x_0, \xi^0)$, we can choose proper local coordinates $x$ at $x_0$ such that $\Lambda$ is locally defined as $\Lambda = \{(H'(\xi), \xi)\}$, with a smooth function $H$ homogeneous of order one. Notice the map

$$(H'(\xi), \xi) \mapsto \xi \tag{1.3}$$

is a diffeomorphism in $\Gamma'$. We can assume $u$ has compact support in $\Gamma'$, otherwise we can apply a cutoff $\Psi$DO of order zero and non-characteristic in $\Gamma_1' \subset \Gamma'$ to it and the difference is smooth outside $\Gamma_1'$. By choosing $L_1, \ldots, L_N$ properly related to $H$, one can show that there is $v(\xi) \in S^{m-n/4}(\mathbb{R}^n)$, a symbol of order $m - n/4$, such that

$$\hat{u}(\xi) = e^{-iH(\xi)} v(\xi) \quad \Rightarrow \quad u(x) = (2\pi)^{-3n/4} \int e^{i\langle x \cdot \xi \rangle - H(\xi)} v(\xi)\, \mathrm{d}\xi, \tag{1.4}$$

where the phase function $x \cdot \xi - H(\xi)$ is a special case of the non-degenerate one. Here the order of $v$ as the symbol does not depends on the cutoff $\Psi$DO that we choose.

If we parameterize $\Lambda$ by a non-degenerate phase function $\phi(x, \theta)$ and this locally gives us the map

$$\begin{aligned} C_\phi = \{(x, \theta);\ \mathrm{d}_\theta \phi(x, \theta) = 0\} &\to \Lambda = \{(x, \xi)\} \\ (x, \theta) &\mapsto (x, \mathrm{d}_x \phi(x, \theta)), \end{aligned} \tag{1.5}$$

which is a diffeomorphism. In this case, the symbol $v(\xi)$ in (1.4) will be transformed into a new amplitude $a(x, \theta) \in S^{m+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$ and $u$ has the local representation

$$u(x) = (2\pi)^{-(n+2N)/4} \int e^{i\phi(x,\theta)} a(x, \theta)\, \mathrm{d}\theta. \tag{1.6}$$

One can show conversely if a distribution is the form of (1.6), then it is a Lagrangian distribution associated with $\Lambda$ defined in Definition 3.

To define the principal symbol of $u$, we define the Hessian matrix

$$\Phi = \begin{pmatrix} \phi''_{xx} & \phi''_{x\theta} \\ \phi''_{\theta x} & \phi''_{\theta\theta} \end{pmatrix}$$

with $\phi$ non-degenerate. (? $\operatorname{rank}\Phi$ ??) For these two local representation, if $\Lambda = \{(H'(\xi), \xi)\}$, one can show

$$v(\xi)|\,\mathrm{d}\xi|^{\frac{1}{2}} - a(x,\theta)e^{i\pi\operatorname{sgn}\Phi/4}|\det\Phi|^{-\frac{1}{2}}|\,\mathrm{d}\xi|^{\frac{1}{2}} \in S^{m+n/4-1}(\Lambda, \Omega^{\frac{1}{2}}),\ \text{for } (x,\theta) \in C_\phi,$$

where $(x,\theta)$ is a function of $\xi$ by the diffeomorphisms in (1.3,1.5). Here $v(\xi)|\,\mathrm{d}\xi|^{\frac{1}{2}}$ is a half density on $\Lambda$. It turns out $a(x,\theta)|\det\Phi|^{-\frac{1}{2}}|$ is also a half density on $C_\phi$. Indeed, Let $\lambda = (\phi'_{x_1}, \ldots, \phi'_{x_n})$ be the local coordinates on $C_\phi$, which can be achieved by (1.3,1.5). We can extend $\lambda$ to a complete coordinate system by adjoining $\mathrm{d}_{\theta_1}\phi, \ldots, \mathrm{d}_{\theta_N}\phi$ near $(x_0, \theta_0)$, where $\theta_0$ is determined from $\xi_0 = \mathrm{d}_\theta\phi(x_0,\theta_0)$. Then we define the one-density $d_C$ on $C_\phi$ by

$$d_C = \Big|\frac{D(x,\theta)}{D(\lambda, \mathrm{d}_\theta\phi)}\Big||\,\mathrm{d}\lambda| = |\det\Phi|^{-1}|\xi|,$$

where $\mathrm{d}\lambda$ is the Lebesgue density. Therefore $a(x,\theta)|\det\Phi|^{-\frac{1}{2}}| = a(x,\theta)d_C^{\frac{1}{2}}$ is a half density on $C_\phi$, which is invariant if we change the local coordinates. For the Maslov factor $e^{i\pi\operatorname{sgn}\Phi/4}$, changing coordinates results in a new degenerate phase function but the Hessian matrix still has the same size, and the factor is always a power of the imaginary unit $i$. Thus, we define the principal symbol of $u$ by

$$v(\xi)|\,\mathrm{d}\xi|^{\frac{1}{2}} \equiv (2\pi)^{n/4}a(x,\theta)d_C^{\frac{1}{2}}e^{i\pi\operatorname{sgn}\Phi/4} \in S^{m+n/4-1}(\Lambda, \Omega_{1/2}),\ \text{for } (x,\theta) \in C_\phi, \quad (1.7)$$

where $S^{m+n/4-1}(\Lambda, \Omega_{1/2})$ is the symbol class containing sections of $\Omega_{1/2}$ over $\Lambda$. One can introduce the Maslov bundle $M_\Lambda$ and regard the principal symbol $\alpha$ as an element in $S^{m+n/4}(\Lambda, M_\Lambda \otimes \Omega_{1/2})/S^{m+n/4-1}(\Lambda, M_\Lambda \otimes \Omega_{1/2})$. Additionally, when $\Lambda$ is a closed Lagrangian submanifold, the mapping $u \mapsto \alpha$ is an isomorphism between $I^m(X, \Lambda; \Omega_{1/2}(X) \otimes E)/I^{m-1}(X, \Lambda; \Omega_{1/2}(X) \otimes E)$.

More generally, the phase $\phi(x,\theta)$ we use to parameterize $\Lambda$ is clean with excess $e$ instead of non-degenerate.We can split $\theta$ variables into two groups, $\theta = (\theta', \theta'')$, where $\theta'$ are the $N-e$ variables such that the Hessian matrix is non-degenerate and $\theta''$ are the

$e$ variables that parameterize the excess set $C_\xi = \{(x,\theta); \phi'_\theta(x,\theta) = 0, \phi'_x(x,\theta) = \xi\}$. In this case, let

$$\Phi = \begin{pmatrix} \phi''_{xx} & \phi''_{x\theta'} \\ \phi''_{\theta' x} & \phi''_{\theta'\theta'} \end{pmatrix}$$

and then the principal symbol is given by

$$v(\xi)|\,\mathrm{d}\xi|^{\frac{1}{2}} \equiv (2\pi)^{n/4} \int_{C_\xi} a(x,\theta) d_C^{\frac{1}{2}} e^{i\pi \operatorname{sgn}\Phi/4}\,\mathrm{d}\theta'' \in S^{m+n/4-1}, \text{ for } (x,\theta) \in C_\phi. \quad (1.8)$$

## 1.4 The calculus of Fourier Integral Operators

In this subsection, we consider the operators whose kernels are Lagrangian distributions. Suppose $X, Y$ are smooth manifolds. Recall the Schwartz kernel theorem which states there is a bijection between the continuous linear map $K : C_0^\infty(Y) \to \mathcal{D}'(X)$ and the distribution $k(x,y) \in \mathcal{D}'(X \times Y)$. The Schwartz kernel theorem is true in the case of densities if the orders match, particularly for the half densities. If we require the kernel $k(x,y) \in I^m(X \times Y, \Lambda)$, where $\Lambda$ is a closed conic Lagrangian of $T^*X \times T^*Y$, then the corresponding operator $K$ is called an Fourier integral operator.

To avoid zero sections, in addition we assume $\Lambda \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ in the following. With this assumption, the operator $K$ maps $C_0^\infty(Y)$ to $C^\infty(X)$ conitnously and therefore can be extended to a continuous map from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.

Recall $\mathrm{WF}(k) \subset \Lambda$. This property combining with Hörmander-Sato Lemma indicates where the wave front set of a distribution with compact support belongs to after we apply the operator $K$ to it. To better describe this, we introduce the concept of the homogeneous **canonical relation**, i.e. a twisted version of $\Lambda$,

$$C = \Lambda' = \{(x,\xi,y,-\eta) \in T^*X \times T^*Y : (x,\xi,y,\eta) \in \Lambda\},$$

which is a conic Lagrangian submanifold w.r.t. the symplectic form $\sigma_X - \sigma_Y$. For such $C$ and a conic set $\Gamma \subset T^*Y$, one can define the composition $C \circ \Gamma = \{(x,\xi) : (x,\xi,y,\eta) \in C, (y,\eta) \in \Gamma\}$. If $u \in \mathcal{E}'(Y)$, then we have $\mathrm{WF}(Ku) \subset C \circ \mathrm{WF}(u)$.

**Definition 4.** If the kernel of an operator $F$ is a Lagrangian distribution in $I^m(X \times Y, C)$, then $F$ is called an **Fourier integral operator** of order $m$ associated with the canonical relation $C$.

A canonical relation is called a (local) canonical graph if it is the graph of a (local) symplectomorphism from $T^*Y$ to $T^*X$. For FIOs associated with conical graphs, one can define elliptic FIOs at least microlocally if their principal symbols are nonvanishing.

**Example 4.** (1) All ΨDOs are Fourier integral operators who canonical relations are the diagonal of $T^*X \times T^*X$.

(2) The pull back induced by a diffeomorphism is an elliptic Fourier integral operator associated with a canonical graph.

For convenience we always assume $C$ is a homogeneous canonical relation from $T^*Y \setminus 0$ to $T^*X \setminus 0$ which is closed in $T^*(X \times Y) \setminus 0$. We have the following projections.

$$\begin{tikzcd} & C \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ T^*X & & T^*Y \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "\varpi_X"'] \arrow[dr, "\varpi_Y"] & \\ X & & Y \end{tikzcd} \tag{1.9}$$

**Proposition 1.** *If $\varpi_X, \varpi_Y$ have subjective differentials, then*

$$\operatorname{rank} \mathrm{d}\pi_X - \dim X = \operatorname{rank} \mathrm{d}\pi_Y - \dim Y.$$

Notice $\dim C = \dim X + \dim Y$. It follows that $\pi_X$ is an immersion, i.e. $\operatorname{rank} \mathrm{d}\pi_X = n_X + n_Y$ if and only if $\pi_Y$ is a submersion, i.e. $\operatorname{rank} \mathrm{d}\pi_Y = 2 \dim Y$.

**Proposition 2.** *If $\operatorname{rank} \mathrm{d}\pi_X = k + \dim X$ (i.e. $\operatorname{rank} \mathrm{d}\pi_Y = k + \dim Y$), then every $A \in I^m(X \times Y, C')$ is continuous from $L^2_c(Y, \Omega_{1/2})$ to $L^2_{loc}(X, \Omega_{1/2})$, provided $m \leq (2k - \dim X - \dim Y)/4$.*

This theorem implies the continuity of the $H^s$ space.

If $A \in I^m(X \times Y, C')$ with principal symbol $\alpha \in S^{m+n/4}(C; M_C \otimes \Omega_{1/2}(C))$, then its adjoint $A^*$ is also an FIO belonging to the class $I^m(Y \times X, (C^{-1})')$. It has the principal symbol $s^*\alpha \in S^{m+n/4}(C^{-1}; M_{C^{-1}} \otimes \Omega_{1/2}(C^{-1}))$, where $s$ is the map $Y \times X \to X \times Y$ interchanging the two factors.

Let $C_1$ be a smooth homogeneous canonical relation from $T^*Y \setminus 0$ to $T^*X \setminus 0$ and $C_2$ another from $T^*Z \setminus 0$ to $T^*Y \setminus 0$ in what follows. Define their composition as

$$C_1 \circ C_2 = \{(x, \xi, z, \zeta) : (x, \xi, y, \eta) \in C_1, (y, \eta, z, \zeta) \in C_2\},$$

which can be treated as the image of $\hat{C} := (C_1 \times C_2) \cap T^*X \times \Delta(T^*Y) \times T^*Z$ under the natural projection

$$\begin{aligned} \Pi : T^*X \times \Delta(T^*Y) \times T^*Z &\to T^*X \times T^*Z, \\ \hat{C} &\to (T^*X \times T^*Z) \setminus 0. \end{aligned} \tag{1.10}$$

We say the composition is **clean** is $\hat{C}$ is a manifold with tangent space equals to the intersection of the tangent spaces of $C_1 \times C_2$ and $T^*X \times \Delta(T^*Y) \times T^*Z$ everywhere. We say it is **proper**, if the map (1.10) is proper. We say it is **connected** if the set $C_\gamma \subset \hat{C}$, defined as the preimage of $\gamma \in C$, is connected. With the proper condition, apparently $C_\gamma$ is a compact manifold with dimension equals to the excess $e$ of the clean intersection. With these definitions, now we have the clean composition calculus of Fourier integral operators.

**Theorem 1** ( [8] Theorem 25.2.3)**.** *Let both of $A_1 \in I^{m_1}(X \times Y, C_1'), A_2 \in I^{m_2}(Y \times Z, C_2')$ are properly supported. Assume the composition $C = C_1 \circ C_2$ is clean, with excess $e$, proper and connected. For $\gamma \in C$, denote by $C_\gamma$ the compact $e$ dimensional fiber over $\gamma$ of the intersection of $C_1 \times C_2$ and $T^*X \times \Delta(T^*Y) \times T^*Z$. Then*

$$A_1 A_2 \in I^{m_1+m_2+e/2}(X \times Z, C')$$

*and the for the principal symbols $a_1, a_2, a$ of $A_1, A_2, A_1A_2$, we have*

$$a = \int_{C_\gamma} a_1 \times a_2.$$

*Here $a_1 \times a_2$ is the density of $C_\gamma$ with values in the fiber of $M_C \otimes \Omega_{1/2}(X \times Z)$.*

# 2. THE BROKEN RAY TRANSFORM

## 2.1 Introduction

This chapter is a slightly modified version of the previous work in [1]. The purpose of this work is to study the integral transform over a general family of broken rays in the plane. A broken ray in the Euclidean space is usually defined as a linear path reflecting from the boundary once, which will be an important example of the broken rays we define, see Section 2.5. In fact, one motivation of this work is the reconstruction of an unknown function from the integral transform over such broken rays in medical imaging. This integral transform is called the V-line transform. It is related to the Single Photon Emission Computed Tomography (SPECT) with Compton cameras in two dimensions, and has been studied in [13, 14].

We define a more general family of broken rays. Suppose $f$ is a distribution with compact support. Roughly speaking, a broken ray $\nu$ is the union of two rays $l_1$ and $l_2$ that are related by a diffeomorphism, as in Figure 2.1. For more details of the definition, see Section 2.2. The broken ray transform

$$Bf(\nu) = \int a(\nu(t), \dot{\nu}(t)) f(\nu(t)) dt \tag{2.1}$$

is a weighted integral of $f$ along $\nu$, where $a$ is a smooth function. One way to think about this is to imagine that there is a curve smoothly connecting $l_1$ and $l_2$, then $B$ becomes an X-ray type of transform over smooth curves. The connecting curve plays no rule in the analysis, if we always assume that $f$ is compactly supported away from it.

The goal is to understand which singularities of $f$ can be recovered from the transform $Bf$, i.e., whether we can recover $f$ up to a smooth error. More specifically, what part of the wave front set $\mathrm{WF}(f)$ can be recovered. Conjugate points naturally exist for broken rays, see Section 2.3. One would expect and we confirm that recovery

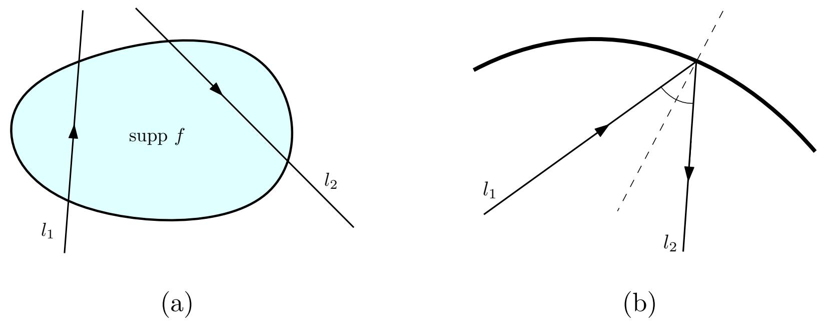



Figure 2.1. Left: a general broken ray, where $l_1$ and $l_2$ are related by a diffeomorphism. Right: a broken ray in the reflection case.

of singularities are affected by the existence of conjugate points on $\nu$. Much work has been done for the class of X-ray type transform with conjugate points [15–18]. In the case of the transform for a generic family of smooth curves [19], if there are no conjugate points, the localized normal operator is an elliptic pseudodifferential operator ($\Psi$DO) of order $-1$. Injectivity and the stability estimates are established, which in particular implies that we can recover the singularities uniquely. When conjugate points exist, however, artifacts may arise, and in some situations they cannot be resolved. A similar situation occurs in synthetic aperture radar imaging [15]; it is impossible to recover $\mathrm{WF}(f)$ if the singularities hit the trajectory of the plane only once, because of the existence of mirror points. On the other hand, if the trajectory is the boundary of a strictly convex domain and we know a priori that $f$ has singularities in a compact set, then we can recover $\mathrm{WF}(f)$ from the global data. However, as shown in [15] this is a global procedure and there is no local reconstruction. In the case of X-ray transforms over geodesic-like families of curves with conjugate points of fold type, a detailed description of the normal operator is given in [16]. Analysis of the normal operator for general conjugate points is done in [20]. Further, [17] shows that regardless of the type of the conjugate points, the geodesic ray transform on Riemannian surfaces is always unstable and we have loss of all derivatives, which leads to the artifacts in the reconstruction near pairs of conjugate points. It is also proved that the attenuated geodesic ray transform is well posed under certain conditions. Most recently, [18] provides a thorough analysis of the stability of attenuated geodesic ray transform and shows what artifacts we can expect when using the Landweber iterative reconstruction for unattenuated problems.

One important example of this setting is the V-line transform. As is shown in Figure 2.1, the diffeomorphism is given by the law of reflection. As mentioned above, we are motivated by the SPECT with Compton cameras in two dimensions. SPECT based on Anger camera is a widely used technique for functional imaging in medical diagnosis and biological research. The using of Compton camera in SPECT is proposed to greatly improve the sensitivity and resolution [21–23]. The gamma photons are emitted proportionally to markers density and then are scattered by two detectors. Photons can be traced back to broken lines. The mathematical model is the cone transform (or conical Radon transform) of an unknown density. Various inversion approaches for certain cases are proposed in [14, 24–35]. The V-line transform can be considered as a special case in two dimensions [13, 14], where each vertex is restricted on a curve and is associated with a single axis. There are also some injectivity and stability results when we allow the rays to reflect from the boundary more than once [36–41]. These reconstructions are from full data and most of them assume specific boundaries at least for the reflection part, for example a flat one or a circle. It also should be mentioned that the broken ray transform or the V-line transform sometimes refers to a different transform from the one we consider in this work, see [42–45]. In their settings, the V-line vertices are inside the object with a fixed axis direction. The integral near the vertices in the support of $f$ makes it possible to recover singularities there. In this work, however, the vertices are always away from support of $f$, which make the recovery more difficult.

Another motivation is the application of parallel ray transform in X-ray luminescence computed tomography (XLCT). A multiple pinhole collimator based on XLCT is proposed in [46] to promote photon utilization efficiency in a single pinhole collimator. In this method, multiple X-ray beams are generated to scan a sample at multiple positions simultaneously, which we mathematically model by the parallel ray transform, see Section 2.6. In fact, we can regard the parallel ray as a ray reflecting off a boundary at infinity.

We are also motivated by the scattering problem for the equation $(-\Delta - \lambda^2 + V)u = 0$ in $\mathbb{R}^2/\Omega$ with Dirichlet or Neumann boundary conditions, where $\Omega$ is a domain with smooth boundary. The recovery of the potential $V$ from the boundary data is related to recover its integral over rays reflected from the boundary in the high frequency limit.

### 2.1.1 Main results

We are inspired by [15, 17, 18] and the main results are

(1) The local problem is locally ill-posed if there are conjugate points, i.e., singularities conormal to the broken rays cannot be recovered uniquely. We describe the microlocal kernel in Theorem 3.

(2) For the V-line transform and the parallel ray transform, the global problem might be well-posed in some cases for most singularities, because singularities can be probed by more than one broken ray. The recovery depends on a discrete dynamical system (a sequence of conjugate covectors) inside the domain, see (2.25). This is a discrete analogue of propagation of singularities as in [15]. If this sequence goes out of the domain, then we can resolve the corresponding singularity.

This chapter is structured as follows. In Section 2.2, we define the broken ray and introduce some notation and assumptions. In Section 2.3 and 2.4, we introduce conjugate points and conjugate covectors along broken rays and give a characterization of them. Then we consider the local problem, i.e., the data $Bf$ is known in a small neighborhood of a fixed broken ray. We show that $B$ is an FIO and the image of two conjugate covectors under its canonical relation are identical. Singularities can be canceled by these conjugate covectors. This implies that we can only reconstruct

$f$ up to an error in the microlocal kernel. We also provide a similar analysis for the numerical result as in [18], if the Landweber iteration is used to reconstruct $f$. In Section 2.5 and 2.6, we apply these conclusions to two cases, the V-line transform and the parallel ray transform, as mentioned above. The conjugate points appearing in the V-line transform coincide with the caustics in geometrical optics, see [47]. Additionally, when the boundary is a circle, we show that there exists conjugate points of fold type as well as cusps. Geometrically, the caustics inside a circle are an interesting problem itself, which can be traced back to the middle of the 19th century [48]. As for the parallel ray transform, the conjugate covectors have simple forms and the sequence of them is given by translation, see (2.28). In both cases, we discuss the local and global recovery of singularities and we perform numerical experiments to illustrate the results. In particular, for rays reflected from a circle, we connect our analysis with the inversion formula derived in [14].

## 2.2 Preliminaries

Throughout this work, we assume $f$ is a distribution supported in a compact set and we use angular brackets to denote the inner product of vectors in $\mathbb{R}^2$. We say a singularity $(x, \xi)$ is recoverable from the broken ray transform if that $Bf$ is smooth implies $(x, \xi) \notin \mathrm{WF}(f)$.

Let $v(\alpha) = (\cos \alpha, \sin \alpha)$ and $w(\alpha) = (-\sin \alpha, \cos \alpha)$. We use $(s, \alpha)$ to parameterize a directed line $\{x \in \mathbb{R}^2 | x \cdot w(\alpha) = s\}$ with the direction $v(\alpha)$ and the unit normal $w(\alpha)$. Note that $(s, \alpha)$ and $(-s, \alpha + \pi)$ belong to the same line but have opposite directions.

Let $\chi : (s_1, \alpha_1) \mapsto (s_2, \alpha_2)$ be a given diffeomorphism. Suppose $l_1$ is a portion of the line $(s_1, \alpha_1)$, which starts from infinity and ends at a point. Suppose $l_2$ is a portion of the line $(s_2, \alpha_2)$, which starts at a point and ends at infinity. A *broken ray* $\nu$ is defined as the union of $l_1$ and $l_2$ if they are related by $(s_2, \alpha_2) = \chi(s_1, \alpha_1)$. We call $l_1$ the incoming part and $l_2$ the outgoing part of $\nu$. We say $\nu$ is *regular* if these two parts do not lie in the same line. In this case, the part $l_1$ and $l_2$ might intersect in the support of $f$ but their conormal bundles are always separated.

We say $\Gamma$ is a *smooth family of broken rays associated with* $\chi$, if

(1) $\Gamma$ is open and each broken ray in $\Gamma$ can be parameterized by its incoming part $(s_1, \alpha_1)$,

(2) there exists a smooth function $q_0(s, \alpha)$ such that the starting point of the outgoing part of each broken ray $(s_1, \alpha_1)$ is given by $q_0(s_1, \alpha_1)$ and satisfies $\langle q_0(s_1, \alpha_1), w(\alpha_2)\rangle = s_2$; the similar is true for the endpoint of the incoming part.

We always assume we are given a smooth family of broken rays.

## 2.3 Conjugate points

In Riemannian geometry, the conjugate vector of a fixed point $p$ is a vector $v$ such that the differential of the exponential map $d_v \exp_p(v)$ is not an isomorphism. The conjugate point is the image of $v$ under the exponential map, for more details see [16, 49]. Conjugate points also exist in the case of broken ray transform, for example, the caustics in geometrical optics, see [47,48]. The light reflected or refracted by a curved surface forms an envelope, which is the conjugate locus of the source. In this section, we define the exponential map and compute the conjugate points for broken rays. We show below that conjugate points on $l_1$ and $l_2$ do not depend on what kind of connecting curves we choose.

There are two different ways to parameterize a line. We can use the Radon parametrization $(s, \alpha)$ as mentioned above, or we can parametrize it by an initial point and an angle. We use the latter one to define the exponential map. Consider a broken ray $\nu_{p,\alpha_1}(t)$

$$\begin{cases} l_1(t) = p + tv(\alpha_1), & -\infty \le t \le t_1, \\ l_2(t) = q_0 + (t - t_2)v(\alpha_2), & t \ge t_2 \ge t_1. \end{cases}$$

whose incoming part $l_1$ passes $p$ and outgoing part $l_2$ starts from $q_0$. Recall that $q_0$ satisfying $\langle q_0, w(\alpha_2)\rangle = s_2$ is chosen to depend on $(s_1, \alpha_1)$ in a smooth way. The time $t_2$ depends on the connecting curve and its parametrization. The analysis below shows that $t_2$ does not influence the conjugate point of $p$ on $l_2$. Observe that the parameterization $(p, \alpha_1)$ gives us a unique Radon parameterization $(s_1, \alpha_1)$ by $s_1 = \langle p, w(\alpha_1)\rangle$. If we fix $p$, then for each $\alpha_1$ the diffeomorphism $\chi$ gives a unique $(s_2, \alpha_2)$, i.e., we have $s_2$, $\alpha_2$, and $q_0$ are all smooth functions of $\alpha_1$ itself. In the following, we use $\frac{d}{d\alpha_1}$ to denote the derivative with respect to $\alpha_1$, when $p$ is fixed and $s_1$ is given by $s_1 = \langle p, w(\alpha_1)\rangle$.

Now define the *exponential map* as $\exp_p(t, \alpha_1) = \nu_{p,\alpha_1}(t)$, for $t \in \mathbb{R}, \alpha_1 \in [0, 2\pi)$. We say $q \in l_2$ is the *conjugate point* of $p$ if there is some $(t, \alpha_1)$ such that the exponential map is not a diffeomorphism for $q = \nu_{p,\alpha_1}(t)$. When $t \geq t_2$, the differential of the exponential map in polar coordinates is represented by the Jacobi matrix $\frac{\partial l_2(t)}{\partial(t,\alpha_1)}$ , where

$$\begin{aligned}\frac{\partial l_2}{\partial \alpha_1} &= (t - t_2)\frac{dv(\alpha_2)}{d\alpha_1} - \frac{dt_2}{d\alpha_1}v(\alpha_2) + \frac{dq_0}{d\alpha_1} \\ &= \Big((t - t_2)\Big(\frac{d\alpha_2}{d\alpha_1}\Big) + \langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle\Big)w(\alpha_2) + \Big(\langle\frac{dq_0}{d\alpha_1}, v(\alpha_2)\rangle - \frac{dt_2}{d\alpha_1}\Big)v(\alpha_2).\end{aligned}$$

Then it has the determinant

$$\det(d\exp_p(tv(\alpha_1))) = \det\begin{bmatrix} \frac{\partial l_2}{\partial t} & \frac{\partial l_2}{\partial \alpha_1}\end{bmatrix} = (t - t_2)\Big(\frac{d\alpha_2}{d\alpha_1}\Big) + \langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle. \tag{2.2}$$

The last equality comes from the observation that $\det[v(\alpha_2)\ w(\alpha_2)] = 1$. Thus, the determinant vanishes if and only if

$$(t - t_2)\frac{d\alpha_2}{d\alpha_1} = -\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle. \tag{2.3}$$

We are finding a solution to equation (2.3) satisfying $t \geq t_2$. There are two cases. If $\frac{d\alpha_2}{d\alpha_1} = 0$, then $\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle$ is zero as well. Otherwise, we must have $\Big(\frac{d\alpha_2}{d\alpha_1}\Big)^{-1}\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle \leq 0$. On the other hand, differentiating $\langle q_0, w(\alpha_2)\rangle = s_2$ with respect to $\alpha_1$ shows

$$\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle + \langle q_0, -v(\alpha_2)\rangle\frac{d\alpha_2}{d\alpha_1} = \frac{ds_2}{d\alpha_1}. \tag{2.4}$$

With the assumption that $\chi$ is a diffeomorphism, $\frac{d\alpha_2}{d\alpha_1}$ and $\frac{ds_2}{d\alpha_1}$ cannot vanish at the same time. This excludes the first case.

Suppose $q$ is the point on $l_2$ at time $t$ such that $d\exp_p(tv_1)$ is not an isomorphism. We have $t - t_2 = \langle q - q_0, v(\alpha_2)\rangle$. By (2.3)(2.4), $q$ should satisfy the equality

$$\langle q, v(\alpha_2)\rangle \frac{d\alpha_2}{d\alpha_1} = -\frac{ds_2}{d\alpha_1}. \tag{2.5}$$

Observe that the projection of $q$ on $v(\alpha_2)$ together with its projection on $w(\alpha_2)$ determines $q$ uniquely. On the contrary, if there exists $q$ on $l_2$ such that the equation (2.5) is true, then the determinant of $d\exp_p(tv(\alpha_1))$ will be zero.

**Proposition 3.** *Suppose $l_1$, $l_2$, and $q_0$ as mentioned above. Let $p$ be a fixed point on $l_1$. Then*

*(a) $p$ has a conjugate point $q$ on $l_2$ if and only if*

$$\Big(\frac{d\alpha_2}{d\alpha_1}\Big)^{-1}\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle \leq 0.$$

*(b) If this occurs, $q$ is uniquely determined by $\langle q, v(\alpha_2)\rangle = -\Big(\frac{d\alpha_2}{d\alpha_1}\Big)^{-1}\frac{ds_2}{d\alpha_1}$.*

*Here we use $\frac{d}{d\alpha_1}$ to denote the derivative with respect to $\alpha_1$ with $p$ fixed and $s_1$ given by $s_1 = \langle p, w(\alpha_1)\rangle$.*

**Remark 1.** If we consider the whole straight line where $l_2$ belongs instead of the ray, then we can always find one and only one conjugate point $q$ satisfying (b), unless $\frac{d\alpha_2}{d\alpha_1} = 0$. The condition (a) is to check whether this $q$ belong to the reflected ray that we define. Additionally, if we perturb $q_0$ a little bit, that is, let $q_0' = q_0 + \epsilon(\alpha_1)v(\alpha_2)$. Then $\frac{dq_0'}{d\alpha_1} = \frac{dq_0}{d\alpha_1} + \epsilon(\alpha_1)w(\alpha_2) + \frac{d\epsilon(\alpha_1)}{d\alpha_1}v(\alpha_2)$. We have

$$\langle\frac{dq_0'}{d\alpha_1}, w(\alpha_2)\rangle = \langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle + \epsilon(\alpha_1).$$

It shows a small enough perturbation of $q_0$ doesn't change the sign of $\langle\frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle$. Therefore the existence of conjugated points is not affected by the choice of $q_0$ in a small neighborhood.

**Remark 2.** Suppose $p$ and $q$ belong to the incoming part $l_1$ and the outgoing part $l_2$ of $\nu$ respectively. If $q$ is the conjugate point of $p$, then we can show $p$ is the conjugate point of $q$ in some sense. Indeed, $q$ is conjugate to $p$ if and only if

$$\langle q, v(\alpha_2)\rangle = -\Big(\frac{d\alpha_2}{d\alpha_1}\Big)^{-1}\frac{ds_2}{d\alpha_1} = -\frac{\frac{\partial s_2}{\partial \alpha_1} - \frac{\partial s_2}{\partial s_1}\langle p, v(\alpha_1)\rangle}{\frac{\partial \alpha_2}{\partial \alpha_1} - \frac{\partial \alpha_2}{\partial s_1}\langle p, v(\alpha_1)\rangle}.$$

Solving $\langle p, v(\alpha_1)\rangle$ out, we have

$$\langle p, v(\alpha_1)\rangle = -\frac{-\frac{\partial s_2}{\partial \alpha_1} - \frac{\partial \alpha_2}{\partial \alpha_1}\langle q, v(\alpha_2)\rangle}{\frac{\partial s_2}{\partial s_1} + \frac{\partial \alpha_2}{\partial s_1}\langle q, v(\alpha_2)\rangle}. \tag{2.6}$$

Now let $\nu'$ be a broken ray passing $p$ and $q$ but with incoming part $(s_2, \alpha_2)$ and associated with $\chi^{-1}$. We list the Jacobian matrix in the following

$$d\chi = \begin{bmatrix} \frac{\partial s_2}{\partial s_1} & \frac{\partial s_2}{\partial \alpha_1} \\ \frac{\partial \alpha_2}{\partial s_1} & \frac{\partial \alpha_2}{\partial \alpha_1} \end{bmatrix}, \qquad d(\chi^{-1}) = (d\chi)^{-1} = \frac{1}{\det(d\chi)}\begin{bmatrix} \frac{\partial \alpha_2}{\partial \alpha_1} & -\frac{\partial s_2}{\partial \alpha_1} \\ -\frac{\partial \alpha_2}{\partial s_1} & \frac{\partial s_2}{\partial s_1} \end{bmatrix}.$$

Notice equation (2.6) exactly means $p$ is the conjugate point of $q$ along $\nu'$.

### 2.4 Microlocal analysis of the local problem

In [17], it is shown that in the geodesic ray transform singularities can be canceled by conjugate points. In this section, we prove the analogous results in Theorem 2 and 3 for the broken ray transform. Recall the definition of a broken ray in Section 2.2. Suppose $\nu$ is a broken ray represented by $(s, \alpha)$. We define the broken ray transform $Bf$ as

$$Bf(s, \alpha) = \int_\nu a(y, s, \alpha) f(y) dl_y, \tag{2.7}$$

where $a(y, s, \alpha)$ is smooth and nonvanishing. Comparing it with (2.1), here we use different parameterization for the weight but still denote it by $a$.

Suppose $f$ has support in a compact subset away from the connecting part. The support of $f$ implies the transform can be interpreted as a sum of Radon transforms over two lines. We can only expect to recover the singularities in their conormal bundles. In the following, we suppose $\nu_0$ is a regular broken ray. For fixed $\nu_0$, we consider $(x_1, \xi^1)$ and $(x_2, \xi^2)$ on its incoming and outgoing part respectively, with $\xi^1$ and $\xi^2$ conormal to them. Let $\Gamma(\nu_0)$ be a small neighborhood of $\nu_0$ and $V^k$ be disjoint small conic neighborhood of $(x_k, \xi^k)$, for $k = 1, 2$. We choose these neighborhoods small enough, such that $V^1$ is disjoint from the conormal bundles of all outgoing parts and $V^2$ is disjoint from that of all incoming parts of broken rays in $\Gamma(\nu_0)$. We project $V^k$ onto $\mathbb{R}^2$ to get the neighborhood $U_k$ of $x_k$. The set $U_k$ might intersect but $V^k$ are always disjoint, for $k = 1, 2$. Figure 2.2 shows a special case when we have disjoint $U_k$.

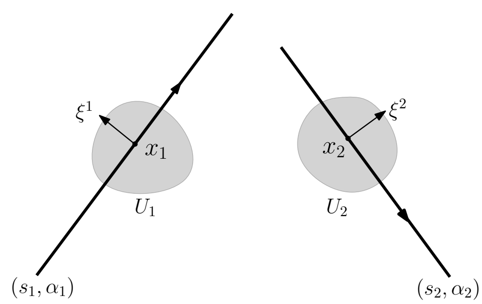



Figure 2.2. The small neighborhood $U_k$ and $(x_k, \xi^k)$, for $k = 1, 2$.

To further localize the problem, we suppose $\mathrm{WF}(f) \subset V^1 \cup V^2$. For convenience, we simply assume $\operatorname{supp} f \subset U_1 \cup U_2$. Let $f_k$ be $f$ restricted to $U_k$ and $B_k$ be $B$ restricted to distributions with wavefront set supported in $V^k$, for $k = 1, 2$. It follows that

$$Bf = B_1 f_1 + B_2 f_2. \tag{2.8}$$

In a small neighborhood, $B_1 f_1$ can be regarded as the Radon transform of $f_1$ and $B_2 f_2$ as the Radon transform performing along the line $(s_2, \alpha_2)$. More precisely, the restricted operators $B_1$ and $B_2$ have the following form up to some smoothing operators

$$B_1 = \phi R \varphi_1, \quad B_2 = \phi \chi^* R \varphi_2,$$

where $R$ is the Radon transform; $\phi(s, \alpha)$ is a smooth cutoff function with $\operatorname{supp} \phi \subset \Gamma(\nu_0)$; $\varphi_k$ are cutoff ΨDOs with essential support in $V^k$, for $k = 1, 2$; the pull back

$\chi^* g(s,\alpha) = g(\chi(s,\alpha))$ is induced by the diffeomorphism $\chi$. We should note that outside $\Gamma(\nu_0)$, there might be another broken ray which carries the singularities $(x_1, \xi^1)$ but with it in the outgoing part. Thus, we actually multiply $\phi$ to $B$ itself as well to make equation (2.8) valid.

To analyze the canonical relation of $B_1$ and $B_2$, we need that of Radon transform. The weighted Radon transform is defined as

$$Rf(s,\alpha) = \int_{\langle w(\alpha), y\rangle = s} \omega(y,\alpha) f(y)\, dy, \tag{2.9}$$

where $w(\alpha) = (-\sin\alpha, \cos\alpha)$, and $\omega(y,\alpha)$ is a smooth function.

