# ON THE GAUDIN AND XXX MODELS
# ASSOCIATED TO LIE SUPERALGEBRAS

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Dedicated to my wife, Dr. Gaojie Fan, who supports me all the time, and my son, Ian Huang, who always cheers me up.

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# ABSTRACT

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We describe a reproduction procedure which, given a solution of the $\mathfrak{gl}_{m|n}$ Gaudin Bethe ansatz equation associated to a tensor product of polynomial modules, produces a family $P$ of other solutions called the population. To a population we associate a rational pseudodifferential operator $R$ and a superspace $W$ of rational functions.

We show that if at least one module is typical then the population $P$ is canonically identified with the set of minimal factorizations of $R$ and with the space of full superflags in $W$. We conjecture that the singular eigenvectors (up to rescaling) of all $\mathfrak{gl}_{m|n}$ Gaudin Hamiltonians are in a bijective correspondence with certain superspaces of rational functions.

We establish a duality of the non-periodic Gaudin model associated with superalgebra $\mathfrak{gl}_{m|n}$ and the non-periodic Gaudin model associated with algebra $\mathfrak{gl}_k$.

The Hamiltonians of the Gaudin models are given by expansions of a Berezinian of an $(m+n)\times(m+n)$ matrix in the case of $\mathfrak{gl}_{m|n}$ and of a column determinant of a $k\times k$ matrix in the case of $\mathfrak{gl}_k$. We obtain our results by proving Capelli type identities for both cases and comparing the results.

We study solutions of the Bethe ansatz equations of the non-homogeneous periodic XXX model associated to super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$. To a solution we associate a rational difference operator $\mathcal{D}$ and a superspace of rational functions $W$. We show that the set of complete factorizations of $\mathcal{D}$ is in canonical bijection with the variety of superflags in $W$ and that each generic superflag defines a solution of the Bethe ansatz equation. We also give the analogous statements for the quasi-periodic supersymmetric spin chains.

# 1. INTRODUCTION

We consider the XXX and the Gaudin models associated to Lie superalgebras $\mathfrak{gl}_{m|n}$. These are well-known fundamental examples of quantum integrable models. The main questions is to describe the eigenvalues and eigenvectors of the corresponding Hamiltonians. We make use of Bethe ansatz method to address this question.

The Hamiltonians of the XXX and the Gaudin models are naturally obtained from the commutative subalgebras of the Yangian and the current algebra respectively, which are called Bethe subalgebras. The Bethe subalgebras are the central objects of our study.

Let us recall the situation in the even case (that is in the case of $n = 0$).

For the Gaudin models the joint eigenvalues of the Bethe subalgebra are identified with Fuchsian scalar differential operators without monodromy and prescribed singularities, see [F04], [MV04]. Such an identification is an example of the geometric Langlands correspondence. Alternatively, the Bethe subalgebra of the Gaudin model acting in an irreducible finite dimensional $\widehat{\mathfrak{gl}}_N$ module, is identified with the coordinate ring of scheme-theoretic intersection of Schubert cells, see [MTV09]. Moreover, the module is identified with the co-regular representation of the coordinate ring. The Bethe subalgebra related to the tensor product of evaluation vector representations is also related to the equivariant cohomology of a certain partial flag variety, see [RSTV11].

For the XXX models associated to Lie algebras, the Bethe subalgebra is described by the transfer matrices corresponding to the auxiliary representations. The eigenvalue of the transfer matrix can be obtained from the $q$-characters of the auxiliary spaces by suitable substitutions of solutions of Bethe ansatz equations, see [FH15], [FJMM17]. In this case the Bethe subalgebra (in the case of vector evaluation modules) also can be identified with the quantum cohomology of the cotangent bundle of a flag variety, see [GRTV13].

In this thesis we make the first steps of obtaining a similar understanding of the Gaudin and XXX models in the supersymmetric case.

## 1.1 Gaudin model

We study the Gaudin model associated to tensor products of polynomial modules over the Lie superalgebra $\mathfrak{gl}_{m|n}$. The main method is the Bethe ansatz; see [MVY14]. It is well-known that the Bethe ansatz method in its straightforward formulation is incomplete – it does not provide the full set of eigenvectors of the Hamiltonians; see [MV07]. Here, we propose a regularization of the Bethe ansatz method, drawing our inspiration from [MV04].

In the case of Lie algebras, the regularization of the Bethe ansatz is obtained by the identification of the spectrum of the model with opers – linear differential operators with appropriate properties [FFR94, R16]. In the case of $\mathfrak{gl}_m$, the opers are reduced to scalar linear differential operators of order $m$ with polynomial kernels. The spaces of polynomials of dimension $m$ obtained this way are intersection points of Schubert varieties whose data is described by the parameters of the Gaudin model. Moreover, the action of the algebra of Gaudin Hamiltonians can be identified with the regular representation of the scheme-theoretic intersection algebra, [MTV09].

We argue that in the case of the Lie superalgebra $\mathfrak{gl}_{m|n}$ one should study rational pseudodifferential operators and appropriate spaces of rational functions which we call $\mathfrak{gl}_{m|n}$ spaces.

Let us describe our findings in more detail. The $\mathfrak{gl}_{m|n}$ Gaudin model depends on the choice of a sequence of polynomial representations, each equipped with distinct complex evaluation parameters. The Bethe ansatz depends on a choice of Borel subalgebra. Such a choice is equivalent to the choice of a parity sequence $\boldsymbol{s} = (s_1, \dots, s_{m+n})$, $s_i \in \{\pm 1\}$. The highest weights of representations and the evaluation parameters are encoded into polynomials $T_i^{\boldsymbol{s}}$ (see (2.9)). A solution of the Bethe ansatz equation is represented by a sequence of monic polynomials $(y_1, \dots, y_{m+n-1})$, so that the roots of $y_i$ are Bethe variables corresponding to the $i$th simple root (see (4.12)).

The key ingredient is the reproduction procedure (see Theorem 2.5.2), which given a solution of the Bethe ansatz equation (BAE) produces a family of new solutions along a simple root. If the simple root is even, then the BAE means that the kernel of the operator

$$\left(\partial - \ln' \frac{T_i^{\boldsymbol{s}} y_{i-1} y_{i+1}}{T_{i+1}^{\boldsymbol{s}} y_i}\right)\left(\partial - \ln' y_i\right)$$

consists of polynomials. Then one shows that all tuples of the form

$$(y_1, \dots, \widetilde{y}_i, \dots, y_{m+n-1}),$$

where $\widetilde{y}_i$ is any (generic) polynomial in the kernel of the differential operator, represent solutions of the BAE. This gives the bosonic reproduction procedure, which was described in [MV04].

If the simple root is odd then the BAE means that $y_i$ divides a certain explicit polynomial $\mathcal{N}$ and it turns out that the tuple $(y_1, \dots, \widetilde{y}_i, \dots, y_{m+n-1})$, $\widetilde{y}_i = \mathcal{N}/y_i$, again satisfies the BAE (if generic). This gives the fermionic reproduction procedure. Moreover, the fermionic reproduction can be rewritten as an equality of rational pseudodiffential operators (assuming $s_i = 1$):

$$\left(\partial - \ln' \frac{T_i^{\boldsymbol{s}} y_{i-1}}{y_i}\right)\left(\partial - \ln' \frac{y_{i+1}}{T_{i+1}^{\boldsymbol{s}} y_i}\right)^{-1} = \left(\partial - \ln' \frac{\widetilde{y}_i}{T_i^{\widetilde{\boldsymbol{s}}} y_{i-1}}\right)^{-1}\left(\partial - \ln' \frac{T_{i+1}^{\widetilde{\boldsymbol{s}}} \widetilde{y}_i}{y_{i+1}}\right),$$

where $\widetilde{\boldsymbol{s}} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$.

The bosonic and fermionic procedures are very different in nature. The bosonic procedure describes a one-parameter family of solutions of the BAE. However, these solutions are not physical: $\deg \widetilde{y}_i$ is large and the corresponding Bethe vector is zero on weight grounds. The fermionic procedure produces only one new solution. Moreover, in contrast to the bosonic case, the new BAE corresponds to a new choice of the Borel subalgebra. If the original solution produced an eigenvector which was singular with respect to the original Borel subalgebra, the new solution produces the eigenvector in the same isotypical component but singular with respect to a new Borel subalgebra. The two eigenvectors are related by the diagonal action of $\mathfrak{gl}_{m|n}$.[1]

The most important feature of the bosonic and fermionic procedures is the conservation of the eigenvalues of the Gaudin Hamiltonians written in terms of the Bethe roots (see Lemma 2.4.5). We call the set of all solutions obtained by repeated applications of the reproduction procedures a *population*.

We define a rational pseudodifferential operator $R$ (see (2.22)). In the standard parity $\boldsymbol{s}_0 = (1, \dots, 1, -1, \dots, -1)$, it has the form: $R = D_{\bar{0}}(D_{\bar{1}})^{-1}$, where $D_{\bar{0}}$, $D_{\bar{1}}$ are scalar differential operators of orders $m$ and $n$ with rational coefficients, given by:

$$D_{\bar{0}} = \left(\partial - \ln' \frac{T_1^{\boldsymbol{s}_0} y_0}{y_1}\right)\left(\partial - \ln' \frac{T_2^{\boldsymbol{s}_0} y_1}{y_2}\right) \dots \left(\partial - \ln' \frac{T_m^{\boldsymbol{s}_0} y_{m-1}}{y_m}\right),$$
$$D_{\bar{1}} = \left(\partial - \ln' \frac{y_{m+n}}{T_{m+n-1}^{\boldsymbol{s}_0} y_{m+n-1}}\right) \dots \left(\partial - \ln' \frac{y_{m+2}}{T_{m+2}^{\boldsymbol{s}_0} y_{m+1}}\right)\left(\partial - \ln' \frac{y_{m+1}}{T_{m+1}^{\boldsymbol{s}_0} y_m}\right).$$

(Here we set $y_0 = y_{m+n} = 1$.) We show that $R$ does not change under reproduction procedures (see Theorem 2.5.3) and, moreover, if at least one weight is typical, then the population is identified with the set of all minimal factorizations of $R$ into linear factors (see Theorem 2.6.9).

Then we study the space $W = V \oplus U$, where $V = \ker D_{\bar{0}}$, $U = \ker D_{\bar{1}}$. We show that if at least one weight is typical, then $U \cap V = 0$. We think of $W$ as a superspace of dimension $m + n$, with even part $V$ and odd part $U$. We identify the population with the space of all full superflags in $W$ (see Theorem 2.6.9).

The operators $D_{\bar{0}}$ and $D_{\bar{1}}$ up to a conjugation coincide with $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ operators. It follows that $W$ consists of rational functions. In other words, $W$ is given by a pair of spaces of polynomials with prescribed ramification conditions linked via polynomials $y_m, T_m, T_{m+1}$. This leads us to a definition of a $\mathfrak{gl}_{m|n}$ *space* (see Section 2.6.3). The Gaudin Hamiltonians acting in tensor products of polynomial modules belong to a

[1]These features are reminiscent of trigonometric Gaudin models and Gaudin with quasi-periodic boundary conditions [MV08], in which the diagonal symmetry is broken. In those cases reproduction produces one new solution, which describes the same eigenvector (up to proportionality) but with respect to a different Borel subalgebra.

natural commutative algebra $\mathcal{B}(\boldsymbol{\lambda})$ of higher Gaudin Hamiltonians. We conjecture that the joint eigenvectors of this algebra $\mathcal{B}(\boldsymbol{\lambda})$ are parametrized by $\mathfrak{gl}_{m|n}$ spaces (see Conjecture 2.7.1).

## 1.2 Duality of supersymmetric Gaudin models

Integrable models associated with finite-dimensional Lie superalgebras have been recently receiving the much deserved attention. While most of the work is done by physicists on the spin-chain side, the theory of the corresponding Gaudin models is also moving forward, see [MR14], [MVY14], [HMVY19]. The duality of various systems is another very important topic which always gets a lot of attention. Here, we discuss the duality of the Gaudin model associated with supersymmetric $\mathfrak{gl}_{m|n}$ to the Gaudin model associated with even $\mathfrak{gl}_k$ acting on the same bosonic-fermionic space.

In the Lie algebra duality setting, the Lie superalgebras $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ both act on the algebra of supersymmetric polynomials $V$ generated by entries of the $(m+n)\times k$ matrix $(x_{i,a})$ where $x_{i,a}$ is even if and only if $i \leq m$. Then each row is identified with the vector representation of $\mathfrak{gl}_k$ and each column with the vector representation of $\mathfrak{gl}_{m|n}$. The two actions are extended to the action on the whole bosonic-fermionic space $V$ of supersymmetric polynomials as differential operators, where they centralize each other, see Section 3.4.1. We chose column evaluation parameters $z_1,\dots,z_k$ for $\mathfrak{gl}_{m|n}$, row evaluation parameters $\Lambda_1,\dots,\Lambda_{m+n}$ for $\mathfrak{gl}_k$ and upgrade the action to the current algebras $\mathfrak{gl}_{m|n}[t]$ and $\mathfrak{gl}_k[t]$ in $V$ so that each row and each column becomes an evaluation module with the corresponding evaluation parameter.

It is well known that the commuting Hamiltonians of the $\mathfrak{gl}_k$ Gaudin system are elements of $U\mathfrak{gl}_k[t]$ given by the coefficients of the column determinant of the $k\times k$ matrix $G = \big(\delta_{a,b}(\partial_u - z_a) - e^{[k]}_{a,b}(u)\big)$, see [T06], where we chose evaluation parameters of columns $z_1,\dots,z_k$ to be the so called boundary parameters of the model.

It is also known that the Hamiltonians of the $\mathfrak{gl}_{m|n}$ Gaudin system are elements of $U\mathfrak{gl}_{m|n}[t]$ given by the coefficients of the Berezinian of the $(m+n)\times(m+n)$ matrix $B = \big(\delta_{i,j}(\partial_v - \Lambda_i) - e^{[m|n]}_{i,j}(v)\big)$, see [MR14], [MM15], and Section 3.3.2. Note that we chose evaluation parameters of rows $\Lambda_1, \dots, \Lambda_{m+n}$ to be the boundary parameters of the model.