**Proposition 4.** *The Radon transform $R$ is an FIO of order $-\frac{1}{2}$ associated with the canonical relation*

$$C_R = \{(\underbrace{\langle y, w(\alpha)\rangle}_{s}, \alpha, \underbrace{\lambda}_{\widehat{s}}, \underbrace{\lambda\langle v(\alpha), y\rangle}_{\widehat{\alpha}}, y, \underbrace{\lambda w}_{\eta})\}.$$

*where $v(\alpha) = (\cos\alpha, \sin\alpha)$ and $w(\alpha)$ as before. Specifically, $C_R$ has two components, corresponding to the choice of the sign of $\lambda$. Each component is a local diffeomorphism. The inverse is also a local diffeomorphism.*

$$C_R^{\pm} : (y,\eta) \mapsto (s, \alpha, \lambda, \lambda\langle v(\alpha), y\rangle) \quad \lambda = \pm|\eta|,\ \alpha = \arg(\pm\frac{\eta}{|\eta|}),\ s = \langle y, w(\alpha)\rangle \tag{2.10}$$

$$C_R^{-1} : (s, \alpha, \widehat{s}, \widehat{\alpha}) \mapsto (y,\eta) \quad y = \frac{\widehat{\alpha}}{\widehat{s}} v(\alpha) + s w(\alpha),\ \eta = \widehat{s} w(\alpha) \tag{2.11}$$

*Proof.* We write the Radon transform as

$$Rf(s,\alpha) = (2\pi)^{-1} \int\int e^{i\lambda(s - \langle w(\alpha), y\rangle)} \omega(y,\alpha) f(y)\, d\lambda dy.$$

The characteristic manifold is $Z = \{(s,\alpha,y) | \Phi(s,\alpha,y) = \lambda(s - \langle w(\alpha), y\rangle) = 0\}$. Then the Lagrangian $\Lambda$ is given by

$$\Lambda = N^* Z == \{(s, \alpha, y, \underbrace{\lambda}_{\Phi_s}, \underbrace{\lambda\langle v(\alpha), y\rangle}_{\Phi_\alpha}, \underbrace{-\lambda w(\alpha)}_{\Phi_y}),\ \langle w(\alpha), y\rangle = s\}.$$

Therefore, the Radon transform is an FIO associated with $\Lambda$ and the canonical relation $C_R$ is obtained by twisting the Lagrangian. The sign of $\lambda$ is chosen corresponding to the orientation of $\eta$ with respect to $w(\alpha)$. It is elliptic at $(y,\eta)$ if and only if $\omega(y,\alpha) \neq 0$ for $\alpha$ such that $w(\alpha)$ is colinear with $\eta$. $\square$

**Lemma 1.** *Suppose* $\chi : (s_1, \alpha_1) \mapsto (s_2, \alpha_2)$ *is a diffeomorphism. Then* $\chi^* : g(s_2, \alpha_2) \mapsto \chi^* g(s_1, \alpha_1) = g(\chi(s_1, \alpha_1))$ *for* $g \in \mathcal{D}'$ *is an FIO whose canonical relation is a diffeomorphism*

$$C_{\chi^*} = \{(s_1, \alpha_1, \widehat{s_1}, \widehat{\alpha_1}, s_2, \alpha_2, \underbrace{(\widehat{s_1}, \widehat{\alpha_1})\Big(d\chi\Big)^{-1}}_{(\widehat{s_2}, \widehat{\alpha_2})}),\ (s_2, \alpha_2) = \chi(s_1, \alpha_1)\}. \tag{2.12}$$

*Proof.* The proof is similar to what we did in last proposition. The induced map $\chi^*$ can be written as the following integral

$$\begin{aligned}
\chi^* g(s_1, \alpha_1) &= \int \delta((s, \alpha) - \chi(s_1, \alpha_1)) g(s, \alpha) ds d\alpha \\
&= (2\pi)^{-2} \int e^{i(\lambda_1(s - s_2) + \lambda_2(\alpha - \alpha_2))} g(s, \alpha) d\lambda_1 d\lambda_2 ds d\alpha,
\end{aligned}$$

where $\chi(s_1, \alpha_1) = (s_2, \alpha_2)$. The characteristic manifold is $Z_{\chi^*} = \{(s, \alpha, s_1, \alpha_1) |\ \phi = \lambda_1(s_2 - s) + \lambda_2(\alpha_2 - \alpha) = 0\}$. The Lagrangian is given by

$$\Lambda_{\chi^*} = \{(s_1, \alpha_1, s, \alpha, \underbrace{(\lambda_1, \lambda_2)(d\chi)}_{\Phi_{s_1, \alpha_1}}, \underbrace{-(\lambda_1, \lambda_2)}_{\Phi_{s, \alpha}}),\ (s, \alpha) = \chi(s_1, \alpha_1)\}.$$

Let $(\lambda_1, \lambda_2)(d\chi) = (\widehat{s_1}, \widehat{\alpha_1})$ and replace $(s, \alpha)$ by $(s_2, \alpha_2)$, we get the canonical relation as is shown above (2.12). $\square$

**Theorem 2.** *We assume* $(x, \xi)$ *and* $(y, \eta)$ *are not conormal to the line joining* $x$ *and* $y$*. Suppose* $V^1$ *is a small enough conical neighborhood of* $(x, \xi)$ *and* $V^2$ *is that of* $(y, \eta)$*. Let* $B_k$ *be* $B$ *restricted to distributions with wavefront set supported in* $V^k$*, for* $k = 1, 2$*. Suppose* $C_k$ *is the canonical relation of* $B_k$*. Then* $C_1(x, \xi) = C_2(y, \eta)$ *if and only if there is a regular broken ray* $\nu$ *joining* $x$ *and* $y$ *such that*

*(a)* $x$ *and* $y$ *are conjugate points along* $\nu$*.*

*(b)* $\xi$ *and* $\eta$ *satisfy* $\xi = \lambda w(\alpha_1), \eta = \frac{\lambda}{\det(d\chi)} \frac{d\alpha_2}{d\alpha_1} w(\alpha_2)$ *for some* $\lambda \neq 0$*, where* $\alpha_1$ *is the angle of the incoming part and* $\alpha_2$ *is that of the outgoing part of* $\nu$*.*

*Proof.* The assumption $(x, \xi)$ and $(y, \eta)$ are not conormal to the line joining $x$ and $y$ is to guarantee that if there is a broken ray that has $(x, \xi)$ and $(y, \eta)$ in its incoming part and outgoing part respectively, then this broken ray is regular. In this way, we can always assume $V^1$ is disjoint from the conormal bundles of all outgoing parts and $V^2$ is disjoint from that of all incoming parts of broken rays in the small neighborhood of the fixed broken ray, which simplifies the problem. Observe that the composition $\chi^* R$ is also an FIO, whose canonical relation $C_{\chi^* R} = C_{\chi^*} \circ C_R$ is a local diffeomorphism. Additionally, the multiplication of cutoff functions does not influence the Lagrangian. Suppose the canonical relation of the restricted operator $B_k$ is called $C_k : (x_k, \xi^k) \mapsto (s_1, \alpha_1, \widehat{s_1}, \widehat{\alpha_1})$, for $k = 1, 2$. As a result, $C_1$ is same as $C_R$ and $C_2$ is same as $C_{\chi^* R}$. Suppose $(s_1, \alpha_1, \widehat{s_1}, \widehat{\alpha_1})$ is the image of $(x_1, \xi^1)$ under $C_R$ and $(s_2, \alpha_2, \widehat{s_2}, \widehat{\alpha_2})$ is that of $(x_2, \xi^2)$. That is, with $s_k$ and $\alpha_k$ given by (2.10), for $k = 1, 2$, we have

$$(\widehat{s_1}, \widehat{\alpha_1}) = \lambda_1(1, \langle x_1, v(\alpha_1)\rangle), \quad (\widehat{s_2}, \widehat{\alpha_2}) = \lambda_2(1, \langle x_2, v(\alpha_2)\rangle).$$

Then from the analysis above, $C_1(x_1, \xi^1) = C_2(x_2, \xi^2)$ if and only if

$$(s_2, \alpha_2) = \chi(s_1, \alpha_1), \quad (\widehat{s_2}, \widehat{\alpha_2}) = (\widehat{s_1}, \widehat{\alpha_1})\Big(d\chi\Big)^{-1}.$$

The first equality says there is a regular broken ray $\nu$ of which $(s_1, \alpha_1)$ and $(s_2, \alpha_2)$ are the incoming and outgoing part. The second condition is equivalent to

$$\lambda_2(1, \langle x_2, v(\alpha_2)\rangle) = \frac{\lambda_1}{\det(d\chi)}\Big(\frac{\partial \alpha_2}{\partial \alpha_1} - \langle x_1, v(\alpha_1)\rangle \frac{\partial \alpha_2}{\partial s_1}, -(\frac{\partial s_2}{\partial \alpha_1} - \langle x_1, v(\alpha_1)\rangle \frac{\partial s_2}{\partial s_1})\Big). \tag{2.13}$$

Notice $\frac{\partial \alpha_2}{\partial \alpha_1} - \langle x_1, v(\alpha_1)\rangle \frac{\partial \alpha_2}{\partial s_1}$ and $\frac{\partial s_2}{\partial \alpha_1} - \langle x_1, v(\alpha_1)\rangle \frac{\partial s_2}{\partial s_1}$ are exactly $\frac{d\alpha_2}{d\alpha_1}$ and $\frac{ds_2}{d\alpha_1}$ if we fixed $x_1$ and consider $s_2, \alpha_2$ as functions of one variable $\alpha_1$. Therefore (2.13) can be written as

$$\lambda_2(1, \langle x_2, v(\alpha_2)\rangle) = \frac{\lambda_1}{\det(d\chi)}(\frac{d\alpha_2}{d\alpha_1}, -\frac{ds_2}{d\alpha_1}).$$

This implies $C_1(x_1, \xi^1) = C_2(x_2, \xi^2)$ if and only if

(a) $\langle x_2, v(\alpha_2)\rangle = -\Big(\frac{d\alpha_2}{d\alpha_1}\Big)^{-1} \frac{ds_2}{d\alpha_1}$, i.e. $x_1$ and $x_2$ are conjugate points along $\nu$, by Proposition 3.

(b) $\lambda_2 = \frac{\lambda_1}{\det(d\chi)} \frac{d\alpha_2}{d\alpha_1}$, with $\xi_1 = \lambda_1 w(\alpha_1)$ and $\xi_2 = \lambda_2 w(\alpha_2)$.

□

**Remark 3.** For $\nu_0$ that is not regular, if $(x, \xi)$ and $(y, \eta)$ are not identical, then we can perform a similar analysis by applying a cutoff ΨDO instead of the cutoff function $\phi$ to $B$.

For $(y, \eta)$ satisfying this theorem, we call it the *conjugate covector* of $(x, \xi)$. Since $C_1$ is a local diffeomorphism, it maps a small neighborhood of $(x, \xi)$ to a small neighborhood of $(s_1, \alpha_1, \widehat{s_1}, \widehat{\alpha_1})$. The similar is true with $C_2$. Then by shrinking $V^1$ and $V^2$ a bit, we can assume $C_1(V^1) = C_2(V^2) \equiv \mathcal{V}$.

**Theorem 3.** *Suppose $(x, \xi)$ and $(y, \eta)$ are conjugate covectors along the broken ray $\nu$. Suppose $f_j \in \mathcal{E}'(U_j)$ with $\mathrm{WF}(f_j) \subset V^j$, for $j = 1, 2$. Then the local data, i.e. the broken ray transform in a small neighborhood of $\nu$*

$$B(f_1 + f_2) \in H^s(\mathcal{V})$$

*if and only if*

$$f_1 + F_{12} f_2 \in H^{s-1/2}(V^1) \Leftrightarrow F_{21} f_1 + f_2 \in H^{s-1/2}(V^2),$$

*where $F_{12} \equiv B_1^{-1} B_2$ and $F_{21} \equiv B_2^{-1} B_1$.*

*Proof.* We follow the arguments in [17]. Notice $B_1$ is a FIO of order $-\frac{1}{2}$ elliptic at $(x, \xi)$. An application of the parametrix $B_1^{-1}$ to $B(f_1 + f_2)$ shows

$$B_1^{-1} B(f_1 + f_2) = f_1 + F_{12} f_2.$$

Then $F_{12} = B_1^{-1} B_2$ is an FIO with canonical relation $C_{12} = C_1^{-1} \circ C_2 : V^2 \to V^1$; and $F_{21} = B_2^{-1} B_1$ is an FIO with canonical relation $C_{21} = C_2^{-1} \circ C_1 : V^1 \to V^2$. □

Thus, given a distribution $f_1$ singular in $V^1$, there exists a distribution $f_2$ singular in $V^2$ such that $B(f_1 + f_2)$ is smooth. One possible choice is $f_2 = -F_{21} f_1$. It is also the only choice up to smooth functions. We can introduce the concept of the microlocal kernel as in [18], which is defined as the space of distributions, modulo smooth functions, whose images by $B$ are smooth functions. Then for any $h$ with $\mathrm{WF}(h) \subset V^1$, we have $h - F_{21}h$ in the microlocal kernel and this describes the later. Therefore the reconstruction for $f = f_1$ always has some error in form of $h - F_{21}h$, for some $h$. In other words, we can recover the singularities of $f$ only up to an error term of the form and therefore they cannot be resolved from the singularities of $Bf$. On the other hand, suppose $(x, \xi) \in \mathrm{WF}(f)$ is conormal to a regular broken ray $\nu_0$ and has no conjugate covectors along it. In the recovery of singularities of $f$ from local data, the covector $(x, \xi)$ is recoverable according to this theorem.

With the notation above, we are going to find out the artifacts arising when we use the backprojection $B^*B$ to reconstruct $f$, if there are conjugate covectors. Without loss of generality, we assume the weight $a(y, s, \alpha) = 1$ in the following. Suppose $\nu_0$ is the broken ray in Theorem 2. In a small neighborhood of $\nu_0$, we have

$$B^*Bf = B_1^*B_1f_1 + B_1^*B_2f_2 + B_2^*B_1f_1 + B_2^*B_2f_2. \tag{2.14}$$

Recall $B_1$ and $B_2$ are defined microlocally. Indeed, on the one hand, the assumption on $\operatorname{supp} f_i$ plays the same role as restricting the operator on $U_k$, for $k = 1, 2$. For simplification, we just ignore them. On the other hand, if we concentrate on the small neighborhood of $\nu_0$, then we exclude the broken ray which has $(x, \xi)$ on its outgoing part. Microlocally $B_1$ is equivalent to the Radon transform operator, which indicates $B_1^*B_1$ is an elliptic ΨDO of order $-1$. Especially, it has the principal symbol $4\pi/|\xi|$. The similar is true for $B_2^*B_2$. As for $B_1^*B_2$ and $B_2^*B_1$, since $(x, \xi)$ and $(y, \eta)$ are conjugate covectors, these two operators are FIOs of order $-1$ associated with canonical relation $C_1^{-1} \circ C_2$ and $C_2^{-1} \circ C_1$ respectively; if there are no conjugate covectors, they are smoothing operators since the canonical relations are empty.

One can follow the same argument in [17,18] to show some properties of the normal operators. In addition, similarly to Radon transform, we can apply a filter to before the backprojection to get a zero order operator. We have

$$B^*\Lambda Bf = B_1^*\Lambda B_1f_1 + B_1^*\Lambda B_2f_2 + B_2^*\Lambda B_1f_1 + B_2^*\Lambda B_2f_2,$$

where $\Lambda = \frac{1}{4\pi}\sqrt{-\partial_s^2}$.

The canonical relation of $B_k^*$ is the inverse of that of $B_k$. Therefore by Egorov's theorem [12], $B_k^*\Lambda B_k$ is a pseudodifferential operator of order zero. We denote the principal symbol of a pseudodifferential operator $P$ by $\sigma_p(P)$. Recall Proposition 4, we have $\sigma_p(\Lambda)\circ C_1 = 1/(4\pi|\xi|)$ and $\sigma_p(\Lambda)\circ C_2 = 1/(4\pi|\eta|)$, where $C_k$ is the conical transformation corresponding to $B_k$ for $k = 1, 2$. Thus, the principal symbol of $B_k^*\Lambda B_k$ equals to $\sigma(B_k^* B_k)(\sigma(\Lambda)\circ C_k) = 1$, which implies

$$B_k^*\Lambda B_k \equiv I \mod \Psi^{-1}, \quad k = 1, 2.$$

This also coincides with the inversion formula for Radon transform. Then with the observation $B_1^*\Lambda B_2 F_{21} = B_1^*\Lambda B_1$ and $F_{21}B_1^*\Lambda B_2 \equiv I$, we have $B_1^*\Lambda B_2 \equiv F_{12}$ up to a lower order. The same is true with $B_2^*\Lambda B_1$. Notice that the calculations are all microlocal and up to order $-1$.

As a result,

$$B^*\Lambda B \equiv \begin{bmatrix} \mathrm{Id} & F_{12} \\ F_{12}^{-1} & \mathrm{Id} \end{bmatrix} := M, \tag{2.15}$$

where we follow the convention to think $f = f_1 + f_2$ as vector functions. It implies when performing the filtered backprojection, the reconstruction has two parts of artifacts, $F_{12}f_2$ in $V^1$ and $F_{21}f_1$ in $V^2$. As in [18], one can show that $F_{12}$ and $F_{21}$ are principally unitary in $H^{-\frac{1}{2}}$, and the artifacts have the same strength as the original distributions.

Next, we consider the numerical reconstruction by using the Landweber iteration as in [18]. For more details of the method, see [50]. We still focus on the local problem, that is, we consider $Bf$ in the small neighborhood of fixed $\nu_0$. With the notation above, we use a slightly different Landweber iteration to solve the equation $Bf = g$, where $g$ denotes the local data and it is assumed be in the range of $B$. We set $\mathcal{L} = \Lambda^{\frac{1}{2}} B$ to have

$$(\mathrm{Id} - (\mathrm{Id} - \gamma\mathcal{L}^*\mathcal{L}))f = \gamma\mathcal{L}^*\Lambda^{\frac{1}{2}} g. \tag{2.16}$$

Then with a small enough and suitable $\gamma > 0$, it can be solved by the Neumann series

$$f = \sum_{k=0}^{+\infty} (\mathrm{Id} - \gamma\mathcal{L}^*\mathcal{L})^k \gamma\mathcal{L}^*\Lambda^{\frac{1}{2}} g.$$

The series converge to the minimal norm solution to $\mathcal{L}f = \Lambda^{\frac{1}{2}}g$. Suppose the original function is $f = f_1 + f_2$. We track the terms of highest order, that is, order zero, to have the approximation sequence

$$f^{(n)} = \sum_{k=0}^{n} (\mathrm{Id} - \gamma M)^k \gamma M f \quad \Rightarrow \quad f^{(n)} = \gamma(\sum_{k=0}^{n} (1-2\gamma)^k) M f.$$

The second equality is from the observation $M^k = 2^{k-1}M$ for $k \geq 1$. The numerical solution is

$$f^{(n)} \to \frac{1}{2} \begin{bmatrix} f_1 + F_{12} f_2 \\ F_{21} f_1 + f_2 \end{bmatrix}, \text{ as } n \to \infty.$$

Therefore, the error equals to $\frac{1}{2}(f_1 - F_{12}f_2) + \frac{1}{2}(f_2 - F_{21}f_1)$, which belongs to the microlocal kernel.

## 2.5 Example 1: the V-line transform

In this section, we apply the conclusions above to the V-line transform. Except in subsection 2.5.4, we suppose the weight function $a(y, s, \alpha) = 1$. First we verify the reflection operator is a diffeomorphism. Then we have the potential cancellation of singularities due to the existence of conjugates points. Especially, we derive an explicit formula to illustrate in which case the conjugate points exist.

### 2.5.1 The Diffeomorphism

Suppose $\Omega$ is a bounded domain with a smooth boundary that can be parameterized by a regular curve $\gamma$. Suppose $\gamma$ is negatively oriented and we choose its arc length parameterization $\gamma(\tau) = (x(\tau), y(\tau))$. By negatively oriented, we mean when we travel on the curve we aways have the curve interior to the right side. The unit tangent vector is $\dot\gamma(\tau) = (\dot x(\tau), \dot y(\tau))$ and unit outward normal is $n(\tau) = (-\dot y(\tau), \dot x(\tau))$, where $\dot f(\tau)$ refers to $\frac{d}{d\tau}f$. We still consider the local problems, and $\gamma$ could be just part of $\partial\Omega$. The *signed curvature* of $\gamma$ is defined as the scalar function $\kappa(\tau)$ such that $\ddot\gamma = \kappa(\tau)n$.

Suppose a ray $(s, \alpha)$ transversally hits $\gamma$ at point $\gamma(\tau_0) = (x(\tau_0), y(\tau_0))$ and then reflects, as is shown in Figure 2.3. In a small neighborhood of such a ray, we have $\tau_0$ is a smooth function of $s$ and $\alpha$ . The proof is simply an application of implicit function theorem. Indeed, since $F(\tau_0, s, \alpha) = \langle w(\alpha), \gamma(\tau_0)\rangle - s = 0$ with $\frac{\partial F}{\partial \tau_0} = \langle w(\alpha), \dot\gamma(\tau_0)\rangle \neq 0$, it follows that $\tau_0$ could be written as a smooth function, say $\tau_0(s, \alpha)$.

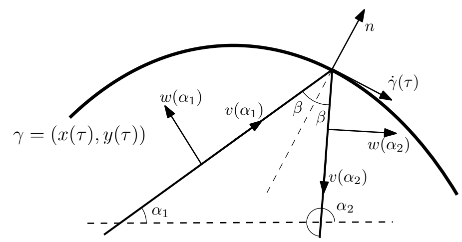



Figure 2.3. A sketch of a broken ray reflected on a smooth boundary and the notation.

Differentiating $F(\tau_0, s, \alpha) = 0$ w.r.t. $s$ and $\alpha$, we get equations of $\frac{\partial \tau_0}{\partial s}$ and $\frac{\partial \tau_0}{\partial \alpha}$. To distinguish $(s, \alpha)$ from the one we use for $l_2$, we replace them by $(s_1, \alpha_1)$ in the following

$$\frac{\partial \tau_0}{\partial s_1} = \frac{1}{\langle w(\alpha_1), \dot\gamma\rangle} \equiv k_s, \qquad \frac{\partial \tau_0}{\partial \alpha_1} = \frac{\langle v(\alpha_1), \gamma(\tau_0)\rangle}{\langle w(\alpha_1), \dot\gamma\rangle} \equiv k_\alpha. \tag{2.17}$$

**Claim 1.** *The reflection operator* $\chi : (s_1, \alpha_1) \mapsto (s_2, \alpha_2)$ *is a local diffeomorphism for* $(s_1, \alpha_1)$ *that hits the boundary transversally.*

*Proof.* As is shown in Figure 2.3, the reflection$\chi$ follows the rules:

$$\begin{cases} \alpha_2 = \alpha_1 + 2\beta + \pi \\ s_2 = \langle \gamma(\tau_0), w(\alpha_2)\rangle \end{cases} \tag{2.18}$$

where $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the incident angle. We use $\beta < 0$ to represent the case when $v(\alpha_1)$ has negative projection along $\dot\gamma$.

Since $\sin\beta = \langle v(\alpha_1), \dot\gamma(\tau_0)\rangle$, it follows that $\beta$ is a smooth function of $s_1$ and $\alpha_1$, which has the derivative

$$\frac{\partial \beta}{\partial s} = \kappa k_s, \qquad \frac{\partial \beta}{\partial \alpha_1} = \kappa k_\alpha - 1,$$

where $\kappa$ is the signed curvature. Then

$$\frac{\partial \alpha_2}{\partial s_1} = 2\kappa k_s, \qquad \frac{\partial \alpha_2}{\partial \alpha_1} = 2\kappa k_\alpha - 1. \tag{2.19}$$

Consequently,

$$\begin{aligned}
\frac{\partial s_2}{\partial s_1} &= \langle \frac{\partial w(\alpha_2)}{\partial s_1}, \gamma(\tau_0)\rangle + \langle w(\alpha_2), \frac{\partial \gamma(\tau_0)}{\partial s_1}\rangle = -\langle v(\alpha_2), \gamma(\tau_0)\rangle \frac{\partial \alpha_2}{\partial s_1} + k_s \langle w(\alpha_2), \dot{\gamma}\rangle, \\
\frac{\partial s_2}{\partial \alpha_1} &= \langle \frac{\partial w(\alpha_2)}{\partial \alpha_1}, \gamma(\tau_0)\rangle + \langle w(\alpha_2), \frac{\partial \gamma(\tau_0)}{\partial \alpha_1}\rangle = -\langle v(\alpha_2), \gamma(\tau_0)\rangle \frac{\partial \alpha_2}{\partial \alpha_1} + k_\alpha \langle w(\alpha_2), \dot{\gamma}\rangle.
\end{aligned}$$

By row reduction, we have

$$\det(d\chi) = \det \begin{bmatrix} \frac{\partial s_2}{\partial s_1} & \frac{\partial s_2}{\partial \alpha_1} \\ \frac{\partial \alpha_2}{\partial s_1} & \frac{\partial \alpha_2}{\partial \alpha_1} \end{bmatrix} = \det \begin{bmatrix} k_s \langle w(\alpha_2), \dot{\gamma}\rangle & k_\alpha \langle w(\alpha_2), \dot{\gamma}\rangle \\ \frac{\partial \alpha_2}{\partial s_1} & \frac{\partial \alpha_2}{\partial \alpha_1} \end{bmatrix}.$$

Thus,

$$\det(d\chi) = \langle w(\alpha_2), \dot{\gamma}\rangle \det \begin{bmatrix} k_s & k_\alpha \\ 2\kappa k_s & 2\kappa k_\alpha - 1 \end{bmatrix} = -\langle w(\alpha_1), \dot{\gamma}\rangle(-k_s) = 1.$$

The determinant of $d\chi$ is nonzero and therefore $\chi$ is a local diffeomorphism. □

### 2.5.2 Conjugate Points

The incoming ray $l_1(t)$ and reflected ray $l_2(t)$ are given in the following

$$\begin{cases} l_1(t) = p + t v(\alpha_1), & 0 \leq t \leq t_1, \\ l_2(t) = \gamma(\tau_0) + (t - t_1) v(\alpha_2), & t \geq t_1, \end{cases}$$

where $q_0 = \gamma(\tau_0)$ is the intersection point on the boundary. Compared with (2.3), now $t_1 = t_2$ and $q_0$ connects $l_1$ and $l_2$. From now on, we use $t_1$ instead of $t_2$. Recall that $\frac{d}{d\alpha_1}$ denotes the derivative with respect to $\alpha_1$ with $p$ fixed and $s_1$ given by $s_1 = \langle p, w(\alpha_1)\rangle$. By equation (2.17)(2.19), a straightforward calculation shows

$$\frac{d\alpha_2}{d\alpha_1} = \frac{2\kappa t_1}{\langle w(\alpha_1), \dot{\gamma}\rangle} - 1, \quad \frac{dq_0}{d\alpha_1} = \frac{t_1}{\langle w(\alpha_1), \dot{\gamma}\rangle}\dot{\gamma},$$

where $t_1 = \langle q_0 - p, v(\alpha_1)\rangle$ is the time or length from $p$ to $q_0$. Plugging these back into (2.2), we have

$$\det(d_v \exp_p(v)) = (t - t_1)\Big(\frac{d\alpha_2}{d\alpha_1}\Big) - t_1.$$

Especially, the matrix is in the following

$$d\exp_p(tv(\alpha_1)) = \Big[\ v(\alpha_2),\ \ ((t-t_1)\big(\tfrac{d\alpha_2}{d\alpha_1}\big) - t_1)w(\alpha_2)\ \Big]. \tag{2.20}$$

**Corollary 1.** *Suppose an incoming ray $l_1$ hits the boundary $\gamma$ transversally at point $\gamma(\tau_0)$ and then reflects. Let $p$ be a fixed point on $l_1$. Then*

*(a) $p$ has a conjugate point $q$ on $l_2$ if and only if $\frac{d\alpha_2}{d\alpha_1} > 0$, more specifically, if and only if*

$$\kappa(\tau_0) < \frac{\langle w(\alpha_1), \dot\gamma(\tau_0)\rangle}{2t_1}. \tag{2.21}$$

*(b) If this happens, $q$ is uniquely determined by $\Delta t_2 = \big(\frac{d\alpha_2}{d\alpha_1}\big)^{-1}\Delta t_1$, where$\Delta t_1 = t_1$ is the time or length from $p$ to $\gamma(\tau_0)$ and $\Delta t_2 = \langle q - \gamma(\tau_0), v(\alpha_2)\rangle$ is that from $\gamma(\tau_0)$ to $q$.*

The statement (a) comes from the observation that the other factor $\langle \frac{dq_0}{d\alpha_1}, w(\alpha_2)\rangle$ in Proposition 3(a) is always negative, as shown in Figure 2.3. This statement has a

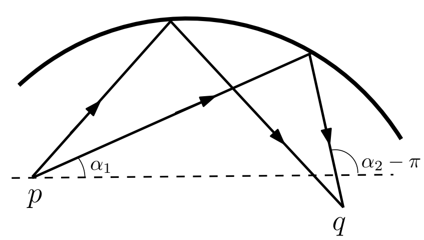



Figure 2.4. Two broken rays intersect when $\alpha_2$ increases as $\alpha_1$ increases.

straightforward geometrical explanation, see Figure 2.4. It says there are conjugate points if and only if $\alpha_2$ increases as $\alpha_1$ increases.

For negatively oriented smooth curve that is the boundary of a convex set, the curvature $\kappa < 0$ and the inner product $\langle w(\alpha_1), \dot\gamma\rangle < 0$. The inequality actually says

$|\kappa(\tau_0)| > \frac{|\langle w(\alpha_1), \dot{\gamma}\rangle|}{2t_1}$. Additionally, observe that $\langle w(\alpha_1), \dot{\gamma}\rangle = -\cos\beta$, where $\beta$ is the incident and the reflected angle. Each component involved in this criterion (2.21) is geometrical and therefore is invariant regardless of what kind of parameterization we choose for the boundary. We should mention that the equation in (b) coincides with the Generalized Mirror Equation in [47], but is in different form and is derived from the perspective of the exponential map.

**Remark 4** (of Theorem 2)**.** For the V-line transform, a broken ray is regular if and only if its incoming part does not hit the boundary perpendicularly.

**Example 5.** Consider a parabolic mirror $-4ay = x^2$, which has the focus at $(0, -a)$. Suppose there is a light source located at the point $p = (0, -d)$. Here $a$ and $d$ are positive constants we are going to choose later. We would like to know in which directions of the light from $p$ there are conjugate points. This example will illustrate the criterion for conjugate points.

Let $\gamma(x) = (x, -\frac{x^2}{4a})$ be the boundary curve. The intersection point is $q_0 = \gamma(x_0)$. Then the incoming ray has the direction along $\overrightarrow{pq_0}$, and $w(\alpha_1), \dot{\gamma}(x_0), \kappa(x_0), t_1$ are calculated directly by definition. After simplification, the criterion (2.21) is equivalent to

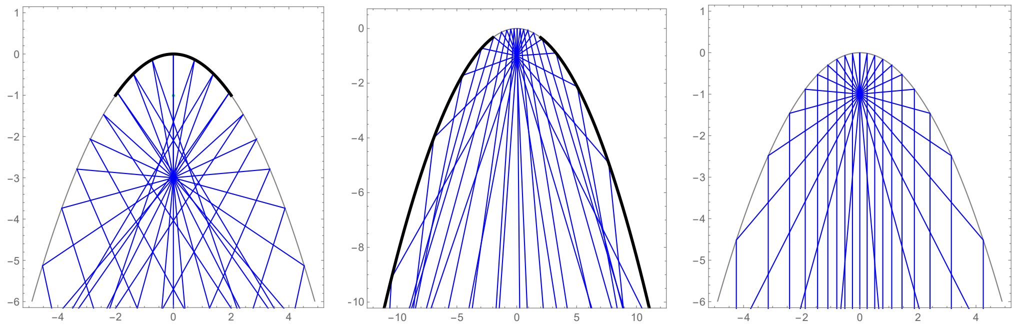
$$(a-d)(\frac{3}{4}x_0^2 - ad) > 0.$$

We have the following three cases.

*case 1:* If $d > a$, then $p$ has conjugate points if and only if the incoming ray hits the boundary at the region $x^2 < \frac{4}{3}ad$, as is shown in Figure 2.5(a).

*case 2:* If $d < a$, then $p$ has conjugate points if and only if the incoming ray hits the boundary at the region $x^2 > \frac{4}{3}ad$, as is shown in Figure 2.5(b).

*case 3:* If $d = a$, then $p$ has no conjugate points for all directions, which coincides with the fact that all rays of light emitting from the focus reflect and travel parallel to the $y$-axis, as is shown in Figure 2.5(c).



(a) (b) (c)

Figure 2.5. In (a) and (b), the bold part is the intersection region where the incoming rays hit there and reflect with conjugate points.

**Example 6.** The second example is to illustrate that we have different types of conjugate points, specifically fold and cusps, if we have a circular mirror with a light source inside. We assume the mirror is centered at the origin $O$ and the source is always not there. Suppose the mirror has radius 1. We follow some notations in the paper [17]. With $p$ fixed, the *tangent conjugate locus* $S(p)$ is the set of all vectors $v$ such that the differential of the exponential map $d_v \exp_p(v)$ is not an isomorphism. By calculations in Section 2.5.2,

$$S(p) = \{t(\cos\alpha_1, \sin\alpha_1), \text{s.t. } F(t,\alpha_1) = (\frac{2\kappa t_1}{\langle w(\alpha_1), \dot{\gamma}(\tau_0)\rangle} - 1)(t - t_1) - t_1 = 0\},$$

where $t_1$ and $\tau_0$ are smooth functions of $\alpha_1$. Now for fixed $v$, we denote the kernel of $d_v\exp_p(v)$ by $N_p(v)$. According to equation (2.20), the differential $d_v\exp_p(v)$ has the matrix form $[v(\alpha_2), 0]$, which indicates that $N_p(v)$ is spanned by $\frac{\partial}{\partial\alpha_1}$. If $N_p(v)$ is transversal to $S(p)$, then we say $v$ is *of fold type*. In this case, $v$ is of fold type when $\frac{\partial F}{\partial\alpha_1} \neq 0$ for all $(t,\alpha_1) \in S(p)$. Otherwise, when there is some $(t,\alpha_1)$ such that $\frac{\partial F}{\partial\alpha_1} = 0$ and it is a simple zero, we have a *cusp*. We show in the following that the cusp exists. A straightforward calculation shows

$$F = (\frac{2t_1}{\cos\beta} - 1)t - 2\frac{t_1^2}{\cos\beta} \Rightarrow \frac{\partial F}{\partial\alpha_1} = \frac{6t_1^2\sin\beta(t_1 - \cos\beta)}{\cos^2\beta(2t_1 - \cos\beta)},$$

where $\cos\beta = -\langle w(\alpha_1), \dot{\gamma}(\tau_0)\rangle$ is a smooth function of $\alpha_1$. If there are conjugate points, we must have $2t_1 - \cos\beta > 0$. The incidence angle $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so we at most have three zeros for $\frac{\partial F}{\partial \alpha}$

- $\beta = 0$, which means the incoming ray and reflected ray coincide. This is a simple zero, because $\frac{d}{d\alpha_1}\sin\beta = t_1 - \cos\beta = t_1 - 1 \neq 0$.
- $\cos\beta - t_1 = 0$ is true for some $\alpha_0$. This happens when $pO$ is perpendicular to the incoming ray. We check $\frac{d}{d\alpha}(\cos\beta - t_1) = \sin\beta \neq 0$. This is also a simple zero.

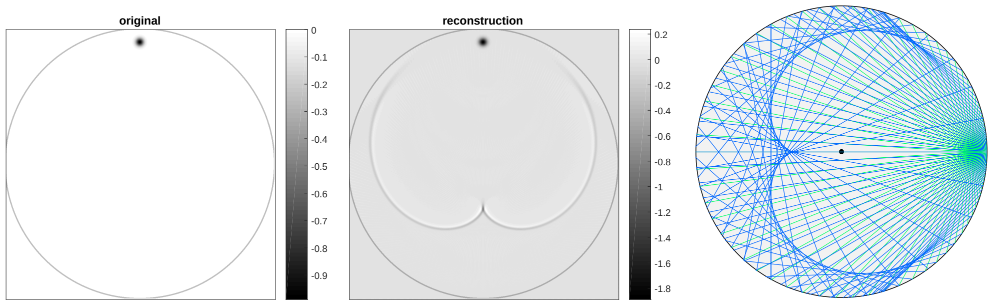
As a result, we have three cusps.

### 2.5.3 Numerical Examples

This subsection aims to illustrate the artifacts arising in the reconstruction by numerical experiments. We say $(x,\xi)$ is *visible* if there is a broken ray $\gamma$ in the family of tomography, such that $(x,\xi)$ is in the conormal bundle of $\gamma$ excluding the connecting part. The fact that $(x,\xi)$ is visible does not necessarily imply that $(x,\xi)$ is recoverable.

**Example 7.** In this example, we use filtered backprojection to recover $f$, which usually serves as the first attempt of reconstruction. Suppose the domain is a disk with radius $R$ and the boundary is negatively oriented. The family of broken rays $\Gamma$ contains any broken ray whose incoming part hits the boundary transversally and has positive projection on it. We choose $f_1$ to be a Gaussian concentrated near a single point, as an approximation of a delta function and $f_2$ to be zero. The support of $f = f_1 + f_2$ is in this disk.

In the code, $Bf$ is parameterized in the coordinate $(x_p, \alpha) \in [-R, R] \times [0, 2\pi)$. Here $(x_p, \alpha)$ refers to the incoming part and we use it to represent the broken ray. This parameterization follows the convention in Radon transform in MATLAB. The radial coordinate $x_p$ is the value along the $x'$-axis, which is oriented at $\alpha$ degree




(a) (b) (c)

Figure 2.6. Artifacts and caustics. Form left to right: $f$, $B^*\Lambda Bf$, and caustics caused by reflected light.

counterclockwise from the $x$-axis. We use the function *radon* to numerically construct our operator $B$ by the following formula

$$Bf(x_p, \alpha) = Rf(x_p, \alpha) + Rf(x'_p, \alpha'),$$

where $(x'_p, \alpha')$ is given by the reflection. Since numerically $Rf$ is known on discrete values of $(x_p, \alpha)$, we use interpolation methods to approximate $Rf(x'_p, \alpha')$. Similarly, $B^*$ is numerically constructed by the function *iradon* and interpolation methods. To better recover $f$, we apply the filter $\Lambda$ to the data before applying the adjoint operator. The plots are shown in the Figure 2.6. We can clearly see the artifacts appear exactly in the location of conjugate points, compared with the caustics caused by a light source. Furthermore, they are expained by equation (2.14).

**Example 8.** This example is to illustrate the reconstruction from local data by Landweber iteration. Assume each $(x, \xi)$ in $\mathrm{WF}(f)$, is visible and is perceived by only one broken ray. Then it has at most one conjugate point. To make it true, we use part of the circle as the reflection boundary. The tomography family $\Gamma$ is the set of all broken rays which comes from the left side with vertices on the boundary.

By [18] , we choose $f$ to be a modified Gaussian with singularities located both in certain space and in direction, that is, a coherent state, as is shown in Figure 2.7(a). We use the Landweber iteration to reconstruct $f$. The artifacts are still there after

100 iterations and the error becomes stable. Then we rotate $f$ or move it to see what happens to the artifacts. Specifically, in (c) and (d), $f$ remains in the location but is rotated by some angles. In (e) and (f), we move $f$ closer to the center and rotate it a bit. As the wave front set of $f$ changes, the artifacts changes and always appear in the location of their conjugate vectors.

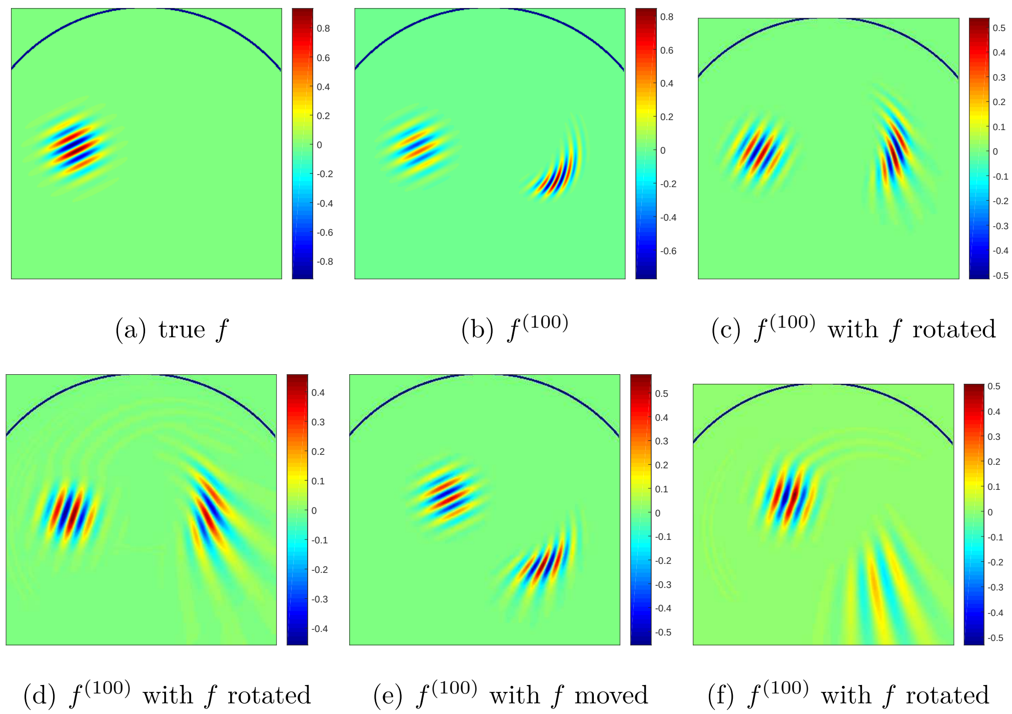



Figure 2.7. Local reconstruction by Landweber iteration.

### 2.5.4 The local problem with non-even weights

Suppose $\nu$ is a regular broken ray parameterized by the incoming part $(s_1, \alpha_1)$. It has the reflected part $(s_2, \alpha_2)$. There is another broken ray $\nu'$ that has the same linear path as $\nu$ but is in opposite direction, which is parametrized by the incoming part $(-s_2, \alpha_2 + \pi)$. In this subsection, instead of working on a neighborhood of $\nu$, we consider the recovery of $f$ from the knowledge of $Bf$ near both $\nu$ and $\nu'$. The conjugate covectors along $\nu$ could also be probed by $\nu'$, which intuitively helps us to recover singularities.

We consider a pair of conjugate covectors $(p_1, \xi^1)$ and $(p_2, \xi^2)$ on the incoming and reflected part of $\nu$ respectively. Suppose they satisfy Theorem 2. A straightforward calculation similar to Remark 2 shows that $(p_2, \xi^2)$ is conjugate to $(p_1, \xi^1)$ along $\nu'$. As proved in [17], we form a similar theorem in the following.

**Theorem 4.** *Suppose $V^k$ are small enough conical neighborhoods of conjugate covectors $(p_k, \xi^k)$, for $k = 1, 2$. Let $f = f_1 + f_2$ with $\mathrm{WF}(f) \subset V^k$. If the weight function for the V-line transform satisfies*

$$\det \begin{bmatrix} a(p_1, s_1, \alpha_1) & a(p_2, s_1, \alpha_1) \\ a(p_1, -s_2, \alpha_2 + \pi) & a(p_2, -s_2, \alpha_2 + \pi) \end{bmatrix} \neq 0,$$

*then $B(f_1 + f_2) \in H^s(\mathcal{V})$ implies $f_k \in H^{s-1/2}(V^k)$, for $k = 1, 2$.*

*Proof.* Let $B_+$ be the broken ray transform restricted in a small neighborhood of $(s_1, \alpha_1)$ and $B_-$ be that in a neighborhood of $(-s_2, \alpha_2 + \pi)$. Now Theorem 2 becomes a $2 \times 2$ system of equations

$$g_+ = B_+ f = B_{+,1} f_1 + B_{+,2} f_2 \in H^s(\mathcal{V}), \tag{2.22}$$

$$g_- = B_- f = B_{-,1} f_1 + B_{-,2} f_2 \in H^s(\mathcal{V}). \tag{2.23}$$

The assumption that the weight function is always nonzero implies that $B_{\pm,1}$ and $B_{\pm,2}$ are elliptic. Applying $B_{+,2} B_{-,2}^{-1}$ to (2.23) and subtracting it from (2.22), we have

$$(Id - Q) f_1 = B_{+,1}^{-1} (g_+ - B_{+,2} B_{-,2}^{-1} g_-), \tag{2.24}$$

where

$$Q = B_{+,1}^{-1} B_{+,2} B_{-,2}^{-1} B_{-,1}.$$

We prove in the following that $Q$ is a $\Psi$DO of order zero with principal symbol not equal to 1. As a result, $Id - Q$ is invertible and we can recover $f_1$ and $f_2$ microlocally from this system.

First we define the operator $N : (s, \alpha) \mapsto (-s, \alpha + \pi)$, which induces a diffeomorphism $N^*$. Recall that $\chi$ is the reflection operator. It is straightforward to check that $\chi N \chi N = Id$. Then we write $Q$ as

$$Q = B_{-,1}^{-1}(B_{-,1}B_{+,1}^{-1}\chi^* N^*)(\chi^* N^* B_{+,2}B_{-,2}^{-1})B_{-,1} \equiv B_{-,1}^{-1}Q_1 Q_2 B_{-,1},$$

where $Q_1$ is the composition inside the first parentheses and $Q_2$ is that inside the second.