The column determinant $\operatorname{cdet} G$ is a differential operator of order $k$ in variable $u$ whose coefficients are power series in $u^{-1}$. The Berezinian $\operatorname{Ber} B$ a pseudodifferential operator in $\partial_v^{-1}$ whose coefficients are power series in $v^{-1}$. Our main result is that after multiplying by simple factors, coefficients of $v^r\partial_v^s$ and of $u^s\partial_u^r$ of the two expansion coincide as differential operators in $V$, see Theorem 3.4.2.

In order to prove our main result we establish two Capelli-like identities, see Propositions 3.4.4 and 3.4.6, which give the normal ordered expansions of the $\operatorname{cdet} G$ and $\operatorname{Ber} B$ acting in $V$. Because of the presence of fermions, those expansions have more terms than the original Capelli identity. However, the main feature is the same: the quantum corrections created by non-commutativity all cancel out and the result is the same as it would be in the supercommutative case.

The expansion of the $\operatorname{cdet} G$ is done by careful accounting of all terms and finding a way to cancel or collect the terms. For the Berezinian expansion we exploit a few tricks. Namely, we represent $\operatorname{Ber} B$ as a Berezinian of a matrix of size $(m+n+k)\times(m+n+k)$ then interchange the rows and columns to reduce the computation to another column determinant. The key property which allows us to do it, is the super version of Manin property of the matrices with some additional property which we call "affine-like". The affine-like property guarantees the existence of various inverse matrices and the Manin property of those inverses, see Section 3.2. In particular, we argue that for such matrices the Berezinian can be defined via quasi-determinants, similar to affine Manin matrices of standard parity treated in [MR14].

Our duality implies that the $\mathfrak{gl}_{m|n}$ Gaudin model has the same remarkable properties as the $\mathfrak{gl}_k$ Gaudin model, see [MTV08b]. Namely, the image of the Bethe algebra is a Frobenius algebra, which can be identified with an appropriate scheme theoretic intersection of Schubert varieties in a Grassmanian. Moreover, the corresponding phase space of the $\mathfrak{gl}_{m|n}$ Gaudin model is a regular representation of this Frobenius algebra. In particular, all joint eigenspaces have dimension one, see Corollary 3.4.3.

The spectrum of Gaudin Hamiltonians is found by the Bethe ansatz, see [MTV06] for the even and [MVY14] for the supersymmetric case. Since the two sets of Hamiltonians actually coincide in $V$, we have a correspondence between solution sets of two very different systems of the Bethe ansatz equations. Moreover, the eigenvectors of $\mathfrak{gl}_k$ model are in a natural bijection with differential operators of order $k$ with quasipolynomial kernels, see [MTV08b], while eigenvectors of $\mathfrak{gl}_{m|n}$ model are conjecturally in a bijection with ratios of differential operators of orders $m$ and $n$, and appropriate superspaces of quasirational functions, cf. [HMVY19].

The duality of the $\mathfrak{gl}_n$ and $\mathfrak{gl}_m$ systems was established in [MTV09b]. The corresponding map between spaces of polynomials is given by an appropriate Fourier transform and it is also identified with the bispectrality property of the KP hierarchy, see [MTV06b]. It is important to understand this map in the supersymmetric case.

We expect that the results of this paper can be extended to the most general duality of Gaudin models associated with $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_{k|l}$. We also expect that a similar duality can be established in the Yangian, see [MTV08], and the quantum setting.

The duality between $\mathfrak{gl}_{1|1}$ and $\mathfrak{gl}_2$ Gaudin models has appeared in [BBK17].

### 1.3 XXX model

The supersymmetric quantum spin chains were introduced back to [Kul85] in 1980s. There is a considerable renewed interests to those models, see [BR08], [BR09], [KSZ08], [HLPRS18], [TZZ15].

We use the method of populations of solutions of the Bethe ansatz equations. It was pioneered in [MV04] in the case of the Gaudin model and then extended to the XXX models constructed from the Yangian associated to $\mathfrak{gl}_n$, see [MV03, MV04, MTV07]. We are helped by the recent work on the populations of the supersymmetric Gaudin model [HMVY19].

Let us describe our findings in more detail. In this paper we restrict ourselves to tensor products of evaluation polynomial $\mathfrak{gl}_{m|n}$-modules. Moreover, we assume that the evaluation parameters are generic, meaning they are distinct modulo $h\mathbb{Z}$ where $h$ is the shift in the super Yangian relations. Note that such tensor products are irreducible $\mathrm{Y}(\mathfrak{gl}_{m|n})$-modules. We also assume that at least one of the participating $\mathfrak{gl}_{m|n}$-modules is typical.

The crucial observation is the reproduction procedure which given a solution of the Bethe ansatz equation and a simple root of $\mathfrak{gl}_{m|n}$, produces another solution, see Theorem 4.4.1.

The reproduction procedure along an even root is given in [MV03]. An even component of a solution of the Bethe ansatz equation gives a polynomial solution of a second order difference equation. The reproduction procedure amounts to trading this solution to any other polynomial solution of the difference equation, see (4.19). We call it the *bosonic reproduction procedure.*

The reproduction procedure along an odd root is different. In fact, an odd component of a solution of the Bethe ansatz equation corresponds to a polynomial which divides some other polynomial, see (4.20). The reproduction procedure changes the divisor to the quotient polynomial with an appropriate shift. We call it the *fermionic reproduction procedure.* The fermionic reproduction procedure looks similar to a mutation in a cluster algebra.

Then the population is the set of all solutions obtained from one solution by recursive application of the reproduction procedure.

Given a solution of the Bethe ansatz equation, we define a rational difference operator of the form $\mathcal{D} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$, where $\mathcal{D}_{\bar{0}}$, $\mathcal{D}_{\bar{1}}$ are linear difference operators of orders $m$ and $n$ with rational coefficients, respectively, see (4.25). The operator $\mathcal{D}$ is invariant under reproduction procedures and therefore it is defined for the population, see Theorem 4.4.3. The idea of considering such an operator is found in [HMVY19] in the case of the Gaudin model. Such an operator in the case of tensor products of vector representations also appears in [Tsu98] in relation to the study of T-systems and analytic Bethe ansatz.

Kernels $V = \ker \mathcal{D}_{\bar{0}}$, $U = \ker \mathcal{D}_{\bar{1}}$ are spaces of rational functions of dimensions $m$ and $n$. Under our assumption, that at least one of the representations is typical, we can show $V \cap U = 0$, see Lemma 4.5.1. We consider superspace $W = V \oplus U$. Then we show that there are natural bijections between three objects: elements of the population of the solutions of the Bethe ansatz equation, superflags in $W$, and complete factorizations of $\mathcal{D}$ into products of linear difference operators and their inverses, see Theorem 4.5.7.

Note that the Bethe ansatz equations depend on the choice of the Borel subalgebra in $\mathfrak{gl}_{m|n}$. The fermionic reproductions change this choice. In general, the Borel subalgebra is determined from the parity of the superflag or, equivalently, from the positions of the inverse linear difference operators in a complete factorization of $\mathcal{D}$.

Thus the solutions of the Bethe ansatz equations correspond to superspaces of rational functions. It is natural to expect that all joint eigenvectors of XXX Hamiltonians correspond to such spaces and that there is a natural correspondence between the eigenvectors of the transfer matrix and points of an appropriate Grassmannian. However, the precise formulation of this correspondence is not established even in the even case, see [MTV07].

We give a few details in the quasi-periodic case as well, see Section 4.6. In this case we also have concepts of reproduction procedure, the population, and the rational difference operator. Then the elements in the population are in a natural bijection with the permutations of the distinguished flags in the space of functions of the form $f(x) = e^{zx} r(x)$, where $r(x) \in \mathbb{C}(x)$ is a rational function and $z \in \mathbb{C}$, see Theorem 4.6.4. A similar picture in the even case is described in [MV08].

# 2. BETHE ANSATZ EQUATION AND RATIONAL PSEUDODIFFERENTIAL OPERATORS

## 2.1 Preliminaries on $\mathfrak{gl}_{m|n}$

Fix $m, n \in \mathbb{Z}_{\geq 0}$. In this section, we will recall some facts about $\mathfrak{gl}_{m|n}$. For details see, for example, [CW12].

### 2.1.1 Lie superalgebra $\mathfrak{gl}_{m|n}$

A *vector superspace* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space. The *parity* of a homogeneous vector $v$ is denoted by $|v| \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. We set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. An element $v$ in $V_{\bar{0}}$ (respectively $V_{\bar{1}}$) is called *even* (respectively *odd*), and we write $|v| = \bar{0}$ (respectively $|v| = \bar{1}$). Let $\mathbb{C}^{m|n}$ be a complex vector superspace, with $\dim(\mathbb{C}^{m|n})_{\bar{0}} = m$ and $\dim(\mathbb{C}^{m|n})_{\bar{1}} = n$. Choose a homogeneous basis $e_i$, $i = 1, \ldots, m+n$, of $\mathbb{C}^{m|n}$ such that $|e_i| = \bar{0}$, $i = 1, \ldots, m$, and $|e_i| = \bar{1}$, $i = m+1, \ldots, m+n$. Set $|i| = |e_i|$.

Let $\boldsymbol{s} = (s_1, \ldots, s_{m+n})$, $s_i \in \{\pm 1\}$, be a sequence such that 1 occurs exactly $m$ times. We call such a sequence a *parity sequence*. We call the parity sequence $\boldsymbol{s}_0 = (1, \ldots, 1, -1, \ldots, -1)$ *standard*. Denote the set of all parity sequences by $S_{m|n}$. The order of $S_{m|n}$ is $\binom{m+n}{m}$. The set $S_{m|n}$ is identified with $\mathfrak{S}_{m+n}/(\mathfrak{S}_m \times \mathfrak{S}_n)$, where $\mathfrak{S}_k$ denotes the permutation group of $k$ letters. We fix a lifting $S_{m|n} = \mathfrak{S}_{m+n}/(\mathfrak{S}_m \times \mathfrak{S}_n) \to \mathfrak{S}_{m+n}$: for each $\boldsymbol{s} \in S_{m|n}$, we define $\sigma_{\boldsymbol{s}} \in \mathfrak{S}_{m+n}$ by

$$\sigma_{\boldsymbol{s}}(i) = \begin{cases} \#\{j \mid j \leq i,\ s_j = 1\} & \text{if } s_i = 1, \\ m + \#\{j \mid j \leq i,\ s_j = -1\} & \text{if } s_i = -1. \end{cases} \tag{2.1}$$

Note that $\sigma_{\boldsymbol{s}_0} = \mathrm{id}$ and $(-1)^{|\sigma_{\boldsymbol{s}}(i)|} = s_i$. (The element $\sigma_{\boldsymbol{s}}$ is sometimes called an *unshuffle*.)

For a parity sequence $\boldsymbol{s} \in S_{m|n}$ and $i = 1, \dots, m+n$, define numbers

$$\boldsymbol{s}_i^+ = \#\{j \mid j > i,\ s_j = 1\}, \qquad \boldsymbol{s}_i^- = \#\{j \mid j < i,\ s_j = -1\}.$$

We have

$$\boldsymbol{s}_i^+ = \begin{cases} m - \sigma_{\boldsymbol{s}}(i) & \text{if } s_i = 1, \\ \sigma_{\boldsymbol{s}}(i) - i & \text{if } s_i = -1, \end{cases} \qquad \boldsymbol{s}_i^- = \begin{cases} i - \sigma_{\boldsymbol{s}}(i) & \text{if } s_i = 1, \\ \sigma_{\boldsymbol{s}}(i) - m - 1 & \text{if } s_i = -1. \end{cases}$$

The *Lie superalgebra* $\mathfrak{gl}_{m|n}$ is spanned by $e_{ij}$, $i, j = 1, \dots, m+n$, with $|e_{ij}| = |i|+|j|$, and the superbracket is given by

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} e_{kj}.$$

The universal enveloping algebra of $\mathfrak{gl}_{m|n}$ is denoted by $U\mathfrak{gl}_{m|n}$.

There is a non-degenerate invariant bilinear form $(\ ,\ )$ on $\mathfrak{gl}_{m|n}$, such that

$$(e_{ab}, e_{cd}) = (-1)^{|a|} \delta_{ad} \delta_{bc}.$$

The *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{gl}_{m|n}$ is spanned by $e_{ii}$, $i = 1, \dots, m+n$. The *weight space* $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$. Let $\epsilon_i$, $i = 1, \dots, m+n$, be a basis of $\mathfrak{h}^*$, such that $\epsilon_i(e_{jj}) = \delta_{ij}$. The bilinear form $(\ ,\ )$ is extended to $\mathfrak{h}^*$ such that $(\epsilon_i, \epsilon_j) = (-1)^{|i|} \delta_{ij}$. The *root system* $\Phi$ is a subset of $\mathfrak{h}^*$ given by

$$\Phi = \{\epsilon_i - \epsilon_j \mid i, j = 1, \dots, m+n \text{ and } i \neq j\}.$$

A root $\epsilon_i - \epsilon_j$ is called *even* (respectively *odd*), if $|i| = |j|$ (respectively $|i| \neq |j|$).