**Claim 2.** *As defined above, $Q_1$ is $\Psi DO$ of order zero with principal symbol $\sigma_1 \circ C_{B_{-,1}}^{-1}$, where $\sigma_1 = a(p_1, -s_2, \alpha_2 + \pi)/a(p_1, s_1, \alpha_1)$. Additionally, $Q_2$ is $\Psi DO$ of order zero with principal symbol $\sigma_2 \circ C_{\chi^* N^* B_{+,2}}^{-1}$, where $\sigma_2 = a(p_2, s_1, \alpha_1)/a(p_2, -s_2, \alpha_2 + \pi)$.*

We will prove this claim below. Assuming it for the moment, by Egorov's theorem $Q$ is a ΨDO with principal symbol

$$\begin{aligned}(\sigma_1 \circ C_{B_{-,1}}^{-1})(\sigma_2 \circ C_{\chi^* N^* B_{+,2}}^{-1}) \circ C_{B_{-,1}} &= \sigma_1(\sigma_2 \circ C_{B_{-,2}}^{-1} C_{B_{-,1}}) \\ &= \frac{a(p_1, -s_2, \alpha_2 + \pi)a(p_2, s_1, \alpha_1)}{a(p_1, s_1, \alpha_1)a(p_2, -s_2, \alpha_2 + \pi)} \neq 1,\end{aligned}$$

which implies that $Id - Q$ is an elliptic ΨDO. Recall that $B_{\pm,i}$ is an FIO of order $-1/2$ for $i = 1, 2$, and therefore $B_{+,1}^{-1}B_{+,2}B_{-,2}^{-1}$ is of order $1/2$. As a result, $f = f_1 + f_2$ can be recovered microlocally by

$$f_1 = (Id - Q)^{-1}B_{+,1}^{-1}(g_+ - B_{+,2}B_{-,2}^{-1}g_-), \quad f_2 = B_{-,2}^{-1}(g_- - B_{-,1}f_1).$$

□

*The proof of the claim.* We connect $B_{\pm,k}$ with the Radon transform restricted to distributions singular in $V^k$ near a certain ray, for $k = 1, 2$. Let $R_{+,k}$ be the Radon transform in $V^k$ near the ray $(s_k, \alpha_k)$. Let $R_{-,k}$ be the Radon transform in $V^k$ near the ray $(-s_k, \alpha_k + \pi)$. We emphasize that the weights of Radon transform here comes from that of the V-line transform as defined in (2.7), which might conflict with the convention. Especially, $R_{+,2}$ is the Radon transform near $(s_2, \alpha_2)$ but has the weight

$a(x, s_1, \alpha_1)$ and $R_{-,1}$ is that near $(-s_1, \alpha_1 + \pi)$ but has the weight $a(x, -s_2, \alpha_2 + \pi)$. It follows that

$$B_{+,1} = R_{+,1}, \quad B_{+,2} = \chi^* R_{+,2}, \quad B_{-,1} = \chi^* R_{-,1}, \quad B_{-,2} = R_{-,2}.$$

Observe that ${R_{+,1}}^{-1} N^* R_{-,1}$ is a ΨDO with principal symbol

$$\sigma_1 = a(x, -s_2, \alpha_2 + \pi)/a(x, s_1, \alpha_1).$$

By Egorov's thoerem, $N^* R_{-,1} {R_{+,1}}^{-1} = R_{+,1}({R_{+,1}}^{-1} N * R_{-,1}){R_{+,1}}^{-1}$ is a ΨDO with principal symbol $\tau_i \circ C^{-1}_{R_{+,1}}$. A similar argument shows that $N^* R_{+,2} R^{-1}_{-,2}$ is a ΨDO with principal symbol $\sigma_2 \circ C^{-1}_{N^* R_{+,2}}$. Consequently, we write $Q_1$ and $Q_2$ as

$$\begin{aligned} Q_1 &= B_{-,1} B^{-1}_{+,1} \chi^* N^* = \chi^* R_{-,1} R^{-1}_{+,1} \chi^* N^* = \chi^* N^* (N^* R_{-,1} R^{-1}_{+,1}) \chi^* N^*, \\ Q_2 &= \chi^* N^* B_{+,2} B^{-1}_{-,2} = N^* R_{+,2} R^{-1}_{-,2}. \end{aligned}$$

Applying Egorov's theorem to the first equation, we have

$$\sigma_p(Q_1) = \sigma_1 \circ C^{-1}_{R_{+,1}} \circ C_{\chi^* N^*} = \sigma_1 \circ C^{-1}_{B_{-,1}}, \quad \sigma_p(Q_2) = \sigma_2 \circ C^{-1}_{N^* R_{+,2}} = \sigma_2 \circ C^{-1}_{B_{-,2}}.$$

The second equality comes from the observation that ${B_{-,2}}^{-1} N^* R_{+,2}$ is a ΨDO. □

**Remark 5.** This condition fails for the attenuated V-line transform that comes from the setting of Compton camera in two dimensions. In that setting, the direction of a broken ray is fixed and we do not have two different directed rays.

### 2.5.5 Global Problems

In this subsection, suppose $\Omega$ is a strictly convex domain with smooth negatively oriented boundary. We consider the V-line transform over all broken rays whose incoming part hits the boundary transversally and has nonnegative projection on it. These rays may reflect from the boundary more than once but here we only consider the one-reflection situation, since we are motivated by the SPECT with Compton camera. We consider the reconstruction of the V-line transform from full data.

Suppose $Bf$ is smooth. We would like to find out whether a given covector $(x_0, \xi^0)$ is in the wave front set of $f$. Assume $(x_0, \xi^0)$ in $\mathrm{WF}(f)$. There are two broken rays in $\Gamma$ that could carry this singularity. One broken ray $\nu_0$ represented by $(s_0, \alpha_0)$ has it in the incoming part, and the other one $\nu_{-1}$ represented by $(s_{-1}, \alpha_{-1})$ has it in the reflected part. Suppose $(x_1, \xi^1)$ and $(x_{-1}, \xi^{-1})$ are its conjugate covectors along $\nu_0$ and $\nu_{-1}$, if they exist. When both $\nu_0$ and $\nu_{-1}$ are regular, we have the following cases.

If at least one of $(x_1, \xi^1)$ and $(x_{-1}, \xi^{-1})$ does not exist, for example $(x_1, \xi^1)$, then the singularity caused by $(x_0, \xi^0)$ in $V^0$ cannot be canceled via $\nu_0$. With the assumption that $Bf$ is smooth, this indicates $(x_0, \xi^0) \in \mathrm{WF}(f)$ impossible.

If both $(x_1, \xi^1)$ and $(x_{-1}, \xi^{-1})$ exist, then the singularities might be canceled by them. We continue to consider $\nu_1$, $\nu_{-2}$ and so on. As a result, we get a sequence of broken rays (we assume they are all regular at this stage) and conjugate covectors. We define

$$\mathcal{M}(x_0, \xi^0) = \{(x_k, \xi^k), \text{ if it exists and is conjugate to } (x_{k'}, \xi^{k'}), \tag{2.25}$$

$$\text{where } k' = k - \operatorname{sgn} k, \text{for } k = \pm 1, \pm 2, \dots\} \tag{2.26}$$

as the set of all conjugate covectors related to $(x_0, \xi^0)$. If $\mathcal{M}(x_0, \xi^0)$ contains finitely many $(x_k, \xi^k)$ whose index $k$ is positive or negative, we say it is *nontrapping* in positive or negative direction. Otherwise, we say it is *trapping*.

Next, let $V^k$ be a small conic neighborhoods of $(x_k, \xi^k) \in \mathcal{M}(x_0, \xi^0)$ and $U_k = \pi(V^k)$. Let $f_k$ be the restriction of $f$ on $U_k$. Now we suppose $\mathcal{M}(x_0, \xi^0)$ is nontrapping, for example, in the positive direction. That is, there exists a maximal integer $k_0$ such that $(x_{k_0}, \xi^{k_0}) \in \mathcal{M}(x_0, \xi^0)$. From analysis above, we assume $k_0 \geq 1$. For $k = 1, \dots, k_0$, by shrinking $V^k$ carefully, we have $C(V^k) = V^{k-1}$. Then the cancellation of singularities shows

$$B_{k-1} f_{k-1} + B_k f_k = 0 \mod C^\infty, \quad k = 1, \dots k_0.$$

Finally we have

$$B_{k_0} f_{k_0} = 0 \mod C^\infty.$$

By applying the diffeomorphism $\chi^*$ and forward substitution, we can show that all $f_k$ must be smooth, for $k = 0, \ldots, k_0$. It is similar if $\mathcal{M}(x_0, \xi^0)$ is nontrapping in negative direction.

The above analysis holds when each $(x_k, \xi^k)$ in $\mathcal{M}(x_0, \xi^0)$ is carried by a regular broken ray. If it is not true, we can still define the sequence of conjugate covectors $\mathcal{M}(x_0, \xi^0)$. If the sequence is nontrapping, then by considering $B$ microlocally and by performing the similar arguments we can show $f$ is smooth. This proves when $\mathcal{M}(x_0, \xi^0)$ is nontrapping, $(x_0, \xi^0)$ is a recoverable singularity.

**Theorem 5.** *Suppose $\Omega$ is a strictly convex domain with smooth boundary. Let $f$ be a distribution supported in $\Omega$. Then $(x_0, \xi^0)$ is recoverable if $\mathcal{M}(x_0, \xi^0)$ is nontrapping. In other words, when $Bf \in C^\infty$, we must have $(x_0, \xi^0) \notin \mathrm{WF}(f)$.*

**Example 9.** As is shown in the Figure 2.8, we use the same domain and family of tomography as in Example 6. Especially, we suppose the disk is centered at the origin for simplification.

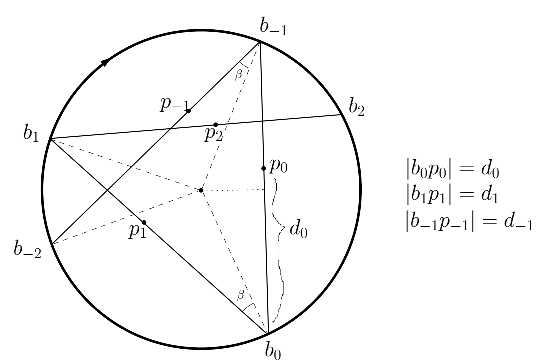



Figure 2.8. Inside a circular mirror, a sequence of broken rays and conjugate points on them.

Considering a point $(p_0, \xi^0)$, we have a sequence of broken rays

$$\ldots, b_{-2}b_{-1}b_0,\ b_{-1}b_0b_1,\ b_0b_1b_2, \ldots\ b_{k-1}b_kb_{k+1}, \ldots,$$

as well as the set $\mathcal{M}(p_0, \xi^0)$.

**Proposition 5.** *In Example 9, we say $(p_0, \xi^0)$ is radial if $p_0$ is the midpoint of a chord such that $\xi_0$ is in its conormal. Then $\mathcal{M}(p_0, \xi^0)$ is trapping if and only if $(p_0, \xi^0)$ is radial.*

*Proof.* Fix a point $p_k$. It might have a conjugate point $p_{k+1}$ along $b_{k-1}b_kb_{k+1}$ or $p_{k-1}$ along $b_{k-2}b_{k-1}b_k$. Let $d_i = |b_kp_k|$ be the distance along the ray from $p_k$ to the boundary point $b_k$. Notice all incidence and reflection angles are equal (call them $\beta$). Then $|b_kb_{k+1}| = 2\cos\beta$ for all integer $k$.

Recall Corollary 1. In this case, we have $\Delta t_1 = d_1$, $\Delta t_2 = 2\cos\beta - d_2$, and $\frac{d\alpha_2}{d\alpha_1} = \frac{2d_1}{\cos\beta} - 1$. Then $p_k$ has a conjugate point $p_{k+1}$ inside the domain if and only if $d_{k+1}$ given by

$$\frac{1}{d_i} + \frac{1}{2\cos\beta - d_{k+1}} = \frac{2}{\cos\beta}$$

has a solution in $(0, 2\cos\beta)$. To simplify, we change the variable that $d_k = \cos\beta(a_k + 1)$. Thus,

$$\frac{1}{1+a_k} + \frac{1}{1-a_{k+1}} = 2 \implies 2a_ia_{k+1} + a_{k+1} - a_k = 0. \tag{2.27}$$

The requirement that $p_k$ is inside the domain means we are finding solutions for $a_k \in (-1, 1)$.

*case 1.* $a_0 = 0$, which implies by $a_k = 0$ for any integer $k$. This is the case when we have $p_0$ at the midpoint of some chord and $\xi^0$ is the conormal of the chord. The same is true with all $(p_k, \xi^k)$. We have a trapping $\mathcal{M}(p_0, \xi^0)$.

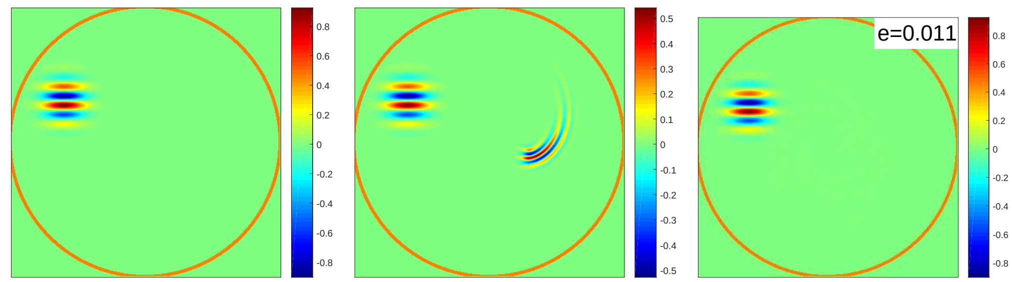
*case 2.* $a_k \neq 0$. Then (2.27) can be reduced to the following iteration formula

$$\frac{1}{a_{k+1}} = \frac{1}{a_k} + 2.$$

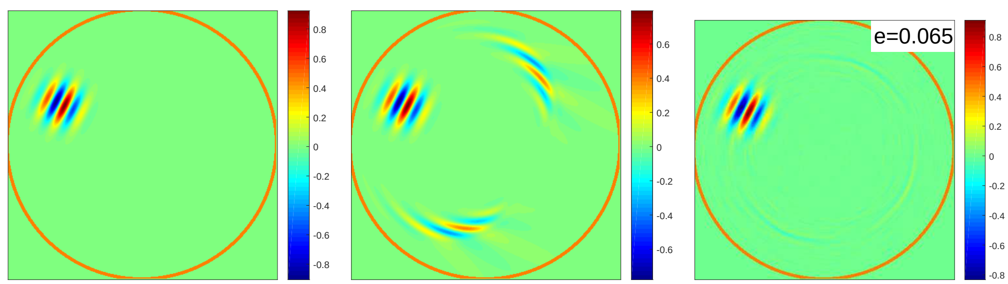
Suppose we start from some $a_0$. Each time, the next $\frac{1}{a_k}$ increases or decreases by 2. With $\frac{1}{a_0} \in (-\infty, -1) \bigcup (1, \infty)$, finally we must have some $\frac{1}{a_k}$ belonging to the interval $(-1, 1)$, which mean $p_k$ goes out of the domain. In this case, $\mathcal{M}(p_0, \xi^0)$ is always nontrapping. $\square$

**Corollary 2.** *Suppose everything as in Example 9. Then $(x_0, \xi^0)$ is recoverable if $(x_0, \xi^0)$ is not radial.*

**Example 10.** With the same set up as above, we first choose $f_1$ to be a modified Gaussian of coherent state whose singularities are not radial. To compare, next we choose $f_2$ to be with radial singularities.




(a) non-radial singularities $f_1$ (b) backprojection $f_1^{(1)}$ (c) $f_1^{(100)}$

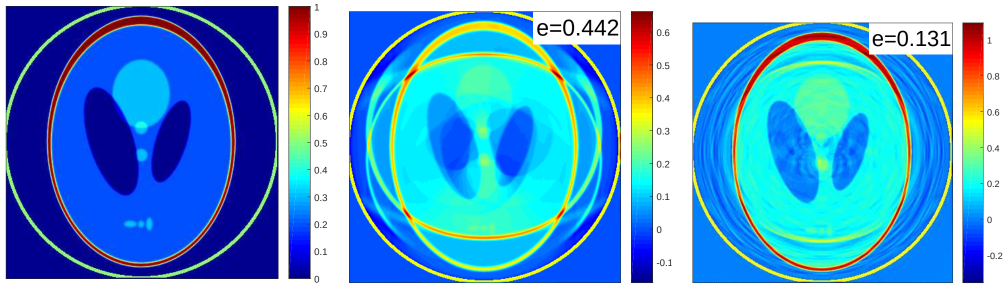



(d) radial singularities $f_2$ (e) backprojection $f_2^{(1)}$ (f) $f_2^{(100)}$

Figure 2.9. Reconstruction of $f_1$ and $f_2$ from global data, where $e = \frac{\|f - f^{(100)}\|_2}{\|f\|_2}$ is the relative error.

As is shown in Figure 2.9, after performing Landweber iteration of 100 steps, all artifacts fade out and the reconstruction has a small error if $f$ has non-radial singularities. On the contrary, if $f$ has radial singularities, the error still decreases as the iteration but in a much slower speed. In these two cases, since $f$ is only supported in a small set, the artifacts arising in the reconstruction may seem not so obvious. However, when $f$ is more complicated, the artifacts might be unignorable. In the following we choose $f_3$ to be a Modified Shepp-Logan phantom.

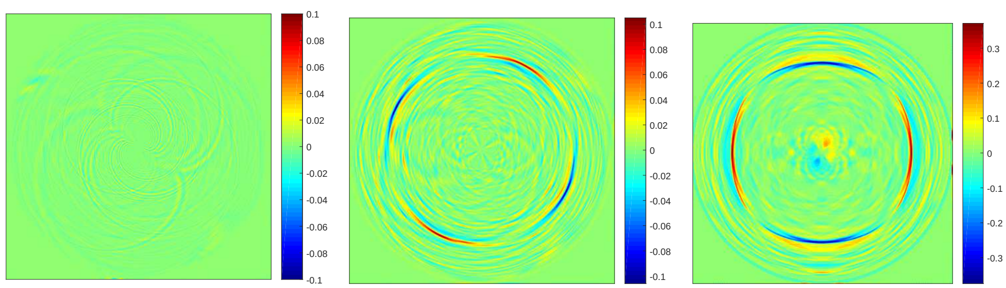
The error plots of these three cases are in Figure 2.11 to better illustrate the difference between radial and non-radial singularities. They also show where the artifacts




(a) true $f_3$ (b) backprojection $f_3^{(1)}$ (c) $f_3^{(100)}$

Figure 2.10. Reconstruction from global data for Modified Shepp-Logan phantom $f_3$, where $e = \frac{\|f-f^{(100)}\|_2}{\|f\|_2}$ is the relative error.

appear (for more details, see 2.5.6). It is clear to see the error of reconstruction is much smaller when we have non-radial singularities than radial ones.



(a) (b) (c)

Figure 2.11. The error plot for the reconstruction of $f_1, f_2, f_3$ in order. The first two has the same range of color bar.

**Example 11.** In this example we consider the reconstruction of the V-line transform in an elliptical domain $Q$ from global data. By [51], the billiard trajectory in an elliptical table has the following cases. If the trajectory crosses one of the focal points, then it converges to the major axis of $Q$. If the trajectory crosses the line segment between the two focal points, then it is tangent to a unique hyperbola, which is determined by the trajectory and shares the same focal points with $Q$. If it does not cross the line segment between the two focal points, then it is tangent to a unique ellipse, which shares the same focal points with $Q$.

In the following numerical experiments, we choose $f$ as a coherent state. It is located and rotated such that the trajectory carrying its singularities falls into the last two cases above. We use Landweber iterations to reconstruct $f$ by iterating 100 steps. As in Figure 2.12, in the reconstruction of the first coherent state, the artifacts disappear as we iterate, since some conjugate points are outside the domain. On the contrary, with conjugate points staying in the domain at least for the first reflection, there is a relative larger error in the reconstruction of the second one. A more complete analysis of the ellipse case is behind the scope of this work.

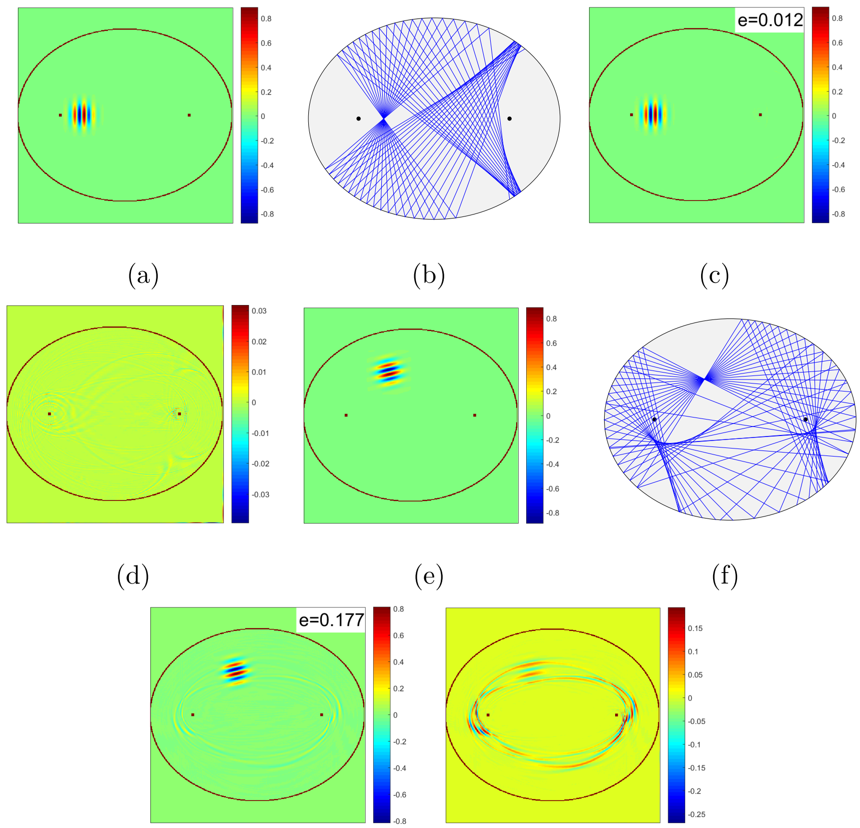
### 2.5.6 Comparison with previous results for a circular domain

This subsection is to connect our analysis to the results in [14]. By expanding $f$ and the data $Bf$ as Fourier series with respect to the angular variable, [14] gives an inversion formula (2.8) for V-line transform with vertices on a circle. The denominator inside the integral has zeros for certain radius $r$ and with noises it could be very unstable. This indicates we can expect certain patterns of the artifacts in the reconstruction. We show these artifacts predicted by (2.8) coincides with the conjugate covectors of radial singularities in the following.

When $(x_0, \xi^0)$ is radial, $\mathcal{M}(x_0, \xi^0)$ is trapping and we have two cases. One is the case that $\mathcal{M}(x_0, \xi^0)$ is a periodic set with period $m$. That is, the broken rays that carry $(x_0, \xi^0)$ after several reflections form a regular polygon of $m$ edges, a convex or star one. The set $P$ of all possible regular polygons can be described by the Schläfli symbol [52],

$$P = \{(m/n),\ p, q \in \mathbb{N},\ 2 \leq 2n < m,\ \gcd(m, n) = 1\}.$$

Here $(m/n)$ refers to a regular polygon with $m$ sides which winds $n$ times around its center. When $n = 1$, it is a convex regular one; otherwise it is a star one. For the




(a) (b) (c) (d) (e) (f) (g) (h)

Figure 2.12. Reconstruction of two coherent states. Left to right: true $f$, the envelopes (caused by trajectories that carry singularities and are reflected only once), $f^{(100)}$ (where $e = \frac{\|f-f^{(100)}\|_2}{\|f\|_2}$), the error.

polygon $(m/n)$, the internal angle equals to $\frac{\pi(m-2n)}{m}$. This implies $|x| = \cos\frac{n\pi}{m}$, where $x$ is the midpoint of one edge. Suppose $Bf$ is smooth. We have

$$B_{i-1}f_{i-1} + B_i f_i = 0 \mod C^\infty, \quad i = 1, \dots, p-2$$
$$B_{m-1}f_{m-1} + B_0 f_0 = 0 \mod C^\infty.$$

By forward substitution, we get

$$(1 + (-1)^{m-1})R_0 f_0 = 0 \mod C^\infty.$$

When $m$ is odd, $f_0$ must be smooth, which implies $f$ is smooth and therefore $(x_0, \xi^0)$ is recoverable. When $m$ is even, it possibly causes artifacts. These artifacts are located at radius $|x| = \cos\frac{(2k+1)\pi}{2l}$, where $m = 2l$ and $n = 2k+1$ with $0 \le 2n < m$. These radius are exactly the positive solution of $s$ such that $\cos(n(\arcsin(s) - \pi/2)) = 0$ in Formula (2.8) in [14].

In the following example, we use the same function as in Figure 2.9 but move them closer to the origin. The plot of error shows the artifacts are centered at the midpoint of each edges of regular stars.

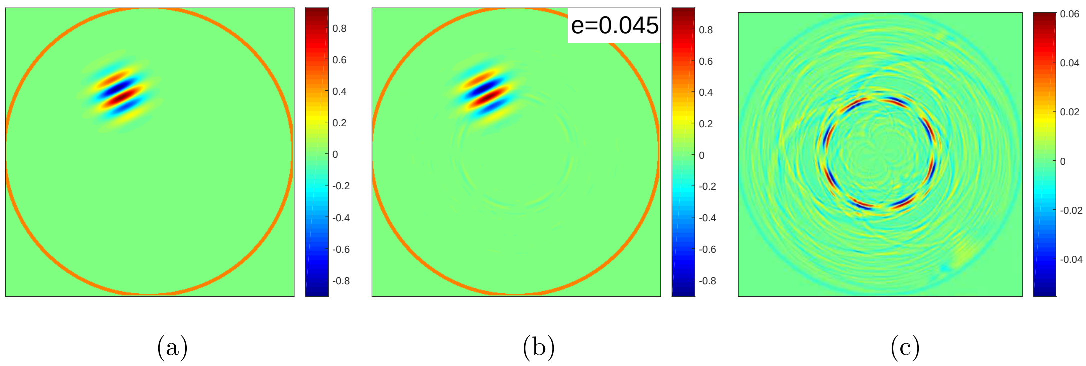



Figure 2.13. Another case of radial singularities. Left to right: true $f$, reconstruction $f^{(100)}$, error for $f$ with radial singularities after 100 iterations. The relative error $e$ is defined as before.

We should mention that in the numerical reconstruction in [14], the regularization (2.12) is used to remove the instabilities caused by these zeros. Therefore the artifacts are removed but on the other hand some true singularities are removed as well. In [33], the regularization is also used in the numerical reconstruction of a smiley phantom but we can still see some artifacts caused by the radial singularities (see Figure 2 in [33]).

### 2.6 Example 2: the Parallel ray transform

We define the parallel ray transform as an integral transform over two or more equidistant parallel rays. The simplest case is the one over two parallel rays and is defined in the following

$$\mathcal{P}f(s,\alpha) = \int_{x\cdot\omega(\alpha)=s} f(x)dx + \int_{x\cdot\omega(\alpha)=s+d} f(x)dx.$$

It can be regarded as one example of the broken ray transform that we defined in Section 2.2, if we suppose the two rays are connected by a smooth curve outside the support of $f$ or simply at the infinity. Additionally, the diffeomorphism $\chi$ is the translation which maps $(s,\alpha)$ to $(s+d,\alpha)$. Following the previous notations and calculations, we have

$$\frac{d\alpha_2}{d\alpha_1} = 1, \quad \frac{ds_2}{ds_1} = -\langle p, v(\alpha_1)\rangle.$$

Suppose $p$ is on the ray $(s_1,\alpha_1)$. By Proposition 3, if $p$ has a conjugate point $q$ on the ray $(s_2,\alpha_2)$, then $q$ is determined by $q = p + w(\alpha_1)d$. By Theorem 2, a singularities $(x,\xi)$ can be canceled by $(y,\eta)$ if and only if $x$ and $y$ are conjugate points and $\xi = \eta$. It is shown in Figure 3 that the artifacts arising when we use the backprojection as the first attempt to recover $f$.

Now we consider the reconstruction by iteration process. Suppose $(x_0,\xi^0) \in \mathrm{WF}(f)$ belongs to the ray $(s_1,\alpha_1)$. It can be canceled by two conjugate covectors $(x_0 \pm d\frac{\xi^0}{|\xi^0|}, \xi^0)$. We follow the same analysis as in the previous section to have

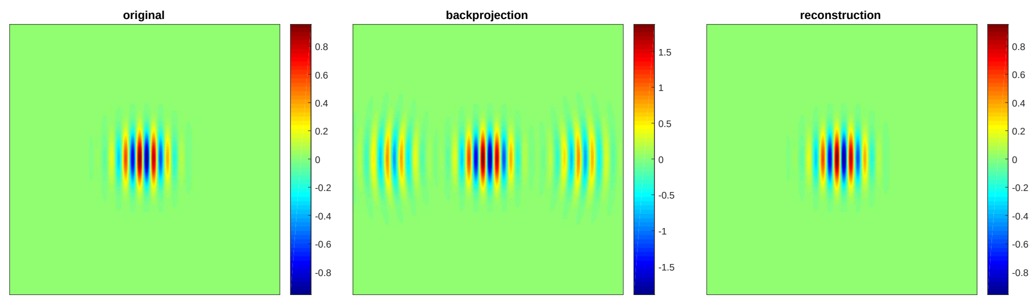
$$\mathcal{M}(x_0,\xi^0) = \{(x_0 + jd\frac{\xi^0}{|\xi^0|}, \xi^0),\ j = \pm1, \pm2, \ldots\}. \tag{2.28}$$

The typology of $\mathcal{M}(x_0,\xi^0)$ is quite clear. It is a discrete set of points which has equal distance. Assume $\mathcal{P}f$ is smooth. Then $(x_0,\xi^0) \in \mathrm{WF}(f)$ implies $\mathcal{M}(x_0,\xi^0) \subset \mathrm{WF}(f)$ by the same argument as before. Thus, we have the following proposition, see also [15].

**Proposition 6.** *Suppose $f \in \mathcal{D}'(\mathbb{R}^2)$ and assume $\mathcal{P}f$ is smooth. Then for any $(x,\xi)$, either $\mathcal{M}(x,\xi) \subset \mathrm{WF}(f)$ or $\mathcal{M}(x,\xi) \cap \mathrm{WF}(f) = \varnothing$.*

In particular, with a prior knowledge that $\mathrm{WF}(f)$ is in a compact set, the singularities are recoverable.

**Corollary 3.** *Suppose $f \in \mathcal{E}'(\mathbb{R}^2)$ and assume $\mathcal{P}f$ is smooth. Then $f$ is smooth.*




(a) (b) (c)

Figure 2.14. Left to right: true $f$, backprojection $f^{(1)}$, $f^{(100)}$.

In the numerical experiment, we use the Landweber iteration to reconstruct $f$. With the assumption that $f \in \mathcal{E}'(\mathbb{R}^2)$, a cutoff operator is performed at every step. After 100 iterations, we get a quite good reconstruction (with $\|f^{(100)} - f\|_\infty = 0.003$).

It should be mentioned that Corollary 3 shows $f$ with singularities in a compact set could be recovered from the global data. This implies when performing the transform, we move the parallel rays around until all of them leave the compact set. In fact, from our analysis above, the condition that the rays leaving at least one side the compact set is enough. On the other hand, the local problem (illumination of a region of interest only) could create artifacts.

# 3. THE INTEGRAL TRANSFORM OVER A GENERIC FAMILY OF SMOOTH CURVES

In the draft [2] , we extend the result for local problems of the broken ray transform to the integral transform over a generic family $\Gamma$ of smooth curves $\gamma(t)$, which are given by $\ddot{\gamma} = G(\gamma, \dot{\gamma})$, where the generator $G(x, v)$ is smooth. The integral transform over this kind of curves without conjugate points is studied in [19]. Injectivity and the stability estimates are established there, which in particular implies that we can recover the singularities uniquely. One example of a generic family of smooth curves with conjugate points is that of magnetic geodesics when there a constant non-zero magnetic field, see [16, 53]. This gives us the circular Radon transform with fixed radius. The main result of this chapter in is Theorem 6 and 7.

## 3.1 Model assumptions

Consider $M = \mathbb{R}^2$ with a Riemannian metric $g$. Let $\Gamma$ be a smooth family of curves satisfying the following properties:

(1) For each $(x, v) \in TM \setminus 0$, there is one unique curve $\gamma(t) \in \Gamma$ passing $x$ in the direction of $v$. That is, we can assume $\gamma(0) = x$ and $\dot{\gamma}(0) = \mu v$, for some smooth function $\mu(x, v) > 0$.

(2) Suppose such $\gamma(t)$ depends smoothly on $(x, v)$, and thus it solves

$$\ddot{\gamma}(t) = G(\gamma, \dot{\gamma}),$$

where the generator $G(x, v)$ is smooth.

By modifying $G$, we can write the equation as

$$D_t \dot{\gamma} = G(\gamma, \dot{\gamma}),$$

where $D_t$ is the covariant derivative along $\dot{\gamma}$. Since $\dot{\gamma}$ never vanishes, we reparameterize these curves such that they have unit speeds. We abuse the notation and use $\gamma_{x,v}(t)$ to denote the smooth curve passing $(x, v) \in SM$, where $SM$ is the circle bundle. This reparameterization will change the generator $G(\gamma, \dot{\gamma})$ and the weight function in the integral transform. Actually, the family of curves with this arc length parameterization solves a new ODE and are called the $\lambda$-geodesics in [54]

$$D_t\dot{\gamma} = \lambda(\gamma, \dot{\gamma})\dot{\gamma}^{\perp},$$

where $\lambda(\gamma, \dot{\gamma}) \in C^{\infty}(SM)$.

### 3.2 Conjugate points and Jacobi fields

Let $\exp_p(t, v) = \gamma_{p,v}(t)$ be the exponential map, where $(p, v) \in SM$. This definition of exponential map uses polar coordinates and is independent of a change of the parameterization for curves in $\Gamma$. We say a point $q = \gamma(t_0)$ is **conjugate** to $p = \gamma(0)$ if the differential $d_{t,v} \exp_p(t, v)$ of the exponential map in polar coordinates is singular at $(t_0, \dot{\gamma(0)})$, as in [55]. Consider the flow $\phi_t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$. Set a fixed curve $\gamma_{p_0,v_0}(t)$ with

$$(p_1, v_1) = \phi_t(p_0, v_0), \quad (p_2, v_2) = \phi_{t+s}(p_0, v_0),$$

where $(p_i, v_i) \in SM$, for $i = 0, 1, 2$. Consider the representation of $(\phi_t)_*$ in local coordinates at $(p_0, v_0)$ given by the Jacobian matrix

$$J_{(p_0,v_0,t)} = \begin{bmatrix} d_p \exp_p(t, v) & d_v \exp_p(t, v) \\ d_p \dot{\exp}_p(t, v) & d_v \dot{\exp}_p(t, v) \end{bmatrix}_{(p_0,v_0,t)}.$$

Suppose $(\alpha_i, \beta_i)$ is the tangent vector at $(p_i, v_i)$, i.e. $\alpha_i \in T_{p_i}M, \beta_i \in T_{v_i}S^1$. The differential of $\phi_t$ satisfies $(\phi_{t+s})_* = (\phi_s)_* \circ (\phi_t)_*$, which implies

$$\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} = J_{(p_1,v_1,s)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = J_{(p_0,v_0,t+s)} \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}, \tag{3.1}$$

whenever

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = J_{(p_0,v_0,t)} \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}. \tag{3.2}$$

We have the following proposition which connects the conjugate points and the Jacobian matrix.

**Proposition 7.** *A point $p_2 = \gamma_{p_0,v_0}(t+s)$ is conjugate to $p_1 = \gamma_{p_0,v_0}(t)$ if and only if there exists $\alpha_0 \in T_{p_0}M$, $\beta_0 \in T_{v_0}S^1$ and $c \in \mathbb{R}$ such that the following equations are satisfied*

$$\begin{cases} d_p \exp_{p_0}(t, v_0)(\alpha_0) + d_v \exp_{p_0}(t, v_0)(\beta_0) = 0, \\ d_p \exp_{p_0}(t+s, v_0)(\alpha_0) + d_v \exp_{p_0}(t+s, v_0)(\beta_0) + c\dot{\gamma}_{p_0,v_0}(s+t) = 0. \end{cases} \tag{3.3}$$

*Proof.* Recall that $p_2$ is conjugate to $p_1$ if only if there exists $c$ and $\beta_1$ such that

$$d_{s,v} \exp_{p_1}(s, v_1)(c, \beta_1) = 0 \Rightarrow d_v \exp_{p_1}(s, v_1)(\beta_1) = -c\dot{\gamma}_{p_1,v_1}(s),$$

This is equivalent to say in equation (3.1) there exist $\beta_1$ and $c$ so that

(a) $\alpha_1 = 0$,

(b) $\alpha_2 = d_p \exp_{p_1}(s, v_1)(\alpha_1) + d_v \exp_{p_1}(s, v_1)(\beta_1) = -c\dot{\gamma}_{p_1,v_1}(s)$.

Since $J_{(p_0,v_0,t)}$ and $J_{(p_0,v_0,t+s)}$ are invertible with $\phi_t$ as a diffeomorphism, then the statement above is equivalent to that there exists $\alpha_0$, $\beta_0$ and $c \in \mathbb{R}$ such that

$$\begin{aligned} \alpha_1 &= d_p \exp_{p_0}(t, v_0)(\alpha_0) + d_v \exp_{p_0}(t, v_0)(\beta_0) = 0, \\ \alpha_2 &= d_p \exp_{p_0}(t+s, v_0)(\alpha_0) + d_v \exp_{p_0}(t+s, v_0)(\beta_0) = -c\dot{\gamma}_{p_0,v_0}(s+t). \end{aligned}$$

□

Proposition 7 indicates that we can define the **Jacobi field** along $\gamma_{p_0,v_0}$ at time $t$

$$J_{(p_0,v_0)}(t) = d_p \exp_{p_0}(t, v_0)(\alpha) + d_v \exp_{p_0}(t, v_0)(\beta) + (c_1 + c_2 t)\dot{\gamma}_{p_0,v_0}(t). \tag{3.4}$$

It is a smooth vector field and can be regarded as the variation field along $\gamma_{p_0,v_0}(t)$. To understand the last term $t\dot{\gamma}_{p_0,v_0}(t)$, it is convenient for us to extend $\Gamma$ such that for any $V = rv \in TM \setminus 0$, with $r > 0, v \in S^1$, there exists a unique curve $\gamma_{x,V}(t)$

belonging to $\Gamma$. Define $\gamma_{x,V}(t) = \gamma_{x,v}(rt)$, which corresponds to our definition of the exponential map. Notice $\gamma_{x,V}(t)$ satisfies the following ODE

$$D_t \dot{\gamma}_{x,V}(t) = |V| \lambda(\gamma_{x,V}(t), \frac{1}{|V|} \dot{\gamma}_{x,V}(t)) \dot{\gamma}_{x,V}^{\perp}(t),$$

which means we can extend $\lambda$ to $C^\infty(TM \setminus 0)$.

The following result is also proved in [56].

**Proposition 8.** *A point $p_2 = \gamma_{p_0,v_0}(t_2)$ is conjugate to $p_1 = \gamma_{p_0,v_0}(t_1)$ if and only if there exists a nonvanishing Jacobi field $J(t)$ such that $J(t_1) = J(t_2) = 0$.*

*Proof.* If $p_2 = \gamma_{p_0,v_0}(t_2)$ is conjugate to $p_1 = \gamma_{p_0,v_0}(t_1)$, then there exists a nonzero $J_{(p_0,v_0)}(t)$ such that it vanishes at $t_1$ and $t_2$. This can be done by letting $c_1 + c_2 t = \lambda \frac{t-t_1}{t_2-t_1}$, according to Proposition 7.

Conversely, suppose there exists a nonzero $J_{(p_0,v_0)}(t)$ satisfying $J_{(p_0,v_0)}(t_1) = J_{(p_0,v_0)}(t_2) = 0$. More precisely, we have the following equations

$$\begin{aligned} d_p \exp_{p_0}(t_1, v_0)(\alpha_0) + d_v \exp_{p_0}(t_1, v_0)(\beta_0) + \lambda_1 \dot{\gamma}_{p_0,v_0}(t_1) = 0, \\ d_p \exp_{p_0}(t_2, v_0)(\alpha_0) + d_v \exp_{p_0}(t_2, v_0)(\beta_0) + \lambda_2 \dot{\gamma}_{p_0,v_0}(t_2) = 0, \end{aligned} \tag{3.5}$$

where $\lambda_1 = c_1 + c_2 t_1$, $\lambda_2 = c_1 + c_2 t_2$. It suffices to find $\alpha_0', \beta_0'$ such that $d_p \exp_{p_0}(t, v_0)(\alpha_0') + d_v \exp_{p_0}(t, v_0)(\beta_0') = \lambda_1 \dot{\gamma}_{p_0,v_0}(t)$, which is proved by Claim 3. In this way, the term of $\dot{\gamma}_{p_0,v_0}(t_1)$ in the first equation is canceled. Therefore, we are in the same situation as in Proposition 7. □

**Claim 3.** *For any constant $c$, we can find $\alpha_0', \beta_0'$ such that $d_p \exp_{p_0}(t, v_0)(\alpha_0') + d_v \exp_{p_0}(t, v_0)(\beta_0') = c \dot{\gamma}_{p_0,v_0}(t)$.*

*Proof.* By [54], consider the infinitesimal generator $F(x, v)$ of the flow $\phi_t$. We define

$$Y(t) = c(d\phi_{-t})(F(\phi_t(x, v))).$$

Differentiating both sides with respect to $t$ implies

$$\dot{Y}(t) = c[F, F] = 0.$$

Thus $Y(t)$ is a constant vector field. By denoting it as $Y = (\alpha_0', \beta_0')$, we have $d\phi_t(Y) = cF$. Suppose $\pi$ is the projection from $SM$ to $M$. Applying $d\pi$ to $d\phi_t(Y)$ claims what we need. □

### 3.3 Microlocal analysis of the local problem

Define the integral transform along curves in $\Gamma$ as

$$I_w f(\gamma) = \int w(\gamma(t), \dot\gamma(t)) f(\gamma(t)) dt, \quad \gamma \in \Gamma.$$

It is an FIO of order $-\frac{1}{2}$ associated with a canonical relation $C$. To better describe the canonical relation, we parametrize the family of such curves near some fixed $\gamma \in \Gamma$. For fixed $\gamma$, there is a hypersurface $H$ such that $\gamma$ hits $H$ transversally at $p_0 = \gamma(0)$ and $v_0 = \dot\gamma(0)$. Suppose $H$ is locally given by $x = 0$ and has local coordinates $y$. Let $\eta$ be the parameterization of the direction $v$.

**Claim 1.** With parameterization of $\Gamma$, the Jacobi field along $\gamma$ can be simplified as

$$J_{(p_0,v_0)}(t) = \beta_1 \frac{\partial\gamma}{\partial y} + \beta_2 \frac{\partial\gamma}{\partial\eta} + c_1 \dot\gamma_{p_0,v_0}(t) + c_2 t \dot\gamma_{p_0,v_0}(t),$$

where $\beta_1, \beta_2, c_1, c_2 \in \mathbb{R}$.

*Proof.* We have shown in Claim 3 that there exists $\alpha, \beta, c$ such that

$$S(t) = d_p \exp_{p_0}(t, v_0)(\alpha) + d_v \exp_{p_0}(t, v_0)(\beta) + c\dot\gamma_{p_0,v_0}(t) = 0,$$

where $c$ is nonzero. We write $d_p \exp_{p_0}(t, v_0)(\alpha) = \alpha_1 \frac{\partial\gamma}{\partial y} + \alpha_2 \frac{\partial\gamma}{\partial x}$. It suffices to show that $\alpha_2 \neq 0$ and therefore $\frac{\partial\gamma}{\partial x}$ is the linear combination of $\frac{\partial\gamma}{\partial y}, \frac{\partial\gamma}{\partial\eta}, \dot\gamma$. Notice we have the initial condition $\frac{\partial\gamma}{\partial\eta}(0) = 0$. This implies $\alpha$ cannot be zero. Moreover, we have

$$0 \equiv \langle S(0), \dot\gamma^\perp(0)\rangle = \alpha_1 \langle \frac{\partial\gamma}{\partial y}, \dot\gamma^\perp(0)\rangle + \alpha_2 \langle \frac{\partial\gamma}{\partial x}, \dot\gamma^\perp(0)\rangle.$$

It follows that $\alpha_2 \neq 0$, otherwise we will have $\alpha_1 = 0$ which conflicts with $\alpha$ nonzero. Indeed, since $\gamma$ hit $H$ transversally, we alawys have $\langle \frac{\partial\gamma}{\partial y}(0), \dot\gamma^\perp(0)\rangle \neq 0$. □

Now we are in the same situation as in [17]. Suppose $x = \gamma_{y,\eta}(t)$ and $\xi$ is the dual variable. Then $(y, \eta, \hat y, \hat\eta, x, \xi) \in C$ if and only if there exists $t$ with $x = \gamma_{y,\eta}(t)$ and

$$\xi_i \dot\gamma^i = 0, \quad \xi_i \frac{\partial\gamma^i}{\partial y} = \hat y, \quad \xi_i \frac{\partial\gamma^i}{\partial\eta} = \hat\eta. \tag{3.6}$$

Inspired by [56], let $e_1(t) = \dot\gamma(t)^\perp, e_2(t) = \dot\gamma(t)$ be a moving frame. We have the corresponding dual basis $e^1(t), e^2(t)$. If we regard $\xi$ as a function of $t$, then the first condition requires $\xi = f(t)e^1(t)$ for some nonvanishing function $f(t)$. Suppose the vector field $\frac{\partial\gamma}{\partial y}$ and $\frac{\partial\gamma}{\partial\eta}$ have the following expansion with respect to $e_1(t)$ and $e_2(t)$

$$\frac{\partial\gamma}{\partial y} = a_1(t)e_1(t) + a_2(t)e_2(t), \quad \frac{\partial\gamma}{\partial\eta} = b_1(t)e_1(t) + b_2(t)e_2(t). \tag{3.7}$$

The second and third conditions in (3.6) implies

$$f(t)a_1(t) = \hat y, \quad f(t)b_1(t) = \hat\eta.$$

If there exists different $(x_1, \xi^1)$ and $(x_2, \xi^2)$ corresponding to the same $(y, \eta)$, then $\hat y b_1(t) - \hat\eta a_1(t) = 0$ is true for some $t = t_1, t_2$ with $t_1, t_2$.

Therefore we define the following vector field

$$c_0(y,\eta,t) = \hat y\frac{\partial\gamma}{\partial\eta} - \hat\eta\frac{\partial\gamma}{\partial\eta} = (\hat y b_1(t) - \hat\eta a_1(t))e_1(t) + (\hat y b_2(t) - \hat\eta a_2(t))e_2(t).$$

Notice $c_0(y,\eta,t_1) = (\hat y b_2(t_1) - \hat\eta a_2(t_1))e_2(t_1) \equiv \lambda_1\dot\gamma(t_1)$ and $c_0(y,\eta,t_2) = (\hat y b_2(t_2) - \hat\eta a_2(t_2))e_2(t_2) \equiv \lambda_2\dot\gamma(t_2)$. Then we can define the following Jacobi field

$$c(y,\eta,t) = \hat y\frac{\partial\gamma}{\partial\eta} - \hat\eta\frac{\partial\gamma}{\partial\eta} - (\lambda_1\frac{t-t_2}{t_1-t_2} + \lambda_2\frac{t-t_1}{t_2-t_1})\dot\gamma(t).$$

Notice $c(y,\eta,t_1) = c(y,\eta,t_2) = 0$, which implies $p_2$ is conjugate to $p_1$.

Conversely, if we have $p_2$ is the conjugate point to $p_1$, then there is a nonzero Jacobi field $J_{(p_0,v_0)}(t)$ so that $J_{(p_0,v_0)}(t_1) = J_{(p_0,v_0)}(t_2) = 0$, more precisely, we have

$$\beta_1\frac{\partial\gamma}{\partial y} + \beta_2\frac{\partial\gamma}{\partial\eta} + c_1\dot\gamma(t) + c_2 t\dot\gamma(t) = 0, \text{ for } t = t_1 < t_2.$$

The projection on $\dot\gamma^\perp(t)$ shows

$$\begin{cases} \beta_1 a_1(t_1) + \beta_2 b_1(t_1) = 0 \\ \beta_1 a_1(t_2) + \beta_2 b_1(t_2) = 0 \end{cases} \iff \begin{bmatrix} a_1(t_1) & b_1(t_1) \\ a_1(t_2) & b_1(t_2) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0. \tag{3.8}$$

Notice $\beta_1, \beta_2$ cannot be both zero, otherwise we have tangent Jocabi field. It implies the matrix in the equation above is singular. To show that (3.6) is true for $t_1, t_2$, we need to find $\xi^1 = f(t_1)e^1(t_1)$ and $\xi^2 = f(t_2)e^1(t_2)$ with nonzero $f(t_1), f(t_2)$ satisfying

$$\begin{cases} f(t_1)a_1(t_1) = f(t_2)a_1(t_2) \\ f(t_1)b_1(t_1) = f(t_2)b_1(t_2) \end{cases} \iff \begin{bmatrix} a_1(t_1) & -a_1(t_2) \\ b_1(t_1) & -b_1(t_2) \end{bmatrix} \begin{bmatrix} f(t_1) \\ f(t_2) \end{bmatrix} = 0. \tag{3.9}$$

First on can see this linear system have nonzero solution, since the matrix is singular by (3.8). To show $f(t_1), f(t_2)$ are nonzero, we need the following claim.

**Claim 4.** *Any nonzero solution to the linear system (3.9) satisfies $f(t_1) \neq 0$ and $f(t_2) \neq 0$.*

*Proof.* We prove it by contradiction. Without loss of generality, we assume $f(t_1) = 0$. The analysis above shows in this case $f(t_2) \neq 0$. Therefore, $a_1(t_2) = b_1(t_2) = 0$. By [56], $a_1(t)$ and $b_1(t)$ satisfy the ODE

$$\ddot{u} + q\dot{u} + ku = 0,$$

where $q(t), k(t)$ are smooth function given there. Then the Wronskian $W(t) = a_1(t)\dot{b_1}(t) - b_1(t)\dot{a_1}(t)$ vanishes for all $t$, since $W(t)$ is independent of $t$ with $W(t_2) = 0$. It implies $a_1(t), b_1(t)$ are linearly dependent. However, this conflicts with the initial conditions $\langle \frac{\partial \gamma}{\partial y}(0), \dot{\gamma}^{\perp}(0)\rangle \neq 0$ and $\frac{\partial \gamma}{\partial \eta}(0) = 0$, since $a_1(t) = \langle \frac{\partial \gamma}{\partial y}, \dot{\gamma}(t)^{\perp}\rangle$ and $b_1(t) = \langle \frac{\partial \gamma}{\partial \eta}, \dot{\gamma}(t)^{\perp}\rangle$. □

Therefore, the canonical relation can be written as

$$C = \{(y, \eta, \lambda a_1(t, y, \eta), \lambda b_1(t, y, \eta), \gamma(t, y, \eta), \lambda\dot{\gamma}^{\perp}(t, y, \eta)),\ (y, \eta) \in BH,\ \lambda \neq 0,\ t \in \mathbb{R}\},$$

where $a_1(t, y, \eta)$ and $b_1(t, y, \eta)$ are projections of $\frac{\partial \gamma(t,y,\eta)}{\partial y}$ and $\frac{\partial \gamma(t,y,\eta)}{\partial \eta}$ onto $\dot{\gamma}^{\perp}(t, y, \eta)$. It is a graph and can be parameterized by $(y, \eta, t, \lambda)$ with $\dim C = 4$.

The group action of $\mathbb{R}$ on $M$ by $\phi_t$ is free and proper. Let $\mathcal{M}$ be the space of curves as we defined before, then $\mathcal{M} = SM/\phi_t$ is a smooth manifold of dimension $2n-2 = 2$. The point-curve relation $Z_0$ is a smooth manifold of dimension $2n-1 = 3$. This is from the coordinate charts and rank of Jacobian matrix.

**Proposition 9.** *The natural projection $\pi_M : C \to T^*M \setminus 0$ is a diffeomorphism.*

*Proof.* For each $(x, \xi)$, with the assumption, there is a unique curve $\gamma$ passing $x$ at time $t$ and conormal to $\xi$. Suppose $\gamma$ hits $H$ transversally with direction parameterized by $\eta$ at time 0. Then $(y, \eta)$ is given by the flow $\phi_{-t}(x, \xi)$ composed with restriction and projection. Thus, $(y, \eta)$ depends on $(x, \xi)$ smoothly and $(\hat{y}, \hat{\eta}) = (\lambda a_1(t, y, \eta), \lambda b_1(t, y, \eta))$ with $\lambda = |\xi|$ also depends on $(x, \xi)$ in a smooth way. □

The projection $\pi_{\mathcal{M}} : C \to T^*\mathcal{M} \setminus 0$ is a local diffeomorphism. We have $\dim C = \dim T^*\mathcal{M} = 4$. The differential $d\pi_{\mathcal{M}}$ has a nonzero determinant, since the Wronskian is nonzero. From the analysis above, $\pi_{\mathcal{M}}$ is a global diffeomorphism if and only if there are no conjugate points.

Define $\mathcal{C}(x,\xi) \equiv \pi_{\mathcal{M}} \circ \pi_M^{-1}(x,\xi) = (y,\eta,\lambda a_1,\lambda b_1)$. We have the following results.

**Theorem 6.** *We have $\mathcal{C}$ is a local diffeomorphism and $\mathcal{C}(p_1,\xi^1) = \mathcal{C}(p_2,\xi^2)$ if and only if there is a curve $\gamma(t,y,\eta)$ joining $p_1$ at $t_1$ and $p_2$ at $t_2$, with $t_2 > t_1$, such that*

*(a) $p_2$ is the conjugate point to $p_1$.*

*(b) $\xi^1 = \lambda_1 \dot{\gamma}^{\perp}(t_1)$ and $\xi^2 = \lambda_2 \dot{\gamma}^{\perp}(t_2)$ with $\lambda_1, \lambda_2$ solving system (3.9).*

Recall the integral transform is defined as

$$I_w f(\gamma) = \int w(\gamma(t),\dot{\gamma}(t)) f(\gamma(t)) dt, \gamma \in \mathcal{M}.$$

**Theorem 7.** *The transform $I_w$ is an FIO of order $-\frac{1}{2}$ associated with the canonical relation $C$. If there are conjugate points, a theorem of cancellation of singularities similar to Theorem 3 can be established; if there are no conjugate points along curves in $\Gamma$, then $I_w$ is elliptic at $(x,\xi)$ if and only if $w(x,v) \neq 0$ for $v$ such that $v^{\perp}$ is collinear with $\xi$.*

# 4. THE CONE TRANSFORM

## 4.1 Introduction

This chapter is a slightly modified version of the previous work in [3]. Let $c(u,\beta,\phi)$ be a circular cone in $\mathbb{R}^3$ with vertex $u$, central axis $\beta$, and opening angle $\phi$, as shown in Figure 4.1. We study the weighted cone transform

$$I_\kappa f(u,\beta,\phi) = \int_{c(u,\beta,\phi)} \kappa f \, \mathrm{d}S, \quad u \in \mathcal{S}, \beta \in S^2, \phi \in (\epsilon, \pi/2-\epsilon)$$

of distributions supported in a domain $M$ in $\mathbb{R}^3$ over cones of which the vertexes are restricted to a smooth surface $\mathcal{S}$, where $\kappa$ is a smooth weight, $S$ is the Euclidean measure on the cone surface, and $\epsilon$ is a small nonnegative number. The goal of this work is to study the microlocal invertibility of this transform.

The cone transform arises in Compton camera imaging dating back to [21–23]. A Compton camera is composed of two detectors: a scatter and an absorber. Both detectors are position and energy sensitive. When incoming gamma photons hit the camera, they have Compton scattering at various angles in the first detector and are completely absorbed in the second one. Photons can be traced back to the surface of cones.

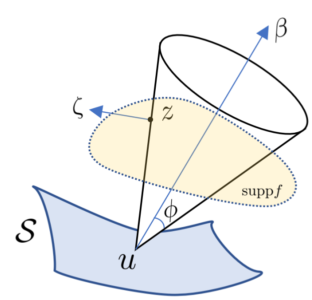



Figure 4.1. The cone $c(u,\beta,\phi)$ with $(z,\zeta)$ conormal to it

From the setting of Compton camera, there are three important points about the transform to be noted here. Firstly, it is natural and unavoidable to have a weight function $\kappa$. Since the integral over the cone is a superposition of the line integrals, even without attenuation, there should be weight $a/|z-u|$ with a smooth function $a$ depending on the cone. Secondly, the probability of scattering angles (i.e. the opening angle $\phi$) governed by the Klein-Nishina distribution excludes angles that are too close to 0 or $\pi/2$. For example, in [23] the scattering angles range from 5° to 75° at 511 keV incident energy and one obtains 5° angular resolution by requiring the energy resolution to be 3.8-18.9 keV. On the other hand, when the scattering angle equals zero, the cone transform reduces to the weighted X-ray transform over a set of rays and when the scattering angle equals $\pi/2$, it reduces to the weighted Radon transform. In both cases the inversion are easier. The limits of scattering angles can be modeled by choosing $\kappa$ supported in such intervals w.r.t. $\phi$. Thirdly, the detectors are located outside the region of interest $M$ so it is reasonable to require the vertexes are restricted to a smooth surface $\mathcal{S}$ that does not intersect $M$.

### 4.1.1 Previous works and main results

A lot of works have been done on the inversion of the cone transform and some of them are [14, 24–35, 57, 58]. For a more complete and detailed list of previous works, see [57]. Some of the inversion formula are for special geometries, or they consider the opening angle $\phi \in [0, \pi]$, or constant weight, or use transform over cones whose vertexes can be everywhere. Most recently, [58] considers a polynomial weight function and computes the normal operator of the cone transform over all cones whose vertexes are in $\mathbb{R}^n$ to show it is a ΨDO. It gives certain integral formula for the amplitude of the normal operator but it is not so clear how to resolve the singularities in the denominator of the integrand to obtain a smooth amplitude. There are also works on a different setting from Compton scattering tomography with transmission modalities, see [59, 60] and their references.

For the cone transform that we define, it is harder to analyze the Schwartz kernel of the normal operator and the normal operator may fail to be a $\Psi$DO in some microlocal region, if Tuy's condition is not satisfied. Instead, we use the clean intersection calculus of Fourier Integral Operators (FIOs, for definition see [12, Def. 25.1.1]) in [12,61] to show under which conditions the microlocalized normal operator is a $\Psi$DO of order $-2$ and it is elliptic with certain nonvanishing assumptions of the weight, see Proposition 13 and 14. This approach was proposed by Guillemin in [62] for the generalized Radon transform. One difficulty to apply Hörmander's clean intersection composition to our transform is that the composition fails to be proper. For this purpose, we modify the clean composition theorem slightly by assuming the microsupport of composed FIO is conically compact instead of the properness condition, following András Vasy's suggestion. Our first result describes the singularities of $f$ that we can recover from the data $I_\kappa f$ in a stable way. We note that this is the intrinsic property of the transform itself no matter what inversion algorithm is used. More specifically, let

$$\mathcal{C}(z_0, \zeta^0) = \{(u, \beta, \phi) | (z_0 - u) \cdot \beta - |z_0 - u| \cos\phi = 0,\ \zeta^0 \cdot (z_0 - u) = 0\}$$

be the set of all cones that are conormal to fixed a $(z_0, \zeta^0) \in T^*M$. Note its dimension equals 2. We can only expect to recover singularities conormal to the cones of which the weight is nonvanishing there. The definition of accessible singularities and Tuy's condition can be found in Section 4.2.

**Theorem 8.** *Suppose $(z_0, \zeta^0)$ is accessible. If $\kappa(u_0, \beta_0, \phi_0, z_0) \neq 0$ for some $(u_0, \beta_0, \phi_0) \in \mathcal{C}(z_0, \zeta^0)$, then $(z_0, \zeta^0)$ is recoverable from local data. In particular, if $I_\kappa f$ is smooth near $(u_0, \beta_0, \phi_0)$, then $f$ is smooth near $(z_0, \zeta^0)$.*

This theorem shows that accessible singularities are recoverable with some nonvanishing assumptions on $\kappa$. The recovery comes from the fact that with the assumptions the normal operator in certain microlocal region is an elliptic $\Psi$DO and therefore it is microlocally invertible. Recall $M$ is a domain in $\mathbb{R}^3$ and let $\mathcal{M} = \mathcal{S} \times S^2 \times (\epsilon, \pi/2 - \epsilon)$

be the family of cones, where $\epsilon$ is a small positive number. We also show the mapping properties of $I_\kappa, I_\kappa^*$ and a microlocal stability estimate when all singularities are accessible in the following theorem.

**Theorem 9.** *Suppose $\mathcal{S}$ satisfy Tuy's condition w.r.t. $M$. Then for any $s \in \mathbb{R}$, we have $I_\kappa : H^s_{loc}(M) \to H^{s+1}_{loc}(\mathcal{M})$, $\quad I^*_\kappa : H^s_{loc}(\mathcal{M}) \to H^{s+1}_{loc}(M)$ are continuous. For $f \in H^s(M), l \in \mathbb{R}$, there exists constant $C_1, C_2, C_{s,l}$ such that*

$$C_1\|f\|_{H^s(M)} - C_{s,l}\|f\|_{H^l(M)} \le \|I_\kappa f\|_{H^{s+1}(\mathcal{M})} \le C_2\|f\|_{H^s(M)}.$$

For surfaces satisfying Tuy's condition, see Example 13. We should mention that assuming $\mathcal{S}, \kappa$ are analytic, one might be able to show the injectivity of $I_\kappa$ by applying the analytic microlocal analysis, see [63]. Once we have the transform $I_\kappa$ is injective on some closed subspace of $H^s(M)$, we can show the microlocal stability actually implies the stability estimate, i.e. we do not have the term $C_{s,l}\|f\|_{H^l(M)}$ in above inequality.

This chapter is structured as follows. Section 4.2 introduces the notations and definitions. Section 4.3 proves that $I_\kappa$ is an FIO and shows in which cases the projection $\pi_{\mathcal{M}}$ is an injective immersion. Section 4.4 presents a slightly modified version of the clean composition theorem. Section 4.5 shows with certain positive assumptions on the weight $\kappa$, the normal operator is an elliptic ΨDO at accessible singularities, which implies Theorem 1. Section 4.6 contains the proof of Theorem 2. Section 4.7 studies the weighted cone transform over cones whose vertexes are restricted to a smooth curve and the opening angle is fixed.

## 4.2 Preliminaries

Here we introduce some notations to be used in the following sections. As mentioned before, let $M$ be an open domain in $\mathbb{R}^3$ and $f \in \mathcal{E}'(M)$ be a distribution with compact support inside $M$. Suppose $\mathcal{S}$ is a smooth regular surface without boundary where the vertexes of cones are located. Throughout this work, we always assume $M$

and $\mathcal{S}$ do not intersect. Let $0 \leq \epsilon < \pi/4$ and $\mathcal{M} = \mathcal{S} \times S^2 \times (\epsilon, \pi/2 - \epsilon)$ be the family of cones that we consider. Notice $\mathcal{M}$ is a smooth manifold.

Since both $u$ and $\beta$ are located in smooth surfaces, we consider them in local coordinates. Suppose $\mathcal{S}$ has the regular parameterization locally given by $u = u(v^1, v^2)$. Let $J_1 = (\frac{\partial u}{\partial v^1}, \frac{\partial u}{\partial v^2}) \equiv (r_1, r_2)$ be the Jacobian matrix. Notice $r_1, r_2$ form a basis for the tangent space at $u$ of $\mathcal{S}$. Suppose $\beta \in S^2$ is locally parameterized by $\beta = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ for $\theta \in (0, \pi)$. Let $J_2 = (\frac{\partial \beta}{\partial \theta}, \frac{\partial \beta}{\partial \psi}) \equiv (\beta_1, \sin\theta\beta_2)$, where $\beta_1 = (\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta)$ and $\beta_2 = (-\sin\psi, \cos\psi, 0)$. Notice $\beta, \beta_1, \beta_2$ form an orthonormal basis in $\mathbb{R}^3$.

**Definition 5.** We say $(z, \zeta) \in T^*M$ is *accessible* by $\mathcal{S}$ if the hyperplane conormal to $(z, \zeta)$ has a non-tangential intersection with $\mathcal{S}$.

If we donate the set of all $(z, \zeta)$ in $T^*M$ that are accessible by $D$, then $D$ is an open set. Observe that $(z, \zeta)$ is accessible implies that exists a cone that it is conormal to, and additionally for any qualified cone $(u, \beta, \phi)$, the covector $\zeta$ cannot be perpendicular to the tangent space at $u$ in $\mathcal{S}$. By $(z, \zeta)$ is conormal to a cone $c(u, \beta, \phi)$, we mean $z$ is on the cone surface and $\zeta$ is conormal to the tangent plane of the cone at $z$.

**Definition 6.** If every $(z, \zeta) \in T^*M$ is accessible, then we say the surface $\mathcal{S}$ satisfying the *Tuy's condition* with respect to $M$.

This coincides with the definition in [35] that Tuy's condition is satisfied if any hyperplane intersecting $M$ has a non-tangential intersection with $\mathcal{S}$.

### 4.3 $I_\kappa$ as an FIO

The weighted cone transform $I_\kappa$ can be written as

$$I_\kappa f(u, \beta, \phi) = \int_{\mathbb{R}^3} \kappa(u, \beta, \phi, z)\delta((z - u) \cdot \beta - |z - u|\cos\phi) f(z)\, \mathrm{d}z,$$
$$u \in \mathcal{S}, \beta \in S^2, \phi \in (\epsilon, \pi/2 - \epsilon)$$

where the distribution $\delta$ has a nonzero factor but we can regard the factor as a part of the weight.

**Proposition 10.** *The weighted cone transform $I_\kappa$ is an FIO of order $-3/2$ associated with the canonical relation*

$$C_I = \{(u, \beta, \phi, \underbrace{{}^{\mathrm{t}}J_1\zeta}_{\hat{u}}, \underbrace{\lambda\,{}^{\mathrm{t}}J_2(z-u)}_{\hat{\beta}}, \underbrace{\lambda|z-u|\sin\phi}_{\hat{\phi}}, z, \zeta),\ \varphi = 0, \lambda \neq 0\},$$

*where $\varphi(u, \beta, \phi, z) = (z-u)\cdot\beta - |z-u|\cos\phi$ and $\zeta = -\lambda(\beta - \frac{z-u}{|z-u|}\cos\phi)$; the vertex $u = u(v^1, v^2)$ locally and $J_1 = \frac{\partial u}{\partial(v^1,v^2)}$ is the Jacobian matrix; the unit vector $\beta = \beta(\theta, \psi)$ is locally parameterized in the spherical coordinates and $J_2 = \frac{\partial \beta}{\partial(\theta,\psi)}$ is the Jacobian matrix.*

*Proof.* We rewrite $I_\kappa$ as the oscillatory integral

$$I_\kappa f(u, \beta, \phi) = (2\pi)^{-1}\int_{\mathbb{R}^3}\int_{\mathbb{R}} e^{i\lambda\varphi(u,\beta,\phi,z)}\kappa(u, \beta, \phi, z) f(z)\,\mathrm{d}\lambda\,\mathrm{d}z,$$

where $\varphi(u, \beta, \phi, z)$ is defined as above. Notice that $\mathrm{d}\varphi \neq 0$. Its Schwartz kernel is a Lagrangian distribution conormal to the characteristics manifold

$$Z = \{(u, \beta, \phi, z) \in \mathcal{M} \times M,\ \varphi = 0\}.$$

Then the conormal bundle is

$$N^*Z = \{(u, \beta, \phi, z, \lambda\,\mathrm{d}_u\varphi, \lambda\,\mathrm{d}_\beta\varphi, \lambda\,\mathrm{d}_\phi\varphi, \lambda\,\mathrm{d}_z\varphi),\ \varphi = 0\}.$$

where we abuse the notation $\mathrm{d}_u$, $\mathrm{d}_\beta$ to denote the differential w.r.t. the parameterization of $u$, $\beta$ and we have

$$\mathrm{d}_z\varphi = \beta - \frac{z-u}{|z-u|}\cos\phi, \quad \mathrm{d}_u\varphi = -\,{}^{\mathrm{t}}J_1\,\mathrm{d}_z\varphi, \tag{4.1}$$

$$\mathrm{d}_\beta\varphi = {}^{\mathrm{t}}J_2(z-u), \quad \mathrm{d}_\phi\varphi = |z-u|\sin\phi. \tag{4.2}$$

Let

$$\Lambda = \{(u, \beta, \phi, z, \lambda\,\mathrm{d}_u\varphi, \lambda\,\mathrm{d}_\beta\varphi, \lambda\,\mathrm{d}_\phi\varphi, \lambda\,\mathrm{d}_z\varphi),\ \varphi = 0, \lambda \neq 0\}.$$

One can show this is a closed conic Lagrangian submanifold of $T^*(\mathcal{M} \times M) \setminus 0$.

Additionally, since we always have $M$ is away from $\mathcal{S}$, from (4.1)(4.2) we get that

$$\mathrm{d}_z\varphi = 0 \Longleftrightarrow \mathrm{d}_u\varphi = \mathrm{d}_\beta\varphi = \mathrm{d}_\phi\varphi = 0 \Longleftrightarrow \sin\phi = 0, \quad \text{when } \varphi = 0.$$

Since $\phi \in (\epsilon, \frac{\pi}{2} - \epsilon)$, therefore the Lagrangian satisfies

$$\Lambda \subset (T^*\mathcal{M} \setminus 0) \times (T^*M \setminus 0).$$

Then the canonical relation $C_I$ is given by the twisted Lagrangian. The order is given by $0 + \frac{1}{2} \times 1 - \frac{1}{4} \times (3+5) = -\frac{3}{2}$ by [12, Proposition 25.1.5]. □

We have $\dim T^*\mathcal{M} = 10$, $\dim T^*M = 6$, and $\dim C_I = 8$. Let $\pi_{\mathcal{M}}$, $\pi_M$ be the natural projection of $C_I$ to $T^*\mathcal{M}, T^*M$ respectively. Note that in the cone transform, neither of these projections can be local diffeomorphism. The following proposition describes the mapping properties of them.

**Proposition 11.** *Recall $D$ is the set of all accessible covectors $(z,\zeta)$ in $T^*M$. Define $C_I(z,\zeta) = \pi_{\mathcal{M}}\pi_M^{-1}(z,\zeta)$ as the set of covectors related with $(z,\zeta)$ by $C_I$. Let $H_{(z,\zeta)}$ be the hyperplane conormal to $(z,\zeta)$. We have the following statements.*

*(a) For each $(z,\zeta) \in D$, the set $C_I(z,\zeta)$ is a surface that can be parametrized by $(u,\phi) \in U_{(z,\zeta)} \times (\epsilon, \pi/2-\epsilon)$, where $U_{(z,\zeta)} = \mathcal{S} \cap H_{(z,\zeta)}$, see Figure 4.2 as examples.*

*(b) For each $(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi}) \in \pi_{\mathcal{M}}\pi_M^{-1}(D)$, there is one unique solution $(z,\zeta)$ for the equation $C_I(z,\zeta) = (u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi})$, which is given by (4.3) and below.*

*(c) The projection $\pi_{\mathcal{M}}$ restricted to $\pi_{\mathcal{M}}^{-1}(D)$ is an injective immersion. In particular, if $\mathcal{S}$ satisfies Tuy's condition, then $\pi_{\mathcal{M}}$ itself is an injective immersion.*

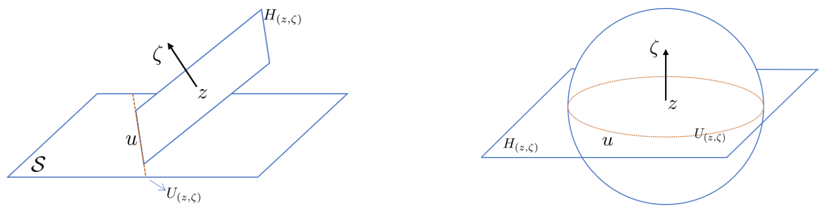
Before the proof, the following are two examples to illustrate its first statement.

**Example 12.** Suppose $\mathcal{S}$ is a plane. W.L.O.G., assume $\mathcal{S} = \{u^3 = 0\}$. Let $z = (z^1, z^2, z^3), \zeta = (\zeta_1, \zeta_2, \zeta_3)$. If $\zeta_1 = \zeta_2 = 0$, then there are no cones that are conormal to $(z,\zeta)$. Here we assume $\zeta_2 \neq 0$. We solve

$$(z-u)\cdot\zeta = \zeta_1(z^1 - v^1) + \zeta_2(z^2 - v^2) + \zeta_3 z^3 = 0$$

to get $v^2 = \frac{1}{\zeta_2}(z\cdot\zeta - v^1\zeta_1)$. Thus, for $(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi}) \in C_I(z,\zeta)$, we have

$$u = (v^1, \frac{1}{\zeta_2}(z\cdot\zeta - v^1\zeta_1), 0), \qquad \beta = \cos\phi\frac{z-u}{|z-u|} - \sin\phi\frac{\zeta}{|\zeta|},$$
$$\phi, v^1 \text{ arbitrary}, \qquad \lambda = \frac{1}{\sin\phi}|\zeta|.$$




(a) (b)

Figure 4.2. Example 1 and 2.

**Example 13.** Suppose $\mathcal{S}$ is a unit sphere. Note in this case $\mathcal{S}$ satisfies Tuy's condition w.r.t. any inside domain away from it. It can be covered by six coordinates charts. We consider the special case that $z = (0,0,0)$ and $\zeta = (0,0,1)$. Then

$$U_{(z,\zeta)} = \{(v^1, v^2, 0),\ v^1v^1 + v^2v^2 = 1\}.$$

In a small neighborhood of $U_{(z,\zeta)}$, the vertex $u$ can be parameterized by one of the following

$$(v^1, \sqrt{1-(v^1)^2-(v^3)^2}, v^3), \quad (v^1, -\sqrt{1-(v^1)^2-(v^3)^2}, v^3),$$
$$(\sqrt{1-(v^2)^2-(v^3)^2}, v^2, v^3), \quad (-\sqrt{1-(v^2)^2-(v^3)^2}, v^2, v^3).$$

Notice that $\partial_{v^3} u\cdot\zeta \neq 0$ and therefore $U_{(z,\zeta)}$ can be parameterized by $v^1$ or $v^2$ from the proof above. This can also be seen from $U_{(z,\zeta)} = \{(v^1, v^2, 0),\ v^1v^1 + v^2v^2 = 1\}$ itself.

*Proof of Proposition 4.* For (a), given $(z,\zeta)$, we are going to find out all possible solutions of $(u,\beta,\phi,\lambda)$ from the canonical relation in Proposition 10. We have some freedom to choose $u \in \mathcal{S}$, but with $\zeta = -\lambda(\beta - \frac{z-u}{|z-u|}\cos\phi)$, the vector $(z-u)$

must be conormal to $\zeta$. This coincides with the fact that the singularity $(z,\zeta)$ can only be possibly detected by the cones that it is conormal to. In other words, the possible choice of $u$ is the set $U_{(z,\zeta)}$. Indeed, the vertex $u$ should satisfy the equation $g(z,\zeta,u)=0$, where $g=(z-u)\cdot\zeta$. The Jacobian matrix is listed in the following,

$$\begin{array}{c|cccc|} & \partial_z & \partial_\zeta & \partial_{v^1} & \partial_{v^2} \\ \hline g & {}^{\mathrm{t}}\zeta & {}^{\mathrm{t}}(z-u) & -r_1\cdot\zeta & -r_2\cdot\zeta \\ \hline \end{array}$$

Figure 4.3. The Jacobian matrix of $g(z,\zeta,u)$.

with $r_1=\partial_{v^1}u$, $r_2=\partial_{v^2}u$. Here $r_1,r_2$ form a basis of the tangent space $T_u\mathcal{S}$. Since $(z,\zeta)$ is accessible, the inner products $r_1\cdot\zeta$ and $r_2\cdot\zeta$ cannot vanish at the same time; otherwise, the covector $\zeta$ is normal to $\mathcal{S}$ at $u$. We simply assume $r_2\cdot\zeta\neq 0$ in a small neighborhood of fixed $u_0\in U_{(z,\zeta)}$. In this neighborhood, the derivative $\partial_{v^2}g\neq 0$. Applying implicit function theorem, we get $v^2$ is a smooth function of $z,\zeta,v^1$ near $u_0$. Locally $U_{(z,\zeta)}$ can be parameterized by $v^1$.

Next, with $u_0$ given, by choosing $\phi$, the axis $\beta$ can be determined by $\beta=\cos\phi\frac{z-u}{|z-u|}-\sin\phi\frac{\zeta}{|\zeta|}$ and $\lambda=\frac{1}{\sin\phi}|\zeta|$. From Proposition 10,

$$\hat{u}={}^{\mathrm{t}}J_1(v_1,v_2)\zeta,\quad \hat{\beta}=\lambda\,{}^{\mathrm{t}}J_2(\beta)(z-u),\quad \hat{\phi}=\lambda|z-u|\sin\phi.$$

These are all smooth functions of $z,\zeta,u,\beta,\phi,\lambda$ and therefore smooth functions of $z,\zeta,v^1,\phi$. The map $(v^1,\phi)\mapsto(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi})\in C_I(z,\zeta)$ is an immersion. Thus, $(v^1,\phi)$ is a local parameterization of $C_I(z,\zeta)$. This proves statement (a). For (b), to recover $(z,\zeta)$ from given $(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi})$, we are solving the following system of equations. Recall $(z-u)/|z-u|=m\in S^2$. We have $\hat{\phi}$ is always nonzero and $\hat{u}$ is nonzero with the assumption that $(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi})\in\pi_{\mathcal{M}}\pi_M^{-1}(D)$. Indeed, $\hat{u}$ vanishes if and only if ${}^{\mathrm{t}}J_1(\beta-m\cos\phi)=(r_1\cdot(\beta-m\cos\phi),r_2\cdot(\beta-m\cos\phi))=0 \implies (r_1\cdot\zeta,r_2\cdot\zeta)=0$. As we stated before, this contradicts with the assumption.

We are solving the system. Divided by $|z-u|$ and $\lambda|z-u|$ respectively, the equations (**??**) and (2.4) give us the projection of $m$ along $\beta$, $\beta_1$, and $\sin\theta\beta_2$. These vectors form a orthogonal basis in $\mathbb{R}^3$ and we can get $m$ from the projection. Plugging back $m$ into equation (**??**), we can solve $\lambda$. Thus, when $\theta \neq 0$ or $\pi$

$$z = u + \frac{\hat{\phi}}{\lambda \sin \phi} m \qquad\qquad \zeta = -\lambda(\beta - \cos \phi m), \tag{4.3}$$

where

$$m = \cos \phi \beta + \frac{\sin \phi}{\hat{\phi}} (\hat{\beta}_1 \beta_1 + \frac{1}{\sin \theta} \hat{\beta}_2 \beta_2) \quad \lambda = \frac{|\hat{u}| \operatorname{sgn} \hat{\phi}}{|\,{}^{\mathrm{t}}J_1 (\beta - m \cos \phi)|}.$$

When $\theta = 0$ or $\pi$, this argument still holds if we choose another regular parameterization of $\beta$ there.

For (c), first we prove $\pi_{\mathcal{M}} : (u, \beta, \phi, \hat{u}, \hat{\beta}, \hat{\phi}, z, \zeta) \mapsto (u, \beta, \phi, \hat{u}, \hat{\beta}, \hat{\phi})$ is an immersion. It suffices to prove that $\operatorname{rank}(\mathrm{d}\pi_{\mathcal{M}}) = \dim C_I = 8$. The canonical relation $C_I$ has a parametrization $(u, \beta, z, \lambda)$, which implies $\dim C_I = 8$. Indeed, we can solve $\phi$ directly from $(u, \beta, z, \lambda)$ and therefore $C_I$ can be represented by them, as in Proposition 10. Recall we have

$$\hat{u} = {}^{\mathrm{t}}J_1 \zeta, \quad \hat{\beta} = \lambda\, {}^{\mathrm{t}}J_2 (z - u), \quad \hat{\phi} = \lambda |z - u| \sin \phi,$$

where $\phi = \frac{z-u}{|z-u|} \cdot \beta$ and $\zeta = -\lambda(\beta - m \cos \phi)$. We list the Jacobian matrix with respect to these variables in Figure 4.4. As before, we write $m = \frac{z-u}{|z-u|}$; the matrix

$$\begin{array}{c|cc|cc|}
 & \partial_u & \partial_\beta & \partial_z & \partial_\lambda \\
\hline
u & J_1 & & 0 & 0 \\
\beta & & J_2 & & \\
\hline
\phi & * & & & \\
\hat{u} & & & & \\
\hat{\beta} & * & & J_3 & \\
\hat{\phi} & * & & & \\
\hline
\end{array}$$

(a)

$$J_3 = \begin{array}{c|cc|}
 & \partial_z & \partial_\lambda \\
\hline
\phi & -\frac{1}{\sin \phi |z-u|} (\beta - m \sin \phi) & 0 \\
\hat{u} & -\frac{\lambda \cos \phi}{|z-u|} ({}^{\mathrm{t}}r_1 - (r_1 \cdot m)\, {}^{\mathrm{t}}m) - \lambda \sin \phi (r_1 \cdot m) \partial_z \phi & r_1 \cdot (\beta - m \cos \phi) \\
 & -\frac{\lambda \cos \phi}{|z-u|} ({}^{\mathrm{t}}r_2 - (r_2 \cdot m)\, {}^{\mathrm{t}}m) - \lambda \sin \phi (r_2 \cdot m) \partial_z \phi & r_2 \cdot (\beta - m \cos \phi) \\
\hat{\beta} & \lambda\, {}^{\mathrm{t}}\beta_1 & \beta_1 \cdot (z - u) \\
 & \lambda\, {}^{\mathrm{t}}\beta_2 \sin \theta & \beta_2 \cdot (z - u) \sin \theta \\
\hat{\phi} & \lambda m \sin \phi + \lambda |z - u| \cos \phi \partial_z \phi & |z - u| \sin \phi \\
\hline
\end{array}$$

(b)

Figure 4.4. The Jacobian matrix of $\pi_{\mathcal{M}}$

$J_1 = (r_1, r_2)$ with $r_1 = \partial_{v^1} u$ and $r_2 = \partial_{v^2} u$, which form a basis of $T_u^* \mathcal{S}$; the matrix $J_2 = (\beta_1, \sin \theta \beta_2)$, where $\beta_1$ and $\beta_2$ are unit vectors orthogonal to $\beta$. Observe that

$\mathrm{rank}J_1 = \mathrm{rank}J_2 = 2$. It suffices to show that the matrix $J_3$ has rank $= 4$. Notice by row reduction, the terms related to $\partial_z\phi$ in the second, third, and sixth row can be canceled by the first row, see Figure 4.5. Next by column reduction, the last column can be simplified if we subtract the inner product of the first three columns with $\frac{1}{\lambda}(z-u)$ from it. Since $(\lambda m\sin\phi)\cdot\beta = \lambda\sin\phi\cos\phi \neq 0$, we only need to show either

$$J_3 \xrightarrow{\text{row reduction}} \begin{bmatrix} -\frac{1}{\sin\phi|z-u|}(\beta - m\sin\phi) & 0 \\ -\frac{\lambda\cos\phi}{|z-u|}({}^{\mathrm{t}}r_1 - (r_1\cdot m)\,{}^{\mathrm{t}}m) & r_1\cdot(\beta - m\cos\phi) \\ -\frac{\lambda\cos\phi}{|z-u|}({}^{\mathrm{t}}r_2 - (r_2\cdot m)\,{}^{\mathrm{t}}m) & r_2\cdot(\beta - m\cos\phi) \\ \lambda\,{}^{\mathrm{t}}\beta_1 & \beta_1\cdot(z-u) \\ \lambda\,{}^{\mathrm{t}}\beta_2\sin\theta & \beta_2\cdot(z-u)\sin\theta \\ \lambda m\sin\phi & |z-u|\sin\phi \end{bmatrix}$$

(a)

$$\xrightarrow{\text{column reduction}} \begin{bmatrix} -\frac{1}{\sin\phi|z-u|}(\beta - m\sin\phi) & 0 \\ -\frac{\lambda\cos\phi}{|z-u|}({}^{\mathrm{t}}r_1 - (r_1\cdot m)\,{}^{\mathrm{t}}m) & r_1\cdot(\beta - m\cos\phi) \\ -\frac{\lambda\cos\phi}{|z-u|}({}^{\mathrm{t}}r_2 - (r_2\cdot m)\,{}^{\mathrm{t}}m) & r_2\cdot(\beta - m\cos\phi) \\ \lambda\,{}^{\mathrm{t}}\beta_1 & 0 \\ \lambda\,{}^{\mathrm{t}}\beta_2\sin\theta & 0 \\ \lambda m\sin\phi & 0 \end{bmatrix}$$

(b)

Figure 4.5. The submatrix after row and column reduction

$r_1\cdot(\beta - m\cos\phi)$ or $r_2\cdot(\beta - m\cos\phi)$ is nonzero. Indeed, if this is not true, we have $\beta - m\cos\phi$ is normal to $\mathcal{S}$ at $u$. By the same arguments as before, this contradicts with the assumption. This proves that $\mathrm{rank}(\mathrm{d}\pi_{\mathcal{M}}) = 8$.

The injectivity of $\pi_{\mathcal{M}}$ comes from the statement (b). In particular, if $\mathcal{S}$ satisfies Tuy's condition, then $R = \{(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi}) \in C_I(z,\zeta) | \forall (z,\zeta)\in T^*M\} = \pi_{\mathcal{M}}(C_I)$. And for every $(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi}) \in \pi_{\mathcal{M}}(C_I)$, there is a unique $(z,\zeta)$ such that

$$\pi_{\mathcal{M}}(u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi},z,\zeta) = (u,\beta,\phi,\hat{u},\hat{\beta},\hat{\phi}).$$

$\square$

## 4.4 Clean Intersection Calculus

In this section, we present the clean intersection calculus stated in [12, Thm. 25.2.3]. We modify it slightly to better suit our problem. Recall a subset of the cotangent bundle is conically compact if its canonical projection on the cosphere bundle is compact. The cosphere bundle can be defined as the quotient space modulo the equivalent relation induced by the group action of $\mathbb{R}^+$. For more details see [64].