### 2.1.2 Root systems

For each parity sequence $\boldsymbol{s} \in S_{m|n}$, define the set of $\boldsymbol{s}$*-positive roots* $\Phi_{\boldsymbol{s}}^+ = \{\epsilon_{\sigma_{\boldsymbol{s}}(i)} - \epsilon_{\sigma_{\boldsymbol{s}}(j)} \mid i, j = 1, \dots, m+n \text{ and } i < j\}$. Define the $\boldsymbol{s}$*-positive simple roots* $\alpha_i^{\boldsymbol{s}} = \epsilon_{\sigma_{\boldsymbol{s}}(i)} - \epsilon_{\sigma_{\boldsymbol{s}}(i+1)}$, $i = 1, \dots, m+n-1$. Define

$$e_{ij}^{\boldsymbol{s}} = e_{\sigma_{\boldsymbol{s}}(i), \sigma_{\boldsymbol{s}}(j)},\ i, j = 1, \dots, m+n.$$

The *nilpotent subalgebra* $\mathfrak{n}_{\boldsymbol{s}}^{+}$ of $\mathfrak{gl}_{m|n}$ (respectively $\mathfrak{n}_{\boldsymbol{s}}^{-}$) associated to $\boldsymbol{s}$, is generated by $\{e_{i,i+1}^{\boldsymbol{s}} \mid i = 1, \ldots, m+n-1\}$ (respectively $\{e_{i+1,i}^{\boldsymbol{s}} \mid i = 1, \ldots, m+n-1\}$). The algebra $\mathfrak{n}_{\boldsymbol{s}}^{+}$ (respectively $\mathfrak{n}_{\boldsymbol{s}}^{-}$) has a basis $\{e_{ij}^{\boldsymbol{s}} \mid i < j\}$ (respectively $\{e_{ij}^{\boldsymbol{s}} \mid i > j\}$). The Borel subalgebra associated to $\boldsymbol{s}$, is $\mathfrak{b}_{\boldsymbol{s}} = \mathfrak{h} \oplus \mathfrak{n}_{\boldsymbol{s}}^{+}$. We call the Borel subalgebra $\mathfrak{b}_{\boldsymbol{s}_0}$ *standard.*

In what follows, many objects depend on a parity sequence $\boldsymbol{s}$. If $\boldsymbol{s}$ is omitted from the notation, then it means the standard parity sequence. For example, we abbreviate $\mathfrak{n}_{\boldsymbol{s}_0}^{+}$, $\mathfrak{n}_{\boldsymbol{s}_0}^{-}$, and $\mathfrak{b}_{\boldsymbol{s}_0}$ to $\mathfrak{n}^{+}$, $\mathfrak{n}^{-}$, and $\mathfrak{b}$, respectively.

**Example 2.1.1.** *Consider the case of $\mathfrak{gl}_{3|3}$. Two possible parity sequences from $S_{3|3}$ are:*

$\boldsymbol{s}_1 = (1, 1, -1, -1, -1, 1)$ *and* $\boldsymbol{s}_2 = (1, -1, 1, -1, 1, -1)$. *We have*

$$\sigma_{\boldsymbol{s}_1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & 3 \end{pmatrix}, \quad \sigma_{\boldsymbol{s}_2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}.$$

*The $\boldsymbol{s}_1$-positive simple roots and $\boldsymbol{s}_2$-positive simple roots are given respectively by*

$$(\alpha_1^{\boldsymbol{s}_1}, \alpha_2^{\boldsymbol{s}_1}, \alpha_3^{\boldsymbol{s}_1}, \alpha_4^{\boldsymbol{s}_1}, \alpha_5^{\boldsymbol{s}_1}) = (\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_5, \epsilon_5 - \epsilon_6, \epsilon_6 - \epsilon_3),$$
$$(\alpha_1^{\boldsymbol{s}_2}, \alpha_2^{\boldsymbol{s}_2}, \alpha_3^{\boldsymbol{s}_2}, \alpha_4^{\boldsymbol{s}_2}, \alpha_5^{\boldsymbol{s}_2}) = (\epsilon_1 - \epsilon_4, \epsilon_4 - \epsilon_2, \epsilon_2 - \epsilon_5, \epsilon_5 - \epsilon_3, \epsilon_3 - \epsilon_6).$$

We have

$$(\alpha_i^{\boldsymbol{s}}, \alpha_j^{\boldsymbol{s}}) = (s_i + s_{i+1})\delta_{i,j} - s_i \delta_{i,j+1} - s_{i+1}\delta_{i+1,j}.$$

The *symmetrized Cartan matrix* associated to $\boldsymbol{s}$, $\big((\alpha_i^{\boldsymbol{s}}, \alpha_j^{\boldsymbol{s}})\big)_{i,j=1}^{m+n-1}$, is described by the blocks

$$\begin{pmatrix} (\alpha_i^{\boldsymbol{s}}, \alpha_i^{\boldsymbol{s}}) & (\alpha_i^{\boldsymbol{s}}, \alpha_{i+1}^{\boldsymbol{s}}) \\ (\alpha_{i+1}^{\boldsymbol{s}}, \alpha_i^{\boldsymbol{s}}) & (\alpha_{i+1}^{\boldsymbol{s}}, \alpha_{i+1}^{\boldsymbol{s}}) \end{pmatrix} = \begin{pmatrix} s_i + s_{i+1} & -s_{i+1} \\ -s_{i+1} & s_{i+1} + s_{i+2} \end{pmatrix}.$$

Explicitly, this block is one of the following cases depending on $(s_i, s_{i+1}, s_{i+2})$:

$$\begin{array}{cccc} (1,1,1) & (1,1,-1) & (1,-1,1) & (-1,1,1) \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, & \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}, \end{array}$$

$$\begin{array}{cccc} (-1,-1,-1) & (-1,-1,1) & (-1,1,-1) & (1,-1,-1) \\ \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, & \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \end{array}$$

### 2.1.3 Representations of $\mathfrak{gl}_{m|n}$

Let $V$ be a $\mathfrak{gl}_{m|n}$ module. Given a parity sequence $\boldsymbol{s} \in S_{m|n}$ and a weight $\lambda \in \mathfrak{h}^*$, a non-zero vector $v^{\boldsymbol{s}}_\lambda \in V$ is called an *$\boldsymbol{s}$-singular vector* of weight $\lambda$ if $\mathfrak{n}^+_{\boldsymbol{s}} v^{\boldsymbol{s}}_\lambda = 0$ and $hv^{\boldsymbol{s}}_\lambda = \lambda(h)v^{\boldsymbol{s}}_\lambda$, for all $h \in \mathfrak{h}$. Denote the subspace of $\boldsymbol{s}$-singular vectors by $V^{\boldsymbol{s}\text{ing}}$. Denote by $V_\lambda$ the subspace of vectors of weight $\lambda$, $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}$. Denote by $V^{\boldsymbol{s}\text{ing}}_\lambda$ the subspace of $\boldsymbol{s}$-singular vectors of weight $\lambda$. Denote the subspaces of $\boldsymbol{s}_0$-singular vectors and of $\boldsymbol{s}_0$-singular vectors of weight $\lambda$ by $V^{sing}$ and $V^{sing}_\lambda$ respectively. Let $L^{\boldsymbol{s}}(\lambda)$ be the *$\boldsymbol{s}$-highest weight irreducible module* of highest weight $\lambda$, generated by the $\boldsymbol{s}$-singular vector $v^{\boldsymbol{s}}_\lambda$. The $\boldsymbol{s}$-singular vector $v^{\boldsymbol{s}}_\lambda \in L^{\boldsymbol{s}}(\lambda)$ is called the *$\boldsymbol{s}$-highest weight vector.* Denote by

$$\lambda_{[\boldsymbol{s}]} = (\lambda_{[\boldsymbol{s}],1}, \dots, \lambda_{[\boldsymbol{s}],m+n}) = \big(\lambda(e^{\boldsymbol{s}}_{11}), \dots, \lambda(e^{\boldsymbol{s}}_{m+n,m+n})\big)$$

the coordinate sequence of $\lambda$ associated to $\boldsymbol{s}$. We also use the notation $L^{\boldsymbol{s}}(\lambda_{[\boldsymbol{s}]})$ for $L^{\boldsymbol{s}}(\lambda)$.

**Example 2.1.2.** *The superspace $\mathbb{C}^{m|n}$ is a $\mathfrak{gl}_{m|n}$ module with the action given by $e_{ij}e_k = \delta_{j,k}e_i$. We have $\mathbb{C}^{m|n} \cong L^{\boldsymbol{s}}(1,0,\dots,0) = L^{\boldsymbol{s}}(\epsilon_{\sigma_{\boldsymbol{s}}(1)})$ for any $\boldsymbol{s} \in S_{m|n}$. The $\boldsymbol{s}$-highest weight vector is $v^{\boldsymbol{s}}_{\epsilon_{\sigma_{\boldsymbol{s}}(1)}} = e_{\sigma_{\boldsymbol{s}}(1)}$. We call $\mathbb{C}^{m|n}$ the* vector representation.

A module $V$ is called a *polynomial module* if it is an irreducible submodule of $(\mathbb{C}^{m|n})^{\otimes n}$ for some $n \in \mathbb{Z}_{\geq 0}$. A highest weight module $L(\lambda)$ with respect to the standard Borel subalgebra $\mathfrak{b}$, is a polynomial module if and only if the weight $\lambda$ satisfies $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all $i$, $\lambda_1 \geq \cdots \geq \lambda_m$, $\lambda_{m+1} \geq \cdots \geq \lambda_{m+n}$, and $\lambda_m \geq \#\{i \mid \lambda_{m+i} \neq 0 \mid i = 1, \ldots, n\}$. A weight $\lambda$ is called a *polynomial weight* if $L(\lambda)$ is a polynomial module. It is known that the category of polynomial modules is a semisimple tensor category.

Let $\mu = (\mu_1 \geq \mu_2 \geq \ldots)$ be a partition: $\mu_i \in \mathbb{Z}_{\geq 0}$ and $\mu_i = 0$ if $i \gg 0$. The partition $\mu$ is called an $(m|n)$*-hook partition* if $\mu_{m+1} \leq N$. Polynomial modules are parametrized by $(m|n)$-hook partitions.

Let $L(\lambda)$ be a polynomial module with highest weight vector $v_\lambda$. Let $\boldsymbol{s}$ be a parity sequence. Then $L(\lambda)$ is isomorphic to an irreducible $\boldsymbol{s}$-highest weight module $L^{\boldsymbol{s}}(\lambda^{\boldsymbol{s}})$. The coordinate sequence $\lambda^{\boldsymbol{s}}_{[\boldsymbol{s}]}$ and the $\boldsymbol{s}$-highest weight vector $v^{\boldsymbol{s}}_\lambda$ can be found recursively as follows.

Let $\boldsymbol{s}^{[i]} = (s_1, \ldots, s_{i+1}, s_i, \ldots, s_{m+n})$ be the parity sequence obtained from $\boldsymbol{s}$ by switching the $i$-th and $(i+1)$-st coordinates. If $s_i \neq s_{i+1}$, then we have

$$\begin{aligned} \lambda^{\boldsymbol{s}^{[i]}}_{[\boldsymbol{s}^{[i]}]} &= (\lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],1}, \ldots, \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i-1}, \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i+1} + \delta, \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i} - \delta, \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i+2}, \ldots, \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],m+n}), \\ v^{\boldsymbol{s}^{[i]}}_{\lambda^{\boldsymbol{s}^{[i]}}} &= (e^{\boldsymbol{s}}_{i+1,i})^{\delta} v^{\boldsymbol{s}}_{\lambda^{\boldsymbol{s}}}, \end{aligned} \tag{2.2}$$

where $\delta = 1$ if $\lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i} + \lambda^{\boldsymbol{s}}_{[\boldsymbol{s}],i+1} \neq 0$ and $\delta = 0$ otherwise.

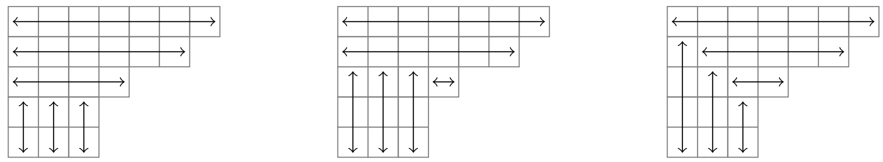
The following example illustrates how the coordinate sequence $\lambda^{\boldsymbol{s}}_{[\boldsymbol{s}]}$ can be found from an $(m|n)$-hook partition, and how the $\boldsymbol{s}$-highest weight vector $v^{\boldsymbol{s}}_\lambda$ is related to the highest weight vector $v_\lambda$.

**Example 2.1.3.** *Let* $\mu = (7, 6, 4, 3, 3)$ *be a* $(3|3)$*-hook partition. Choose some parity sequences:*

$$\boldsymbol{s}_0 = (1, 1, 1, -1, -1, -1), \quad \boldsymbol{s}_1 = (1, 1, -1, -1, -1, 1), \quad \boldsymbol{s}_2 = (1, -1, 1, -1, 1, -1).$$

*The highest weights and the highest weight vectors for those choices can be read as:*

Fig. 2.1. From (3|3)-hook partitions to highest weights



$$\lambda^{\boldsymbol{s}_0}_{[\boldsymbol{s}_0]} = (7,6,4,2,2,2) \quad \lambda^{\boldsymbol{s}_1}_{[\boldsymbol{s}_1]} = (7,6,3,3,3,1) \quad \lambda^{\boldsymbol{s}_2}_{[\boldsymbol{s}_2]} = (7,4,5,3,2,2)$$

$$v^{\boldsymbol{s}_0}_{\lambda^{\boldsymbol{s}_0}} = v_\lambda, \qquad v^{\boldsymbol{s}_1}_{\lambda^{\boldsymbol{s}_1}} = e_{63}e_{53}e_{43}v_\lambda, \qquad v^{\boldsymbol{s}_2}_{\lambda^{\boldsymbol{s}_2}} = e_{53}e_{42}e_{43}v_\lambda.$$

Another way to find $\lambda^{\boldsymbol{s}}_{[\boldsymbol{s}]}$ from $\lambda$ is given below in Theorem 2.6.2.