**Proposition 12.** *Let $X$, $Y$, $Z$ be three $C^\infty$ manifolds. Suppose $C_1$ from $(T^*Y \setminus 0)$ to $(T^*X \setminus 0)$ is a $C^\infty$ homogeneous canonical relation closed in $T^*(X \times Y) \setminus 0$ and $C_2$ another from $(T^*Z \setminus 0)$ to $(T^*Y \setminus 0)$. Let*

$$A_1 \in I^{m_1}(X \times Y, C_1'), \quad A_2 \in I^{m_2}(Y \times Z, C_2')$$

*be properly supported FIOs with principal symbol $\alpha_1$ and $\alpha_2$ respectively. Assume we have*

*(A1) $C_1 \times C_2$ intersects $T^*X \times \Delta(T^*Y) \times T^*Z$ cleanly with excess $e$. We call the intersection $\hat{C}$.*

*(A2) there exist conically compact subsets $K_1, K_2$ of $C_1, C_2$ respectively, such that the microsupport of $A_1, A_2$ is in some open set of $K_1, K_2$ respectively.*

*(A3) the inverse image $C_\gamma$ under the projection $\hat{C} \to T^*(X \times Z) \setminus 0$ of any $\gamma \in C \equiv C_1 \circ C_2$ is connected.*

*Then there exists an open set $\mathcal{O}$ in $T^*X \times T^*Z$ such that $\mathcal{O} \cap C$ is a conic Lagrangian submanifold and*

$$A_1 A_2 \in I^{m_1+m_2+e/2}(X \times Z, (\mathcal{O} \cap C)')$$

*and for the principal symbol $\alpha$ of $A_1 A_2$ we have*

$$\alpha = \int_{C_\gamma} \alpha_1 \times \alpha_2, \tag{4.4}$$

*where $\alpha_1 \times \alpha_2$ is the density on $C_\gamma$ as is defined in [12, Theorem 25.2.3].*

**Remark 6.** If we compare this proposition with Theorem 25.2.3, the difference is that Hörmander assumes the restricted projection $\Pi : \hat{C} \to T^*(X \times Z) \setminus 0$ is proper instead of (A2) above. The properness of restricted $\Pi$ has the following implications in the original proof. First, since by [65, Theorem 21.2.14] the restricted $\Pi$ has constant rank, if it is proper and additionally has connected fibers (i.e. the preimage of a single point is connected, that is assumption (A3)), then the image $C = \Pi(\hat{C})$ is an embedded submanifold, see Lemma 2 and its remark. Second, since a continuous map is proper if and only if it is closed and has compact fibers (i.e. the preimage of a single point is compact), we have $C$ is a closed submanifold and $C_\gamma$ is compact. The later implies the integral $\int_{C_\gamma} \alpha_1 \times \alpha_2$ is well-defined.

*Proof of Proposition 12: part 1.* We prove in the following that there exists an open set $\mathcal{O} \subset T^*X \times T^*Z$ such that $\mathcal{O} \cap C$ is an embedded submanifold of $T^*X \times T^*Z$.

Let $\Pi$ be the natural projection

$$\Pi : T^*X \times \Delta(T^*Y) \times T^*Z \to T^*X \times T^*Z.$$

Since $\Pi$ is an open map, we can choose $\mathcal{O}$ by choosing an open subset $\hat{\mathcal{O}}$ of $T^*X \times \Delta(T^*Y) \times T^*Z$ by condition (2). More specifically, recall $\hat{C} = (C_1 \times C_2) \cap (T^*X \times \Delta(T^*Y) \times T^*Z)$ and we define

$$\hat{K} = (K_1 \times K_2) \cap (T^*X \times \Delta(T^*Y) \times T^*Z),$$
$$\hat{W} = (\mathrm{WF}(A_1) \times \mathrm{WF}(A_2)) \cap (T^*X \times \Delta(T^*Y) \times T^*Z)$$

Let $\hat{\mathcal{O}}$ be an open set such that $\hat{W} \subset \hat{\mathcal{O}} \cap \hat{C} \subset \hat{K} \subset \hat{C}$. Then we have $\mathcal{O} = \Pi(\hat{\mathcal{O}})$ is an open subset.

We can show $\mathcal{O} \cap C$ is an embedded submanifold in two ways. On the one hand, since $\Pi$ restricted on $\hat{K}$ is proper and $\hat{\mathcal{O}} \cap \hat{C} \subset \hat{K}$, by the proof of Lemma 2, each point in $\mathcal{O} \cap C$ has a submanifold coordinate chart. On other other hand, we can prove it by the following claim.

**Claim 5.** *The restricted projection* $\Pi : \hat{\mathcal{O}} \cap \hat{C} \to \mathcal{O}$ *is a closed map.*

Also by Lemma 2, we have $\mathcal{O} \cap C$ is an embedded submanifold of $\mathcal{O}$ and therefore that of $T^*X \times T^*Z$.

Then, by Claim 6, the wave front set of $A$ is contained in $\Pi(\hat{W})$ and thus is contained in $\mathcal{O} \cap C$. Now we can define the Fourier integral distributions on the embedding Lagrangian submanifold $C \cap \mathcal{O}$ by [12, Lemma 25.1.2] and its remark. Although $\mathcal{O} \cap C$ is not necessarily closed, by the remark in [66, p. 147], we can require the symbols (the amplitude in any specific representation) vanishing outside a closed conic subset and have the same conclusions about the principal symbols. $\square$

*Proof of Proposition 12: part 2.* We follow the proof of Hörmander and skip most of it here. One can first show $A_1A_2$ can be written as a sum of an FIO associated with $\mathcal{O} \cap C$ and a smoothing operator, and then compute the principal symbol. The only difference is that we still need to verify that the integration in (4.4) is well-defined. We claim that the principal symbol $\alpha_1, \alpha_2$ have conically compact support. Indeed, consider the local representation of $A_1, A_2$

$$A_1(x,y) = (2\pi)^{-(n_X+n_Y+2N_1)/4} \int e^{i\phi(x,y,\theta)} a_1(x,y,\theta)\, \mathrm{d}\theta,$$
$$A_2(y,z) = (2\pi)^{-(n_Y+n_Z+2N_2)/4} \int e^{i\psi(y,z,\tau)} a_2(y,z,\tau)\, \mathrm{d}\tau,$$

where $\phi, \psi$ are non-degenerate phase functions and $a_1, a_2$ are amplitudes. For more details see [12, Theorem 25.2.3]. Then the principal symbols of $A_1, A_2$ are

$$\alpha_1 = a_1(x,y,\theta) e^{\pi i/4 \operatorname{sgn} H_\phi} d_{C_1}^{\frac{1}{2}}, \quad \alpha_2 = a_2(y,z,\tau) e^{\pi i/4 \operatorname{sgn} H_\psi} d_{C_2}^{\frac{1}{2}}, \tag{4.5}$$

according to [12, p. 14]. Since $A_1, A_2$ have conically compact microsupport respectively, it suffices to show that the principal symbol vanishes outside the microsupport. Indeed, for fixed $(x_0, y_0, \xi^0, \eta^0) \notin \mathrm{WF}(A_1)$ in the canonical relation, there exists a small conic neighborhood of $(x_0, y_0, \xi^0, \eta^0)$ such that the local representation of $A_1$ above is smooth. By [67, Lemma 4.1], there is $a_{1,\infty} \in S^{-\infty}$ such that $a_1 = a_{1,\infty}$ on $\Sigma_\phi = \{(x,y,\theta),\ \phi'_\theta = 0\}$. On the other hand, the principal symbol $\alpha_1$ defined in (4.5) only depends the amplitude restricted to the the manifold $\Sigma_\phi$. Therefore, it vanishes in this neighborhood.

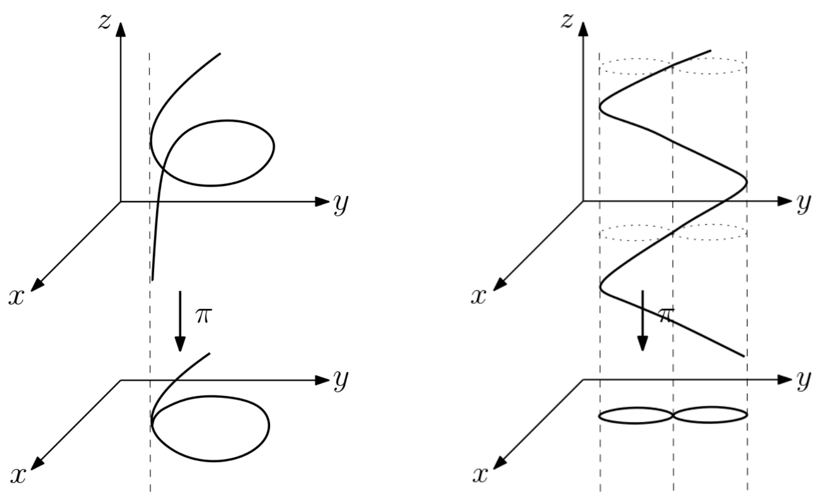
Suppose $\gamma = (x, \xi, z, \zeta)$. We integrate $\alpha_1 \times \alpha_2$ over the closed set $C_\gamma \cap \operatorname{supp}(\alpha_1 \times \alpha_2)$ and it is contained in $(\pi_X^{-1}(x,\xi) \times \pi_Z^{-1}(z,\zeta)) \cap (\operatorname{supp} \alpha_1 \times \operatorname{supp} \alpha_2)$, where $\pi_X, \pi_Z$ are the natural projection from $C_1, C_2$ to $T^*X, T^*Z$ respectively. By Lemma 3, the later is compact and therefore this integral is well defined. $\square$

The following are lemmas that we mention above and their proofs.

**Lemma 2.** *Let $f : X \to Y$ be a smooth map with constant rank. If $f$ is closed and the preimage $f^{-1}(y)$ of any $y \in f(X)$ is connected, then $f(X)$ is an embedded submanifold of $Y$.*

**Remark 7.** In particular, this lemma holds if we assume $f$ is proper instead of simply closed. A slightly different version of this lemma can be found in [68]. It claims the proof can be found in Appendix C.3 in [65]. The proof we write in the following is based on the outlines of Hörmander and we borrow most of it from an online answer that proves the case when $f$ is proper.

**Remark 8.** It is necessary to assume that $f$ is closed. Consider projection $\pi$ from $\mathbb{R}^3$ to the $xy$ plane. Let $\nu_1$ be a smooth curve as is shown in Figure 4.6 (a), whose image under projection is the figure 6. Here $f$ is the projection $\pi$ restricted on $\nu_1$. The preimage for each point in $\pi(\nu_1)$ under $f$ is connected but $f$ is not closed. The image $f(\nu_1)$ is not an embedded submanifold. It is also necessary to assume that each fiber




(a) (b)

Figure 4.6. Counterexamples

of $f$ is connected. A counterexample shown in Figure 4.6 (b) is constructed from the immersed manifold "figure 8". By lifting it into $\mathbb{R}^3$ and smoothly extended the two ends, we get a smooth curve $\nu_2$ in $\mathbb{R}^3$ and $f$ is defined as before. Notice $f$ is a closed map but the preimage of certain point is not connected.

*Proof of Lemma 2.* We followed the suggestions posted in [69] to prove this lemma. The proof can be divided into three steps.

Step 1. By [65, C 3.3], the constant rank of $f$ shows $f(X)$ is locally a submanifold. That is, for every $x \in X$ and $y = f(x)$, there exists an open neighborhood $U_x \subset X$ of $x$ and an open neighborhood $V_y \subset Y$ of $y$ such that $f(U_x) \subset V_y$ is a submanifold of $V_y$. It suffices to prove that there is an open $W_y \subset V_y$ such that $f(X) \cap W_y = f(U_x) \cap W_y$. Then we will have $f(X)$ is a submanifold of $Y$.

Step 2. Since $f^{-1}(y)$ is connected, for any $x, x' \in f^{-1}(y)$, we can show $f$ maps the neighborhood of $x, x'$ to the "same" neighborhood of $f(y)$ by defining an equivalence relation. More specifically, we say $x \sim x'$ if for any open neighborhood $O_x$ of $x$ and $O_{x'}$ of $x'$, there exists open neighborhood $U_x \subset O_x$ and $U_{x'} \subset O_{x'}$ such that $f(U_x) = f(U_{x'})$. Observe that each equivalence class is an open set, and different classes are disjoint. Therefore, if there are more than one equivalence class, then they form a partition of $f^{-1}(y)$, which contradicts with the connectedness.

Step 3. Back to what we want to prove, assume there is no such $W_y$ exists. Then there exists a sequence $y_k \in f(X)$ converging to $y$ with each point distinct from $y$ but $y_k \notin f(U_x)$. Pick arbitrary preimage $x_k$ of $y_k$. When $f$ is closed instead of proper, we have two cases. If $\{x_k\}$ has limit points in $X$, then we can choose a subsequence $x_{k_j} \to x'$, which is forced to be in $f^{-1}(y)$. By Step 2, there should be $U_{x'}$ and $\tilde{U}_x \subset U_x$ such that $f(U_{x'}) = f(\tilde{U}_x)$. For large enough $k$, we have $x_k \in U_{x'}$ and therefore $y_k \in f(U_{x'}) \subset f(U_x)$, which contradicts the assumption. If $\{x_k\}$ has no limit points, then $\{x_k\}$ is a closed subset of $X$. The set $\{y_k\} = f(\{x_k\})$ has a limit point $y$ which is not contained in $\{y_k\}$. This contradicts with the assumption that $f$ is closed. $\square$

**Claim 6.** *If the microsupport* $\mathrm{WF}(A_1)$ *and* $\mathrm{WF}(A_2)$ *are conically compact, then the wave front set* $\mathrm{WF}(A)$ *is conically compact and is contained in* $\Pi(\hat{W})$.

*Proof.* By [70, Theorem 8.2.14], since $\mathrm{WF}(A_1), \mathrm{WF}(A_1)$ are away from the zero sections, then we have

$$\mathrm{WF}'(A) \subset \mathrm{WF}'(A_1) \circ \mathrm{WF}'(A_2),$$

where $\mathrm{WF}'(\cdot)$ is the twisted relation. Hence, $\mathrm{WF}'(A)$ is a closed conic set contained in the image of the projection from the intersection $(\mathrm{WF}'(A_1) \times \mathrm{WF}'(A_2)) \cap (T^*X \times T^*\Delta(Y) \times T^*Z)$ to $T^*X \times T^*Z$. The intersection is conically compact so it is closed. Then the projection restricted there is continuous and it maps compact set to compact set. Moreover, it commutes with the multiplication by positive scalars in the covariant variables, and therefore the image of the intersection is conically compact. Thus, we have $\mathrm{WF}'(A)$ is conically compact. $\square$

*Proof of Claim 5.* Since $\hat{\mathcal{O}} \cap \hat{C}$ is an open set in $\hat{K}$, any closed subset $S$ is the intersection of $\hat{\mathcal{O}}$ with some closed subset $S_0$ of $\hat{K}$. Notice $\hat{K}$ is conically compact and $\Pi$ preserves the fiber, which implies $\Pi$ restricted to $\hat{K}$ is proper (by Lemma 3) and therefore is closed. It follows that $\Pi(S_0)$ is closed and therefore $\Pi(S) = \Pi(S_0) \cap \mathcal{O}$ is a closed subset of $\mathcal{O}$. $\square$

**Lemma 3.** *Suppose $C_1$ from $(T^*Y \setminus 0)$ to $(T^*X \setminus 0)$ is a $C^\infty$ homogeneous canonical relation closed in $T^*(X \times Y) \setminus 0$. Let $K$ be a closed subset of $C_1$ with the projection to $Y$ compact. Then the projection $\pi_X : K \to T^*X$ is proper.*

*Proof.* By [11, Prop. 2.17], a continuous map between two topological manifolds is proper if and only if it maps sequence that escapes to infinity to sequence that escapes to infinity. We say a sequence escaping to infinity if every compact subset contains at most finitely many elements of this sequence. In our case, we prove by contradiction. Assume there is a sequence $\{(x_n, \xi^n, y_n, \eta^n)\} \subset K$ escaping to infinity but its image $\{(x_n, \xi^n)\}$ does not. That is, there is a compact set in $T^*X$ that contains infinitely many elements of $\{(x_n, \xi^n)\}$. By compactness, we can choose a subsequence $\{(x_i, \xi^i)\}$ convergent to $(x_0, \xi^0)$. Additionally, since $K$ has compact projection on $Y$, there is a convergent subsequence $\{(x_j, \xi^j, y_j, \eta^j)\}$ such that we also have $y_j$ converges to $y_0$. For large enough $j$, there is a conic neighborhood $V$ of $(x_0, y_0)$ such that $\{(x_j, \xi^j, y_j, \eta^j)\}$ is contained in $V$. In local coordinates, since $\{(x_j, \xi^j, y_j, \eta^j)\}$ escapes to infinity, W.L.O.G we can assume $|\eta^j| > j$ for each $j$. Then by homogeneity, we have $(x_j, \frac{1}{|\eta^j|}\xi^j, y_j, \frac{1}{|\eta^j|}\eta^j) \in K$ and there is a subsequence convergent to $(x_j, 0, y_j, \eta^0)$ with $\eta^0 \in S^{n_Y}$. This contradicts with the assumption that $C_1$ has no zero section. $\square$

### Clean Composition

A special case is when we compose the operator with its adjoint. As the following lemma shows, in this case with certain condition the composition is clean.

**Lemma 4.** *Suppose $C_2$ from $(T^*X \setminus 0)$ to $(T^*Y \setminus 0)$ is a $C^\infty$ homogeneous canonical relation closed in $T^*(Y \times X) \setminus 0$. Let $A_2 \in I^m(Y \times X, C_2)$ be a properly supported FIO associated with $C_2$. If the projection $\pi_Y : C_2 \to T^*Y$ is an injective immersion, then the composition $A_2^* A_2$ is clean. In particular, the canonical relation of the composition map is the identity.*

We follow the same arguments in [71].

*Proof.* From [12, Thm. 25.2.2], we have $A_2^* \in I^m(X \times Y, (C_2^{-1})')$. By definition, the composition is clean if $\hat{C}_2 \equiv (C_2^{-1} \times C_2) \bigcap (T^*X \times \Delta(T^*X) \times T^*X)$ is a smooth manifold and its tangent space equals to the intersection of the tangent space of the intersecting manifolds. Indeed, let $\gamma_k = (x_k, \xi^k, y_k, \eta^k) \in C_2$, $k = 1, 2$ and $s : X \times Y \to Y \times X$ be the interchanging map. We have

$$(s^*\gamma_1, \gamma_2) \in \hat{C} \Leftrightarrow \pi_Y(\gamma_1) = \pi_Y(\gamma_2).$$

Since the projection is injective, then $\gamma_1 = \gamma_2$. This implies

$$\hat{C} = \{(s^*\gamma, \gamma), \ \gamma \in C_2\}, \quad T_{(s^*\gamma,\gamma)}\hat{C} = \{(s^*\gamma, s_*\delta_\gamma, \gamma, \delta_\gamma), \ \delta_\gamma \in T_\gamma C_2\}.$$

On the other hand, we have $(s_*\delta_{\gamma_1}, \delta_{\gamma_2}) \in T_{(s^*\gamma_1,\gamma_2)}(C_2^{-1} \times C_2)$ is contained in the tangent space of $T^*X \times \Delta(T^*X) \times T^*X$, if and only if

$$\pi_Y(\gamma_1) = \pi_Y(\gamma_2), \quad \mathrm{d}\pi_Y(\delta_{\gamma_1}) = \mathrm{d}\pi_Y(\delta_{\gamma_2}).$$

Since $\pi_Y$ is an injective immersion, it follows that $\gamma_1 = \gamma_2$ and $\delta_{\gamma_1} = \delta_{\gamma_2}$, which proves the lemma. $\square$

**Remark 9.** In the setting of this lemma, the connectedness condition (A3) in Proposition 12 is not needed. Observe that from the proof of Proposition 12 and Lemma 2, the connectedness of $C_\gamma$ is required to guarantee that the composition $C_1 \circ C_2$ does not intersect itself. However, in this case we have the composition is the diagonal of $T^*X \times T^*X$, which is automatically not self-intersecting.

## 4.5 The normal operator $I_\kappa^* I_\kappa$ as a ΨDO

In order to apply the clean composition theorem in Proposition 12 to $I_\kappa^* I_\kappa$, we need the composition satisfying three assumptions (A1), (A2), (A3).

For (A1), by Proposition 11 and Lemma 4, if $T^*M$ is accessible, then the composition $I_\kappa{}^* I_\kappa$ is clean. For (A3), see the remark after Lemma 4. In most case in application, the surface $\mathcal{S}$ is a plane, which makes the situation simpler. As for (A2), we would like to show that with certain assumptions on the support of $\kappa$ (or by choosing proper smooth cutoff functions), we can find a compact subset $K$ of $C_I$ such that the microsupport of $I_\kappa$ (or multiplied by cutoffs) is contained in some open subset of $K$.

**Lemma 5.** *Let $\chi_1(z), \chi_2(u,\phi)$ be smooth cutoff functions with compact supports. Then $\chi_2 I_\kappa \chi_1$ is a Lagrangian distribution with conically compact microsupport support in $C_I$. Additionally, there exists a compact set $K \subset C_I$ such that* $\mathrm{WF}(\chi_2 I_\kappa \chi_1)$ *is contained in some open subset of $K$. In particular, these statements are automatically true for $I_\kappa$ itself when $\kappa$ has compact support in $\mathcal{M} \times M$.*

*Proof.* As is shown in the proof of Propostion 11 (c), we have $(u, \beta, \hat{\phi}, z)$ is a parameterization of $C_I$. Thus, the map

$$\begin{aligned} F_0 : &\mathcal{S} \times S^2 \times \mathbb{R} \setminus 0 \times M \to C_I \\ &(u, \beta, \hat{\phi}, z) \mapsto (u, \beta, \phi, \hat{u}, \hat{\beta}, \hat{\phi}, z, \zeta) \end{aligned}$$

is a continuous submersion. Since we have continuous functions map compact sets to compact sets and submersions are open maps that map open sets to open sets, this implies $F_0$ maps compact (or open) set in $\mathcal{S} \times S^2 \times \mathbb{R} \setminus 0 \times M$ to compact (or open) set in $C_I$.

In fact, we have $F_0$ maps conically compact (or open) set to conically compact (or open) set. Indeed, suppose we have a conically compact neighborhood in $\mathcal{S} \times S^2 \times \mathbb{R} \setminus 0 \times M$, we can modify it to a compact neighborhood by restricting $|\hat{\phi}|$. Then the image of the compact set is compact in $C_I$ and therefore is compact in $C_I$ restricted to the cosphere bundle. Since $\hat{u}, \hat{\beta}, \hat{\phi}, \zeta$ is homogeneous of order 1 w.r.t. $|\hat{\phi}|$, we have the image of the conically compact (or open) set is conically compact (or open).

For the first statement of this lemma, notice that $\beta \in S^2$ is compact and we have the compact supports w.r.t $z, u$. Since $\phi \in (\epsilon, \pi/2 - \epsilon)$ might not be the whole range of $\phi = \cos^{-1}(\beta \cdot \frac{z-u}{|z-u|})$, we additionally assume we have compact support w.r.t $\phi$. To show the existence of the compact set $K$, observe that for any compact set $K'$ in $M, \mathcal{S}$ or $(\epsilon, \pi/2 - \epsilon)$, we can find a larger compact set such that $K'$ is in some open set of this larger compact set. By the arguments above, it proves the second statement. $\square$

With the lemma above, assuming $\kappa$ has compact support, we can apply Proposition 12 to $I_\kappa^* I_\kappa$ to show it is a $\Psi$DO. Moreover, with additional assumptions it is an elliptic $\Psi$DO, according to the formula (4.4) for composed principal symbols. Indeed, by [12, Thm. 25.2.2] the principal symbol $\sigma(I_\kappa^*)$ of the adjoint operator $I_\kappa^*$ equals to $s^*\sigma(I_\kappa)^*$, where $s$ is the interchange map $s : M \times \mathcal{M} \to \mathcal{M} \times M$. It follows that over the fiber $C_\gamma$, the term $\sigma(I_\kappa^*) \times \sigma(I_\kappa)$ is always nonnegative and its integral is positive if $\kappa \neq 0$ somewhere.

**Proposition 13.** *Assume $T^*M$ is accessible. Suppose the weight function $\kappa(u, \beta, \phi, z)$ has compact support in $\mathcal{M} \times M$. If we have $\kappa(u_0, \beta_0, \phi_0, z_0) \neq 0$ for some $(u_0, \beta_0, \phi_0) \in \mathcal{C}(z_0, \zeta^0)$, then ${I_\kappa}^* I_\kappa$ is a $\Psi$DO of order $-2$ elliptic at $(z_0, \zeta^0)$.*

**Remark 10.** The order is calculated by $(-3/2) + (-3/2) + e/2$, where the excess $e$ equals to the dimension of $C_\gamma$. By Proposition 11, we have $C_\gamma \cong C_I(z, \zeta)$, of which the dimension is 2.

If it is not true that the whole cotangent bundle $T^*M$ is accessible, then we can have a microlocal version of Proposition 13 by choosing proper cutoff ΨDOs. More specifically, for a fixed accessible covector $(z_0, \zeta^0)$, there exists a conic neighborhood $\Gamma_0$ of $(z_0, \zeta^0)$ such that each covector in this neighborhood is accessible. By choosing a cutoff ΨDO which is supported in this neighborhood, then we can prove the following proposition.

**Proposition 14.** *Suppose $(z_0, \zeta^0)$ is accessible. Suppose $\kappa(u_0, \beta_0, \phi_0, z_0) \neq 0$ for some covector $(u_0, \beta_0, \phi_0, \hat{u}^0, \hat{\beta}^0, \hat{\phi}^0)$ in $C_I(z_0, \zeta^0)$. Then there exists a ΨDO $P(z, D_z)$ of order zero elliptic at $(z_0, \zeta^0)$ with microsupport in a conically compact neighborhood that is accessible, such that for any $Q = Q(u, \beta, \phi, D_u, D_\beta, D_\phi)$ that is a ΨDO of order zero elliptic at $(u_0, \beta_0, \phi_0, \hat{u}^0, \hat{\beta}^0, \hat{\phi}^0)$ with conically compact support, the microlocalized normal operator $(QI_\kappa P)^* QI_\kappa P$ is a ΨDO of order $-2$ elliptic at $(z_0, \zeta^0)$.*

Theorem 1 is the direct result of this proposition and the following example is a special case when we have unrecoverable singularities.

**Example 14.** Let the surface of vertices $\mathcal{S} = \{(x^1, x^2, 0)|\ x^1x^1 + x^2x^2 < R^2\}$ be an open disk with radius $R$ in the $xy$ plane. Let $M$ be an open domain above $xy$ plane.

Consider a covector $(z, \zeta) \in T^*M$ with $\zeta_1$ or $\zeta_2$ nonzero. There is a cone $c(u, \beta, \phi)$ with vertex $u = (u^1, u^2, 0)$ in $\mathcal{S}$ conormal to $(z, \zeta)$ only if

$$\begin{aligned} &(z-u)\cdot\zeta = 0 & &\Rightarrow & &u^1\zeta_1 + u^2\zeta_2 = z\cdot\zeta \\ &u^1u^1 + u^2u^2 < R^2 & & & &u^1u^1 + u^2u^2 < R^2 \end{aligned}$$

If we have $\zeta_2 \neq 0$, we can solve $u^2 = \frac{z\cdot\zeta - v^1\zeta_1}{\zeta_2}$ from the first equation, and plug it in the second one to have

$$u^1u^1 + \frac{(z\cdot\zeta - v^1\zeta_1)^2}{(\zeta_2)^2} < R^2.$$

For simplification, we denote $u^1$ by $t$ to get

$$((\zeta_1)^2 + (\zeta_2)^2)t^2 - 2\zeta_1(z\cdot\zeta)t + (z\cdot\zeta)^2 - (\zeta_2)^2R^2 < 0.$$

This is a parabola opening to the top. There exists a solution of $t$ if and only if

$$\Delta = b^2 - 4ac = 4\Big(((\zeta_1)^2(z\cdot\zeta)^2 - (((\zeta_1)^2 + (\zeta_2)^2)((z\cdot\zeta)^2 - (\zeta_2)^2R^2)\Big) > 0,$$

which implies

$$((\zeta_1)^2 + (\zeta_2)^2)R^2 - (z\cdot\zeta)^2 > 0. \tag{4.6}$$

If $\zeta_2 = 0$, then we must have $\zeta_1 \neq 0$. By symmetry we should get the same result. Notice, if we can find such $u$ in $\mathcal{S}$ that $(z-u)\cdot\zeta = 0$, then we can construct a cone $c(u,\beta,\phi)$ conormal to $(z,\zeta)$ by properly choosing $\beta$ and $\phi$. Thus, the set of the unrecoverable singularities is

$$\{(z,\zeta)|\ ((\zeta_1)^2 + (\zeta_2)^2)R^2 - (z\cdot\zeta)^2 < 0\}.$$

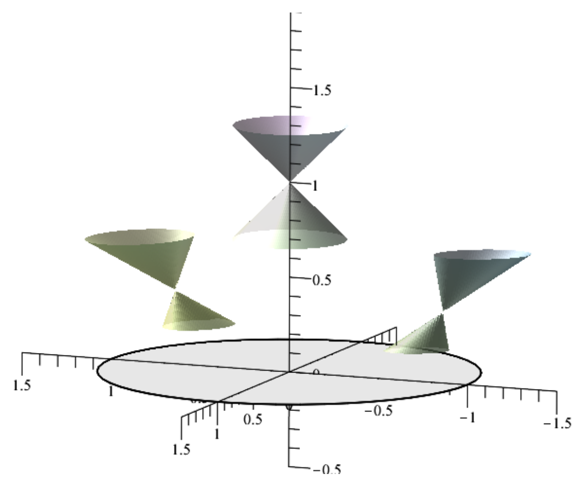



Figure 4.7. unrecoverable singularities at three points

## 4.6 Proof of Theorem 2

For the $L^2$ estimates, we have the following restatement of [66, Theorem 4.1.9 and Thmeorem 4.3.2], which indicates the mapping properties in the general case.

**Proposition 15.** *Let $C$ be a homogeneous canonical relation from $T^*M$ to $T^*\mathcal{M}$ such that*

*(1) the maps $C \to M$ and $C \to \mathcal{M}$ have subjective differentials,*

*(2) the projections $\pi_M : C \to T^*M$ and $\pi_{\mathcal{M}} : C \to T^*\mathcal{M}$ have constant rank.*

*Let $k_M = \operatorname{rank}(\mathrm{d}\pi_M) - \dim M$ and $k_{\mathcal{M}} = \operatorname{rank}(\mathrm{d}\pi_{\mathcal{M}}) - \dim \mathcal{M}$ respectively. Then $k_M = k_{\mathcal{M}} \equiv k$. Suppose $A \in I^m(\mathcal{M} \times M, C')$ is properly supported. Then*

$$A : H^s_{loc}(M) \to H^{s+t}_{loc}(\mathcal{M}), \quad A^* : H^s_{loc}(\mathcal{M}) \to H^{s+t}_{loc}(M)$$

*are continuous, for any $t \leq (2k - \dim M - \dim \mathcal{M})/4 - m$ and $s \in \mathbb{R}$.*

*Proof.* This corollary is a direct result of the theorems above. Let $\Lambda^\sigma$ be the ΨDO of order $\sigma$ such that $\Lambda^\sigma u = \int e^{ix\cdot\xi}(1 + |\xi|^2)^{\frac{\sigma}{2}} \hat{u}(\xi)\,\mathrm{d}\xi$. We apply [66, Thm.4.3.2] to $\tilde{A} = \Lambda^{s+t} A \Lambda^{-s} \in I^{m+t}(\mathcal{M} \times M, C')$. Then $\tilde{A} : L^2_{loc} M \to L^2_{loc}(\mathcal{M})$ continuously, for any $m + t \leq (2k - \dim M - \dim \mathcal{M})/4$. Notice if $C$ satisfies the conditions (1)(2) in above corollary, then $C^{-1}$ satisfies them as well. Since $A^* \in I^m(M \times \mathcal{M}, (C^{-1})')$, we can apply the same argument. □

Now we consider the conical Radon transform and its canonical relation $C_I$. We have the following claim by Proposition 15.

**Claim 7.** *When $T^*M$ is accessible, the canonical relation $C_I$ satisfies condition (1) and (2). In particular, we have $k = \dim M = 3$ and the inequality in Corollary 15 is $t \leq 1$.*

*Proof.* By Proposition 11, the projection $\pi_{\mathcal{M}}$ is an injective immersion, which implies $\pi_M$ is a submersion. Thus, $k = \dim(T^*M) - \dim M = \dim M$. From the proof of Proposition 11, we have $C_I$ is parameterized by $(u, \beta, z, \hat{\phi})$. It is obvious that $C_I \to M$ has subjective differential. For the projection $C_I \to \mathcal{M}$, the Jacobian is in the following, where $* = \frac{\partial \phi}{\partial z}$. Since $\phi$ is solved from $(z - u) \cdot \beta - |z - u| \cos\phi = 0$, differentiating w.r.t. $z$ we have

$$\frac{\partial \phi}{\partial z} = -\frac{1}{|z - u| \sin\phi}(\beta - \frac{z - u}{|z - u|} \cos\phi).$$

Thus, the vector $\frac{\partial \phi}{\partial z}$ is nonzero and the differential of projection has rank equals $\dim \mathcal{M}$. □

$$
\begin{array}{c|cc|c|c|}
 & \partial_u & \partial_\beta & \partial_z & \partial_{\hat{\phi}} \\
\hline
u & J_1 & & & \\
 & & & 0 & 0 \\
\beta & & J_2 & & \\
\phi & \# & & * & 0 \\
\hline
\end{array}
$$

Figure 4.8. The Jacobian matrix of the projection $C_I \to \mathcal{M}$

By the claim above, we can prove Theorem 2 in the following based on similar arguments in [16, 72]

*Proof of Theorem 2.* By Corollary 15 and the claim above, we have the second inequality and $\|I_\kappa^* g\|_{H^{s+1}(M)} \le C\|g\|_{H^s(\mathcal{M})}$, for some constant $C$. From Proposition 13, we have $C_1\|f\|_{H^s(M)} - C_{s,l}\|f\|_{H^l(M)} \le \|I_\kappa^* I_\kappa f\|_{H^{s+2}(M)}$. Combining these two we have desired result. $\square$

*Another proof of the estimate for $s = -1$.* First we prove $I_\kappa : H^s_{\bar{M}} \to H^{s+1}(\mathcal{M})$ is bounded, for $s \ge 0$, where $H^s_{\bar{M}}$ is the space of all $u \in H^s(\mathbb{R}^3)$ supported in $\bar{M}$. Indeed, we have

$$\|I_\kappa f\|^2_{H^{s+1}(\mathcal{M})} \le C \sum_{|\alpha| \le 2s+2} |(\partial^\alpha_{(u,\beta,\phi)} I_\kappa f, I_\kappa f)_{L^2(M)}| = C \sum_{|\alpha| \le 2s+2} |(I_\kappa^* \partial^\alpha_{(u,\beta,\phi)} I_\kappa f, f)_{L^2(M)}|.$$

Here we have

$$\partial^\alpha_{(u,\beta,\phi)} I_\kappa f = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \int \left(\partial^{\alpha_1}_{(u,\beta,\phi)}\kappa\right) \delta^{(|\alpha_2|)}(g) \left(\partial^{\alpha_2+\alpha_3}_{(u,\beta,\phi)} g\right) f(z)\, \mathrm{d}z,$$

where $\alpha, \alpha_1, \alpha_2, \alpha_3$ are multi-indexes and $g = (z-u)\cdot\beta - |z-u|\cos\phi$. Since we always have $|\partial_z g| \ne 0$, locally we can write $\delta^{(1)}(g)$ as $\frac{1}{\partial_{z_j} g}\partial_{z_j}\delta(g)$, where $j$ is the index such that $\partial_{z_j} g \ne 0$. Therefore by integration by parts and the induction, we get the similar integral transform of derivatives of $f$ up to order $|\alpha|$ with a new weight function. This implies $I_\kappa^* \partial^\alpha_{(u,\beta,\phi)} I_\kappa f$ is a $\Psi$DO of order $|\alpha| - 2$. Thus, we have the estimates

$$\|I_\kappa f\|^2_{H^{s+1}(\mathcal{M})} \le C\|f\|_{H^s(M)}.$$

When $s = -1$, the proof is simplified. Then by duality, we have $I_\kappa^* : H_{\bar{\mathcal{M}}}^{-s-1} \to H^{-s}(M)$ is bounded, for $(H^{s+1}(\mathcal{M}))^* = H_{\bar{\mathcal{M}}}^{-s-1}$ and $(H^s(M))^* = H_{\bar{M}}^s$. Since we always assume $\kappa(u, \beta, \phi, z)$ is compactly supported, we have $I_\kappa f$ has support in $\mathcal{M}$. Thus,

$$\|I_\kappa^* I_\kappa f\|_{H^{-s}(M)} \leq C\|I_\kappa f\|_{H^{-s-1}(\mathcal{M})}.$$

Combining these two inequality and the ellipticity of $I_\kappa^* I_\kappa$ , we have for $l < -1$,

$$C_1\|f\|_{H^{-1}} - C_l\|f\|_{H^l(M)} \leq \|I_\kappa f\|_{L^2(\mathcal{M})} \leq C_2\|f\|_{H^{-1}}.$$

We abuse the notation $C, C_1, C_2$ to denote different constants. □

## 4.7 Restricted Cone Transform

In this section, suppose $\mathcal{S}_0$ is a smooth regular curve that is parameterized by $u(t)$. For fixed $\phi_0 \in (\epsilon, \pi/2 - \epsilon)$, we define the restricted cone transform as

$$I_{\kappa,\phi_0} f(u, \beta) = \int \kappa(u, \beta, z)\delta((z - u) \cdot \beta - |z - u| \cos\phi_0) f(z)\, \mathrm{d}z.$$

We have the following corollaries.

**Corollary 4.** *The restricted cone transform $I_{\kappa,\phi_0}$ is an FIO of order $-1$ associated with the canonical relation*

$$C_{\phi_0} = \{(u, \beta, \underbrace{u'(t) \cdot \zeta}_{\hat{u}}, \underbrace{\lambda\, {}^{\mathrm{t}}J_2(z - u)}_{\hat{\beta}}, z, \zeta),\ \ \varphi_{\phi_0} = 0\},$$

*where $\varphi_{\phi_0}(u, \beta, z) = (z - u) \cdot \beta - |z - u| \cos\phi_0$ and $\zeta = -\lambda(\beta - \frac{z-u}{|z-u|} \cos\phi_0)$; the vertex $u = u(t)$; the unit vector $\beta$ is parameterized in the spherical coordinates and $J_2$ is the Jacobian matrix defined as before.*

**Corollary 5.** *Let $D_0$ be the set of all $(z, \zeta)$ in $T^*M$ that are accessible w.r.t. $\mathcal{S}_0$. Then $D_0$ is an open set. Let $R_0 = \{(u, \beta, \phi, \hat{u}, \hat{\beta}, \hat{\phi}) \in C_I(z, \zeta) | (z, \zeta)$ is accessible$\}$. We have the following properties.*

(a) *For every* $(u,\beta,\hat{u},\hat{\beta}) \in R_0$*, there is one unique solution* $(z,\zeta)$ *for the equation* $C_{\phi_0}(z,\zeta) = (u,\beta,\hat{u},\hat{\beta})$*, which is given by (4.7) and (4.8) .*

(b) *The projection* $\pi_{\mathcal{M}}$ *restricted to* $\pi_{\mathcal{M}}^{-1}(D_0)$ *is an injective immersion. In particular, if* $\mathcal{S}_l$ *satisfies Tuy's condition, then* $\pi_{\mathcal{M}}$ *itself is an injective immersion.*

*Proof.* For (a), notice we have $\lambda|z-u| = \frac{\hat{\phi}}{\sin\phi} = \frac{1}{\sin\phi}|(\hat{\beta}_1, \frac{1}{\sin\theta}\hat{\beta}_2)|$. By (4.3) and its proof, when $\theta \neq 0$ or $\pi$,

$$z = u + \frac{|(\hat{\beta}_1, \frac{1}{\sin\theta}\hat{\beta}_2)|}{\lambda\sin\phi}m, \quad \zeta = -\lambda(\beta - \cos\phi m) \tag{4.7}$$

where

$$m = \cos\phi\beta + \frac{\sin\phi}{|(\hat{\beta}_1, \frac{1}{\sin\theta}\hat{\beta}_2)|}(\hat{\beta}_1\beta_1 + \frac{1}{\sin\theta}\hat{\beta}_2\beta_2).$$

We still need to solve $\lambda$ from $(u,\beta,\hat{u},\hat{\beta})$. Note that $\hat{u} = u'(t)\cdot\zeta = -\lambda u'(t)\cdot(\beta - \frac{z-u}{|z-u|}\cos\phi_0) \neq 0$, since $(z,\zeta)$ is accessible. Therefore, we have

$$\lambda = -\frac{\hat{u}}{u'(t)\cdot(\beta - \frac{z-u}{|z-u|}\cos\phi_0)}. \tag{4.8}$$

For (b), to prove that $\pi_{\mathcal{M}}$ restricted to $\pi_{\mathcal{M}}^{-1}(D_0)$ is an immersion, it suffices to show that $C_{\phi_0}$ is parameterized by $(u,\beta,\hat{u},\hat{\beta})$. Indeed, from (b), $(z,\zeta)$ can be represented by $(u,\beta,\hat{u},\hat{\beta})$. By writing $z = u + \rho\cos\phi_0\beta + \rho\sin\phi_0(\sin\alpha\beta_1 + \cos\alpha\beta_2)$ and performing a same argument as in the proof of Proposition 4, we can show it is a parameterization and therefore the rank of the differential of $\pi_{\mathcal{M}}$ equals $\dim C_{\phi_0} = 6$. □

**Remark 11.** Based on similar arguments, we can also show the same results hold if we only restrict the vertexes on a smooth curve $\mathcal{S}_0$ without $\phi$ fixed. Additionally, one can show an analog of recovery of singularities in both cases as in Theorem 1.

# 5. RAYLEIGH WAVES AND STONELEY WAVES

## 5.1 Introduction

Rayleigh waves in linear elasticity are a type of surface waves. They are first studied by Lord Rayleigh in [73] and can be the most destructive waves in an earthquake. They propagate along a traction-free boundary and decay rapidly into the media. By geophysical literatures, Rayleigh waves have a retrograde elliptical particle motion for shallow depth in the case of flat boundary and homogeneous media, see [**?**, **?**]. Stoneley waves are a type of interface waves that propagate along the interface between two different solids. They are first predicted in [74]. Roughly speaking, Rayleigh waves can be regarded as a special (limit) case of Stoneley waves. Both geophysical and mathematical works have been done for these two kinds of waves, see [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, 5, 73, 75, 75–77] and their references. Most geophysical works on them are considering specific situations, for example, the case of flat boundaries, plane waves, or homogeneous media. The propagation phenomenon of Rayleigh waves in an isotropic elastic system is first studied by Michael Taylor in [76] from a microlocal analysis point of view. Kazuhiro Yamamoto in [**?**] shows the existence of Stoneley waves as the propagation of singularities in two isotopic media with smooth arbitrary interfaces. Sönke Hansen in [**?**] derives the Rayleigh quasimodes by the spectral factorization methods for inhomogeneous anisotropic media with curved boundary and then in [**?**] shows the existence of Rayleigh waves by giving ray series asymptotic expansions in the same setting. In particular, the author derives the transport equation satisfied by the leading amplitude which represents the term of highest frequency. Most recently in [4], the authors describe the microlocal behavior of solutions to the transmission problems in isotropic elasticity with curved interfaces. Surface waves are briefly mentioned there as possible solutions of evanescent type which propagate on the boundary.

In this work, we construct the microlocal solutions of Rayleigh waves and Stoneley waves, describe their microlocal behaviors, and compute the direction of their polarization explicitly, for an isotropic elastic system with variable coefficients and a curved boundary. In particular, we show the retrograde elliptical particle motion in both waves as an analog to the flat case. Essentially, the existence of Rayleigh waves comes from the nonempty kernel of the principal symbol of the Dirichlet-to-Neumann map (DN map) $\Lambda$ in the elliptic region. In Section 3, based on the analysis in [4], one can see the Rayleigh waves are corresponding to the solution to $\Lambda u = l$ on the boundary, where $l$ is a source microlocally supported in the elliptic region. Next, inspired by the diagonalization of the Neumann operator for the case of constant coefficients in [5], we diagonalize $\Lambda$ microlocally up to smoothing operators by a symbol construction in [6]. The DN map $\Lambda$ is a matrix-valued pseudodifferential operator with the principal symbol $\sigma_p(\Lambda)$ in (5.3) and the way to diagonalize it is not obvious. One can see the diagonalization is global and it gives us a system of one hyperbolic equation and two elliptic equations on the boundary with some metric. The solution to this system applied by a ΨDO of order zero serves as the Dirichlet boundary condition on the timelike boundary $\mathbb{R} \times \partial\Omega$ of the elastic system, and then the Rayleigh wave can be constructed in the same way as the construction of the parametrix for elliptic evolution equations, as the Cauchy data is microlocally supported in the elliptic region. The wave front set and the direction of the microlocal polarization of the Rayleigh waves on the boundary can be derived during the procedure and they explain the propagation of Rayleigh waves and show a retrograde elliptical particle motion. These results are based on the diagonalization of the DN map. In Section 3.3, we derive the microlocal Rayleigh waves on the boundary if we have the Cauchy data at $t = 0$. The polarization is given in Theorem 10 and the leading terms show a retrograde elliptical motion of the particles, same as that of the case of homogeneous media in [?,?]. In Section 3.4, the inhomogeneous problem, i.e., when there is a source on the boundary, is studied and the microlocal solution and polarization are in 11. In the second part of this work, Stoneley waves are analyzed in a similar way with a more complicated system on the boundary. The main results of Stoneley waves for Cauchy problems is in 13 and for inhomogeneous problems is in 14. The microlocal Stoneley waves derived there have a similar pattern as that of Rayleigh waves and the leading terms show a retrograde elliptical motion of the particles as well.

## 5.2 Preliminaries

Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary. Suppose the density $\rho$ and the Lamé parameters $\mu, \nu$ are smooth functions depending on the space variable $x$, and even the time $t$.

In this section, we recall some notations and results in [4]. For a fixed point $x_0$ on the boundary, one can choose the semigeodesic coordinates $x = (x', x^3)$ such that the boundary $\partial\Omega$ is locally given by $x^3 = 0$. For this reason, we view $u$ as a one form and write the elastic system in the following invariant way in presence of a Riemannian metric $g$. Let $\nabla$ be the covariant differential in Riemannian geometry. We define the symmetric differential $\mathrm{d}^s$ and the divergence $\delta$ as

$$(\mathrm{d}^s u)_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i), \quad (\delta v)_i = \nabla^j v_{ij}, \quad \delta u = \nabla^i u_i,$$

where $u$ is a covector field and $v$ is a symmetric covariant tensor field of order two. The stress tensor is given by

$$\sigma(u) = \lambda(\delta u) g + 2\mu \, \mathrm{d}^s u.$$

Then the operator $E$ and the normal stress are

$$Eu = \rho^{-1}\delta\sigma(u) = \rho^{-1}(\mathrm{d}(\lambda\delta u) + 2\delta(\mu \, \mathrm{d}^s u)), \quad Nu = \sum_j \sigma_{ij}(u)\nu^j|_{\partial\Omega},$$

where $\nu^j$ is the outer unit normal vector on the boundary. The elastic wave equation can be written as

$$u_{tt} = Eu$$

and near some fixed $(x_0, \xi^0)$ one can decouple this system up to smoothing operators by a ΨDO $U$ of order zero such that

$$U^{-1}EU = \begin{pmatrix} c_s^2\Delta_g + A_s & 0 \\ 0 & c_p^2\Delta_g + A_p \end{pmatrix} \mod \Psi^{-\infty},$$

where $A_s$ is a $2 \times 2$ matrix ΨDO of order one, $A_p$ is a scalar ΨDO of order one, and the S wave and P wave speed are

$$c_s = \sqrt{\mu/\rho}, \quad c_p = \sqrt{(\lambda + 2\mu)/\rho}.$$

Let $w = (w^S, w^P) = U^{-1}u$. Then the elastic system decouples into two wave equations. This decoupling indicates that the solution $u$ has a natural decomposition into the S wave and P wave modes.

### 5.2.1 The boundary value problem in the elliptic region

When we solve the boundary value problems for the elastic system, the construction of the microlocal outgoing solution depends on where the wave front set of the Cauchy data is. The Rayleigh waves happen when there is a free boundary and the singularities of the Cauchy data on the boundary are in the **elliptic region** $\tau^2 < c_s^2|\xi'|_g^2$.

In this case, given the boundary data $f = u|_{x^3=0}$, first we get $w|_{x^3=0}$ by considering the restriction operator $U_{out}$ of the ΨDO $U$ to the boundary, which maps $w|_{x^3=0}$ to $f$. It is shown that $U_{out}$ is an elliptic one and therefore it is microlocally invertible in [4]. Then we seek the outgoing microlocal solution $w$ to the two wave equations with the boundary data, see [4, (47)] Since the wave front set of the boundary data is in the elliptic region, the Eikonal equations have no real valued solutions. Instead, the microlocal solution is constructed by a complex valued phase function, see [4, §5.3] for more details. After we construct $w$, we have $u = U^{-1}w$ as the microlocal solution to the elastic system.

## 5.3 Rayleigh Wave

### 5.3.1 Diagonalization of the DN map

The main goal of this section is to construct the microlocal solution of Rayleigh waves and to analyze their microlocal polarization. We follow Denker's notation to denote the vector-valued distributions on a smooth manifold $X$ with values in $\mathbb{C}^N$ by $\mathcal{D}'(X, \mathbb{C}^N)$. Similarly $\mathcal{E}'(X, \mathbb{C}^N)$ is the set of distributions with compact support in $X$ with values in $\mathbb{C}^N$.

Suppose only for a limited time $0 < t < T$ there is a source on the boundary $\Gamma = \mathbb{R}_t \times \mathbb{R}^2$. We are solving $Nu = l$ on $\Gamma$ in the elliptic region, where $u$ is an outgoing solution to the elastic wave equation, i.e.

$$\begin{cases} u_{tt} - Eu = 0 & \text{in } \mathbb{R}_t \times \mathbb{R}^3, \\ Nu = l & \text{on } \Gamma, \\ u|_{t<0} = 0, \end{cases} \tag{5.1}$$

with $l(t, x') \in \mathcal{E}'((0, T) \times \mathbb{R}^2, \mathbb{C}^3)$ microlocally supported in the elliptic region. This is equivalent to solving the boundary value problem $u|_\Gamma = f$, with $f$ being the outgoing solution on $\Gamma$ to

$$\Lambda f = l, \quad f|_{t<0} = 0, \tag{5.2}$$

where $\Lambda$ is the **Dirichlet-to-Neumann map** (DN map) which maps the Dirichlet boundary data $u|_\Gamma$ to the Neumann boundary data $Nu|_\Gamma$. More specifically, as long as we solve $f$ from (5.2), we can construct the microlocal outgoing solution $u$ to (5.1) as an evanescent mode with a complex valued phase function. The construction of the microlocal solution to the boundary value problem with wave front set in the elliptic region is well studied in [4, §5.3]. The main task of this section is to solve (5.2) microlocally and study the microlocal polarization of its solution.

**Proposition 16.** *In the elliptic region, the DN map $\Lambda$ is a matrix $\Psi DO$ with principal symbol*

$$\sigma_p(\Lambda) = \frac{i}{|\xi'|_g^2 - \alpha\beta} \begin{pmatrix} \mu(\alpha-\beta)\xi_2^2 + \beta\rho\tau^2 & -\mu\xi_1\xi_2(\alpha-\beta) & -i\mu\xi_1\theta \\ -\mu\xi_1\xi_2(\alpha-\beta) & \mu(\alpha-\beta)\xi_2^2 + \beta\rho\tau^2 & -i\mu\xi_2\theta \\ i\mu\theta\xi_1 & i\mu\theta\xi_2 & \alpha\rho\tau^2 \end{pmatrix} \equiv \frac{i}{|\xi'|_g^2 - \alpha\beta} N_1, \tag{5.3}$$

*where*

$$\alpha = \sqrt{|\xi'|_g^2 - c_s^{-2}\tau^2}, \quad \beta = \sqrt{|\xi'|_g^2 - c_p^{-2}\tau^2}, \quad \theta = 2|\xi'|_g^2 - c_s^{-2}\tau^2 - 2\alpha\beta. \tag{5.4}$$

*Proof.* In [4] the restriction $U_{out}$ of $U$ to the boundary maps $w|_{x^3=0}$ to $f = u|_{x^3=0}$ and it is an elliptic ΨDO with principal symbol and the parametrix

$$\sigma_p(U_{out}) = \begin{pmatrix} 0 & -i\alpha & \xi_1 \\ i\alpha & 0 & \xi_2 \\ -\xi_2 & \xi_1 & i\beta \end{pmatrix}, \quad \sigma_p(U_{out}^{-1}) = -\frac{i}{\alpha(|\xi'|_g^2 - \alpha\beta)} \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 - \alpha\beta & -i\alpha\xi_2 \\ -(\xi_2^2 - \alpha\beta) & \xi_1\xi_2 & i\alpha\xi_1 \\ i\alpha\xi_1 & i\alpha\xi_2 & -\alpha^2 \end{pmatrix}.$$

The operator $M_{out}$ that maps $w|_{x^3=0}$ to the Neumann boundary data $l = Nu|_{x^3=0}$ is an ΨDO with principal symbol

$$\sigma_p(M_{out}) = \begin{pmatrix} -\mu\xi_1\xi_2 & \mu(\xi_1^2 + \alpha^2) & 2i\mu\beta\xi_1 \\ -\mu(\xi_2^2 + \alpha^2) & \mu\xi_1\xi_2 & 2i\mu\beta\xi_2 \\ -2i\mu\alpha\xi_2 & 2i\mu\alpha\xi_1 & -2\mu|\xi'|_g^2 + \rho\tau^2 \end{pmatrix}.$$

Then the principal symbol of the DN map

$$\sigma_p(\Lambda) = \sigma_p(M_{out}U_{out}^{-1}) = \sigma_p(M_{out})\sigma_p(U_{out}^{-1})$$

can be computed and is given in (5.3). □

In the following, we first diagonalize $\sigma_p(\Lambda)$ in the sense of matrix diagonalization and then we microlocally decouple the system $\Lambda f = l$ up to smoothing operators.

By [5], first we have

$$V_0^*(t,x',\tau,\xi')N_1(t,x',\tau,\xi')V_0(t,x',\tau,\xi') = \begin{pmatrix} \beta\rho\tau^2 & -i|\xi'|_g\mu\theta & 0 \\ i|\xi'|_g\mu\theta & \alpha\rho\tau^2 & 0 \\ 0 & 0 & \mu\alpha(|\xi'|_g^2-\alpha\beta), \end{pmatrix}. \tag{5.5}$$

where

$$V_0(t,x',\tau,\xi') = \begin{pmatrix} \xi_1/|\xi'|_g & 0 & -\xi_2/|\xi'|_g \\ \xi_2/|\xi'|_g & 0 & \xi_1/|\xi'|_g \\ 0 & 1 & 0 \end{pmatrix}.$$

Then let

$$m_1(t,x',\tau,\xi') = \frac{(\alpha+\beta)\rho\tau^2-\sqrt{\varrho}}{2}, \qquad m_2(t,x',\tau,\xi') = \frac{(\alpha+\beta)\rho\tau^2+\sqrt{\varrho}}{2},$$
$$m_3(t,x',\tau,\xi') = \mu\alpha(|\xi'|_g^2-\alpha\beta), \qquad \text{with } \varrho = (\alpha-\beta)^2\rho^2\tau^4 + 4|\xi'|_g^2\mu^2\theta^2 > 0.$$

Notice we always have the following equalities

$$m_1+m_2 = (\alpha+\beta)\rho\tau^2, \qquad m_1m_2 = \alpha\beta\rho^2\tau^4 - |\xi'|_g^2\mu^2\theta^2. \tag{5.6}$$

We conclude that the principal symbol $\sigma_p(\Lambda)$ can be diagonalized

$$W^{-1}(t,x',\tau,\xi')\sigma_p(\Lambda)W(t,x',\tau,\xi') = \frac{i}{|\xi'|_g^2-\alpha\beta}\begin{pmatrix} m_1(t,x',\tau,\xi') & 0 & 0 \\ 0 & m_2(t,x',\tau,\xi') & 0 \\ 0 & 0 & m_3(t,x',\tau,\xi') \end{pmatrix}$$

by a matrix

$$W(t,x',\tau,\xi') = V_0(t,x',\tau,\xi')V_1(t,x',\tau,\xi'),$$

where

$$V_1(t,x',\tau,\xi') = \begin{pmatrix} i|\xi'|_g\mu\theta/k_1 & i|\xi'|_g\mu\theta/k_2 & 0 \\ (\beta\rho\tau^2-m_1)/k_1 & (\beta\rho\tau^2-m_2)/k_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with

$$k_j = \sqrt{(\beta\rho\tau^2-m_j)^2+|\xi'|_g^2\mu^2\theta^2}, \quad \text{for } j=1,2. \tag{5.7}$$

More specifically,

$$W(t,x',\tau,\xi') = \begin{pmatrix} i\mu\theta\xi_1/k_1 & i\mu\theta\xi_1/k_2 & -\xi_1/|\xi| \\ i\mu\theta\xi_2/k_1 & i\mu\theta\xi_2/k_2 & \xi_2/|\xi| \\ (\beta\rho\tau^2 - m_1)/k_1 & (\beta\rho\tau^2 - m_2)/k_2 & 0 \end{pmatrix} \tag{5.8}$$

is an unitary matrix. Here $m_1, m_2, m_3$ are the eigenvalues of $N_1(t,x',\tau,\xi')$ smoothly depending on $t,x',\tau,\xi'$. The eigenvalues $\widetilde{m}_j(t,x',\tau,\xi')$ of $\sigma_p(\Lambda)$ are given by $\widetilde{m}_j(t,x',\tau,\xi') = im_j(t,x',\tau,\xi')/(|\xi'|_g^2 - \alpha\beta)$, for $j = 1,2,3$. Notice that $m_2, m_3$ are always positive. It follows that only $m_1(t,x',\tau,\xi')$ could be zero and this happens if and only if the determinant of the $2\times 2$ blocks in (5.5) equals zero, i.e.

$$\begin{aligned} 0 &= \alpha\beta\rho^2\tau^4 - |\xi'|_g^2\mu^2\theta^2 = \alpha\beta\rho^2\tau^4 - |\xi'|_g^2\mu^2(|\xi'|_g^2 + \alpha^2 - 2\alpha\beta)^2 \\ &= (|\xi'|_g^2 - \alpha\beta)\underbrace{(4\mu^2\alpha\beta|\xi'|_g^2 - (\rho\tau^2 - 2\mu|\xi'|_g^2)^2)}_{R(\tau,\xi')}. \end{aligned} \tag{5.9}$$

Notice the elliptic region has two disconnected comportments $\pm\tau > 0$. We consider the analysis for $\tau > 0$ and the other case is similar. Define $s = \tau/|\xi'|_g$ and let

$$\begin{aligned} &a(s) = \frac{\alpha}{|\xi'|_g} = \sqrt{1 - c_p^{-2}s^2}, \quad b(s) = \frac{\beta}{|\xi'|_g} = \sqrt{1 - c_s^{-2}s^2}, \\ &\theta(s) = \frac{\theta}{|\xi'|_g^2} = 2 - c_s^{-2}s^2 - 2a(s)b(s), \quad k_j(s) = \frac{k_j}{|\xi'|_g^3}, \quad \text{for } j = 1,2. \end{aligned} \tag{5.10}$$

Then (5.9) is equivalent to

$$R(s) = \frac{R(\tau,\xi')}{|\xi'|_g^4} = 4\mu^2 a(s)b(s) - (\rho s^2 - 2\mu)^2 = 0. \tag{5.11}$$

It is well-known that at fixed point $(t,x')$, there exists a unique simple zero $s_0$ satisfying $R(s) = 0$ for $0 < s < c_s < c_p$. This zero $s_0$ corresponds to a wave called the Rayleigh wave and it is called the **Rayleigh speed** $c_R \equiv s_0 < c_s < c_p$. Rayleigh waves are first studied in [73]. Since $s_0$ is simple, i.e., $R'(s_0) \neq 0$, by the implicit function theorem we have the root of $R(s) = 0$ can be written as a smooth function $s_0(t,x')$ near a small neighborhood of the fixed point. Then we can write $m_1(t,x',\tau,\xi')$ as a product of $(s - s_0(t,x'))$ and an elliptic factor, i.e.

$$\widetilde{m}_1(t,x',\tau,\xi') = \frac{i}{|\xi'|_g - \alpha\beta} m_1(t,x',\tau,\xi') = e_0(t,x',\tau,\xi') i(\tau - c_R(t,x')|\xi'|_g), \tag{5.12}$$

where $e_0$ is nonzero and homogeneous in $(\tau, \xi')$ of order zero. There is a characteristic variety

$$\Sigma_R = \{(t, x', \tau, \xi'),\ \tau^2 - c_R^2(t, x')|\xi'|_g^2 = 0\}$$

corresponding to $\widetilde{m}_1 = 0$. In order to fully decouple the system up to smoothing operators, we want the three eigenvalues to be distinct. Notice this is not necessary in our situation, since with $m_2, m_3 > 0$ one can always decouple the system into a hyperbolic one and an elliptic system near $s_0$.

**Claim 2.** Near $s = c_R$, the eigenvalues $m_1(t, x', \tau, \xi'), m_2(t, x', \tau, \xi'), m_3(t, x', \tau, \xi')$ are distinct.

*Proof.* We already have $m_1 \neq m_2$. Additionally, one can show that $m_2 > m_3$ is always true by the following calculation

$$\begin{aligned}
m_2 - m_3 =& |\xi'|_g{}^3 \Big( \frac{(a+b)\rho s^2 + \sqrt{(a-b)^2\rho^2 s^4 + 4\mu^2(1+a^2-2ab)^2}}{2} - \mu a(1-ab) \Big) \\
>& |\xi'|_g{}^3 \Big( \frac{(a+b)\mu(1-a^2) + 2\mu(1+a^2-2ab)}{2} - \mu a(1-ab) \Big) \\
=& \frac{\mu}{2}\Big( (b-a) + a^2(b-a) + (a-b)^2 + 1 - b^2 \Big) > 0,
\end{aligned}$$

where $a(s), b(s)$ are defined in (5.10). The values of $m_1$ and $m_3$ might coincide but near $\Sigma_R$ they are separate, since $m_1$ is close to zero while $m_3 = \mu\alpha(|\xi'|_g^2 - \alpha\beta) > 0$ has a positive lower bound. Therefore, near $\Sigma_R$ we have three distinct eigenvalues. $\square$

Let $\widetilde{W}(t, x', D_t, D_{x'})$ be an elliptic ΨDO of order zero as constructed in [6] with the principal symbol equal to $W(t, x', \tau, \xi')$. Let the operators $e_0(t, x', D_t, D_{x'}) \in \Psi^0$ with symbol $e_0(t, x', \tau, \xi')$ and $\widetilde{m}_j(t, x', D_t, D_{x'}) \in \Psi^1$ with symbols $\frac{i}{|\xi'|_g - \alpha\beta} m_j(t, x', \tau, \xi')$, for $j = 2, 3$. Near some fixed $(t, x', \tau, \xi') \in \Sigma_R$, the DN map $\Lambda$ can be fully decoupled as

$$\widetilde{W}^{-1}\Lambda\widetilde{W} = \begin{pmatrix} e_0(\partial_t - ic_R(t, x')\sqrt{-\Delta_{x'}}) + r_1 & 0 & 0 \\ 0 & \widetilde{m}_2 + r_2 & 0 \\ 0 & 0 & \widetilde{m}_3 + r_3 \end{pmatrix} \mod \Psi^{-\infty},$$

where $r_1(t, x', D_t, D_{x'}), r_2(t, x', D_t, D_{x'}), r_3(t, x', D_t, D_{x'}) \in \Psi^0$ are the lower order term. If we define

$$r(t, x', D_t, D_x') = e_0^{-1} r_1 \in \Psi^0, \quad \lambda(t, x', D_{x'}) = c_R(t, x')\sqrt{-\Delta_{x'}} \in \Psi^1 \tag{5.13}$$

in what follows, then the first entry in the first row can be written as $e_0(\partial_t - i\lambda(t, x', D_{x'}) + r(t, x', D_t, D_x'))$.

**Remark 12.** Each entry of the matrix $\sigma_p(\Lambda)$ is homogeneous in $(\tau, \xi')$ of order 1 and that of $W(t, x', \tau, \xi')$ is homogeneous of order 0. The operator $e_0(t, x', D_t, D_x')$ has a homogeneous symbol, which implies its parametrix will have a classical one. After the diagonalization of the system, the operator $r_1(t, x', D_t, D_x')$ have a classical symbol, and so does $r(t, x', D_t, D_x')$.

Now let

$$h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \widetilde{W}^{-1} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \widetilde{W}^{-1} f, \qquad \tilde{l} = \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \widetilde{W}^{-1} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \widetilde{W}^{-1} l, \tag{5.14}$$

where $u_j$ is the component of any vector valued distribution $u$ for $j = 1, 2, 3$. Solving $\Lambda f = l \mod C^\infty$ is microlocally equivalent to solving the following system

$$\begin{cases} (\partial_t - i c_R(t, x')\sqrt{-\Delta_{x'}} + r(t, x', D_t, D_x'))h_1 = e_0^{-1}\tilde{l}_1, & \mod C^\infty, \\ (\widetilde{m}_2 + r_2)h_2 = \tilde{l}_2, & \mod C^\infty, \\ (\widetilde{m}_3 + r_3)h_3 = \tilde{l}_3, & \mod C^\infty. \end{cases} \tag{5.15}$$

In the last two equations, the operators $\widetilde{m}_j + r_j$ are elliptic so we have $h_j = (\widetilde{m}_j + r_j)^{-1}\tilde{l}_j \mod C^\infty$, for $j = 2, 3$. The first equation is a first-order hyperbolic equation with lower order term.

### 5.3.2 Inhomogeneous hyperbolic equation of first order

For convenience, in this subsection we use the notation $x$ instead of $x'$. Suppose $x \in \mathbb{R}^n$.

**Definition 7.** Let $\lambda(t, x, D_x) \in \Psi^1$ be an elliptic operator with a classical symbol smoothly depending on a parameter $t$ and the lower term $r(t, x, D_t, D_x) \in \Psi^0$ with a classical symbol.

In this subsection we are solving the inhomogeneous hyperbolic equation

$$\begin{cases} (\partial_t - i\lambda(t, x, D_x + r(t, x, D_t, D_x)))w = g(t, x), \quad t > 0 \\ w(0, x) = 0. \end{cases} \tag{5.16}$$

where $g(t, x) \in \mathcal{E}'((0, T) \times \mathbb{R}^n)$ with microsupport in the elliptic region.

**Remark 13.** If the density $\rho$ and the Lamé parameters $\lambda, \nu$ are time-dependent, then $\lambda, r$ depend on $t, x$. Otherwise, the eigenvalues $m_1, m_2, m_3$ only depends on $x, \xi, \tau$, and therefore we have $s_0(x), c_R(x)$ and $\lambda(x, \xi), r(x, D_t, D_x)$ instead of the functions and operators above.

Generally, the operator $\partial_t - i\lambda(t, x, D_x)$ is not a $\Psi$DO unless the principal symbol of $\lambda$ is smooth in $\xi$ at $\xi = 0$. However, since we only consider the elliptic region, we can always multiply it by a cutoff $\Psi$DO whose microsupport is away from $\xi = 0$ and this gives us a $\Psi$DO. Therefore, by the theorem of propagation of singularities by Hörmander, we have $\mathrm{WF}(w) \subset \mathrm{WF}(g) \cup C_F \circ \mathrm{WF}(g)$ if $w$ is the solution to (5.16), where $C_F$ is given by the flow of $H_{\partial_t - i\lambda(t,x,D_x)}$, for a more explicit form see (5.27).

### Homogeneous equations

We claim the homogeneous first-order hyperbolic equation with lower terms given an initial condition

$$\begin{cases} (\partial_t - i\lambda(t, x, D_x) + r(t, x, D_t, D_x))v = 0, \quad \mod C^\infty \\ v(0, x) = v_0(x) \in \mathcal{E}'(\mathbb{R}^n), \end{cases} \tag{5.17}$$

has a microlocal solution by the geometric optics construction

$$v(t, x) = \int a(t, x, \xi) e^{i\varphi(t,x,\xi)} \hat{v}_0(\xi) \, \mathrm{d}\xi, \qquad \mod C^\infty, \tag{5.18}$$

where we require $a(t,x,\xi) \in S^0$ and $\varphi(t,x,\xi)$ is a phase function that is smooth, real valued, homogeneous of order one in $\xi$ with $\nabla_x\varphi \neq 0$ on the conic support of $a$. These assumptions guarantees the oscillatory integral (5.18) is a well-defined Lagrangian distribution. The procedure presented in the following is based on the construction in [6, VIII.3].

If we suppose

$$(\partial_t - i\lambda(t,x,D_x) + r(t,x,D_t,D_x))v(t,x) = \int c(t,x,\xi)e^{i\varphi(t,x,\xi)}\hat{v}_0(\xi)\,\mathrm{d}\xi,$$

then we have

$$c(t,x,\xi) = i\varphi_t a + \partial_t a - ib + d,$$

where

$$b = e^{-i\varphi}\lambda(ae^{i\varphi}), \quad d = e^{-i\varphi}r(ae^{i\varphi})$$

have the asymptotic expansions according to the Fundamental Lemma. In the following we use the version given in [67].

**Lemma 6.** *[67, Theorem 3.1] Suppose $\phi(x,\theta)$ is a smooth, real-valued function for $x \in \Omega, \theta \in S^n$ and the gradient $\nabla_x\phi \neq 0$. Suppose $P(x,D_x)$ is a pseudodifferential operator of order $m$. Write $\phi(x,\theta) - \phi(y,\theta) = (x-y)\cdot\nabla\phi(y,\theta) - \phi_2(x,y)$, where $\phi_2(x,y) = \mathcal{O}(|x-y|^2)$. Then we have the asymptotic expansion*

$$e^{-i\rho\phi}P(x,D_x)(a(x)e^{i\rho\phi}) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha P(x,\rho\nabla_x\phi)\mathcal{R}_\alpha(\phi;\rho,D_x)a, \ \textit{for } \rho > 0, \tag{5.19}$$

*where we use the notation*

$$\mathcal{R}_\alpha(\phi;\rho,D_x)a = D_y^\alpha\{e^{i\rho\phi_2(x,y)}a(y)\}|_{y=x}. \tag{5.20}$$

**Remark 14.** Indeed, we can write

$$\phi(x,\theta)-\phi(y,\theta) = (x-y)\cdot\nabla\phi(y,\theta)+(x-y)^T\Big(\int_0^1\int_0^1(t\nabla_y^2\phi(y+st(x-y),\theta))\,\mathrm{d}s\,\mathrm{d}t\Big)(x-y),$$

which implies $-\phi_2(x,y)$ equals to the last term above. One can show $\mathcal{R}_\alpha(\phi;\rho,D_x)a$ is a polynomial w.r.t. $\rho$ with degree $\leq \lfloor|\alpha|/2\rfloor$. In particular, we have

$$\mathcal{R}_\alpha(\phi;\rho,D_x)a = D_x^\alpha a(x), \quad \text{for } |\alpha| = 1,$$

$$\mathcal{R}_\alpha(\phi;\rho,D_x)a = D_x^\alpha a(x) - i\rho D_x^\alpha\phi(x,\theta)a(x), \quad \text{for } |\alpha| = 2.$$

From Lemma 6, we have the following asymptotic expansions of $b, d$ by writing $(\tau, \xi) = \rho\theta, \varphi = \rho\phi(x,\theta)$, with $\rho > 0, \theta \in S^{n+1}$. We use the notation $\lambda^{(\alpha)} = \partial_\xi^\alpha$ and $r^{(\alpha)} = \partial^\alpha_{(\tau,\xi)} r$ to have

$$\begin{aligned} b &\sim \sum_\alpha \frac{1}{\alpha!}\lambda^{(\alpha)}(t,x,\rho\nabla_x\phi)\mathcal{R}_\alpha(\phi;\rho,D_x)a(t,x,\rho\theta) \\ &\sim \underbrace{\lambda(t,x,\rho\nabla_x\phi)a}_{\text{order } \leq 1} + \sum_{|\alpha|=1}\underbrace{\lambda^{(\alpha)}(t,x,\rho\nabla_\phi)D_x^\alpha a}_{\text{order } \leq 0} \\ &\quad + \sum_{|\alpha|=2}\frac{1}{\alpha!}(\underbrace{\lambda^{(\alpha)}(t,x,\rho\nabla_x\phi)D_x^\alpha a}_{\text{order } \leq -1} - i\underbrace{\lambda^{(\alpha)}(t,x,\rho\nabla_x\phi)(D_x^\alpha\varphi)a}_{\text{order } \leq 0}) + \underbrace{\dots}_{\text{order } \leq -1}, \end{aligned}$$

and

$$\begin{aligned} d &\sim \sum_\alpha \frac{1}{\alpha!}r^{(\alpha)}(t,x,\rho\nabla_{t,x}\phi)\mathcal{R}_\alpha(\phi;\rho,D_{t,x})a(t,x,\rho\theta) \\ &\sim \underbrace{r(t,x,\rho\nabla_{t,x}\phi)a}_{\text{order } \leq 0} + \sum_{|\alpha|=1}\underbrace{r^{(\alpha)}(t,x,\rho\nabla_{t,x}\phi)D_{t,x}^\alpha a}_{\text{order } \leq -1} + \sum_{|\alpha|=2}\frac{1}{\alpha!}(\underbrace{r^{(\alpha)}(t,x,\rho\nabla_{t,x}\phi)D_{t,x}^\alpha a}_{\text{order } \leq -2} \\ &\quad - i\underbrace{r^{(\alpha)}(t,x,\rho\nabla_{t,x}\phi)(D_{t,x}^\alpha\varphi)a)}_{\text{order } \leq -1} + \underbrace{\dots}_{\text{order } \leq -2}. \end{aligned}$$

Indeed for each fixed $\alpha$, the order of each term in the asymptotic expansion of $b$ is no more than $1 - |\alpha| + \lfloor |\alpha|/2 \rfloor$, that of $d$ is no more than $0 - |\alpha| + \lfloor |\alpha|/2 \rfloor$.

To construct the microlocal solution, we are finding proper $a(t,x,\xi)$ in form of $\sum_{j\leq 0} a_j(t,x,\xi)$, where $a_j \in S^{-j}$ is homogeneous in $\xi$ of degree $-j$. We also write $b, c, d$ as asymptotic expansion such that

$$c(t,x,\xi) \sim \sum_{j\leq 1} c_j(t,x,\xi) = (i\varphi_t - i\lambda(t,x,\nabla_x\varphi))a + \sum_{j\leq 0}(\partial_t a_j - ib_j + d_j),$$

where we separate the term of order 1 since it gives us the eikonal equation

$$c_1 = i(\varphi_t - \lambda_1(t,x,\nabla_x\varphi))a = 0 \quad \Rightarrow \quad \varphi_t = \lambda_1(t,x,\nabla_x\varphi), \qquad \text{with } \varphi(0,x,\xi) = x\cdot\xi \tag{5.21}$$

for the phase function. Then equating the zero order terms in $\xi$ we have

$$\partial_t a_0 - ib_0 + d_0 = c_0 = 0 \Rightarrow Xa_0 - \gamma a_0 = 0, \qquad \text{with } a_0(0,x,\xi) = 1, \tag{5.22}$$

where we set $X = \partial_t - \nabla_\xi \lambda_1 \cdot \nabla_x$ be the vector field and

$$\gamma = (i\lambda_0(t, x, \nabla_x \varphi) + \sum_{|\alpha|=2} \frac{1}{\alpha!} \lambda_1^{(\alpha)} D_x^\alpha \varphi - r_0(t, x, \lambda_1, \nabla_x \varphi).$$

Then for lower order terms, i.e. $j \leq -1$, we have

$$0 = c_j = Xa_j - \gamma a_j - e_j, \qquad \text{with } a_j(0, x, \xi) = 0, \tag{5.23}$$

where $e_j$ is expressible in terms of $\varphi, a_0, a_{-1}, \ldots, a_{j-1}, \lambda, r$. This finishes the construction in (5.18).

**Remark 15.** This construction of microlocal solution is valid in a small neighborhood of $t = 0$, since the Eikonal equation is locally solvable. However, we can find some $t_0 > 0$ such that the solution $v$ is defined and use the value at $t_0$ as the Cauchy data to construct a new solution for $t > t_0$, for the same arguments see [4, §3.1]

**Inhomogeneous equations when $r(t, x, D_t, D_x) = 0$.**

Now we are going to solve the inhomogeneous equation with zero initial condition. A simpler case would be when the lower order term $r(t, x, D_t, D_x)$ vanishes, i.e.

$$\begin{cases} (\partial_t - i\lambda(t, x, D_x))w = g(t, x), & t > 0 \\ w(0, x) = 0. \end{cases} \tag{5.24}$$

In this way the microlocal solution can be obtained by the Duhamel's principle. Indeed, let the phase function $\varphi(t, x, \xi)$, the amplitude $a(t, x, \xi)$ to be constructed for solutions to the homogeneous first order hyperbolic equation with an initial condition as in (5.17). More specifically, suppose the phase $\varphi(t, x, \xi)$ solves the eikonal equation (5.21) and the amplitude $a(t, x, \xi) = a_0 + \sum_{j \leq -1} a_j$ solves the transport equation (5.22) and (5.23) with $\gamma = (i\lambda_0 + \sum_{|\alpha|=2} \frac{1}{\alpha!} \lambda_1^{(\alpha)} D_x^\alpha \varphi)$. Then the solution to (5.24) up to a smooth error is given by

$$w(t, x) = \int H(t - s) a(t - s, x, \xi) e^{i(\varphi(t-s,x,\xi) - y\cdot\xi)} l(s, y) \, \mathrm{d}y \, \mathrm{d}\xi \, \mathrm{d}s \equiv L_{\varphi,a} l(t, x), \tag{5.25}$$

where we define $L_{\varphi,a}$ as the solution operator to (5.24) with the phase $\varphi$ and the amplitude $a$. Here, the kernel of $L_{\varphi,a}$

$$k_L(t,x,s,y) = H(t-s)\int a(t-s,x,\xi)e^{i(\varphi(t-s,x,\xi)-y\cdot\xi)}l(s,y)\,\mathrm{d}\xi$$

can be formally regarded as the product

$$k_L(t,x,s,y) = H(t-s)k_F(t,x,s,y) \tag{5.26}$$

of a conormal distribution $H(t-s)$ and a Lagrangian distribution where $k_F(t,x,s,y)$ is analyzed by the following claim.

**Claim 3.** The kernel $k_F$ defined by (5.26) is a Lagrangian distribution associated with the canonical relation

$$C_F = \{(\underbrace{t,x,\varphi_t(t-s,x,\xi),\varphi_x(t-s,x,\xi),s,y,\varphi_t(t-s,x,\xi),\xi}_{t,x,\hat{t},\hat{x},s,y,\hat{s},\hat{y}}),\ \text{with } y=\varphi_\xi(t-s,x,\xi)\}. \tag{5.27}$$

It is the kernel of an FIO $F_{\varphi,a}$ of order $-\frac{1}{2}$.

**Remark 16.** In Euclidean case, we have $\varphi(t,x,\xi) = t|\xi| + x\cdot\xi$. Then

$$C_F = \{(t,x,|\xi|,\xi,s,y,|\xi|,\xi),\ \text{with } x = y-(t-s)\frac{\xi}{|\xi|}\}.$$

Further, we show that $k_L$ is a distribution kernel such that the microlocal solution (5.25) is well-defined for any $g(t,x)\in\mathcal{E}'((0,T)\times\mathbb{R}^n)$ supported in the elliptic region.

**Proposition 17.** *The kernel $k_L$ is a well-defined distribution, with the twisted wave front set satisfying* $\mathrm{WF}'(k_L)\subset C_0\cup C_\Delta\cup C_F$, *where*

$$C_0 = \{(\underbrace{t,x,\mu,0,s,y,\mu,0}_{t,x,\hat{t},\hat{x},s,y,\hat{s},\hat{y}}),t=s,\mu\neq 0\},\quad C_\Delta = \{(\underbrace{t,x,\tau,\xi,s,y,\tau,\xi}_{t,x,\hat{t},\hat{x},s,y,\hat{s},\hat{y}}),\ t=s,x=y\}.$$

*If $g(t,x)\in\mathcal{E}'((0,T)\times\mathbb{R}^n)$ microlocally supported in the elliptic region, then $L_{\varphi,a}$ is a distribution with* $\mathrm{WF}(L_{\varphi,a}g)\subset\mathrm{WF}(g)\cup C_F\circ\mathrm{WF}(g)$, *where $C_F$ is defined in (5.27).*

*Proof.* We use [70, Theorem 8.2.10], with the assumption that the principal symbol of $\lambda(t,x,D_x)$ is homogeneous in $\xi$ of order one. By [66, Theorem 2.5.14], for $g(t,x) \in \mathcal{E}'((0,T)\times\mathbb{R}^n)$, since $\mathrm{WF}(k_L)$ has no zero sections, we have

$$\mathrm{WF}(L_{\varphi,a}g) \subset C_0 \circ \mathrm{WF}(g) \cup \mathrm{WF}(g) \cup C_F \circ \mathrm{WF}(g).$$

In particular, the first term in the right side is ignorable in general. However, if we assume there is no $(t,x,\mu,0) \in \mathrm{WF}(g)$, which is true if $g$ is microlocally supported in the elliptic region, then for $w$ satisfying equation (5.24), the wave front set $\mathrm{WF}(w) \subset \mathrm{WF}(g) \cup C_F \circ \mathrm{WF}(g)$. Especially for $t \geq T$, we have $\mathrm{WF}(w) \subset C_F \circ \mathrm{WF}(g)$. □

**Inhomogeneous equations with nonzero $r(t,x,D_t,D_x)$.**

When the lower oder term $r(t,x,D_t,D_x)$ is nonzero, the Duhamel's principle does not work any more. Instead, we can use the same iterative procedure as in [67, Section 5] to construct an operator $\tilde{e}(t,x,D_t,D_x) \in \Psi^{-1}$ such that

$$\partial_t - i\lambda(t,x,D_x) + r(t,x,D_t,D_x) = (I-\tilde{e})(\partial_t - i\lambda(t,x,D_x) + \sum_{j\leq 0}\tilde{r}_j(t,x,D_x)) \mod \Psi^{-\infty}. \tag{5.28}$$

Here each $\tilde{r}_j(t,x,D_x)$ has a classical symbol so does their sum. In particular, the principal symbol of $\tilde{r}_0(t,x,D_x)$ is $r_0(t,x,\lambda_1(t,x,\xi),\xi)$. The similar trick is performed for $\lambda$-pseudodifferential operators in [5].

In this way, the microlocal solution to the inhomogeneous hyperbolic equation can be written as

$$w(t,x) = L_{\varphi,a}(I-\tilde{e})^{-1}g \tag{5.29}$$

$$= \int H(t-s)a(t-s,x,\xi)e^{i(\varphi(t-s,x,\xi)-y\cdot\xi)}((I-\tilde{e})^{-1}g)(s,y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}s \tag{5.30}$$

where $L_{\varphi,a}$ is the solution operator of the inhomogeneous first order hyperbolic equation $(\partial_t - i\lambda(t,x,D_x) + \sum_{j=0}\tilde{r}_j(t,x,D_x))v = g$ with zero initial condition. Since $(I-\tilde{e})$ is an elliptic ΨDO with principal symbol equal to 1, we have the same conclusion for the wave front set of $w$ as the simpler case.