Define the $\boldsymbol{s}$-*Weyl weight*

$$\rho^{\boldsymbol{s}} = \frac{1}{2} \sum_{\substack{\alpha \in \Phi^+_{\boldsymbol{s}} \\ \alpha \text{ is even}}} \alpha - \frac{1}{2} \sum_{\substack{\beta \in \Phi^+_{\boldsymbol{s}} \\ \beta \text{ is odd}}} \beta.$$

A weight $\lambda$ is called *typical* if $(\lambda + \rho^{\boldsymbol{s}_0}, \alpha) \neq 0$, for any odd root $\alpha$. Otherwise $\lambda$ is called *atypical*. The module $L(\lambda)$ is *typical* if $\lambda$ is typical and *atypical* otherwise. If $\lambda$ is a polynomial weight, then $\lambda$ is typical if and only if $\lambda(e_{mm}) \geq N$. Let $\mu = (\mu_1, \mu_2, \dots)$ be the $(m|n)$-hook partition that parametrizes $L(\lambda)$. Then $L(\lambda)$ is typical if and only if $\mu_m \geq n$. In Example 2.1.3, all weights are typical.

## 2.2 Rational pseudodifferential operators and flag varieties

We establish some generalities about ratios of differential operators.

### 2.2.1 Rational pseudodifferential operators

We recall some results from [CDSK12] and [CDSK12b].

Let $\mathcal{K}$ be a differential field of characteristic zero, with the derivation $\partial$. The main example for this paper is the field of complex-valued rational functions $\mathcal{K} = \mathbb{C}(x)$.

Consider the division ring of *pseudodifferential operators* $\mathcal{K}((\partial^{-1}))$. An element $A \in \mathcal{K}((\partial^{-1}))$ has the form

$$A = \sum_{j=-\infty}^{m} a_j \partial^j, \ a_j \in \mathcal{K}, \ m \in \mathbb{Z}.$$

One says that $A$ has *order* $m$, $\mathrm{ord}A = m$, if $a_m \neq 0$. One says that $A$ is *monic* if $a_m = 1$.

We have the following relations in $\mathcal{K}((\partial^{-1}))$:

$$\partial\partial^{-1} = \partial^{-1}\partial = 1,$$
$$\partial^r a = \sum_{j=0}^{\infty} \binom{r}{j} a^{(j)} \partial^{r-j},\ a \in \mathcal{K},\ r \in \mathbb{Z},$$

where $a^{(j)}$ is the $j$-th derivative of $a$ and $a^{(0)} = a$.

All nonzero elements in $\mathcal{K}((\partial^{-1}))$ are invertible. The inverse of $A$ is given by

$$A^{-1} = \partial^{-m} \sum_{r=0}^{\infty} \Big( - \sum_{j=-\infty}^{-1} a_m^{-1} a_{j+m} \partial^j \Big)^r a_m^{-1}.$$

The algebra of differential operators $\mathcal{K}[\partial]$ is a subring of $\mathcal{K}((\partial^{-1}))$.

Let $D \in \mathcal{K}[\partial]$ be a monic differential operator. The differential operator $D$ is called *completely factorable over* $\mathcal{K}$ if $D = d_1 \dots d_m$, where $d_i = \partial - a_i$, $a_i \in \mathcal{K}$, $i = 1, \dots, m$.

Denote $\{u \in \mathcal{K} \mid Du = 0\}$ by $\ker D$. Clearly, if $\dim(\ker D) = \mathrm{ord}D$, then $D$ is completely factorable over $\mathcal{K}$; see also Section 2.2.2.

The division subring $\mathcal{K}(\partial)$ of $\mathcal{K}((\partial^{-1}))$, generated by $\mathcal{K}[\partial]$, is called the *division ring of rational pseudodifferential operators* and elements in $\mathcal{K}(\partial)$ are called *rational pseudodifferential operators.*

Let $R$ be a rational pseudodifferential operator. If we can write $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ for some $D_{\bar{0}}, D_{\bar{1}} \in \mathcal{K}[\partial]$, then this is called a *fractional factorization* of $R$. A fractional factorization $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ is called *minimal* if $D_{\bar{1}}$ is monic and has the minimal possible order.

**Proposition 2.2.1** ( [CDSK12b]). *Let $R \in \mathcal{K}(\partial)$ be a rational pseudodifferential operator. Then the following is true.*

1. *There exists a unique minimal fractional factorization of $R$.*

2. *Let* $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ *be the minimal fractional factorization. If* $R = \widetilde{D}_{\bar{0}} \widetilde{D}_{\bar{1}}^{-1}$ *is a fractional factorization, then there exists* $D \in \mathcal{K}[\partial]$ *such that* $\widetilde{D}_{\bar{0}} = D_{\bar{0}} D$ *and* $\widetilde{D}_{\bar{1}} = D_{\bar{1}} D$.

3. *Let* $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ *be a fractional factorization such that* $\dim(\ker D_{\bar{0}}) = \mathrm{ord} D_{\bar{0}}$ *and* $\dim(\ker D_{\bar{1}}) = \mathrm{ord} D_{\bar{1}}$. *Then* $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ *is the minimal fractional factorization of* $R$ *if and only if* $\ker D_{\bar{0}} \cap \ker D_{\bar{1}} = 0$.

$\square$

We call $R$ an $(m|n)$-*rational pseudodifferential operator* if for the minimal fractional factorization $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ we have $\mathrm{ord}(D_{\bar{0}}) = m$ and $\mathrm{ord}(D_{\bar{1}}) = n$.

Let $R$ be a monic $(m|n)$-rational pseudodifferential operator. Let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. The form $R = d_1^{s_1} \dots d_{m+n}^{s_{m+n}}$, where $d_i = \partial - a_i$, $a_i \in \mathcal{K}$, $i = 1, \dots, m+n$, is called the *complete factorization with the parity sequence* $\boldsymbol{s}$. We denote the set of all complete factorizations of $R$ by $\mathcal{F}(R)$ and the set of all complete factorizations of $R$ with parity sequence $\boldsymbol{s}$ by $\mathcal{F}^{\boldsymbol{s}}(R)$.

Let $R_1 = (\partial - a)(\partial - b)^{-1}$ and $R_2 = (\partial - c)^{-1}(\partial - d)$ be two $(1|1)$-rational pseudodifferential operators. Here $a, b, c, d \in \mathcal{K}$, $a \neq b$, and $c \neq d$. Then $R_1 = R_2$ if and only if

$$\begin{cases} c = b + \ln'(a-b), \\ d = a + \ln'(a-b), \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} a = d - \ln'(c-d), \\ b = c - \ln'(c-d), \end{cases} \tag{2.3}$$

where $\ln'(f) = f'/f$ stands for the logarithmic derivative.

Let $R$ be an $(m|n)$-rational pseudodifferential operator. Let $R = d_1^{s_1} \dots d_{m+n}^{s_{m+n}}$, $d_i = \partial - a_i$, be a complete factorization. Suppose $s_i \neq s_{i+1}$. Then $d_i \neq d_{i+1}$. We use equation (2.3) to construct $\widetilde{d}_i$ and $\widetilde{d}_{i+1}$ such that $d_i^{s_i} d_{i+1}^{s_{i+1}} = \widetilde{d}_i^{s_{i+1}} \widetilde{d}_{i+1}^{s_i}$. That gives a complete factorization of $R = d_1^{s_1} \dots \widetilde{d}_i^{s_{i+1}} \widetilde{d}_{i+1}^{s_i} \dots d_{m+n}^{s_{m+n}}$ with the new parity sequence $\widetilde{\boldsymbol{s}} = \boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$.

Repeating this procedure, we obtain a canonical identification of the set $\mathcal{F}^{\boldsymbol{s}}(R)$ of complete factorizations of $R$ with parity sequence $\boldsymbol{s}$ with the set $\mathcal{F}^{\boldsymbol{s}_0}(R)$ of complete factorizations of $R$ with parity sequence $\boldsymbol{s}_0$.

### 2.2.2 Complete factorizations of rational pseudodifferential operators and flag varieties

Let $W = W_{\bar{0}} \bigoplus W_{\bar{1}}$ be a vector superspace with $\dim(W_{\bar{0}}) = m$ and $\dim(W_{\bar{1}}) = n$. A full flag in $W$ is a chain of subspaces $\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{m+n} = W\}$ such that $\dim F_i = i$. Any basis $\{w_1, \dots, w_{m+n}\}$ of $W$ generates a full flag by the rule $F_i = \mathrm{span}(w_1, \dots, w_i)$. (By basis, we mean always ordered basis.) A full flag is called a *full superflag* if it is generated by a homogeneous basis. We denote by $\mathcal{F}(W)$ the set of all full superflags.

If $m = 0$ or $n = 0$, then every full flag is a full superflag. Thus, in this case $\mathcal{F}(W)$ is the usual flag variety.

To a given homogeneous basis $\{w_1, \dots, w_{m+n}\}$ of $W$, we associate a parity sequence $\boldsymbol{s} \in S_{m|n}$ by the rule $s_i = (-1)^{|w_i|}$, $i = 1, \dots, m+n$. We say a full superflag $\mathcal{F}$ has parity sequence $\boldsymbol{s}$ if it is generated by a homogenous basis associated to $\boldsymbol{s}$. We denote by $\mathcal{F}^{\boldsymbol{s}}(W)$ the set of all full superflags of parity $\boldsymbol{s}$.

The following lemma is obvious.

**Lemma 2.2.2.** *We have*

$$\mathcal{F}(W) = \bigsqcup_{\boldsymbol{s} \in S_{m|n}} \mathcal{F}^{\boldsymbol{s}}(W), \qquad \mathcal{F}^{\boldsymbol{s}}(W) = \mathcal{F}(W_{\bar{0}}) \times \mathcal{F}(W_{\bar{1}}). \qquad \square$$

Let $R$ be a monic $(m|n)$-rational pseudodifferential operator over $\mathcal{K}$. Let $R = D_{\bar{0}} D_{\bar{1}}^{-1}$ be the minimal fractional factorization of $R$. Assume that $\dim(\ker D_{\bar{0}}) = m$, and $\dim(\ker D_{\bar{1}}) = n$.

Let $V = W_{\bar{0}} = \ker D_{\bar{0}}$, $U = W_{\bar{1}} = \ker D_{\bar{1}}$, $W = W_{\bar{0}} \oplus W_{\bar{1}}$.

Given a basis $\{v_1, \dots, v_m\}$ of $V$, a basis $\{u_1, \dots, u_n\}$ of $U$, and a parity sequence $\boldsymbol{s} \in S_{m|n}$, define a homogeneous basis $\{w_1, \dots, w_{m+n}\}$ of $W$ by the rule $w_i = v_{\boldsymbol{s}_i^+ + 1}$ if $s_i = 1$ and $w_i = u_{\boldsymbol{s}_i^- + 1}$ if $s_i = -1$. Conversely, any homogeneous basis of $W$ gives a basis of $V$, a basis of $U$, and a parity sequence $\boldsymbol{s}$.

**Example 2.2.3.** *If* $\boldsymbol{s} = (1, -1, -1, 1, 1, -1, 1, -1)$*, then*

$$\{w_1, \dots, w_8\} = \{v_4, u_1, u_2, v_3, v_2, u_3, v_1, u_4\}.$$

Given a basis $\{v_1,\dots,v_m\}$ of $V$, a basis $\{u_1,\dots,u_n\}$ of $U$, and a parity sequence $\boldsymbol{s}\in S_{m|n}$, define $d_i=d_i(\boldsymbol{s},\{v_1,\dots,v_m\},\{u_1,\dots,u_n\})=\partial-a_i$ where

$$a_i=\ln'\frac{\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+ +1},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-})}{\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-})}\qquad\text{if } s_i=1, \tag{2.4}$$

$$a_i=\ln'\frac{\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^- +1})}{\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-})}\qquad\text{if } s_i=-1, \tag{2.5}$$

where the Wronskian is given by the standard formula

$$\mathrm{Wr}\,(f_1,\dots,f_r)=\det\left(f_j^{(i-1)}\right)_{i,j=1}^{r}.$$

If two bases $\{v_1,\dots,v_m\}$, $\{\widetilde{v}_1,\dots,\widetilde{v}_m\}$ generate the same full flag of $V$ and two bases $\{u_1,\dots,u_n\}$, $\{\widetilde{u}_1,\dots,\widetilde{u}_n\}$ generate the same full flag of $U$, then the coefficients $a_i$ computed from $v_j,u_j$ and from $\widetilde{v}_j,\widetilde{u}_j$ coincide.

**Proposition 2.2.4.** *We have a complete decomposition of $R$ with parity $\boldsymbol{s}$: $R=d_1^{s_1}\dots d_{m+n}^{s_{m+n}}$.*

*Proof.* If $\boldsymbol{s}=\boldsymbol{s}_0$ is standard, then the statement of the proposition is well known: see for example the Appendix in [MV04].

Let $\boldsymbol{s}$ and $\widetilde{\boldsymbol{s}}$ differ only in positions $i,i+1$: $s_j=\widetilde{s}_j$ for $j\neq i,i+1$ and $s_i=-s_{i+1}=-\widetilde{s}_i=\widetilde{s}_{i+1}$. Then we have $d_j=\widetilde{d}_j$ for $j\neq i,i+1$. In addition $d_i^{s_i}d_{i+1}^{s_{i+1}}=\widetilde{d}_i^{\,\widetilde{s}_i}\widetilde{d}_{i+1}^{\,\widetilde{s}_{i+1}}$ follows from the Wronski identity

$$\begin{aligned}&\mathrm{Wr}\,\Big(\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+ +1},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-}),\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^- +1})\Big)\\&=\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+ +1},u_1,u_2,\dots,u_{\boldsymbol{s}_i^- +1})\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-}).\end{aligned}$$

□

We identify full superflags in $W$ with complete factorizations of $R$. Namely, by Proposition 2.2.4 we have a map: $\rho:\mathcal{F}(W)\to\mathcal{F}(R)$ and $\rho^{\boldsymbol{s}}:\mathcal{F}^{\boldsymbol{s}}(W)\to\mathcal{F}^{\boldsymbol{s}}(R)$.