**Proposition 18.** *Assume $g(t,x') \in \mathcal{E}'((0,T)\times\mathbb{R}^2)$ microlocally supported in the elliptic region. Then the inhomogeneous first-order hyperbolic equation (5.16) admits a unique microlocal solution given by (5.30), where the phase function $\varphi(t,x,\xi)$ and the amplitude $a(t,x,\xi)$ are constructed for the operator $(\partial_t - i\lambda(t,x,D_x) + \sum_{j=0}\tilde{r}_j(t,x,D_x))$ in (5.28), as in Subsection 5.3.2. More specifically, the phase $\varphi(t,x,\xi)$ solves the eikonal equation (5.21); the amplitude $a(t,x,\xi) = a_0 + \sum_{j\le -1} a_j$ solves the transport equation (5.22) and (5.23) with $\gamma = (i\lambda_0(t,x,\xi) + \sum_{|\alpha|=2}\frac{1}{\alpha!}\lambda_1^{(\alpha)} D_x^\alpha \varphi - r_0(t,x,\lambda_1,\xi))$.*

*Proof.* To verify the parametrix, we still need to show that if $w$ is the solution to (5.16) with $g \in C^\infty$ and $w(0,x) \in C^\infty$, then $w \in C^\infty$ as well. By (5.28) it suffices to show this is true when the lower order term $r(t,x,D_t,D_x)$ can be reduced to the form $r(t,x,D_x)$ or vanishes. One can verify that the operator $\partial_t - i\lambda(t,x,D_x) + r(t,x,D_x)$ is symmetric hyperbolic as is defined in [6]. Then following the same arguments there, by a standard hyperbolic estimates, one can show $w$ is smooth. □

### 5.3.3 The Cauchy problem and the polarization

In this subsection, before assuming the source $l \in \mathcal{E}'((0,T)\times\Gamma)$ and solving the inhomogeneous equation (5.2) with zero initial condition, we first assume that the source exists for a limited time for $t<0$ and we have the Cauchy data $f|_{t=0}$ at $t=0$, i.e.

$$\Lambda f = 0, \text{ for } t>0, \quad f|_{t=0} \text{ given }. \tag{5.31}$$

Recall the diagonalization of $\Lambda$ in (5.14, 5.15). The homogeneous equation $\Lambda f = 0$ implies

$$f = \widetilde{W} h = \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} h_1 \mod C^\infty \Leftarrow h_2 = h_3 = 0 \mod C^\infty, \tag{5.32}$$

where $h_1$ solves the homogeous first-order hyperbolic equation in (5.15). Notice in this case the hyperbolic operator is $\partial_t - ic_R(t,x')|\xi'|_g + r(t,x',D_t,D_{x'})$ with $r$ given by (5.13). If we have the initial condition $h_{1,0} \equiv h_1|_{t=0}$, then by the construction in Subsection 5.3.2, then

$$h_1(t,x') = \int a(t,x',\xi')e^{i\varphi(t,x',\xi')}\hat{h}_{1,0}(\xi')\,\mathrm{d}\xi' \quad \mathrm{mod}\ C^\infty, \tag{5.33}$$

where the phase function $\varphi$ solves the eikonal equation (5.21); the amplitude $a = a_0 + \sum_{j\leq -1} a_j$ solves the transport equation (5.22) and (5.23) with $\gamma = (\sum_{|\alpha|=2}\Big(\frac{1}{\alpha!}c_R(t,x')D^\alpha_{\xi'}|\xi'|_g D^\alpha_{x'}\varphi - r_0(t,x',\lambda_1,\nabla_{x'}\varphi)\Big)$.

To find out how the initial condition of $f$ is related to that of $h$, we plug (5.33) into (5.32), use the Fundamental Lemma in Lemma 6, and set $t = 0$. Since $\varphi(0,x',\xi') = x'\cdot\xi'$, after these steps we get three ΨDOs of order zero, of which the symbols can be computed from the Fundamental Lemma, such that

$$f|_{t=0} = \begin{pmatrix}\widetilde{W}_{11,0}\\ \widetilde{W}_{21,0}\\ \widetilde{W}_{31,0}\end{pmatrix} h_{1,0} \quad \mathrm{mod}\ C^\infty. \tag{5.34}$$

In particular, the principal symbols are

$$\sigma_p(\widetilde{W}_{11,0}) = \sigma_p(W_{11})(0,x',c_R|\xi'|_g,\xi'), \qquad \sigma_p(\widetilde{W}_{21,0}) = \sigma_p(W_{21})(0,x',c_R|\xi'|_g,\xi'),$$
$$\sigma_p(\widetilde{W}_{31,0}) = \sigma_p(W_{31})(0,x',c_R|\xi'|_g,\xi')$$

and by (5.8) they are elliptic ΨDOs. This indicates not any arbitrary initial conditions can be imposed for (5.31). Instead, to have a compatible system, we require that there exists some distribution $h_0$ such that $f(0,x')$ can be written in form of (5.34).

**Theorem 10.** *Suppose $f(0,x')$ satisfies (5.34) with some $h_{1,0} \in \mathcal{E}'$. Then microlocally the homogeneous problem with Cauchy data (5.31) admits a unique microlocal solution*

$$
\begin{aligned}
f &= \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} \int a(t,x',\xi') e^{i\varphi(t,x',\xi')} \hat{h}_{1,0}(\xi') \,\mathrm{d}\xi' \mod C^\infty \\
&= \int \underbrace{\begin{pmatrix} i\mu\theta(c_R) \nabla_{x_1}\varphi / |\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R) \nabla_{x_2}\varphi / |\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2(t,x') \end{pmatrix} \frac{a_0(t,x',\xi') e^{i\varphi(t,x',\xi')}}{k_1(c_R)}}_{\mathcal{P}} \hat{h}_{1,0}(\xi') \,\mathrm{d}\xi' + \text{ lower order terms,}
\end{aligned}
\tag{5.35}
$$

*where $c_R$ is the Rayleigh speed, $b(s), \theta(s), k_1(s)$ are defined in (5.10), and $a(t,x',\xi')$ is the amplitude from the geometric optics construction with the highest order term $a_0(t,x',\xi')$.*

With $f$, one can construct the real displacement $u$ as an evanescent mode. Notice this theorem gives us a local representation of $f$ in the sense of Remark 15.

To understand the polarization of microlocal solution in the theorem above, if we write the real and imaginary part of the term $\mathcal{P}$ separately, then we have

$$
\Re(\mathcal{P}) = \frac{|a_0(t,x',\xi')|}{k_1(c_R)} \begin{pmatrix} \mu\theta(c_R)\cos(\varphi + \upsilon + \pi/2) \nabla_{x_1}\varphi / |\nabla_{x'}\varphi|_g \\ \mu\theta(c_R)\cos(\varphi + \upsilon + \pi/2) \nabla_{x_2}\varphi / |\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2(t,x') \sin(\varphi + \upsilon + \pi/2) \end{pmatrix},
$$

$$
\Im(\mathcal{P}) = i\frac{|a_0(t,x',\xi')|}{k_1(c_R)} \begin{pmatrix} i\mu\theta(c_R)\cos(\varphi + \upsilon) \nabla_{x_1}\varphi / |\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R)\cos(\varphi + \upsilon) \nabla_{x_2}\varphi / |\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2(t,x') \sin(\varphi + \upsilon) \end{pmatrix},
$$

where we assume $\upsilon = \arg(a_0(t,x',\xi'))$. It follows that the solution in (5.35) can be regarded as a superposition of $\Re(\mathcal{P})$ and $\Im(\mathcal{P})$. We are going to show that each of them has a **retrograde elliptical motion** in the following sense.

Take the real part $\Re(\mathcal{P})$ as an example, by assuming $\hat{h}_{1,0}(\xi')$ is real. Then the real part of the displacement $f(t,x')$ on the boundary is the superposition of

$\Re(\mathcal{P}) \equiv (p_1, p_2, p_3)^T$. On the one hand, we have the following equation satisfied by the components

$$\frac{p_1^2}{|\mu\theta(c_R)|^2} + \frac{p_2^2}{|\mu\theta(c_R)|^2} + \frac{p_3^2}{|b(c_R)\rho c_R^2|^2} = \frac{|a_0(t, x', \xi')|^2}{k_1^2(c_R)}$$

which describes an ellipsoid.

On the other hand, the rotational motion of the particle has the direction opposite to its orbital motion, i.e. the direction of the propagation of singularities. Indeed, if we consider the Euclidean case, we have $a_0(t, x', \xi') = 1$ and $\varphi(t, x', \xi') = t|\xi'| + x' \cdot \xi'$. It follows that $v = 0$. Locally the singularities propagate along the path $-\frac{\xi'}{|\xi|}t + y'$ in the direction of $(-\xi', 0)$ on the boundary $x^3 = 0$, where $y'$ is the initial point. W.O.L.G. assume $\xi^1, \xi^2 > 0$ in what follows. In this case, we have

$$\Re(\mathcal{P}) = \frac{1}{k_1(c_R)} \begin{pmatrix} \mu\theta(c_R)\cos(t|\xi'| + x' \cdot \xi' + \pi/2)\xi_1/|\xi'| \\ \mu\theta(c_R)\cos(t|\xi'| + x' \cdot \xi' + \pi/2)\xi_2/|\xi'| \\ b(c_R)\rho c_R^2 \sin(t|\xi'| + x' \cdot \xi' + \pi/2) \end{pmatrix}.$$

Notice $t|\xi'| + x' \cdot \xi' + \pi/2$ is the angle between the vector $(p_1, p_2, p_3)^T$, as the direction of strongest singularity of the solution, and the plane $x^3 = 0$ as in Figure 5.1. At each fixed point, the polarized vector $(p_1, p_2, p_3)^T$ rotates in the direction that the angle increases as $t$ increases, i.e. the clockwise direction, while the singularities propagates in the counterclockwise direction. Therefore, we have a retrograde motion.

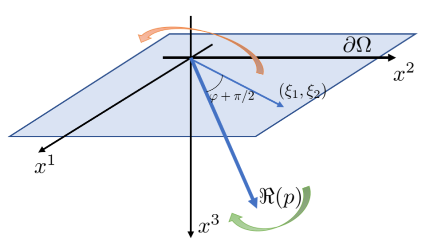



Figure 5.1. Propagation of the wave and the rotation of the polarization.

In the general case, notice by (5.22) the leading amplitude satisfies

$$Xa_0 - \gamma a_0 = 0, \qquad \text{with } a_0(0, x', \xi') = 1,$$

where

$$X = \partial_t - \nabla_{\xi'}\lambda_1 \cdot \nabla_{x'}, \quad \gamma = (\sum_{|\alpha|=2} \frac{1}{\alpha!}\lambda_1^{(\alpha)} D_{x'}^{\alpha}\varphi - r_0(t, x', \lambda_1, \nabla_{x'}\varphi),$$

with $\lambda_1 = c_R|\xi'|_g$. In the following assume the Rayleigh speed $c_R$ only depends on the space variable $x'$ and we write $\tilde{g} = c_R^{-2}g$. Let

$$\gamma = \gamma_1(t, x', \xi') + \gamma_2(t, x', \xi')i$$

be the decomposition of the real and imaginary part. We emphasize that the imaginary part $\gamma_2(t, x', \xi')$ comes from that of $r_0(t, x', \lambda_1, \nabla_{x'}\varphi)$, the lower order term in the decoupled system, and it is a classical symbol of order zero.

The integral curve of $X$ is the unit speed geodesic $(t, \sigma_{y', -\xi'/|\xi'|_{\tilde{g}}}(t))$ and for simplification we omit the initial point $y'$ and the initial direction $-\xi'/|\xi'|_{\tilde{g}}$ to write it as $(t, x'(t))$ for the moment. Then the solution of the transport equation is

$$a_0(t, x'(t), \xi') = e^{-\int_0^t \gamma(s, x'(s), \xi')\,\mathrm{d}s} = e^{-\int_0^t \gamma_1(s, x'(s), \xi')\,\mathrm{d}s} \cdot e^{-i\int_0^t \gamma_2(s, x'(s), \xi')\,\mathrm{d}s},$$

which implies $\upsilon = -\int_0^t \gamma_2(s, x'(s), \xi')\,\mathrm{d}s$ and $\upsilon$ is a classical symbol of order zero. To find out at each fixed point $x'$ how the polarization rotates as the time changes, we compute the time derivative

$$(\varphi + \upsilon)_t = |\nabla_{x'}\varphi(t, x', \xi')|_{\tilde{g}} + \upsilon_t, \tag{5.36}$$

which is positive for $\xi'$ large enough since $\upsilon_t$ is a symbol of order zero. This indicates for large $\xi'$ the angle $\varphi + \upsilon$ increases as $t$ increases and the rotation of the polarization vector is still clockwise, if we assume at the fixed point $\nabla_{x'}\varphi$ points in the direction that $x^1, x^2$ increase.

In this case, the singularities propagates along the null characteristics of $\partial_t - \lambda_1$. Particularly, the wave propagates along the geodesics $\sigma_{y', -\xi'/|\xi'|_{\tilde{g}}}(t)$, where $y'$ is the initial point. By Remark 17, along the geodesics we have $\nabla_{x'}\varphi = -|\xi'|_{\tilde{g}}\tilde{g}\dot{\sigma}_{y', -\xi'/|\xi'|_{\tilde{g}}}(t)$, which is exactly the opposite direction where the wave propagates. Since we assume $\nabla_{x'}\varphi$ points in the direction that $x^1, x^2$ increase, the wave propagates in the counter-clockwise direction. Therefore, we have a retrograde elliptical motion as before.

**Remark 17.** Notice in (5.36), the phase function $\varphi$ is always the dominated term for large $\xi'$. By (5.21), the phase function $\varphi$ satisfies the Eikonal equation

$$\varphi_t = |\nabla_{x'}\varphi|_{\tilde{g}}, \qquad \text{with } \varphi(0,x,\xi) = x \cdot \xi.$$

By [Geometric Optics], we solve it locally by the method of characteristics. Let $H(t,x,\tau,\eta) = \tau - c_R|x'|_{\tilde{g}}$ be the Hamiltonian. First we find the Hamiltonian curves by solving the system

$$\begin{aligned} \dot{t}(s) &= H_\tau = 1, & \dot{x'}(s) &= H_{\xi'} = -g^{-1}\xi'/|\xi'|_{\tilde{g}}, \\ \dot{\tau}(s) &= -H_t = 0, & \dot{\eta'}(s) &= -H_{x'} = (\partial_{x'} g^j k)\xi_j\xi_k/|\xi'|_{\tilde{g}}, \end{aligned}$$

where $s$ is the parameter and we set $\eta(s) = \nabla_{x'}\varphi(t,x',\xi')$. This system corresponds to the unit speed geodesic flow

$$x'(t) = \sigma_{y',-\xi'/|\xi'|_{\tilde{g}}}(t), \quad \eta'(t) = -|\xi'|_{\tilde{g}}\tilde{g}\dot{\sigma}_{y',-\xi'/|\xi'|_{\tilde{g}}}(t),$$

where $y' = x'(0)$.

### 5.3.4 The inhomogeneous problem

In this subsection, we solve the inhomogeneous problem (5.2). We apply Proposition 18 to the first equation in (5.15) with zero initial condition. Recall $\tilde{l}_1$ is defined in (5.14) and $e_0$ in (5.12). Then the first equation with with zero initial condition has a unique microlocal solution

$$h_1(t,x') = \int H(t-s)a(t-s,x',\xi')e^{i(\varphi(t-s,x',\xi')-y'\cdot\xi')}((I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1)(s,y')\,\mathrm{d}y'\,\mathrm{d}\xi'\,\mathrm{d}s, \tag{5.37}$$

where the phase function $\varphi(t,x',\xi')$ and the amplitude $a(t,x',\xi')$ are given by Proposition 18 with the hyperbolic operator being $\partial_t - ic_R(t,x')\sqrt{-\Delta_{x'}} + r(t,x',D_t,D'_x)$. We can also write the solution as $h_1 = L_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1$ by (5.29). This proves the following proposition.

**Theorem 11.** *Suppose $l(t,x') \in \mathcal{E}'((0,T)\times\mathbb{R}^2,\mathbb{C}^3)$ microlocally supported in the elliptic region. The inhomogeneous system (5.2) with zero initial condition at $t=0$ admits a unique microlocal solution*

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \widetilde{W} \begin{pmatrix} L_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1 \\ (\widetilde{m}_2+r_2)^{-1}\tilde{l}_2 \\ (\widetilde{m}_3+r_3)^{-1}\tilde{l}_3 \end{pmatrix}, \text{ where } \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \widetilde{W}^{-1} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \mod C^\infty. \tag{5.38}$$

Recall that we assume the microsupport of $l(t,x)$ is supported in $(0,T)\times\mathbb{R}^2$. Since $\widetilde{W}^{-1} = (\widetilde{W}^{-1})_{ij}$ for $i,j=1,2,3$ is a matrix-valued ΨDO, so does the microsupport of $\tilde{l}(t,x)$. Similarly, the microsupport of $h_2, h_3, (I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1$ are supported in $(0,T)\times\mathbb{R}^2$ as well.

In the following, we are going to find the polarization of the microlocal solution (5.38) outside the microsupport of $l(t,x')$. In other words, we only consider the solution when the source vanishes and we always assume $t\geq T$. With this assumption, the Heaviside function in $k_L$ is negligible, i.e.

$$h_1(t,x) = F_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1,$$

where $F_{\varphi,a}$ is the FIO defined in (3) associated with the canonical relation $C_F$. Additionally, we have $h_2 \in C^\infty,\ h_2 \in C^\infty$ when $t \geq T$. Then for $t\geq T$, the microlocal solution (5.38) is

$$f = \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} F_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1 \mod C^\infty, \tag{5.39}$$

where $\tilde{l}_1 = (\widetilde{W}^{-1})_{11}l_1 + (\widetilde{W}^{-1})_{12}l_2 + (\widetilde{W}^{-1})_{13}l_3$. To find out the leading term of the amplitude of the solution, we need to find the leading term $a_0$ of the amplitude of $h_1$, the solution to the following transport equation

$$(\partial_t - \nabla_{\xi'}\lambda_1\cdot\nabla_{x'})a_0 - (\sum_{|\alpha|=2}\frac{1}{\alpha!}\lambda_1^{(\alpha)}D_{x'}^\alpha\varphi - r_0(t,x',\lambda_1,\nabla_{x'}\varphi))a_0 = 0, \text{ with } a_0(0,x',\xi') = 1.$$

Here the zero order term $r_0(t,x,\lambda_1,\xi)$ of the $\Psi$DO $r(t,x',D_t,D_{x'})$ is involved. The procesure of computing $r_0$ is in the Appendix A. Then by Lemma 6, the leading term is given by

$$f \approx \int \begin{pmatrix} \sigma_p(\widetilde{W}_{11})(t,x,\partial_t\varphi,\nabla_{x'}\varphi) \\ \sigma_p(\widetilde{W}_{21})(t,x,\partial_t\varphi,\nabla_{x'}\varphi) \\ \sigma_p(\widetilde{W}_{31})(t,x,\partial_t\varphi,\nabla_{x'}\varphi) \end{pmatrix} a_0(t-s,x,\xi)e^{i\varphi}\sigma_p(e_0^{-1})(t,x,\partial_t\varphi,\nabla_x\varphi)\hat{\hat{l}}_1(s,\xi')\,\mathrm{d}s\,\mathrm{d}\xi'.$$

Recall $\sigma_p(\widetilde{W}) = \sigma_p(W)$ given in (5.8) and the definition of $e_0$ in (5.12). The leading term equals to

$$\begin{aligned}
&[\frac{a_0(t-s,x',\xi')\sigma_p(e_0^{-1})(t,x,\partial_t\varphi,\nabla_{x'}\varphi)}{\sqrt{(\beta\rho\tau^2-m_1)^2+|\xi'|_g^2\mu^2\theta^2}} \begin{pmatrix} i\mu\theta\xi_1/|\xi'|_g \\ i\mu\theta\xi_2/|\xi'|_g \\ \beta\rho\tau^2 \end{pmatrix}]\,|_{\tau=\partial_t\varphi,\ \xi'=\nabla_{x'}\varphi} \\
=&\frac{a_0(t-s,x',\xi')\sigma_p(e_0^{-1})(t,x,\partial_t\varphi,\nabla_{x'}\varphi)}{\sqrt{b(s_0)^2\rho^2c_R^2(t,x')+\mu^2\theta^2(s_0)}} \begin{pmatrix} i\mu\theta(c_R)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2(t,x') \end{pmatrix} \\
=&\iota(c_R)a_0(t-s,x',\xi') \begin{pmatrix} i\mu\theta(c_R)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2 \end{pmatrix} := \mathcal{P}
\end{aligned}$$

with

$$\iota(c_R) \equiv \frac{\sigma_p(e_0^{-1})(t,x',\partial_t\varphi,\nabla'_x\varphi)}{\sqrt{b(c_R)^2\rho^2c_R^2(t,x')+\mu^2\theta^2(c_R)}} = \frac{1}{R'(c_R)}\sqrt{\frac{a(c_R)+b(c_R)}{b(c_R)}}$$

where we use the notation $a(s), b(s), \theta(s), R(s)$ defined in (5.10) and combine (5.6,5.9). Further, taking the phase function into consideration, we have the real and the imaginary part of the integrand equal to

$$\Re(\mathcal{P}) = \iota(c_R)|a_0(t-s,x,\xi)| \begin{pmatrix} \mu\theta(c_R)\cos(\varphi+\upsilon)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ \mu\theta(c_R)\cos(\varphi+\upsilon)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2\sin(\varphi+\upsilon) \end{pmatrix},$$

$$\Im(\mathcal{P}) = i\iota(c_R)|a_0(t-s,x,\xi)| \begin{pmatrix} \mu\theta(c_R)\sin(\varphi+\upsilon)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ \mu\theta(c_R)\sin(\varphi+\upsilon)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ -b(c_R)\rho c_R^2\cos(\varphi+\upsilon) \end{pmatrix},$$

where we set $\varphi = \varphi(t-s,x,\xi) - y\cdot\xi$ and write $\upsilon = \arg(a_0)$. This implies a retrograde elliptical motion of $f$ as we stated before.

Moreover, if we expand the term $\hat{\hat{l}}_1$, then the microlocal solution in (5.39) can be written as

$$f = \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} F_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\begin{pmatrix} \widetilde{W}_{11}^{-1} & \widetilde{W}_{12}^{-1} & \widetilde{W}_{13}^{-1} \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \mod C^\infty.$$

Similarly, by the Fundamental Lemma we compute the leading term in the amplitude

$$\begin{aligned}
&a_0(t-s,x,\xi)\begin{pmatrix} \sigma_p(\widetilde{W}_{11}) \\ \sigma_p(\widetilde{W}_{21}) \\ \sigma_p(\widetilde{W}_{31}) \end{pmatrix}\sigma_p(e_0^{-1})\begin{pmatrix} \sigma_p(\widetilde{W}_{11}^{-1}) & \sigma_p(\widetilde{W}_{12}^{-1}) & \sigma_p(\widetilde{W}_{13}^{-1}) \end{pmatrix}(t,x,\partial_t\varphi,\nabla_x\varphi) \\
&= [a_0(t-s,x,\xi)\frac{\sigma_p(e_0^{-1})}{k_1^2}\begin{pmatrix} i\mu\theta\xi_1/|\xi'|_g \\ i\mu\theta\xi_2/|\xi'|_g \\ \beta\rho\tau^2 - m_1 \end{pmatrix}\begin{pmatrix} -i\mu\theta\xi_1 & -i\mu\theta\xi_2 & \beta\rho\tau^2-m_1 \end{pmatrix}]\,|_{\tau=\partial_t\varphi,\ \xi'=\nabla_{x'}\varphi} \\
&= \frac{a_0(t-s,x,\xi)}{b(c_R)\rho c_R^2 R'(c_R)}\begin{pmatrix} i\mu\theta(c_R)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2 \end{pmatrix}\times \qquad (5.40) \\
&\qquad \begin{pmatrix} -i\mu\theta(c_R)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g & -i\mu\theta(c_R)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g & b(c_R)\rho c_R^2 \end{pmatrix},
\end{aligned}$$

where the last equality comes from that $\frac{\sigma_p(e_0^{-1})}{k_1^2} = \frac{(\tau - c_R|\xi'|_g)m_2}{(\beta\rho\tau^2 - m_1)R(\tau,\xi')(m_2-m_1)}$. Therefore, we have the following theorem.

**Theorem 12.** *Assume everything in Theorem 11. For $t \geq T$, the displacement on the boundary equals to*

$$f = \int \frac{a_0(t-s,x,\xi)e^{i\varphi}}{b(c_R)\rho c_R^2 R'(c_R)} \begin{pmatrix} i\mu\theta(c_R)\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ i\mu\theta(c_R)\nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ b(c_R)\rho c_R^2 \end{pmatrix} (-i\mu\theta(c_R)\hat{l}_1(s,\xi')\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g +$$
$$- i\mu\theta(c_R)\hat{l}_2(s,\xi')\nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g + b(c_R)\rho c_R^2\hat{l}_3(s,\xi'))\,\mathrm{d}s\,\mathrm{d}\xi' + \textit{ lower order terms }. \tag{5.41}$$

This theorem describes the microlocal polarization up to lower order terms of the displacement of Rayleigh waves on the boundary, when there is a source $(l_1, l_2, l_3)^T$ microlocally supported in the elliptic region with compact support. Up to lower order terms, the displacement can also be regarded as a supposition of the real part and imaginary part of the leading term and each of them has a elliptical retrograde motion as we discussed before. Indeed, the leading term has a similar pattern of that in Theorem 10, except different scalar functions in each component.

### 5.3.5 Flat case with constant coefficients

In this subsection, we suppose the boundary $\Gamma$ is flat given by $x^3 = 0$ with Euclidean metric. Suppose the parameters $\lambda, \mu, \rho$ are constants.

In this case, by using the partition of unity, the elastic wave equation can be fully decoupled as

$$U_0^{-1}EU_0 = \begin{pmatrix} -c_s^2\Delta I_2 & 0 \\ 0 & -c_p^2\Delta \end{pmatrix}$$

and the solution is $u = U_0 w$, where

$$w(t,x) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \int e^{i(t\tau + x'\cdot\xi')} \begin{pmatrix} e^{-\alpha x^3}\hat{f}_1(\tau,\xi') \\ e^{-\alpha x^3}\hat{f}_2(\tau,\xi') \\ e^{-\beta x^3}\hat{f}_3(\tau,\xi') \end{pmatrix} \mathrm{d}\tau\,\mathrm{d}\xi'$$

with $f_i$ as the solution to $\Lambda f = l$, where

$$\alpha = \sqrt{|\xi'|_g^2 - c_s^{-2}\tau^2}, \quad \beta = \sqrt{|\xi'|_g^2 - c_p^{-2}\tau^2}.$$

To solve $\Lambda f = l$, we perform the same procedure as before. By verifying $\sigma_0(\Lambda) = 0$, we have

$$W^{-1}\Lambda W = \begin{pmatrix} e_0(\partial_t - ic_R\sqrt{-\Delta_{x'}}) & 0 & 0 \\ 0 & \widetilde{m}_2 & 0 \\ 0 & 0 & \widetilde{m}_3 \end{pmatrix},$$

where $e_0$ is a ΨDO of order zero with symbol

$$\sigma(e_0)^{-1} = \frac{(|\xi'|_g^2 - \alpha\beta)(\tau - c_R|\xi'|)}{m_1},$$

where we use some notation as before

$$\theta = (2 - c_s^{-2})|\xi'|^2 - 2\alpha\beta, \qquad m_1 = \frac{(\alpha+\beta)\rho\tau^2 - \sqrt{\varrho}}{2}, \text{ with } \varrho = (\alpha-\beta)^2\rho^2\tau^4 + 4\mu^2\theta^2|\xi'|^2.$$

Combining the microlocal solution (5.18) to the homogeneous hyperbolic equation and that of the inhomogeneous one with zero initial condition in (5.63), we have the solution to the inhomogeneous one with arbitrary initial condition

$$\begin{cases} (\partial_t - ic_R\sqrt{-\Delta_{x'}})h_1 = e_0^{-1}(W^{-1}l)_1 \equiv g, \quad t > 0 \\ h_1(0, x') = h_0(x') \end{cases}$$

has the following solution

$$h_1(t,x') = \int e^{i(tc_R|\xi'|_g + x'\cdot\xi')}\hat{h}_0(\xi')\,\mathrm{d}\xi' + \int H(t-s)e^{i((t-s)c_R|\xi'|_g + x'\cdot\xi')}\hat{g}(s,\xi')\,\mathrm{d}s\,\mathrm{d}\xi' \tag{5.42}$$

and one can show

$$\hat{h}_1(\tau,\xi') = \delta(c_R|\xi'| - \tau)\hat{h}_0(\xi') + \hat{H}(\tau - c_R|\xi'|)\hat{g}(\tau,\xi')$$

by directly taking the Fourier transform. Since all ΨDOs involved here have symbols free of $x'$, then they are Fourier multipliers and we have

$$\hat{g}(\tau,\xi') = \sigma(e_0^{-1})\sum_k \sigma(W^{-1})_{1k}\hat{l}_k(\tau,\xi'),$$

where $\sigma(W)$ given in (5.8) is unitary and therefore $\sigma(W^{-1}) = \sigma(W^*) = \sigma(W)^*$.

The last two equations after we diagonalize the DN map have the following solutions

$$\hat{h}_2(\tau,\xi') = \sigma(\tilde{m}_2)^{-1}\sum_k \sigma(W^{-1})_{2k}\hat{l}_k(\tau,\xi'), \quad \hat{h}_3(\tau,\xi') = \sigma(\tilde{m}_3)^{-1}\sum_k \sigma(W^{-1})_{3k}\hat{l}_k(\tau,\xi').$$

Thus, the displacement on the boundary is given by $\hat{f}(\tau,\xi') = \sigma(W)\hat{h}(\tau,\xi')$.

**Example 15.** In the following assume we have a time-periodic source

$$l(t,x') = \begin{pmatrix} A_1, & A_2, & A_3 \end{pmatrix}^T e^{ipt}\delta(x_1),$$

where $p$ is a positive number and $A_1, A_2, A_3$ are constants. This gives us a line source on the boundary.

Furthermore, we assume $A_1 = A_2 = 0$. In this case $\mathcal{F}_{x'}l_3(s,\xi') = A_3 e^{ips}\delta(\xi_2)$. Since the amplitude $a_0(t-s,x',\xi') \equiv 1$, by (5.41,5.42) the displacement away from the support of the source up to lower order terms equals to

$$\begin{aligned}
f(t,x') &= \int \frac{e^{i(tc_R|\xi'|+x'\cdot\xi')}}{\sqrt{b(c_R)(b(c_R)+a(c_R))\rho c_R^2}} \begin{pmatrix} i\mu\theta(c_R)\xi_1/|\xi'| \\ i\mu\theta(c_R)\xi_2/|\xi'| \\ b(c_R)\rho c_R^2 \end{pmatrix} \hat{h}_0(\xi')\,\mathrm{d}\xi' + \text{lower order terms} \\
&\quad + \int H(t-s)\frac{-ie^{i((t-s)c_R|\xi'|+x'\cdot\xi')}}{b(c_R)\rho c_R^2 R'(c_R)} \begin{pmatrix} i\mu\theta(c_R)\xi_1/|\xi'| \\ i\mu\theta(c_R)\xi_2/|\xi'| \\ b(c_R)\rho c_R^2 \end{pmatrix} b(c_R)\rho c_R^2 A_3 e^{ips}\delta(\xi_2)\,\mathrm{d}s\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \\
&= \int \frac{e^{i(tc_R|\xi_1|+x_1\xi_1)}}{\sqrt{b(c_R)(b(c_R)+a(c_R))\rho c_R^2}} \begin{pmatrix} i\mu\theta(c_R)\xi_1/|\xi_1| \\ 0 \\ b(c_R)\rho c_R^2 \end{pmatrix} \hat{h}_0(\xi')\,\mathrm{d}\xi' + \text{lower order terms} \\
&\quad + \frac{iA_3}{R'(c_R)} \begin{pmatrix} i\mu\theta(c_R)\Big(e^{i(t+\frac{x_1}{c_R})p} - e^{i(t-\frac{x_1}{c_R})p}\Big) \\ 0 \\ b(c_R)\rho c_R^2\Big(e^{i(t+\frac{x_1}{c_R})p} + e^{i(t-\frac{x_1}{c_R})p}\Big) \end{pmatrix} + \frac{A_3}{R'(c_R)} \int \frac{e^{i(pt+x_1\cdot\xi_1)}}{p-c_R|\xi_1|} \begin{pmatrix} i\mu\theta(c_R)\xi_1/|\xi_1| \\ 0 \\ b(c_R)\rho c_R^2 \end{pmatrix} \mathrm{d}\xi_1.
\end{aligned}$$

If we choose the Cauchy data as the inverse Fourier transform of the tempered distribution $\hat{h}_{1,0}(\xi') = -\frac{A_3\sqrt{b(c_R)(b(c_R)+a(c_R))}\rho c_R^2}{R'(c_R)} e^{i(p-c_R|\xi_1|)}/(p-c_R|\xi_1|)$, then we have

$$f(t,x') = \frac{iA_3}{R'(c_R)}\begin{pmatrix} i\mu\theta(c_R)\Big(e^{i(t+\frac{x_1}{c_R})p} - e^{i(t-\frac{x_1}{c_R})p}\Big) \\ 0 \\ b(c_R)\rho c_R^2\Big(e^{i(t+\frac{x_1}{c_R})p} + e^{i(t-\frac{x_1}{c_R})p}\Big)\end{pmatrix} + \text{ lower order terms,}$$

which coincides with the results in [Lamb].

**Example 16.** In this example assume we have $l = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}\delta(t)\delta(x_1)$, where $A_1, A_2, A_3$ are constants. This gives us a line source on the boundary.

In this case, since $\hat{l}_3(s,\xi') = A_3\delta(s)\delta(\xi_2)$ and $A_1 = A_2 = 0$, by (5.41) the displacement for $t > 0$ equals to

$$f = \int \frac{e^{i((t-s)c_R|\xi'|+x'\cdot\xi')}}{b(c_R)\rho c_R^2 R'(c_R)}\begin{pmatrix} i\mu\theta(c_R)\xi_1/|\xi'| \\ i\mu\theta(c_R)\xi_2/|\xi'| \\ b(c_R)\rho c_R^2\end{pmatrix} b(c_R)\rho c_R^2 A_3\delta(s)\delta(\xi_2)\,\mathrm{d}s\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \mod C^\infty$$

$$= \int \frac{A_3}{R'(c_R)} e^{i(tc_R|\xi_1|+x_1\cdot\xi_1)}\begin{pmatrix} i\mu\theta(c_R)\operatorname{sgn}\xi_1 \\ 0 \\ b(c_R)\rho c_R^2\end{pmatrix}\mathrm{d}\xi_1 = \begin{pmatrix} iA_3\mu\theta(c_R)I_1/R'(c_R) \\ 0 \\ A_3 b(c_R)\rho c_R^2 I_2/R'(c_R)\end{pmatrix} \mod C^\infty,$$

where

$$I_1 = \int e^{i(tc_R|\xi_1|+x'\cdot\xi_1)}\operatorname{sgn}\xi_1\,\mathrm{d}\xi_1 = \int e^{i(\xi_1(tc_R+x_1))}H(\xi_1)\,\mathrm{d}\xi_1 - \int e^{i(\xi_1(-tc_R+x_1))}H(-\xi_1)\,\mathrm{d}\xi_1$$
$$= \pi(\delta(-tc_R+x_1) - \delta(tc_R+x_1)) + i(\text{p.v.}\frac{1}{tc_R+x_1} - \text{p.v.}\frac{1}{-tc_R+x_1}),$$
$$I_2 = \int e^{i(tc_R|\xi_1|+x'\cdot\xi_1)}\,\mathrm{d}\xi_1 = \int e^{i(\xi_1(tc_R+x_1))}H(\xi_1)\,\mathrm{d}\xi_1 + \int e^{i(\xi_1(-tc_R+x_1))}H(-\xi_1)\,\mathrm{d}\xi_1$$
$$= -\pi(\delta(-tc_R+x_1) + \delta(tc_R+x_1)) + i(\text{p.v.}\frac{1}{-tc_R+x_1} + \text{p.v.}\frac{1}{tc_R+x_1}).$$

### 5.4 Stoneley waves

In this section, we assume $\Gamma$ is an interface between two domains $\Omega_-, \Omega_+$. Locally, $\Gamma$ can be flatten as $x^3 = 0$ and $\Omega_+$ is the positive part. For the density and Lamé parameters, we have $\rho_+, \lambda_+, \mu_+$ in $\Omega_+$ and $\rho_-, \lambda_-, \mu_-$ in $\Omega_-$, which are functions smooth up to $\Gamma$. Let $u^\pm$ be $u$ restricted to $\Omega_\pm$.

Suppose there are no incoming waves but boundary sources $l, q \in \mathcal{E}'((0,T) \times \mathbb{R}^2, \mathbb{C}^3)$ microlocally supported in the elliptic region, i.e. we are finding the outgoing microlocal solution $u^\pm$ for the elastic equation with transmission conditions

$$\begin{cases} \partial_t^2 u^\pm - Eu^\pm = 0 & \text{in } \mathbb{R}_t \times \Omega_\pm, \\ [u] = l, \quad [Nu] = q & \text{on } \Gamma, \\ u|_{t<0} = 0, \end{cases} \tag{5.43}$$

where $[v]$ denote the jump of $v$ from the positive side to the negative side across the surface $\Gamma$. By (9.11) in [4], with no incoming waves the transmission conditions can be written in the form of

$$\begin{pmatrix} U^+_{out} & -U^-_{out} \\ M^+_{out} & -M^-_{out} \end{pmatrix} \begin{pmatrix} w^+_{out} \\ w^-_{out} \end{pmatrix} = \begin{pmatrix} l \\ q \end{pmatrix} \Longrightarrow \begin{pmatrix} I & -I \\ \Lambda^+ & -\Lambda^- \end{pmatrix} \begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} l \\ q \end{pmatrix}, \tag{5.44}$$

if we set

$$f^+ = U^+_{out} w^+_{out} = u^+|_\Gamma, \quad f^- = U^-_{out} w^-_{out} = u^-|_\Gamma.$$

This implies that if we can solve $f^\pm$ from (5.44), then the solution $u_\pm$ to (5.43) can be solved by constructing microlocal outgoing solutions to the boundary value problems with Dirichlet b.c. $f^+, f^-$ in $\Omega_+, \Omega_-$ respectively. Since $x^3$ has positive sign in $\Omega_+$ and negative sign in $\Omega_-$, to have evanescent modes in both domain, we choose $\xi^s_{3,\pm}, \xi^p_{3,\pm}$ with opposite signs

$$\xi^s_{3,\pm} = \pm i\alpha_\pm \equiv \pm i\sqrt{|\xi'|^2_g - c^{-2}_{s,\pm}\tau^2}, \quad \xi^p_{3,\pm} = \pm i\beta_\pm \equiv \pm i\sqrt{|\xi'|^2_g - c^{-2}_{p,\pm}\tau^2},$$

where

$$c_{s,\pm} = \sqrt{\mu_\pm/\rho_\pm}, \quad c_{p,\pm} = \sqrt{(\lambda_\pm + 2\mu_\pm)/\rho_\pm}.$$

Then the principal symbols $\sigma_p(\Lambda^{\pm})$ are

$$\frac{i}{|\xi'|_g^2-\alpha_\pm\beta_\pm}\begin{pmatrix} \pm(\mu_\pm(\alpha_\pm-\beta_\pm)\xi_2^2+\beta_\pm\rho_\pm\tau^2) & \pm\mu_\pm\xi_1\xi_2(\beta_\pm-\alpha_\pm) & -i\mu_\pm\xi_1\theta_\pm \\ \pm\mu_\pm\xi_1\xi_2(\beta_\pm-\alpha_\pm) & \pm(\mu_\pm(\alpha_\pm-\beta_\pm)\xi_2^2+\beta_\pm\rho_\pm\tau^2) & -i\mu_\pm\xi_2\theta_\pm \\ i\mu_\pm\theta_\pm\xi_1 & i\mu_\pm\theta_\pm\xi_2 & \pm\alpha_\pm\rho_\pm\tau^2 \end{pmatrix},$$

where

$$\theta_\pm=|\xi'|_g^2+\alpha_\pm^2-2\alpha_\pm\beta_\pm=2|\xi'|_g^2-c_{s,\pm}^{-2}\tau^2-2\alpha_\pm\beta_\pm.$$

To solve (5.44), first we multiply the equation by an invertible matrix to have

$$\begin{pmatrix} I & 0 \\ -\Lambda^+ & I \end{pmatrix}\begin{pmatrix} I & -I \\ \Lambda^+ & -\Lambda^- \end{pmatrix}\begin{pmatrix} f^+ \\ f^- \end{pmatrix}=\begin{pmatrix} l \\ q-\Lambda^+ l \end{pmatrix}.$$

$$\Rightarrow\begin{pmatrix} I & -I \\ 0 & \Lambda^+-\Lambda^- \end{pmatrix}\begin{pmatrix} f^+ \\ f^- \end{pmatrix}=\begin{pmatrix} l \\ q-\Lambda^+ l \end{pmatrix}\Rightarrow\begin{cases} f^+=l+f^- \\ (\Lambda^+-\Lambda^-)f^-=q-\Lambda^+ l \end{cases}$$

In the following, first we solve $f^-$ from

$$(\Lambda^+-\Lambda^-)f^-=q-\Lambda^+ l \tag{5.45}$$

microlocally and then we have $f^+$. This gives the microlocal outgoing solution to (5.44).

Recall the calculation before, the principal symbol of the DN map can be partially diagonalized by the matrix $V_0$. By the same trick, we have

$$V_0^*\sigma_p(\Lambda^+-\Lambda^-)V_0=\begin{pmatrix} iM & 0 \\ 0 & i(\mu_+\alpha_++\mu_-\alpha_-) \end{pmatrix}$$

where

$$M=\frac{1}{|\xi'|_g^2-\alpha_+\beta_+}\begin{pmatrix} \beta_+\rho_+\tau^2 & -i|\xi'|_g\mu_+\theta_+ \\ i|\xi'|_g\mu_+\theta_+ & \alpha_+\rho_+\tau^2 \end{pmatrix}-\frac{1}{|\xi'|_g^2-\alpha_-\beta_-}\begin{pmatrix} -\beta_-\rho_-\tau^2 & -i|\xi'|_g\mu_-\theta_- \\ i|\xi'|_g\mu_-\theta_- & -\alpha_-\rho_-\tau^2 \end{pmatrix}.$$

Let $s=\frac{\tau}{|\xi'|_g}$ as before. We follow the similar argument as in [75] to show the $2\times 2$ matrix $M$ has two distinct eigenvalues and only one of them could be zero for $0<s<\min c_{s,\pm}$. Define the following functions of $s$ related to $\alpha_\pm,\beta_\pm,\theta_\pm$

$$a_\pm(s)=\sqrt{1-c_{s,\pm}^{-2}s^2},\quad b_\pm(s)=\sqrt{1-c_{p,\pm}^{-2}s^2},\quad \theta_\pm(s)=2-c_{s,\pm}^{-2}-2a_\pm(s)b_\pm(s). \tag{5.46}$$

Set

$$
\begin{aligned}
N_\pm(s) &= \frac{1}{1-a_\pm b_\pm}\begin{pmatrix} b_\pm(s)\rho_\pm s^2 & -i\mu_\pm\theta_\pm(s) \\ -i\mu_\pm\theta_\pm(s) & a_\pm(s)\rho_\pm s^2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{b_\pm(s)\rho_\pm s^2}{1-a_\pm b_\pm} & -i(2\mu_\pm - \frac{\rho_\pm s^2}{1-a_\pm b_\pm}) \\ i(2\mu_\pm - \frac{\rho_\pm s^2}{1-a_\pm b_\pm}) & \frac{a_\pm(s)\rho_\pm s^2}{1-a_\pm b_\pm} \end{pmatrix},
\end{aligned}
$$

and it follows the matrix $M$ can be represented by

$$
\begin{aligned}
M &= |\xi'|_g \begin{pmatrix} \frac{b_+(s)\rho_+ s^2}{1-a_+b_+} + \frac{b_-(s)\rho_- s^2}{1-a_-b_-} & -i(2(\mu_+-\mu_-) - (\frac{\rho_+ s^2}{1-a_+b_+} - \frac{\rho_- s^2}{1-a_-b_-})) \\ i(2(\mu_+-\mu_-) - (\frac{\rho_+ s^2}{1-a_+b_+} - \frac{\rho_- s^2}{1-a_-b_-})) & \frac{a_+(s)\rho_+ s^2}{1-a_+b_+} + \frac{a_-(s)\rho_- s^2}{1-a_-b_-} \end{pmatrix} \\
&= |\xi'|_g (N_+(s) + N_-^T(s)) \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \qquad (5.47)
\end{aligned}
$$

where we denote the entry of $M$ by $M_{ij}$ for $i,j = 1,2$ and the two eigenvalues by $m_1(s)$ and $m_2(s)$. Notice

$$
\begin{aligned}
\varrho &= \sqrt{(M_{11}-M_{22})^2 + 4M_{12}M_{21}} \\
&= (\frac{(b_+(s)-a_+(s))\rho_+ s^2}{1-a_+b_+} - \frac{(b_-(s)-a_-(s))\rho_- s^2}{1-a_-b_-})^2 + (2(\mu_+-\mu_-) - (\frac{\rho_+ s^2}{1-a_+b_+} - \frac{\rho_- s^2}{1-a_-b_-}))^2
\end{aligned}
$$

is always nonnegative. The eigenvalues of $M$ can be written in the following specific form

$$
\begin{aligned}
m_1(t,x',\tau,\xi') &= \frac{M_{11}+M_{22}-\sqrt{\varrho}}{2} \equiv |\xi'|_g m_1(s), \qquad (5.48) \\
m_2(t,x',\tau,\xi') &= \frac{M_{11}+M_{22}+\sqrt{\varrho}}{2} \equiv |\xi'|_g m_2(s),
\end{aligned}
$$

where only $m_1(s)$ could be zero. More precisely, we have $m_1(s)$ vanishes if and only if the equation

$$
\begin{aligned}
S(s) = ((\rho_+a_- + \rho_-a_+)(\rho_+b_- + \rho_-b_+) - (\rho_+-\rho_-)^2)s^4 - 4(\mu_+-\mu_-)^2(1-a_+b_+)(1-a_-b_-) \\
+ 4(\mu_+-\mu_-)(\rho_+(1-a_-b_-) - \rho_-(1-a_+b_+))s^2 \qquad (5.49)
\end{aligned}
$$

is satisfied for some $s_0$. If such $s_0$ exists, it corresponds to the propagation speed $c_{ST}$ of the so called Stoneley waves, first proposed in [74]. We call $c_{ST}$ the **Stoneley**

**speed** and it is a simple zero by Proposition 19 in the following. This proposition of uniqueness of Stoneley waves is proved by the definiteness of $N_\pm(s)$ and first appears in [77] and then is used in [75]. Here we present a slightly different one.