**Proposition 2.2.5.** *The maps $\rho$, $\rho^{\boldsymbol{s}}$ are bijections.*

*Proof.* Clearly, $\rho^{\boldsymbol{s}_0}$ is a bijiection. We have a canonical bijection between $\mathcal{F}^{\boldsymbol{s}}(W)$ and $\mathcal{F}^{\boldsymbol{s}_0}(W)$. We have a canonical bijection between $\mathcal{F}^{\boldsymbol{s}}(R)$ and $\mathcal{F}^{\boldsymbol{s}_0}(R)$. These two bijections are compatible with $\rho^{\boldsymbol{s}}$ and $\rho^{\boldsymbol{s}_0}$. The proposition follows. □

## 2.3 Bethe ansatz

We recall some facts about the Gaudin model associated to $\mathfrak{gl}_{m|n}$; see, for example, [MVY14].

### 2.3.1 Gaudin Hamiltonians

Let $(V_1,\dots,V_N)$ be a sequence of $\mathfrak{gl}_{m|n}$ modules. Let $\boldsymbol{z} = (z_1,\dots,z_N)$ be a sequence of pairwise distinct complex numbers. Consider the tensor product $V = \bigotimes_{k=1}^N V_k$. The *Gaudin Hamiltonians* $\mathcal{H}_r \in \mathrm{End}(V)$, $r = 1,\dots,N$, are given by

$$\mathcal{H}_r = \sum_{\substack{k=1\\k\neq r}}^{N} \frac{\sum_{a,b=1}^{m+n} e_{ab}^{(r)} e_{ba}^{(k)} (-1)^{|b|}}{z_r - z_k},$$

where $e_{ab}^{(k)} = \underbrace{1\otimes\cdots\otimes 1}_{k-1} \otimes e_{ab} \otimes \underbrace{1\otimes\cdots\otimes 1}_{n-k}$, $k = 1,\dots,N$.

The proof of the following properties (which are well-known in the case of $\mathfrak{gl}_m$) can be found in [MVY14].

**Lemma 2.3.1.** *We have:*

1. *the Gaudin Hamiltonians mutually commute, $[\mathcal{H}_r, \mathcal{H}_k] = 0$, for all $r, k$;*
2. *the Gaudin Hamiltonians commute with the diagonal $\mathfrak{gl}_{M|N}$ action, $[\mathcal{H}_k, X] = 0$, for all $k$ and all $X \in \mathfrak{gl}_{m|n}$;*
3. *the sum of the Gaudin Hamiltonians is zero, $\sum_{k=1}^N \mathcal{H}_k = 0$;*
4. *if $V_k, k = 1,\dots,N$, are polynomial modules, then for generic $z_k, k = 1,\dots,N$, the Gaudin Hamiltonians are diagonalizable;*
5. *if $V_k, k = 1,\dots,N$, are vector representations, then the joint spectrum of the Gaudin Hamiltonians is simple for generic $\boldsymbol{z}$.* □

### 2.3.2 Bethe ansatz equation

We fix a parity sequence $\boldsymbol{s} \in S_{m|n}$, a sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ of $\mathfrak{gl}_{m|n}$ weights, and a sequence $\boldsymbol{z} = (z_1, \ldots, z_N)$ of pairwise distinct complex numbers. We call $(\lambda^{(k)})^{\boldsymbol{s}}$ the *weight at the point* $z_k$ *with respect to* $\boldsymbol{s}$ and denote it by $\lambda^{(\boldsymbol{s},k)}$. Denote $\lambda^{(\boldsymbol{s},k)}(e^{\boldsymbol{s}}_{ii}) = \lambda^{(\boldsymbol{s},k)}_{[\boldsymbol{s}],i}$ by $\lambda^{(\boldsymbol{s},k)}_i$.

Let $\boldsymbol{l} = (l_1, \ldots, l_{m+n-1})$ be a sequence of non-negative integers. Define $l = \sum_{i=1}^{m+n-1} l_i$. Let $c : \{1, \ldots, l\} \to \{1, \ldots, m+n-1\}$ be the *colour function*,

$$c(j) = r, \text{ if } \sum_{i=1}^{r-1} l_i < j \leq \sum_{i=1}^{r} l_i.$$

Let $\boldsymbol{t} = (t_1, \ldots, t_l)$ be a collection of variables. We say that $t_j$ has *colour* $c(j)$. Define the *weight at* $\infty$ *with respect to* $\boldsymbol{s}$*,* $\boldsymbol{\lambda}$*, and* $\boldsymbol{l}$ by

$$\lambda^{(\boldsymbol{s},\infty)} = \sum_{k=1}^{N} \lambda^{(\boldsymbol{s},k)} - \sum_{i=1}^{m+n-1} \alpha^{\boldsymbol{s}}_i l_i.$$

The *Bethe ansatz equation* (BAE) associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, is a system of algebraic equations on variables $\boldsymbol{t}$:

$$-\sum_{k=1}^{N} \frac{(\lambda^{(\boldsymbol{s},k)}, \alpha^{\boldsymbol{s}}_{c(j)})}{t_j - z_k} + \sum_{\substack{r=1\\ r\neq j}}^{l} \frac{(\alpha^{\boldsymbol{s}}_{c(r)}, \alpha^{\boldsymbol{s}}_{c(j)})}{t_j - t_r} = 0, \; j = 1, \ldots, l. \tag{2.6}$$

The BAE is a system of equations for $\boldsymbol{t}$ and we call the single equation (2.6) *the Bethe ansatz equation for* $\boldsymbol{t}$ *related to* $t_j$.

Note that if $\boldsymbol{t}$ is a solution of the BAE and $(\alpha^{\boldsymbol{s}}_{c(r)}, \alpha^{\boldsymbol{s}}_{c(j)}) \neq 0$ for some $j \neq r$, then $t_j \neq t_r$. Also if $(\lambda^{(\boldsymbol{s},k)}, \alpha^{\boldsymbol{s}}_{c(j)}) \neq 0$ for some $k$ and $j$, then $t_j \neq z_k$.

In addition, we impose the following condition. Suppose $(\alpha^{\boldsymbol{s}}_i, \alpha^{\boldsymbol{s}}_i) = 0$. Choose $j$ such that $c(j) = i$ and consider the equation related to $t_j$ as an equation for one variable when all variables $t_r$ with $c(r) \neq i$ are fixed. This equation does not depend on the choice of $j$. Suppose $t$ is a solution of this equation of multiplicity $a$. Then we require that the number of $t_j$ such that $c(j) = i$ and $t_j = t$ is at most $a$. This condition will be important in what follows; cf. especially Lemma 2.4.3, Theorem 2.5.2, and Conjecture 2.7.3.

The group $\mathfrak{S}_{\boldsymbol{l}} = \mathfrak{S}_{l_1} \times \cdots \times \mathfrak{S}_{l_{m+n-1}}$ acts on $\boldsymbol{t}$ by permuting the variables of the same colour.

We do not distinguish between solutions of the BAE in the same $\mathfrak{S}_{\boldsymbol{l}}$-orbit.

### 2.3.3 Weight function

Let $\lambda^{(k)}$, $k = 1, \ldots, N$, be polynomial $\mathfrak{gl}_{m|n}$ weights. Let $v_k^{\boldsymbol{s}} = v^{\boldsymbol{s}}_{\lambda^{(\boldsymbol{s},k)}}$ be an $\boldsymbol{s}$-highest weight vector in the irreducible $\mathfrak{gl}_{m|n}$ module $L(\lambda^{(k)})$. Consider the tensor product $L(\boldsymbol{\lambda}) = \bigotimes_{k=1}^{N} L(\lambda^{(k)})$. The *weight function* is a vector $w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t})$ in $L(\boldsymbol{\lambda})$ depending on parameters $\boldsymbol{z} = (z_1, \ldots, z_N)$ and variables $\boldsymbol{t} = (t_1, \ldots, t_l)$. The weight function $w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t})$ is constructed as follows (see [MVY14]).

Let an *ordered partition* of $\{1, \ldots, l\}$ into $n$ parts be a sequence

$$\boldsymbol{I} = (i_1^1, \ldots, i_{p_1}^1; \ldots; i_1^N, \ldots, i_{p_N}^N),$$

where $p_1 + \cdots + p_N = l$ and $\boldsymbol{I}$ is a permutation of $(1, \ldots, l)$. Let $P(l, N)$ be the set of all such ordered partitions.

Denote $F^{\boldsymbol{s}}_{c(r)} = e^{\boldsymbol{s}}_{c(r)+1,c(r)}$. To each ordered partition $\boldsymbol{I} \in P(l, N)$, associate a vector $F^{\boldsymbol{s}}_{\boldsymbol{I}} v \in L(\boldsymbol{\lambda})$ and a rational function $w_{\boldsymbol{I}}(\boldsymbol{z}, \boldsymbol{t})$,

$$F^{\boldsymbol{s}}_{\boldsymbol{I}} v = F^{\boldsymbol{s}}_{c(i_1^1)} \ldots F^{\boldsymbol{s}}_{c(i_{p_1}^1)} v_1^{\boldsymbol{s}} \otimes \cdots \otimes F^{\boldsymbol{s}}_{c(i_1^N)} \ldots F^{\boldsymbol{s}}_{c(i_{p_N}^n)} v_N^{\boldsymbol{s}},$$

$$w_{\boldsymbol{I}}(\boldsymbol{z}, \boldsymbol{t}) = w_{\{i_1^1, \ldots, i_{p_1}^1\}}(z_1, \boldsymbol{t}) \ldots w_{\{i_1^N, \ldots, i_{p_N}^N\}}(z_N, \boldsymbol{t}),$$

where for $\{i_1, \ldots, i_r\} \subset \{1, \ldots, l\}$,

$$w_{\{i_1, \ldots, i_r\}}(z, \boldsymbol{t}) = \frac{1}{(t_{i_1} - t_{i_2}) \ldots (t_{i_{r-1}} - t_{i_r})(t_{i_r} - z)}.$$

Define

$$(-1)^{|\boldsymbol{I}|} = \prod_{r=1}^{l} \prod_{\substack{j>r \\ \boldsymbol{I}(j)<\boldsymbol{I}(r)}} (-1)^{|F^{\boldsymbol{s}}_{c(r)}| \cdot |F^{\boldsymbol{s}}_{c(j)}|}.$$

Then the weight function $w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t})$ is

$$w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t}) = \sum_{\boldsymbol{I} \in P(l,N)} (-1)^{|\boldsymbol{I}|} w_{\boldsymbol{I}}(\boldsymbol{z}, \boldsymbol{t}) F^{\boldsymbol{s}}_{\boldsymbol{I}} v. \tag{2.7}$$

We have the following theorem.

**Theorem 2.3.2** ( [MVY14]). *If $\boldsymbol{\lambda}$ is a sequence of polynomial weights and $\boldsymbol{t}$ is a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then the vector $w^{\boldsymbol{s}}(\boldsymbol{z},\boldsymbol{t}) \in L(\boldsymbol{\lambda})$ is a joint eigenvector of the Gaudin Hamiltonians, $\mathcal{H}_k w^{\boldsymbol{s}}(\boldsymbol{z},\boldsymbol{t}) = E_k w^{\boldsymbol{s}}(\boldsymbol{z},\boldsymbol{t})$, $k = 1,\dots,N$, where the eigenvalues $E_k$ are given by*

$$E_k = \sum_{\substack{r=1\\ r\neq k}}^{N} \frac{\left(\lambda^{(\boldsymbol{s},k)},\lambda^{(\boldsymbol{s},r)}\right)}{z_k - z_r} + \sum_{j=1}^{l} \frac{\left(\lambda^{(\boldsymbol{s},k)},\alpha^{\boldsymbol{s}}_{c(j)}\right)}{t_j - z_k}. \tag{2.8}$$

*Moreover, the vector $w^{\boldsymbol{s}}(\boldsymbol{z},\boldsymbol{t})$ belongs to $(L(\boldsymbol{\lambda}))^{\mathrm{sing}}_{\lambda^{(\boldsymbol{s},\infty)}}$.* □

If $\boldsymbol{t}$ is a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then the value of the weight function $w^{\boldsymbol{s}}(\boldsymbol{z},\boldsymbol{t})$ is called a *Bethe vector.*

### 2.3.4 Polynomials representing solutions of the BAE

Fix a parity sequence $\boldsymbol{s} \in S_{m|n}$. Let $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(N)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z} = (z_1,\dots,z_N)$ be a sequence of pairwise distinct complex numbers.

Define a sequence of polynomials $\boldsymbol{T}^{\boldsymbol{s}} = (T^{\boldsymbol{s}}_1,\dots,T^{\boldsymbol{s}}_{m+n})$ associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$ and $\boldsymbol{z}$,

$$T^{\boldsymbol{s}}_i(x) = \prod_{k=1}^{N}(x-z_k)^{\lambda_i^{(\boldsymbol{s},k)}},\ i = 1,\dots,m+n. \tag{2.9}$$

Note that $T^{\boldsymbol{s}}_i(T^{\boldsymbol{s}}_{i+1})^{-s_i s_{i+1}}$ is a polynomial for all $i = 1,\dots,m+n$.

Let $\boldsymbol{l} = (l_1,\dots,l_{m+n-1})$ be a sequence of non-negative integers. Let $\boldsymbol{t} = (t_1,\dots,t_l)$ be a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$. Define a sequence of polynomials $\boldsymbol{y} = (y_1,\dots,y_{m+n-1})$ by

$$y_i(x) = \prod_{j,\ c(j)=i}(x-t_j),\ i = 1,\dots,m+n-1. \tag{2.10}$$

We say the *sequence of polynomials $\boldsymbol{y}$ represents $\boldsymbol{t}$.*

We consider each polynomial $y_i(x)$ up to a multiplication by a non-zero number. We also do not consider zero polynomials $y_i(x)$. Thus, the sequence $\boldsymbol{y}$ defines a point in the direct product $\mathbb{P}(\mathbb{C}[x])^{m+n-1}$ of $m+n-1$ copies of the projective space associated to the vector space of polynomials in $x$. We also have $\deg y_i = l_i$.

A sequence of polynomials $\boldsymbol{y}$ is *generic with respect to* $\boldsymbol{s}$, $\boldsymbol{\lambda}$, *and* $\boldsymbol{z}$, if it satisfies the following conditions:

1. if $s_i s_{i+1} = 1$, then $y_i(x)$ has only simple roots;
2. if $(\alpha_i^{\boldsymbol{s}}, \alpha_j^{\boldsymbol{s}}) \neq 0$ and $i \neq j$, then $y_i(x)$ and $y_j(x)$ have no common roots;
3. all roots of $y_i(x)$ are different from the roots of $T_i^{\boldsymbol{s}}(x)(T_{i+1}^{\boldsymbol{s}}(x))^{-s_i s_{i+1}}$.

If $\boldsymbol{y}$ represents a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then $\boldsymbol{y}$ is generic with respect to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$.

## 2.4 Reproduction procedure for $\mathfrak{gl}_2$ and $\mathfrak{gl}_{1|1}$

We recall the reproduction procedure for $\mathfrak{gl}_2$, see [MV04], and define its analogue for $\mathfrak{gl}_{1|1}$.

### 2.4.1 Reproduction procedure for $\mathfrak{gl}_2$

Consider the case of $m = 2$ and $n = 0$. We write $\mathfrak{gl}_{2|0} \cong \mathfrak{gl}_{0|2} \cong \mathfrak{gl}_2$. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)}) = ((p_1, q_1), \dots, (p_N, q_N))$ be a sequence of polynomial $\mathfrak{gl}_2$ weights: $p_k, q_k \in \mathbb{Z}$, $p_k \geq q_k \geq 0$, $k = 1, \dots, N$. Let $\boldsymbol{z} = (z_1, \dots, z_N)$ be a sequence of pairwise distinct complex numbers. We have

$$T_1 = \prod_{k=1}^{N} (x - z_k)^{p_k}, \qquad T_2 = \prod_{k=1}^{N} (x - z_k)^{q_k}.$$

Let $p = \deg T_1$ and $q = \deg T_2$.

Let $l$ be a non-negative integer. Let $\boldsymbol{t} = (t_1, \dots, t_l)$ be a collection of variables. The Bethe ansatz equation associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$ and $l$, is given by

$$-\sum_{k=1}^{N} \frac{p_k - q_k}{t_j - z_k} + \sum_{\substack{r=1 \\ r \neq j}}^{l} \frac{2}{t_j - t_r} = 0, \ j = 1, \dots, l. \tag{2.11}$$

One can reformulate the BAE (2.11) and construct a family of new solutions of the BAE as follows.

**Lemma 2.4.1** ( [MV04]). *Let $y$ be a degree $l$ polynomial generic with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{z}$.*

1. *The polynomial $y \in \mathbb{C}[x]$ represents a solution of the BAE* (2.11) *associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$ and $l$, if and only if there exists a polynomial $\widetilde{y} \in \mathbb{C}[x]$, such that*

$$\mathrm{Wr}\,(y, \widetilde{y}) = T_1 T_2^{-1}. \tag{2.12}$$

2. *If $\widetilde{y}$ is generic, then $\widetilde{y}$ represents a solution of the BAE associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$ and $\widetilde{l}$, where $\widetilde{l} = \deg \widetilde{y}$.*

□

Explicitly, the polynomial $\widetilde{y}$ in Lemma 2.4.1 is given by

$$\widetilde{y}(x) = c_1 y(x) \int T_1(x) T_2^{-1}(x) y^{-2}(x) dx + c_2 y(x), \tag{2.13}$$

where $c_1$ is some non-zero complex number and $c_2 \in \mathbb{C}$ is arbitrary. The BAE (2.11) guarantees that the integrand has no residues and therefore $\widetilde{y}$ is a polynomial. All but finitely many $\widetilde{y}$ are generic with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{z}$, and therefore represent solutions of the BAE (2.11).

Thus, from the polynomial $y$, we construct a family of polynomials $\widetilde{y}$. Following [MV04], we call this construction the $\mathfrak{gl}_2$ *reproduction procedure*.

Let $P_y$ be the closure of the set containing $y$ and all $\widetilde{y}$ in $\mathbb{P}(\mathbb{C}[x])$. We call $P_y$ the $\mathfrak{gl}_2$ *population originated at $y$*. The set $P_y$ is identified with the projective line $\mathbb{C}P^1$ with projective coordinates $(c_1 : c_2)$.

The weight at infinity associated to $\boldsymbol{\lambda}, l$, is $\lambda^{(\infty)} = (p-l, q+l)$. Assume the weight $\lambda^{(\infty)}$ is dominant, meaning $2l \le p - q$. Then the weight at infinity associated to $\boldsymbol{\lambda}, \widetilde{l}$, is

$$\widetilde{\lambda}^{(\infty)} = (p - \widetilde{l}, q + \widetilde{l}) = (q + l - 1, p - l + 1) = s \cdot \lambda^{(\infty)},$$

where $s \in \mathfrak{S}_2$ is the non-trivial $\mathfrak{gl}_2$ Weyl group element, and the dot denotes the shifted action.

Let $\widetilde{y} = \prod_{r=1}^{\widetilde{l}}(x - \widetilde{t}_r)$ and $\widetilde{\boldsymbol{t}} = (\widetilde{t}_1, \dots, \widetilde{t}_{\widetilde{l}})$. If $y$ is generic, then by Lemma 2.4.1, $\widetilde{\boldsymbol{t}}$ is a solution of the BAE (2.11). By Theorem 2.3.2, the value of the weight function $w(\boldsymbol{z}, \widetilde{\boldsymbol{t}})$ is a singular vector. However, $\widetilde{\lambda}^{(\infty)}$ is not dominant and therefore $w(\boldsymbol{z}, \widetilde{\boldsymbol{t}}) = 0$ in $L(\boldsymbol{\lambda})$. So, in a $\mathfrak{gl}_2$ population only the unique smallest degree polynomial corresponds to an actual eigenvector in $L(\boldsymbol{\lambda})$.

Consider formula (4.10) for the eigenvalues $E_k$ of the Gaudin Hamiltonians. It is clear that

$$\ln' y(z_k) = \ln' \widetilde{y}(z_k), \; k = 1, \dots, N,$$

which implies that the eigenvalues $E_k$ for the solution $\boldsymbol{t}$ of the BAE are equal to those for the solution $\widetilde{\boldsymbol{t}}$. That fact can be reformulated in the following form.

Define a differential operator

$$D(y) = \left(\partial - \ln' \frac{T_1}{y}\right)(\partial - \ln' T_2\, y).$$

The operator $D(y)$ does not depend on a choice of polynomial $y$ in a population, $D(y) = D(\widetilde{y})$.

### 2.4.2 Reproduction procedure for $\mathfrak{gl}_{1|1}$

Consider the case of $m = n = 1$. We have $S_{1|1} = \{(1, -1), (-1, 1)\}$. Let $\boldsymbol{s}$ and $\widetilde{\boldsymbol{s}} = \boldsymbol{s}^{[1]}$ be two different parity sequences. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ be a sequence of polynomial $\mathfrak{gl}_{1|1}$ weights. For each $k = 1, \dots, N$, let us write $\lambda_{[\boldsymbol{s}]}^{(\boldsymbol{s},k)} = (p_k, q_k)$, where $p_k, q_k \in \mathbb{Z}_{\geq 0}$ and if $p_k = 0$ then $q_k = 0$. Note that $\lambda^{(k)}$ is atypical if and only if it is zero, $p_k = q_k = 0$, which happens if and only if $p_k + q_k = 0$. Let $\boldsymbol{z} = (z_1, \dots, z_N)$ be a sequence of pairwise distinct complex numbers.

Let

$$\widetilde{p}_k = \begin{cases} q_k + 1 & \text{if } \; p_k + q_k \neq 0, \\ 0 & \text{if } \; p_k + q_k = 0, \end{cases} \qquad \widetilde{q}_k = \begin{cases} p_k - 1 & \text{if } \; p_k + q_k \neq 0, \\ 0 & \text{if } \; p_k + q_k = 0. \end{cases}$$

Equation (2.9) becomes

$$T_1^{\boldsymbol{s}} = \prod_{k=1}^{N}(x-z_k)^{p_k}, \quad T_2^{\boldsymbol{s}} = \prod_{k=1}^{N}(x-z_k)^{q_k},$$

$$T_1^{\widetilde{\boldsymbol{s}}} = \prod_{\substack{k=1\\ p_k+q_k\neq 0}}^{N}(x-z_k)^{q_k+1} = \prod_{k=1}^{N}(x-z_k)^{\widetilde{p}_k}, \quad T_2^{\widetilde{\boldsymbol{s}}} = \prod_{\substack{k=1\\ p_k+q_k\neq 0}}^{N}(x-z_k)^{p_k-1} = \prod_{k=1}^{n}(x-z_k)^{\widetilde{q}_k}.$$

Let $p = \deg T_1^{\boldsymbol{s}}$, $q = \deg T_2^{\boldsymbol{s}}$. Similarly, let $\widetilde{p} = \deg T_1^{\widetilde{\boldsymbol{s}}}$, $\widetilde{q} = \deg T_2^{\widetilde{\boldsymbol{s}}}$.

Let $M = \#\{k \mid p_k + q_k \neq 0\}$ be the number of typical modules. Then $\widetilde{p} = q + M$ and $\widetilde{q} = p - M$.

Let $l$ be a non-negative integer. Let $\boldsymbol{t} = (t_1, \dots, t_l)$ be a collection of variables. The Bethe ansatz equation associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $l$, takes the form:

$$\sum_{k=1}^{N} \frac{p_k + q_k}{t_j - z_k} = 0, \ j = 1, \dots, l. \tag{2.14}$$

The Bethe ansatz equation (2.14) can be written in the form

$$\ln'\left(T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}}\right)(t_j) = 0.$$

Note that $T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}} = T_1^{\widetilde{\boldsymbol{s}}} T_2^{\widetilde{\boldsymbol{s}}}$. Thus, in the case of $\mathfrak{gl}_{1|1}$, the BAEs (2.14) associated to $\boldsymbol{s}$ and $\widetilde{\boldsymbol{s}}$ coincide.

Define a map $\pi$ from non-zero rational functions $\mathbb{C}(x)$ to monic polynomials in $\mathbb{C}[x]$ with distinct roots. For any nonzero rational function $f(x)$, $\pi(f)(z) = 0$ if and only if $f(z) = 0$ or $(1/f)(z) = 0$.

**Example 2.4.2.** *We have* $\pi\left(x^5(x-1)^4(x-3)^{-1}(x+6)^{-2}\right) = x(x-1)(x-3)(x+6)$.

The polynomial $\pi(f)$ is the minimal monic denominator of the rational function $\ln'(f)$ of smallest possible degree.

We call the sequence of polynomial $\mathfrak{gl}_{1|1}$ weights $\boldsymbol{\lambda}$ *typical* if at least one of the weights $\lambda^{(k)}$ is typical. Then $\boldsymbol{\lambda}$ is typical if and only if $p+q \neq 0$. Also $\boldsymbol{\lambda}$ is not typical if and only if $T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}} = 1$.

We reformulate the BAE (2.14) and construct a new solution as follows.

**Lemma 2.4.3.** *Let $y$ be a polynomial of degree $l$. Let $\boldsymbol{\lambda}$ be typical.*

1. *The polynomial $y$ represents a solution of the BAE* (2.14) *associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $l$, if and only if there exists a polynomial $\widetilde{y}$, such that*

$$y \cdot \widetilde{y} = \ln'(T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}})\, \pi(T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}}). \tag{2.15}$$

2. *The polynomial $\widetilde{y}$ represents a solution of the BAE* (2.14) *associated to $\widetilde{\boldsymbol{s}}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\widetilde{l}$, where $\widetilde{l} = \deg \widetilde{y} = M - 1 - l$.* □

From the polynomial $y$, we construct a unique polynomial $\widetilde{y}$. We call this construction the $\mathfrak{gl}_{1|1}$ *reproduction procedure.*

Let $P_y$ be the set containing $y$ and $\widetilde{y}$. The set $P_y$ is called the $\mathfrak{gl}_{1|1}$ *population originated at $y$.*

The weight at infinity associated to $\boldsymbol{s}, \boldsymbol{\lambda}$, and $l$ is $\lambda^{(\boldsymbol{s},\infty)}_{[\boldsymbol{s}]} = (p-l, q+l)$. The weight at infinity associated to $\widetilde{\boldsymbol{s}}, \boldsymbol{\lambda}$ and $\widetilde{l}$ is $\widetilde{\lambda}^{(\widetilde{\boldsymbol{s}},\infty)}_{[\widetilde{\boldsymbol{s}}]} = (\widetilde{p} - \widetilde{l}, \widetilde{q} + \widetilde{l}) = (q + l + 1, p - l - 1)$. Thus we have $\lambda^{(\boldsymbol{s},\infty)} = \widetilde{\lambda}^{(\widetilde{\boldsymbol{s}},\infty)} + \alpha^{\boldsymbol{s}}$. In particular, both $y$ and $\widetilde{y}$ correspond to actual eigenvectors of the Gaudin Hamiltonians.

**Remark 2.4.4.** *If $\boldsymbol{\lambda}$ is not typical, then all participating representations are one-dimensional and the situation is trivial. In particular, we have $y(x) = 1$. In this case we can define $\widetilde{y} = 1$. We do not discuss this case any further.*

### 2.4.3 Motivation for $\mathfrak{gl}_{1|1}$-reproduction procedure

We show that in parallel to the $\mathfrak{gl}_2$ reproduction procedure, the eigenvalues of the Gaudin Hamiltonians corresponding to polynomials in the same $\mathfrak{gl}_{1|1}$ population are the same.