**Proposition 19.** *For $0 < s < \min c_{s,\pm}$, the eigenvalues $m_1(s), m_2(s)$ decrease as $s$ increases. Only $m_1(s)$ can be zero. This happens when there is some $s_0$ such that*

*Particularly, if such $s_0$ exists, it is unique and is a simple zero of $m_1(s)$.*

*Proof.* We claim the matrix $N_\pm(s)$ and their transposes satisfy

(a) the limit $N_\pm(0) \equiv \lim_{s\to 0} N_\pm(s)$ exist and are positive definite,

(b) for $0 < s < \min c_{s,\pm}$, the derivative $N'_\pm(s)$ is negative definite,

(c) for $0 < s < \min c_{s,\pm}$, the trace $\mathrm{Tr}(N_\pm(s))$ is always positive.

Then $M$ satisfy these conditions as well, which indicates its eigenvalues decrease as $s$ increases but their sum is always positive, i.e. at most one of them could be zero. The monotonic decreasing of eigenvalues implies the zero should be a simple one.

Now we prove the claim. For (a), we compute

$$\begin{aligned}\lim_{s\to 0}\frac{\rho_\pm s^2}{1-a_\pm b_\pm} &= \lim_{s\to 0}\frac{2\rho_\pm s}{-a'_\pm b_\pm - a_\pm b'_\pm} = \lim_{s\to 0}\frac{2\rho_\pm}{-a''_\pm b_\pm - 2a'_\pm b'_\pm - a_\pm b''_\pm}\\ &= \frac{2\mu_\pm(2\mu_\pm+\lambda_\pm)}{3\mu_\pm+\lambda_\pm} \equiv c_\pm.\end{aligned}$$

Assuming $\mu_\pm, \lambda_\pm > 0$, we have $\frac{4}{3}\mu_\pm < c_\pm < 2\mu_\pm$. Then $\mathrm{Tr}(N_\pm(0)) = 2c_\pm > 0$ and $\det(N_\pm(0)) = 4\mu_\pm(c_\pm - \mu_\pm) > 0$.

To prove (b), for convenience we change the variable $\iota = s^2$ and $\widetilde{N}_\pm(\iota) = N_\pm(\sqrt{\iota})$ with $s > 0$. Then it is sufficient to show $\widetilde{N}'_\pm(\iota) > 0$. Indeed we have

$$\widetilde{N}_\pm(\iota) = \begin{pmatrix} \tilde{b}_\pm(\iota)\kappa_\pm(\iota) & -i(2\mu_\pm - \kappa_\pm(\iota)) \\ i(2\mu_\pm - \kappa_\pm(\iota)) & \tilde{a}_\pm(\iota)\kappa_\pm(\iota) \end{pmatrix},$$

where we set

$$\tilde{a}_\pm(\iota) = \sqrt{1 - c_{s,\pm}^{-2}\iota}, \quad \tilde{b}_\pm(\iota) = \sqrt{1 - c_{p,\pm}^{-2}\iota}, \quad \kappa_\pm(\iota) = \frac{\rho_\pm \iota}{1 - \tilde{a}_\pm \tilde{b}_\pm}.$$

Then

$$\widetilde{N}'_{\pm}(\iota) = \begin{pmatrix} \tilde{b}'_{\pm}(\iota)\kappa_{\pm}(\iota) + \tilde{b}_{\pm}(\iota)\kappa'_{\pm}(\iota) & i2\kappa'_{\pm}(\iota)) \\ -i2\kappa'_{\pm}(\iota)) & \tilde{a}'_{\pm}(\iota)\kappa_{\pm}(\iota) + \tilde{a}_{\pm}(\iota)\kappa'_{\pm}(\iota) \end{pmatrix}$$

and

$$\begin{aligned} \kappa'_{\pm} &= \frac{\rho_{\pm}}{(1-\tilde{a}_{\pm}\tilde{b}_{\pm})}(1 - \frac{\tilde{a}_{\pm}}{2\tilde{b}_{\pm}} - \frac{\tilde{b}_{\pm}}{2\tilde{a}_{\pm}}), \\ \tilde{a}'_{\pm}(\iota)k(\iota) &= \frac{-\rho_{\pm}c_{s,\pm}^{-2}\iota}{2\tilde{a}_{\pm}(1-\tilde{a}_{\pm}\tilde{b}_{\pm})} = \frac{\rho_{\pm}}{2(1-\tilde{a}_{\pm}\tilde{b}_{\pm})}(\tilde{a}_{\pm} - \frac{1}{\tilde{a}_{\pm}}). \end{aligned}$$

Therefore, the determinant and the transpose of $\widetilde{N}'_{\pm}(\iota)$ are

$$\begin{aligned} \det(\widetilde{N}'_{\pm}(\iota)) &= (\kappa'_{\pm})^2\tilde{a}_{\pm}\tilde{b}_{\pm} + \kappa_{\pm}\kappa'_{\pm}\tilde{a}'_{\pm}\tilde{b}_{\pm} + \kappa_{\pm}\kappa'_{\pm}\tilde{a}_{\pm}\tilde{b}'_{\pm} - (\kappa'_{\pm})^2 \\ &= \frac{\rho_{\pm}^2}{(1-\tilde{a}_{\pm}\tilde{b}_{\pm})^2}\frac{1}{2\tilde{a}_{\pm}\tilde{b}_{\pm}}(\tilde{a}_{\pm} - \tilde{b}_{\pm})^2 > 0, \\ \mathrm{Tr}(\widetilde{N}'_{\pm}(\iota)) &= \kappa'_{\pm}(\tilde{a}_{\pm} + \tilde{b}_{\pm}) + \kappa_{\pm}(\tilde{a}'_{\pm} + \tilde{b}'_{\pm}) \\ &= -\frac{\rho_{\pm}(\tilde{a}_{\pm} + \tilde{b}_{\pm})}{2\tilde{a}_{\pm}\tilde{b}_{\pm}(1-\tilde{a}_{\pm}\tilde{b}_{\pm})^2}((\tilde{a}_{\pm} - \tilde{b}_{\pm})^2 + (\tilde{a}_{\pm}\tilde{b}_{\pm} + 1)^2) < 0, \end{aligned}$$

which indicates $\widetilde{N}'_{\pm}(\iota)$ is negative definite. For (c), obviously we have

$$\mathrm{Tr}(N_{\pm})(s) = \frac{(a_{\pm}(s) + b_{\pm}(s))\rho_{\pm}s^2}{1 - a_{\pm}b_{\pm}} > 0.$$

□

If $S(s) \neq 0$ for all $0 < s < \min c_{s,\pm}$, then $\Lambda + -\Lambda_-$ is an elliptic ΨDO and the microlocal solution to (5.45) is $f^- = (\Lambda_+ - \Lambda_-)^{-1}(l_2 - \Lambda_+ l_1)$. The singularities does not propagate. Otherwise, if there exists $0 < c_{ST} < \min c_{s,\pm}$ such that $m_1(c_{ST}) = 0$, then (5.44) has a nontrivial microlocal solution that propagates singularities, analogously to the case of Rayleigh waves.

In the following suppose there exists a $c_{ST}$ such that $S(c_{ST}) = 0$. By Proposition 19, this zero is simple so by the implicit function theorem it is a smooth function $s_{ST}(t, x')$ in a small neighborhood of a fixed point. Then we can write $\tilde{m}_1(t, x', \tau, \xi')$ as a product similar to what we have before

$$\widetilde{m}_1(t, x', \tau, \xi') = im_1(t, x', \tau, \xi') = e_0(t, x', \tau, \xi')i(\tau - c_{ST}(t, x')|\xi'|_g). \tag{5.50}$$

To decouple the system as what we did in Section 5.3, we need the following claim. Notice even without this claim, the analysis still holds since with only $\widetilde{m}_1$ vanishing the last two eigenvalues always give us an elliptic 2 by 2 system.

**Claim 4.** The three eigenvalues $\widetilde{m}_1(t,x',\tau,\xi'), \widetilde{m}_1(t,x',\tau,\xi'), \widetilde{m}_1(t,x',\tau,\xi')$ of the matrix $\sigma_p(\Lambda_+ - \Lambda_-)$ are distinct near $s = c_{ST}$.

*Proof.* Obviously near $s_0$ we have $\widetilde{m}_1 \neq \widetilde{m}_2$ and $\widetilde{m}_1 \neq \widetilde{m}_3$. The values of $\widetilde{m}_2, \widetilde{m}_3$ may coincide but near $s_0$ they are separate by the following calculation

$$
\begin{aligned}
&\widetilde{m}_1(t,x',\tau,\xi') + \widetilde{m}_2(t,x',\tau,\xi') - \widetilde{m}_3(t,x',\tau,\xi') = \mathrm{Tr}(M) - i(\mu_+\alpha_+ + \mu_-\alpha_-) \\
&= i|\xi'|_g \sum_{\nu=\pm} \frac{(a_\nu(s)+b_\nu(s))\rho_\nu s^2 - \mu_\nu a_\nu(1-a_\nu b_\nu)}{1-a_\nu b_\nu} \\
&= i|\xi'|_g \sum_{\nu=\pm} \frac{(a_\nu(s)+b_\nu(s))\rho_\nu s^2 - \mu_\nu a_\nu(s) + \mu_\nu b_\nu(s)(1-\frac{\rho_\nu}{\mu_\nu}s^2)}{1-a_\nu(s)b_\nu(s)} \\
&= i|\xi'|_g \sum_{\nu=\pm} \frac{a_\nu(s)\rho_\nu s^2 + \mu_\nu(b_\nu(s)-a_\nu(s))}{1-a_\nu b_\nu},
\end{aligned}
$$

where the imaginary part is always positive, since by (5.46) we have $0 < a_\nu < b_\nu < 1$ and $\widetilde{m}_1 = 0$ at $s = c_{ST}$. ☐

More specifically, this time we define

$$
V_1(t,x',\tau,\xi') = \begin{pmatrix} M_{21}/k_1 & M_{21}/k_2 & 0 \\ (M_{11}-\widetilde{m}_1)/k_1 & (M_{11}-\widetilde{m}_2)/k_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
$$

where $M_{ij}$ is the entry of $M$ in (5.47), for $i,j = 1,2$ and we define

$$
k_j = \sqrt{|M_{11}-\widetilde{m}_j|^2 + |M_{21}|^2}, \quad k_j(s) = k_j/|\xi'|_g^2. \tag{5.51}
$$

Then

$$W(t,x',\tau,\xi') = V_0(t,x',\tau,\xi')V_1(t,x',\tau,\xi')$$
$$= \begin{pmatrix} M_{21}\xi_1/(|\xi'|_g k_1) & -M_{12}\xi_1/(|\xi'|_g k_2) & -\xi_2/|\xi'|_g \\ M_{21}\xi_2/(|\xi'|_g k_1) & -M_{12}\xi_2/(|\xi'|_g k_2) & \xi_1/|\xi'|_g \\ (M_{11}-\widetilde{m}_1)/k_1 & (M_{11}-\widetilde{m}_2)/k_2 & 0 \end{pmatrix}. \tag{5.52}$$

Let the operators $e_0(t,x',D_t,D_{x'}) \in \Psi^0$ with symbol $e_0(t,x',\tau,\xi')$ in (5.50) and $\widetilde{m}_j(t,x',D_t,D_{x'}) \in \Psi^1$ with symbols $\tilde{m}_j(t,x',\tau,\xi')$, for $j=2,3$. By [6], there exists an elliptic $\Psi$DO of order zero $\widetilde{W}(t,x',D_t,D_{x'})$ with the principal symbol equal to $W(t,x',\tau,\xi')$, such that near some fixed $(t,x',\tau,\xi') \in \Sigma_R$, the operator $\Lambda^+ - \Lambda^-$ can be fully decoupled as

$$\widetilde{W}^{-1}(\Lambda^+ - \Lambda^-)\widetilde{W} = \begin{pmatrix} e_0(\partial_t - ic_{ST}(t,x')\sqrt{-\Delta_{x'}}) + r_1 & 0 & 0 \\ 0 & \widetilde{m}_2 + r_2 & 0 \\ 0 & 0 & \widetilde{m}_3 + r_3 \end{pmatrix} \mod \Psi^{-\infty},$$

where $r_1(t,x',D_t,D_{x'}), r_2(t,x',D_t,D_{x'}), r_3(t,x',D_t,D_{x'}) \in \Psi^0$ are the lower order term. If we define

$$r(t,x',D_t,D_x') = e_0^{-1} r_1 \in \Psi^0, \tag{5.53}$$

in what follows, then the first entry in the first row can be written as

$$e_0(\partial_t - ic_{ST}(t,x')\sqrt{-\Delta_{x'}} + r(t,x',D_t,D_x')). \tag{5.54}$$

### 5.4.1 The Cauchy problem and the polarization

In this subsection, similar to Subsection 5.3.3, we first assume that the source exists for a limited time for $t<0$ and we have the Cauchy data $f^-|_{t=0}$ at $t=0$, i.e.

$$(\Lambda_+ - \Lambda_-)f^- = 0, \text{ for } t>0, \quad f^-|_{t=0} \text{ given }. \tag{5.55}$$

Recall the diagonalization of $\Lambda_+ - \Lambda_-$ in before. Let

$$h^- = \begin{pmatrix} h_1^- \\ h_2^- \\ h_3^- \end{pmatrix} = \widetilde{W}^{-1} \begin{pmatrix} f_1^- \\ f_2^- \\ f_3^- \end{pmatrix} = \widetilde{W}^{-1} f^-.$$

The homogeneous equation $(\Lambda_+ - \Lambda_-)f^- = 0$ implies

$$f^- = \widetilde{W} h^- = \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} h_1^- \mod C^\infty \Leftarrow h_2^- = h_3^- = 0 \mod C^\infty, \tag{5.56}$$

where $h_1^-$ solves the homogeneous first-order hyperbolic equation in (5.54). Notice in this case the hyperbolic operator is $\partial_t - ic_{ST}(t,x')|\xi'|_g + r(t,x',D_t,D_{x'})$ with $r$ given by (5.53). If we have the initial condition $h_{1,0}^- \equiv h_1^-|_{t=0}$, then by the construction in Subsection 5.3.2, then

$$h_1^-(t,x') = \int a(t,x',\xi') e^{i\varphi(t,x',\xi')} \hat{h}_{1,0}^-(\xi') \,\mathrm{d}\xi' \mod C^\infty, \tag{5.57}$$

where the phase function $\varphi$ solves the eikonal equation (5.21); the amplitude $a = a_0 + \sum_{j\le -1} a_j$ solves the transport equation (5.22) and (5.23) with $\gamma = (\sum_{|\alpha|=2} \Big(\frac{1}{\alpha!} c_R(t,x') D_{\xi'}^\alpha |\xi'|_g D_{x'}^\alpha \varphi - r_0(t,x',\lambda_1,\nabla_{x'}\varphi)\Big)$.

By the same analysis before, the initial condition of $f^-$ is related to that of $h^-$ by

$$f^-|_{t=0} = \begin{pmatrix} \widetilde{W}_{11,0} \\ \widetilde{W}_{21,0} \\ \widetilde{W}_{31,0} \end{pmatrix} h_{1,0}^- \mod C^\infty, \tag{5.58}$$

where the principal symbols are

$$\sigma_p(\widetilde{W}_{11,0}) = \sigma_p(W_{11})(0,x',c_R|\xi'|_g,\xi'), \qquad \sigma_p(\widetilde{W}_{21,0}) = \sigma_p(W_{21})(0,x',c_R|\xi'|_g,\xi'),$$
$$\sigma_p(\widetilde{W}_{31,0}) = \sigma_p(W_{31})(0,x',c_R|\xi'|_g,\xi').$$

We have the following theorem as an analog to Theorem 10.

**Theorem 13.** *Suppose $f^-(0,x')$ satisfies (5.58) with some $h^-_{1,0} \in \mathcal{E}'$. Then microlocally the homogeneous problem with Cauchy data (5.55) admits a unique microlocal solution*

$$
\begin{aligned}
f^- &= \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} \int a(t,x',\xi') e^{i\varphi(t,x',\xi')} \hat{h}^-_{1,0}(\xi')\, \mathrm{d}\xi' \quad \mathrm{mod}\ C^\infty \\
&= \int \underbrace{\begin{pmatrix} i\zeta_1 \nabla_{x_1}\varphi / |\nabla_{x'}\varphi|_g \\ i\zeta_1 \nabla_{x_2}\varphi / |\nabla_{x'}\varphi|_g \\ \zeta_2 \end{pmatrix} \frac{a_0(t,x',\xi') e^{i\varphi(t,x',\xi')}}{k_1(c_{ST})}}_{\mathcal{P}} \hat{h}^-_{1,0}(\xi')\, \mathrm{d}\xi' + \text{ lower order terms},
\end{aligned}
\tag{5.59}
$$

*where we define*

$$
\begin{aligned}
\zeta_1 &= (2(\mu_+ - \mu_-) - (\frac{\rho_+ c_{ST}^2}{1 - a_+(c_{ST})b_+(c_{ST})} - \frac{\rho_- c_{ST}^2}{1 - a_-(c_{ST})b_-(c_{ST})})), \\
\zeta_2 &= (\frac{b_+(c_{ST})\rho_+ c_{ST}^2}{1 - a_+(c_{ST})b_+ v} + \frac{b_-(c_{ST})\rho_- c_{ST}^2}{1 - a_-(c_{ST})b_-(c_{ST})})
\end{aligned}
$$

*as smooth functions w.r.t. $t, x', \xi'$ with $c_{ST}$ be the Stoneley speed, $a_+(s), b_+(s)$ defined in (5.46), $k_1(s)$ defined in (5.51).*

This theorem describes the microlocal polarization of the displacement of a Stoneley wave on the intersurface. Up to lower order terms, the displacement $f^-$ can also be regarded as a supposition of $\Re(\mathcal{P})$ and $\Im(\mathcal{P})$, each of which has a elliptical retrograde motion as we discussed before. Indeed, the leading term of $f$ has a similar pattern of that of Rayleigh waves in Theorem (10), except different scalar functions in each component.

### 5.4.2 The inhomogeneous problem

In this subsection, first we are going to microlocally solve the inhomogeneous problem

$$
(\Lambda_+ - \Lambda_-) f^- = q - \Lambda^+ l, \text{ for } t > 0, \quad f^-|_{t=0} = 0, \tag{5.60}
$$

and the then the microlocal solution to (5.44) and (5.43) can be derived as we stated before. We perform the same analysis for the Rayleigh wave in the previous section.

Let

$$h^- = \begin{pmatrix} h_1^- \\ h_2^- \\ h_3^- \end{pmatrix} = \widetilde{W}^{-1} \begin{pmatrix} f_1^- \\ f_2^- \\ f_3^- \end{pmatrix} = \widetilde{W}^{-1} f^-, \qquad \tilde{l} = \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \widetilde{W}^{-1}(l_2 - \Lambda^+ l_1), \tag{5.61}$$

where $u_j$ is the component of any vector valued distribution $u$ for $j = 1, 2, 3$. Solving $\Lambda f = l \mod C^\infty$ is microlocally equivalent to solving the following system

$$\begin{cases} (\partial_t - ic_{ST}(t,x')\sqrt{-\Delta_{x'}} + r(t,x',D_t,D'_x))h_1^- = e_0^{-1}\tilde{l}_1, & \mod C^\infty, \\ (\widetilde{m}_2 + r_2)h_2^- = \tilde{l}_2, & \mod C^\infty, \\ (\widetilde{m}_3 + r_3)h_3^- = \tilde{l}_3, & \mod C^\infty. \end{cases} \tag{5.62}$$

In the last two equations, the operators $\widetilde{m}_j + r_j$ are elliptic so we have $h_j^- = (\widetilde{m}_j + r_j)^{-1}\tilde{l}_j, \mod C^\infty$ for $j = 2, 3$. The first equation is a first-order inhomogeneous hyperbolic equation with lower order term, which can be solved by Duhamel's principle. We apply Proposition 18 to have

$$\begin{aligned} h_1^-(t,x') &= \int H(t-s)a(t-s,x',\xi')e^{i(\varphi(t-s,x',\xi')-y'\cdot\xi')}((I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1)(s,y')\,\mathrm{d}y'\,\mathrm{d}\xi'\,\mathrm{d}s \\ &= L_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1, \end{aligned} \tag{5.63}$$

where the phase function $\varphi(t,x',\xi')$ and the amplitude $a(t,x',\xi')$ are given by Proposition 18 with the hyperbolic operator being $\partial_t - ic_{ST}(t,x')\sqrt{-\Delta_{x'}} + r(t,x',D_t,D'_x)$; and $\tilde{e}$ is defined in (5.14) for the new hyperbolic operator. We also write the solution as $h_1^- = L_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1$ by (5.29). This proves the following theorem, as an analog to Theorem 11, 12.

**Theorem 14.** *Suppose $l(t,x'), q(t,x') \in \mathcal{E}'((0,T)\times\mathbb{R}^2,\mathbb{C}^3)$ microlocally supported in the elliptic region. The inhomogeneous system (5.60) with zero initial condition at $t=0$ admits a unique microlocal solution*

$$\begin{pmatrix} f_1^- \\ f_2^- \\ f_3^- \end{pmatrix} = \widetilde{W} \begin{pmatrix} L_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1 \\ (\widetilde{m}_2 + r_2)^{-1}\tilde{l}_2 \\ (\widetilde{m}_3 + r_3)^{-1}\tilde{l}_3 \end{pmatrix}, \ \textit{where} \ \begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \widetilde{W}^{-1}(q - \Lambda^+ l) \mod C^\infty, \tag{5.64}$$

*where $\widetilde{W}$ has the principal symbol in (5.52), $e_0$ defined in (5.50), and $\tilde{e}$ is constructed as in (5.28). Then the microlocal solution to the transmission problem (5.43) can be constructed as evanescent modes from the boundary value $f^-$ and $f^+ = l + f^-$. In particular, for $t \geq T$, the displacement on the boundary has the leading term*

$$\begin{aligned} f^- &= \begin{pmatrix} \widetilde{W}_{11} \\ \widetilde{W}_{21} \\ \widetilde{W}_{31} \end{pmatrix} F_{\varphi,a}(I-\tilde{e})^{-1}e_0^{-1}\tilde{l}_1 \\ &= \int \begin{pmatrix} i\zeta_1 \nabla_{x_1}\varphi/|\nabla_{x'}\varphi|_g \\ i\zeta_1 \nabla_{x_2}\varphi/|\nabla_{x'}\varphi|_g \\ \zeta_2 \end{pmatrix} \frac{e^{i\varphi}a_0(t-s,x',\xi')}{m_1'(c_{ST})k_1(c_{ST})} \hat{\tilde{l}}_1(s,\xi')\,\mathrm{d}s\,\mathrm{d}\xi' + \ \textit{lower order terms}\ , \end{aligned}$$

*where we define*

$$\begin{aligned} \zeta_1 &= (2(\mu_+ - \mu_-) - (\frac{\rho_+ c_{ST}^2}{1 - a_+(c_{ST})b_+(c_{ST})} - \frac{\rho_- c_{ST}^2}{1 - a_-(c_{ST})b_-(c_{ST})})), \\ \zeta_2 &= (\frac{b_+(c_{ST})\rho_+ c_{ST}^2}{1 - a_+(c_{ST})b_+ v} + \frac{b_-(c_{ST})\rho_- c_{ST}^2}{1 - a_-(c_{ST})b_-(c_{ST})}) \end{aligned}$$

*as smooth functions w.r.t. $t, x', \xi'$ with $c_{ST}$ be the Stoneley speed, $m_1(s), k_1(s)$ defined in (5.48, 5.51).*

### 5.5 Appendix: the lower order term $r_0$.

In this section, we are going to write out the explicit form of the principal symbol of $r$. Recall the decoupling procedure in [6]. First we have

$$W^{-1}\sigma_p(\Lambda)W = \begin{pmatrix} \widetilde{m}_1 & 0 & 0 \\ 0 & \widetilde{m}_2 & 0 \\ 0 & 0 & \widetilde{m}_3 \end{pmatrix} + R \equiv G + R \mod \Psi^{-\infty},$$

where $R \in \Psi^0$. Then we decouple terms of order zero by finding a $K_1$ in form of

$$K_1 = \begin{pmatrix} 0 & K_{12} & K_{13} \\ K_{21} & 0 & K_{23} \\ K_{31} & K_{32} & 0 \end{pmatrix} \in \Psi^{-1}$$

such that

$$(1+K_1)(G+R)(1+K_1)^{-1} = G + (K_1G - GK_1 + A) + \cdots$$

and the off diagonal terms of $(K_1G - GK_1 + A)$ vanish. There exists a unique solution for $K_1$ since all eigenvalues are distinct. After this step, the diagonal terms $R_{jj}, j = 1,2,3$ in $R$ remains and they form the zero order terms of the decoupled system.

We introduce the notations $P^{(\alpha)} = iD^{\alpha}_{(\tau,\xi)}P$ and $P_{(\alpha)} = D^{\alpha}_{(t,x)}P$ for a matrix-valued symbol $P(t,x,\tau,\xi)$ in the following. Moreover, the principal symbol of $R_{jj}$ are the second highest order term in the asymptotic expansion of the product

$$\begin{aligned}\sigma(W^*\Lambda W) = \sigma(W^*(\Lambda W)) &\sim \sum_{\alpha\geq 0} \frac{i^{|\alpha|-1}}{\alpha!}\sigma(W^{-1})^{(\alpha)}\sigma(\Lambda W)_{(\alpha)} \\ &\sim \sum_{\alpha\geq 0} \frac{i^{|\alpha|-1}}{\alpha!}\sigma(W^{-1})^{(\alpha)}(\sum_{\beta\geq 0}\sigma(\Lambda)^{(\beta)}\sigma(W)_{(\beta)})_{(\alpha)},\end{aligned}$$

where the product of symbols are multiplication of matrices. Observe that any terms that are multiples of $\widetilde{m}_1$ vanish along $\Sigma_R$. It follows that

$$\begin{aligned}\sigma_p(R_{11}) = &\sum_{|\alpha|=1}\sum_k \sigma(W^*)^{(\alpha)}_{1k}((\sigma_p(\Lambda)\sigma(W))_{(\alpha)})_{k1} + \sigma_p(W^*)_{1k}(\sigma_p(\Lambda)^{(\alpha)}\sigma(W)_{(\alpha)}))_{kj} \\ &+ \sum_k \sigma_p(W^*)_{1k}(\sigma_{p-1}(\Lambda)\sigma(W))_{k1},\end{aligned}$$

where we use $\sigma_{p-1}$ to denote the symbol of second highest order. By analyzing the pattern of each matrix, one can show that the first and second term are real. Whether the third one is real or not depends on the pattern of $\sigma_{p-1}(\Lambda)$.

**Remark 18.** The second highest order term $\sigma_{p-1}(\Lambda)$ can be computed by

$$\begin{aligned}\sigma_{p-1}(\Lambda) = \sigma_{p-1}(M_{out})\sigma_p(U_{out})^{-1} - \sigma_p(\Lambda)\sigma_{p-1}(U_{out})\sigma_p(U_{out})^{-1} \\ - (\sum_{|\alpha|=1} \sigma_p(\Lambda)^{(\alpha)}\sigma_p(U_{out})_{(\alpha)})\sigma_p(U_{out})^{-1},\end{aligned}$$

where $\sigma_{p-1}(M_{out}), \sigma_{p-1}(U_{out})$ can be regarded as the composition of a ΨDO with an FIO with complex phase function. Indeed, we consider the geometric optics construction of $w(t,x)$ from the boundary value problem, which is a Lagrangian distribution with complex phase function. By the Fundamental Lemma for the complex phase in [67], one can find the composition is a ΨDO if restricted on the boundary $x_3 = 0$ and we have $\sigma_{p-1}(M_{out})$. The same procedure works for $\sigma_{p-1}(U_{out})$.

Notice the second term in $\sigma_{p-1}(\Lambda)$ multiplied by $\sigma_p(W^*)$ on the left will vanish along $\Sigma_R$ in the first row and the pattern of the third one will give a real term after conjugated by $\sigma_p(W)$.

Since we write $\widetilde{m}_1 \sim e_0(\partial_t - ic_R(t,x')\sqrt{-\Delta_{x'}})$ in the principal symbol level, there is an extra term in $r_1$ besides $R_{11}$. Let $p = i(\tau - c_R(t,x')|\xi'|)$. Then we have

$$\sigma(r_1) = \sigma_p(R_{11}) - \sum_{|\alpha|=1} e_0^{(\alpha)} p_{(\alpha)},$$

and

$$r_0 \equiv \sigma_p(r) = \sigma_p(e_0^{-1} r_1) = \sigma_p(e_0)^{-1}(\sigma_p(R_{11}) - \sum_{|\alpha|=1} e_0^{(\alpha)} p_{(\alpha)}).$$

# REFERENCES


[1] Yang Zhang. Artifacts in the inversion of the broken ray transform in the plane. *Inverse Problems & Imaging*, 14(1):1–26, 2020.

[2] Yang Zhang. (In preparation) The X-ray transform for a generic family of curves with conjugate points.

[3] Yang Zhang. Recovery of singularities for the weighted cone transform appearing in compton camera imaging. *Inverse Problems*, 36(2):025014, jan 2020.

[4] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. The transmission problem in linear isotropic elasticity.

[5] Plamen Stefanov. Lower bound of the number of the rayleigh resonances for arbitrary body. *Indiana University Mathematics Journal*, 49(1):0–0, 2000.

[6] Michael Eugene Taylor. *Pseudodifferential Operators (PMS-34)*. Princeton University Press, 2017.

[7] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[8] Lars Hörmander. *The Analysis of Linear Partial Differential Operators IV*. Springer Berlin Heidelberg, 2009.

[9] François Treves. *Introduction to Pseudodifferential and Fourier Integral Operators, Volume 2*. Springer US, 1980.

[10] Lars Hörmander. Fourier integral operators. i. *Acta Mathematica*, 127(0):79–183, 1971.

[11] John M. Lee. *Introduction to Smooth Manifolds*. Springer New York, 2003.

[12] Lars Hörmander. *The Analysis of Linear Partial Differential Operators IV*. Springer Berlin Heidelberg, 2009.

[13] Markus Haltmeier, Sunghwan Moon, and Daniela Schiefeneder. Inversion of the attenuated v-line transform with vertices on the circle. *IEEE Transactions on Computational Imaging*, pages 1–1, 2017.

[14] Sunghwan Moon and Markus Haltmeier. Analytic inversion of a conical Radon transform arising in application of Compton cameras on the cylinder. *SIAM Journal on Imaging Sciences*, 10(2):535–557, 2017.

[15] Plamen Stefanov and Gunther Uhlmann. Is a curved flight path in SAR better than a straight one? *SIAM J. Appl. Math.*, 73(4):1596–1612, 2013.

[16] Plamen Stefanov and Gunther Uhlmann. The geodesic X-ray transform with fold caustics. *Anal. PDE*, 5(2):219–260, 2012.

[17] François Monard, Plamen Stefanov, and Gunther Uhlmann. The geodesic ray transform on Riemannian surfaces with conjugate points. *Comm. Math. Phys.*, 337(3):1491–1513, 2015.

[18] Sean Holman, Francois Monard, and Plamen Stefanov. The attenuated geodesic x-ray transform. 08 2017.

[19] Bela Frigyik, Plamen Stefanov, and Gunther Uhlmann. The X-ray transform for a generic family of curves and weights. *J. Geom. Anal.*, 18(1):89–108, 2008.

[20] Sean Holman and Gunther Uhlmann. On the microlocal analysis of the geodesic x-ray transform with conjugate points. 2015.

[21] R W. Todd, J M. Nightingale, and D B. Everett. A proposed Îş camera. 251:132–134, 09 1974.

[22] D.B. Everett, J.S. Fleming, R.W. Todd, and J.M. Nightingale. Gamma-radiation imaging system based on the Compton effect. *Proceedings of the Institution of Electrical Engineers*, 124(11):995, 1977.

[23] Manbir Singh. An electronically collimated gamma camera for single photon emission computed tomography. part i: Theoretical considerations and design criteria. *Medical Physics*, 10(4):421–427, Jul 1983.

[24] Roman Basko, Gengsheng L Zeng, and Grant T Gullberg. Application of spherical harmonics to image reconstruction for the Compton camera. *Physics in Medicine & Biology*, 43(4):887, 1998.

[25] L. C. Parra. Reconstruction of cone-beam projections from Compton scattered data. *IEEE Transactions on Nuclear Science*, 47(4):1543–1550, Aug 2000.

[26] T Tomitani and M Hirasawa. Image reconstruction from limited angle Compton camera data. *Physics in Medicine & Biology*, 47(12):2129, 2002.

[27] Bruce Smith. Reconstruction methods and completeness conditions for two Compton data models. *JOSA A*, 22(3):445–459, 2005.

[28] VoichiÅča Maxim, Mirela FrandeÅ§, and RÃľmy Prost. Analytical inversion of the Compton transform using the full set of available projections. *Inverse Problems*, 25(9):095001, 2009.

[29] Rim Gouia-Zarrad and Gaik Ambartsoumian. Exact inversion of the conical Radon transform with a fixed opening angle. *Inverse Problems*, 30(4):045007, mar 2014.

[30] Fatma Terzioglu. Some inversion formulas for the cone transform. *Inverse Problems*, 31(11):115010, 2015.

[31] Peter Kuchment and Fatma Terzioglu. Three-dimensional image reconstruction from Compton camera data. *SIAM Journal on Imaging Sciences*, 9(4):1708–1725, 2016.

[32] Chang-Yeol Jung and Sunghwan Moon. Inversion formulas for cone transforms arising in application of Compton cameras. *Inverse Problems*, 31(1):015006, 2015.

[33] Daniela Schiefeneder and Markus Haltmeier. The Radon transform over cones with vertices on the sphere and orthogonal axes. *arXiv*, 2016.

[34] Sunghwan Moon. On the determination of a function from its conical Radon transform with a fixed central axis. *SIAM Journal on Mathematical Analysis*, 48(3):1833–1847, jan 2016.

[35] Fatma Terzioglu and Peter Kuchment. Inversion of weighted divergent beam and cone transforms. *Inverse Problems and Imaging*, 11(6):1071–1090, sep 2017.

[36] Joonas Ilmavirta. Broken ray tomography in the disc. *Inverse Problems*, 29(3):035008, 2013.

[37] Mark Hubenthal. The broken ray transform on the square. *Journal of Fourier Analysis and Applications*, 20(5):1050–1082, Oct 2014.

[38] Joonas Ilmavirta. On the broken ray transform. *arXiv preprint arXiv:1409.7500*, 2014.

[39] Joonas Ilmavirta. A reflection approach to the broken ray transform. *MATHEMATICA SCANDINAVICA*, 117(2):231, dec 2015.

[40] Mark Hubenthal. The broken ray transform in $n$ dimensions with flat reflecting boundary. *Inverse Problems and Imaging*, 9(1):143–161, jan 2015.

[41] Joonas Ilmavirta and Mikko Salo. Broken ray transform on a riemann surface with a convex obstacle. *Communications in Analysis and Geometry*, 24(2):379–408, 2016.

[42] Marcela Morvidone, M.K. Nguyen, Tuong Truong, and H Zaidi. On the V-Line Radon transform and its imaging applications. 2010, 07 2010.

[43] Lucia Florescu, Vadim A Markel, and John C Schotland. Inversion formulas for the broken-ray Radon transform. *Inverse Problems*, 27(2):025002, 2011.

[44] Gaik Ambartsoumian. Inversion of the V-line Radon transform in a disc and its applications in imaging. *Computers & Mathematics with Applications*, 64(3):260–265, aug 2012.

[45] R Krylov and A Katsevich. Inversion of the broken ray transform in the case of energy-dependent attenuation. *Physics in Medicine and Biology*, 60(11):4313–4334, may 2015.

[46] Wei Zhang, Dianwen Zhu, Michael Lun, and Changqing Li. Multiple pinhole collimator based x-ray luminescence computed tomography. *Biomed. Opt. Express*, 7(7):2506–2523, Jul 2016.

[47] Jeffrey A. Boyle. Using rolling circles to generate caustic envelopes resulting from reflected light. *The American Mathematical Monthly*, 122(5):452, 2015.

[48] A. Cayley. A memoir upon caustics. *Philosophical Transactions of the Royal Society of London*, 147(0):273–312, jan 1857.

[49] Frank W Warner. The conjugate locus of a riemannian manifold. *American Journal of Mathematics*, 87(3):575–604, 1965.

[50] PLAMEN STEFANOV and YANG YANG. Multiwave tomography with reflectors: Landweber's iteration. *Inverse Problems & Imaging*, 11(2), 2017.

[51] Sunwoo Park. An introduction to dynamical billiards.

[52] H.S.M. Coxeter. *Introduction to Geometry*. Wiley, 1961.

[53] Nurlan S. Dairbekov, Gabriel P. Paternain, Plamen Stefanov, and Gunther Uhlmann. The boundary rigidity problem in the presence of a magnetic field. *Adv. Math.*, 216(2):535–609, 2007.

[54] Will J. Merry and Gabriel P. Paternain. *Inverse Problems in Geometry and Dynamics Lecture notes*. incomplete version, 2011.

[55] Plamen Stefanov and Gunther Uhlmann. Boundary rigidity and stability for generic simple metrics. *J. Amer. Math. Soc.*, 18(4):975–1003, 2005.

[56] Yernat M. Assylbekov and Nurlan S. Dairbekov. The x-ray transform on a general family of curves on finsler surfaces. *The Journal of Geometric Analysis*, May 2017.

[57] Fatma Terzioglu, Peter Kuchment, and Leonid Kunyansky. Compton camera imaging and the cone transform: a brief overview. *Inverse Problems*, 34(5):054002, apr 2018.

[58] Fatma Terzioglu. Some analytic properties of the cone transform. *Inverse Problems*, 35(3):034002, jan 2019.

[59] GaÃńl Rigaud and Bernadette N Hahn. 3d compton scattering imaging and contour reconstruction for a class of radon transforms. *Inverse Problems*, 34(7):075004, may 2018.

[60] James W. Webber, , and Sean Holman and. Microlocal analysis of a spindle transform. *Inverse Problems & Imaging*, 13(2):231–261, 2019.

[61] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. In *Mathematics Past and Present Fourier Integral Operators*, pages 243–283. Springer Berlin Heidelberg, 1994.

[62] Victor Guillemin. On some results of Gel'fand in integral geometry, 1985.

[63] Bela Frigyik, Plamen Stefanov, and Gunther Uhlmann. The x-ray transform for a generic family of curves and weights. *Journal of Geometric Analysis*, 18(1):89–108, nov 2007.

[64] François Treves. *Introduction to Pseudodifferential and Fourier Integral Operators*. Springer US, 1980.

[65] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[66] Lars Hörmander. Fourier integral operators. i. *Acta Mathematica*, 127(0):79–183, 1971.

[67] François Treves. *Introduction to Pseudodifferential and Fourier Integral Operators, Volume 2*. Springer US, 1980.

[68] Victor Guillemin and Shlomo Sternberg. The moment map revisited. *Journal of Differential Geometry*, 69(1):137–162, jan 2005.

[69] Max (https://math.stackexchange.com/users/2633/max). Question about image of proper smooth map of constant rank. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/625526 (version: 2014-01-03).

[70] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer Berlin Heidelberg, 2003.

[71] Matti Lassas, Lauri Oksanen, Plamen Stefanov, and Gunther Uhlmann. The light ray transform on lorentzian manifolds. in preparation.

[72] Yernat Assylbekov and Plamen Stefanov. A sharp stability estimate for the geodesic ray transform.

[73] Lord Rayleigh. On waves propagated along the plane surface of an elastic solid. *Proceedings of the London Mathematical Society*, s1-17(1):4–11, nov 1885.

[74] Elastic waves at the surface of separation of two solids. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 106(738):416–428, oct 1924.

[75] P. Chadwick and P. Borejko. Existence and uniqueness of stoneley waves. *Geophysical Journal International*, 118(2):279–284, aug 1994.

[76] Michael E. Taylor. Rayleigh waves in linear elasticity as a propagation of singularities phenomenon. 1979.

[77] Considerations of the existence of interfacial (stoneley) waves in bonded anisotropic elastic half-spaces. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 402(1822):153–166, dec 1985.