Let $y = \prod_{r=1}^{l}(x - t_r)$, $\widetilde{y} = \prod_{r=1}^{\widetilde{l}}(x - \widetilde{t}_r)$. Let $\boldsymbol{t} = (t_1, \dots, t_l)$, $\widetilde{\boldsymbol{t}} = (\widetilde{t}_1, \dots, \widetilde{t}_{\widetilde{l}})$.

Let $h_k = p_k + q_k$, $k = 1, \dots, N$. Let $\mathcal{N}(T)$ be the monic polynomial proportional to $\ln'(T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}})\, \pi(T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}})$.

From Theorem 2.3.2, we have

$$\mathcal{H}_k w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t}) = E_k w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t})$$

and

$$\mathcal{H}_k w^{\widetilde{\boldsymbol{s}}}(\boldsymbol{z}, \widetilde{\boldsymbol{t}}) = \widetilde{E}_k w^{\widetilde{\boldsymbol{s}}}(\boldsymbol{z}, \widetilde{\boldsymbol{t}}),$$

where

$$E_k = s_1 \sum_{\substack{r=1\\ r\neq k}}^{N} \frac{p_k p_r - q_k q_r}{z_k - z_r} + s_1 \sum_{j=1}^{l} \frac{h_k}{t_j - z_k}, \qquad \widetilde{E}_k = \widetilde{s}_1 \sum_{\substack{r=1\\ r\neq k}}^{N} \frac{\widetilde{p}_k \widetilde{p}_r - \widetilde{q}_k \widetilde{q}_r}{z_k - z_r} + \widetilde{s}_1 \sum_{j=1}^{\widetilde{l}} \frac{h_k}{\widetilde{t}_j - z_k}. \tag{2.16}$$

**Lemma 2.4.5.** *The eigenvalues $E_k$ and $\widetilde{E}_k$, $k = 1, \dots, N$, of the Gaudin Hamiltonians are the same.*

*Proof.* Set $t_{l+r} = \widetilde{t}_r$, $r = 1, \dots, \widetilde{l}$.

If $p_k + q_k = 0$, then $E_k = \widetilde{E}_k = 0$. Without loss of generality, assume $p_k + q_k \neq 0$, $k = 1, \dots, M$, $M > 0$, and $p_k + q_k = 0$, $k = M+1, \dots, N$, and consider $E_1 - \widetilde{E}_1$. We have

$$s_1(E_1 - \widetilde{E}_1) = \sum_{k=2}^{M} \frac{h_1 + h_k}{z_1 - z_k} + \sum_{r=1}^{M-1} \frac{h_1}{t_r - z_1}. \tag{2.17}$$

The polynomial $\mathcal{N}(T)(x)$ is

$$\mathcal{N}(T)(x) = \prod_{k=1}^{M-1}(x - t_k) = (h_1 + \cdots + h_M)^{-1} \sum_{k=1}^{M} h_k (x - z_1) \dots \widehat{(x - z_k)} \dots (x - z_M).$$

Evaluate the function $\ln'(\mathcal{N}(T))$ at $z_1$ and we have

$$\ln'(\mathcal{N}(T))(z_1) = \sum_{r=1}^{M-1} \frac{1}{z_1 - t_r} = \sum_{k=2}^{M} \frac{h_1 + h_k}{h_1(z_1 - z_k)}.$$

Thus, the right-hand side of (2.17) is zero. □

**Corollary 2.4.6.** *We have $e_{21}^{\boldsymbol{s}} w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t}) = c w^{\widetilde{\boldsymbol{s}}}(\boldsymbol{z}, \widetilde{\boldsymbol{t}})$, for some non-zero constant $c$.*

*Proof.* It follows from the results of [MVY14] that for generic $\boldsymbol{z}$, the Gaudin Hamiltonians $\mathcal{H}_k$ acting in $(L(\boldsymbol{\lambda}))^{\boldsymbol{s}\mathrm{ing}} = (\otimes_k L(\lambda^k))^{\boldsymbol{s}\mathrm{ing}}$ have joint simple spectrum. Moreover, for generic $\boldsymbol{z}$, $w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t}) \neq 0$ and $w^{\widetilde{\boldsymbol{s}}}(\boldsymbol{z}, \widetilde{\boldsymbol{t}}) \neq 0$.

Therefore, $w^{\boldsymbol{s}}(\boldsymbol{z}, \boldsymbol{t})$ and $w^{\widetilde{\boldsymbol{s}}}(\boldsymbol{z}, \widetilde{\boldsymbol{t}})$ belong to the same irreducible two-dimensional submodule of $L(\boldsymbol{\lambda})$. Moreover, their weights are related by $\lambda^{(\boldsymbol{s},\infty)} = \widetilde{\lambda}^{(\widetilde{\boldsymbol{s}},\infty)} + \alpha^{\boldsymbol{s}}$. The corollary follows. □

Define a rational pseudodifferential operator:

$$R^{\boldsymbol{s}}(y) = \left(\partial - s_1 \ln' \frac{T_1^{\boldsymbol{s}}}{y}\right)^{s_1} \left(\partial - s_2 \ln'(T_2^{\boldsymbol{s}} y)\right)^{s_2}.$$

**Lemma 2.4.7.** *If $\boldsymbol{\lambda}$ is typical, then $R^{\boldsymbol{s}}(y)$ is a $(1|1)$-rational pseudodifferential operator. If $\boldsymbol{\lambda}$ is not typical, then $R^{\boldsymbol{s}}(y) = 1$.*

*Let $\boldsymbol{\lambda}$ be typical. The rational pseudodifferential operator does not depend on a choice of a polynomial in a population: $R^{\boldsymbol{s}}(y) = R^{\widetilde{\boldsymbol{s}}}(\widetilde{y})$.*

*Proof.* The lemma is proved by a direct computation. □

## 2.5 Reproduction procedure for $\mathfrak{gl}_{m|n}$

We define the reproduction procedure and populations in the general case.

### 2.5.1 Reproduction procedure

Let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z} = (z_1, \dots, z_N)$ be a sequence of pairwise distinct complex numbers. Let $\boldsymbol{T}^{\boldsymbol{s}}$ be a sequence of polynomials associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, see (2.9). Denote $\pi\left(T_i^{\boldsymbol{s}}(T_{i+1}^{\boldsymbol{s}})^{-s_i s_{i+1}}\right)$ by $\pi_i^{\boldsymbol{s}}$.

For $i \in \{1, \dots, m+n-1\}$, set $\boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$.

**Lemma 2.5.1.** *If $s_i = s_{i+1}$, then $\boldsymbol{T}^{\boldsymbol{s}^{[i]}} = \boldsymbol{T}^{\boldsymbol{s}}$ and if $s_i \neq s_{i+1}$, then*

$$\boldsymbol{T}^{\boldsymbol{s}^{[i]}} = (T_1^{\boldsymbol{s}}, \dots, T_{i+1}^{\boldsymbol{s}} \pi_i^{\boldsymbol{s}}, T_i^{\boldsymbol{s}} (\pi_i^{\boldsymbol{s}})^{-1}, \dots, T_{m+n}^{\boldsymbol{s}}).$$

*Proof.* This follows from (2.2). □

Let $\boldsymbol{l} = (l_1, \dots, l_{m+n-1})$ be a sequence of nonnegative integers.

We reformulate the BAE (2.6) and construct a family of new solutions as follows. By convention, we set $y_0 = y_{m+n} = 1$.

**Theorem 2.5.2.** *Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials generic with respect to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, such that $\deg y_k = l_k$, $k = 1, \dots, m+n-1$.*

1. *The sequence $\boldsymbol{y}$ represents a solution of the BAE* (2.6) *associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, if and only if for each $i = 1, \dots, m+n-1$, there exists a polynomial $\widetilde{y}_i$, such that*

$$\mathrm{Wr}\,(y_i, \widetilde{y}_i) = T_i^{\boldsymbol{s}} \left(T_{i+1}^{\boldsymbol{s}}\right)^{-1} y_{i-1}y_{i+1} \quad \text{if } s_i = s_{i+1}, \tag{2.18}$$

$$y_i\,\widetilde{y}_i = \ln' \left(\frac{T_i^{\boldsymbol{s}} T_{i+1}^{\boldsymbol{s}} y_{i-1}}{y_{i+1}}\right) \pi_i^{\boldsymbol{s}} y_{i-1}y_{i+1} \quad \text{if } s_i \neq s_{i+1}. \tag{2.19}$$

2. *Let $i \in \{1, \dots, m+n-1\}$ be such that $\widetilde{y}_i \neq 0$. Then if*

$$\boldsymbol{y}^{[i]} = (y_1, \dots, \widetilde{y}_i, \dots, y_{m+n-1})$$

*is generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, then $\boldsymbol{y}^{[i]}$ represents a solution of the BAE associated to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $\boldsymbol{l}^{[i]}$, where $\boldsymbol{l}^{[i]} = (l_1, \dots, \widetilde{l}_i, \dots, l_{m+n-1})$, $\widetilde{l}_i = \deg \widetilde{y}_i$.*

*Proof.* Part *(1)* follows from Lemma 2.4.1 and Lemma 2.4.3.

We prove Part *(2)*. Let $y_r = \prod_{j=1}^{l_r}(x - t_j^{(r)})$, $r = 1, \dots, m+n-1$, and $\widetilde{y}_i = \prod_{j=1}^{\widetilde{l}_i}(x - \widetilde{t}_j^{(i)})$. Let $\boldsymbol{t} = (t_j^{(r)})_{r=1,\dots,m+n-1}^{j=1,\dots,l_r}$ and $\widetilde{\boldsymbol{t}} = (\widetilde{t}_j^{(r)})_{r=1,\dots,m+n-1}^{j=1,\dots,\widetilde{l}_r}$, where we set $l_r = \widetilde{l}_r, t_j^{(r)} = \widetilde{t}_j^{(r)}$ if $r \neq i$. The tuple $\boldsymbol{t}$ satisfies the BAE associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $\boldsymbol{l}$. We prove the Bethe ansatz equation for $\widetilde{\boldsymbol{t}}$ associated to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $\boldsymbol{l}^{[i]}$. The BAE for $\widetilde{\boldsymbol{t}}$ related to $\widetilde{t}_j^{(i)}$ holds by Lemma 2.4.1 and Lemma 2.4.3. The BAEs for $\widetilde{\boldsymbol{t}}$ and $\boldsymbol{t}$ related to $t_j^{(r)}$, $|r-i| > 1$, are the same. We treat the non-trivial cases.

Consider the case of $s_i = s_{i+1}$. Dividing (2.18) by $y_i\widetilde{y}_i$ and evaluating at $x = t_j^{(i\pm1)}$, we obtain

$$\sum_{a=1}^{l_i} \frac{1}{t_j^{(i\pm1)} - t_a^{(i)}} = \sum_{a=1}^{\widetilde{l}_i} \frac{1}{t_j^{(i\pm1)} - \widetilde{t}_a^{(i)}}.$$

Thus, the BAE for $\widetilde{\boldsymbol{t}}$ related to $t_j^{(i\pm1)}$ follows from the BAE for $\boldsymbol{t}$ related to $t_j^{(i\pm1)}$.

Consider the case of $s_i = -s_{i+1} = 1$. The argument depends on $s_{i-1}$, $s_{i+2}$. Consider for example the case of $s_{i-1} = -s_{i+2} = 1$.

We prove the BAE for $\widetilde{\boldsymbol{t}}$ related to $t_j^{(i-1)}$:

$$-\sum_{k=1}^{N} \frac{\lambda_{i-1}^{(\boldsymbol{s},k)} + \lambda_{i+1}^{(\boldsymbol{s},k)} + \delta}{t_j^{(i-1)} - z_k} + \sum_{r=1}^{l_{i-2}} \frac{-1}{t_j^{(i-1)} - t_r^{(i-2)}} + \sum_{r=1}^{\widetilde{l}_i} \frac{1}{t_j^{(i-1)} - \widetilde{t}_r^{(i)}} = 0, \tag{2.20}$$

where $\delta = 1$ if $\lambda_i^{(\boldsymbol{s},k)} + \lambda_{i+1}^{(\boldsymbol{s},k)} \neq 0$ and $\delta = 0$ otherwise.

The BAE for $\boldsymbol{t}$ related to $t_j^{(i-1)}$ is

$$-\sum_{k=1}^{N} \frac{\lambda_{i-1}^{(\boldsymbol{s},k)} - \lambda_i^{(\boldsymbol{s},k)}}{t_j^{(i-1)} - z_k} + \sum_{r=1}^{l_{i-2}} \frac{-1}{t_j^{(i-1)} - t_r^{(i-2)}} + \sum_{r=1}^{l_i} \frac{-1}{t_j^{(i-1)} - t_r^{(i)}} + \sum_{\substack{r=1\\ r\neq j}}^{l_{i-1}} \frac{2}{t_j^{(i-1)} - t_r^{(i-1)}} = 0. \tag{2.21}$$

Take the logarithmic derivative of equation (2.19) for $y_i$ and evaluate it at $t_j^{(i-1)}$. The left-hand side is

$$\ln'(y_i\widetilde{y}_i)\Big|_{x=t_j^{(i-1)}} = \sum_{r=1}^{l_i} \frac{1}{t_j^{(i-1)} - t_r^{(i)}} + \sum_{r=1}^{\widetilde{l}_i} \frac{1}{t_j^{(i-1)} - \widetilde{t}_r^{(i)}}$$

and the right-hand side is

$$\begin{aligned} &\ln'\left(\ln'\left(T_i^{\boldsymbol{s}}T_{i+1}^{\boldsymbol{s}}y_{i-1}y_{i+1}^{-1}\right)\pi_i^{\boldsymbol{s}}y_{i-1}y_{i+1}\right)\Big|_{x=t_j^{(i-1)}} \\ &\qquad = \Big(\ln'(T_i^{\boldsymbol{s}}T_{i+1}^{\boldsymbol{s}})\pi_i^{\boldsymbol{s}}y'_{i-1}y_{i+1} \\ &\qquad\qquad\qquad + (\pi_i^{\boldsymbol{s}}y'_{i-1}y_{i+1})' - \pi_i^{\boldsymbol{s}}y'_{i-1}y'_{i+1}\Big)/(\pi_i^{\boldsymbol{s}}y'_{i-1}y_{i+1})\Big|_{x=t_j^{i-1}} \\ &\qquad = \sum_{k=1}^{N} \frac{\lambda_i^{(\boldsymbol{s},k)} + \lambda_{i+1}^{(\boldsymbol{s},k)} + \delta}{t_j^{(i-1)} - z_k} + \sum_{\substack{r=1\\ r\neq j}}^{l_{i-1}} \frac{2}{t_j^{(i-1)} - t_r^{(i-1)}}. \end{aligned}$$

(Note here that the $t_j^{(i-1)}$ are all distinct, by the assumption that $\boldsymbol{y}^{[i]}$ is generic.) The difference of the right-hand side and the left-hand side is exactly the difference between (2.20) and (2.21).

The BAE for $\widetilde{\boldsymbol{t}}$ related to $t_j^{(i+1)}$ is proved by a similar computation.

All other cases are similar, we omit further details. □

If $s_i = s_{i+1}$, then starting from $\boldsymbol{y}$ we construct a family of new sequences $\boldsymbol{y}^{[i]}$, isomorphic to $\mathbb{C}$, by using (2.18). We call this construction the *bosonic reproduction procedure in i-th direction.* If $s_i \neq s_{i+1}$, and $T_i^{\boldsymbol{s}}T_{i+1}^{\boldsymbol{s}}y_{i-1} \neq cy_{i+1}$, $c \in \mathbb{C}^\times$, then starting from $\boldsymbol{y}$ we construct a single new sequence $\boldsymbol{y}^{[i]}$ by using (2.19). We call this construction the *fermionic reproduction procedure in i-th direction.* From the definition of fermionic reproduction procedure, $(\boldsymbol{y}^{[i]})^{[i]} = \boldsymbol{y}$.

If $\boldsymbol{y}^{[i]}$ is generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}^{[i]}$, and $\boldsymbol{z}$, then by Theorem 2.5.2, we can apply the reproduction procedure again.

Bosonic reproduction procedures fix parity sequences, while fermionic reproductions procedures change parity sequences. Denote by

$$P_{(\boldsymbol{y},\boldsymbol{s})} \subset (\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times S_{m|n}$$

the closure of the set of all pairs $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{s}})$ obtained from the initial pair $(\boldsymbol{y}, \boldsymbol{s})$ by repeatedly applying all possible reproductions. We call $P_{(\boldsymbol{y},\boldsymbol{s})}$ the $\mathfrak{gl}_{m|n}$ *population of solutions of the BAE associated to* $\boldsymbol{s}$*,* $\boldsymbol{z}$*,* $\boldsymbol{\lambda}$*, and* $\boldsymbol{l}$*, originated at* $\boldsymbol{y}$. By definition, $P_{(\boldsymbol{y},\boldsymbol{s})}$ decomposes as a disjoint union over parity sequences,

$$P_{(\boldsymbol{y},\boldsymbol{s})} = \bigsqcup_{\widetilde{\boldsymbol{s}} \in S_{M|N}} P^{\tilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})}, \qquad P^{\widetilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})} = P_{(\boldsymbol{y},\boldsymbol{s})} \cap \left( (\mathbb{P}(\mathbb{C}[x]))^{m+n-1} \times \{\widetilde{\boldsymbol{s}}\} \right).$$

### 2.5.2 Rational pseudodifferential operator associated to population

We define a rational pseudodifferential operator which does not change under the reproduction procedure.

Let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. Let $\boldsymbol{z} = (z_1, \dots, z_N)$ be a sequence of pairwise distinct complex numbers. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. The sequence $\boldsymbol{T}^{\boldsymbol{s}} = (T_1^{\boldsymbol{s}}, \dots, T_{m+n}^{\boldsymbol{s}})$ is given by (2.9).

Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials. Recall our convention that $y_0 = y_{m+n} = 1$. Define a rational pseudodifferential operator $R$ over $\mathbb{C}(x)$,

$$R^{\boldsymbol{s}}(\boldsymbol{y}) = \left(\partial - s_1 \ln' \frac{T_1^{\boldsymbol{s}} y_0}{y_1}\right)^{s_1} \left(\partial - s_2 \ln' \frac{T_2^{\boldsymbol{s}} y_1}{y_2}\right)^{s_2} \times \dots \\ \times \left(\partial - s_{m+n} \ln' \frac{T_{m+n}^{\boldsymbol{s}} y_{m+n-1}}{y_{m+n}}\right)^{s_{m+n}}. \tag{2.22}$$

The following theorem is the main result of this section.

**Theorem 2.5.3.** *Let* $P$ *be a* $\mathfrak{gl}_{m|n}$ *population. Then the rational pseudodifferential operator* $R^{\boldsymbol{s}}(\boldsymbol{y})$ *does not depend on the choice of* $\boldsymbol{y}$ *in* $P$.

*Proof.* We want to show

$$\left(\partial - s_i \ln' \frac{T_i^{\boldsymbol{s}} y_{i-1}}{y_i}\right)^{s_i} \left(\partial - s_{i+1} \ln' \frac{T_{i+1}^{\boldsymbol{s}} y_i}{y_{i+1}}\right)^{s_{i+1}}$$
$$= \left(\partial - s_{i+1} \ln' \frac{T_i^{\boldsymbol{s}^{[i]}} y_{i-1}}{\widetilde{y}_i}\right)^{s_{i+1}} \left(\partial - s_i \ln' \frac{T_{i+1}^{\boldsymbol{s}^{[i]}} \widetilde{y}_i}{y_{i+1}}\right)^{s_i}.$$

We have four cases, $(s_i, s_{i+1}) = (\pm 1, \pm 1)$. The cases of $s_i = s_{i+1}$ are proved in [MV04].

Consider the case of $s_i = -s_{i+1} = 1$. We want to show

$$\left(\partial - \ln' \frac{T_i^{\boldsymbol{s}} y_{i-1}}{y_i}\right) \left(\partial - \ln' \frac{y_{i+1}}{T_{i+1}^{\boldsymbol{s}} y_i}\right)^{-1}$$
$$= \left(\partial - \ln' \frac{\widetilde{y}_i}{T_{i+1}^{\boldsymbol{s}} \pi_i^{\boldsymbol{s}} y_{i-1}}\right)^{-1} \left(\partial - \ln' \frac{T_i^{\boldsymbol{s}} (\pi_i^{\boldsymbol{s}})^{-1} \widetilde{y}_i}{y_{i+1}}\right).$$

This equation is proved by a direct computation using (2.3) and (2.19). We only note that the rational function $T_i^{\boldsymbol{s}} T_{i+1}^{\boldsymbol{s}} y_{i-1} y_{i+1}^{-1}$ is not constant by the assumption that the reproduction is possible.

The case of $s_i = -s_{i+1} = -1$ is similar. $\square$

We denote the rational pseudodifferential operator corresponding to a population $P$ by $R_P$.

It is known that the Gaudin Hamiltonians acting in $L(\boldsymbol{\lambda})$ can be included in a natural commutative algebra $\mathcal{B}(\boldsymbol{\lambda})$ of higher Gaudin Hamiltonians, see [MR14]. We expect that similar to the even case, the rational pseudodifferential operator $R^{\boldsymbol{s}}(\boldsymbol{y})$ encodes the eigenvalues of the algebra $\mathcal{B}(\boldsymbol{\lambda})$ acting on the Bethe vector corresponding to $\boldsymbol{y}$. Then, Theorem 2.5.3 would assert that the formulas for the eigenvalues of $\mathcal{B}(\boldsymbol{\lambda})$ do not depend on the choice of $\boldsymbol{y}$ in the population.

Here we show that the eigenvalues (4.10) of the (quadratic) Gaudin Hamiltonians do not change under the $\mathfrak{gl}_{m|n}$ reproduction procedure. Denote the eigenvalues of the Gaudin Hamiltonians given in (4.10) by $E_k(\boldsymbol{y})$, $k = 1, \ldots, N$. Note that $E_k(\boldsymbol{y})$ is defined only if $y_i(z_k) \neq 0$ whenever $T_i^{\boldsymbol{s}} (T_{i+1}^{\boldsymbol{s}})^{-s_i s_{i+1}}$ vanishes at $x = z_k$. We call such sequences $\boldsymbol{y}$ *$k$-admissible.*

**Lemma 2.5.4.** *Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials such that there exists a sequence of polynomials $\widetilde{\boldsymbol{y}}$ satisfying* (2.18) *if $s_i = s_{i+1}$ or* (2.19) *if $s_i = -s_{i+1}$. Suppose that $\boldsymbol{y}$ and $\widetilde{\boldsymbol{y}}$ are $k$-admissible. Then $E_k(\boldsymbol{y}) = E_k(\widetilde{\boldsymbol{y}})$.*

*Proof.* In the case of $s_i = s_{i+1}$, the lemma follows from $\ln' y_i(z_k) = \ln' \widetilde{y}_i(z_k)$, $k = 1, \dots, N$.

In the case of $s_i \neq s_{i+1}$, the lemma follows from taking logarithmic derivative of the equation (2.19) for $y_i$ and evaluating at $x = z_k$, $k = 1, \dots, N$, cf. proof of Lemma 2.4.5. We only note that by (2.19) the polynomial $y_{i-1}y_{i+1}$ does not vanish at $z_k$ if $T_iT_{i+1}$ does and $y_i$, $\widetilde{y}_i$ do not. □

### 2.5.3 Example of a population

In what follows, we study the structure of a population.

Consider $\mathfrak{gl}_{2|1}$. We have three parity sequences, $\boldsymbol{s}_0 = (1, 1, -1)$, $\boldsymbol{s}_1 = (1, -1, 1)$, and $\boldsymbol{s}_2 = (-1, 1, 1)$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, where $\lambda^{(i)} = (1, 1, 0)$, for $i = 1, 2, 3$. Let $\boldsymbol{z} = (1, \omega, \omega^2)$, where $\omega$ is a primitive cubic root of unity. We have $\boldsymbol{T} = \boldsymbol{T}^{\boldsymbol{s}_0} = (x^3 - 1, x^3 - 1, 1)$. Let $\boldsymbol{y} = (y_1, y_2) = (1, 1)$.

1. First, apply the bosonic reproduction procedure in the first direction to $\boldsymbol{y}$. We have $\boldsymbol{s}_0^{[1]} = \boldsymbol{s}_0$, $\boldsymbol{T}^{\boldsymbol{s}_0} = \boldsymbol{T}$, and $\boldsymbol{y}^{[1]} = (y_1^{[1]}, y_2^{[1]}) = (x - c, 1)$, where $c \in \mathbb{C}P^1$. At $c = \infty$, $\boldsymbol{y}^{[1]} = (1, 1) = \boldsymbol{y}$.

2. Second, apply the fermionic reproduction procedure in the second direction to $\boldsymbol{y}^{[1]}$. We have $(\boldsymbol{s}_0)^{[2]} = \boldsymbol{s}_1$ and $\boldsymbol{T}^{\boldsymbol{s}_1} = (x^3 - 1, x^3 - 1, 1)$. We have $(\boldsymbol{y}^{[1]})^{[2]} = (x - c, 4x^3 - 3cx^2 - 1)$.

3. Third, apply the fermionic reproduction procedure in the first direction to $(\boldsymbol{y}^{[1]})^{[2]}$. We have $(\boldsymbol{s}_1)^{[1]} = \boldsymbol{s}_2$ and $\boldsymbol{T}^{\boldsymbol{s}_2} = ((x^3 - 1)^2, 1, 1)$. We have $((\boldsymbol{y}^{[1]})^{[2]})^{[1]} = (2x^4 + x, 4x^3 - 3cx^2 - 1)$.

It is easy to check that all further reproduction procedures cannot create a new sequence. Therefore the $\mathfrak{gl}_{2|1}$-population $P_{(1,1)}$ is the union of three $\mathbb{C}P^1$, $P^{\boldsymbol{s}_0}_{(1,1)} = \{(x-c,1) \mid c \in \mathbb{C}P^1\}$, $P^{\boldsymbol{s}_1}_{(1,1)} = \{(x-c, 4x^3-3cx^2-1) \mid c \in \mathbb{C}P^1\}$, and $P^{\boldsymbol{s}_2}_{(1,1)} = \{(2x^4+x, 4x^3-3cx^2-1) \mid c \in \mathbb{C}P^1\}$.

We have the following representations for the rational pseudodifferential operator of the population: $R_P = R^{\boldsymbol{s}_0} = R^{\boldsymbol{s}_1} = R^{\boldsymbol{s}_2}$:

$$\begin{aligned} R_P &= \left(\partial - \frac{3x^2}{x^3-1}\right)\left(\partial - \frac{3x^2}{x^3-1}\right)\partial^{-1} = \left(\partial - \frac{3x^2}{x^3-1}\right)\left(\partial - \frac{2x^3-3cx^2+1}{x^4-cx^3-x+c}\right)\partial^{-1} \\ &= \left(\partial - \ln'\frac{x^3-1}{x-c}\right)\left(\partial - \ln'\frac{4x^3-3cx^2-1}{(x^3-1)(x-c)}\right)^{-1}\left(\partial - \ln'(4x^3-3cx^2-1)\right) \\ &= \left(\partial - \ln'\frac{2x^4+x}{(x^3-1)^2}\right)^{-1}\left(\partial - \ln'\frac{2x^4+x}{4x^3-3cx^2-1}\right)\left(\partial - \ln'(4x^3-3cx^2-1)\right). \end{aligned}$$

## 2.6 Populations and flag varieties

We call a sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ of polynomial $\mathfrak{gl}_{m|n}$ weights *typical* if at least one of the $\lambda^{(k)}$, $k = 1, \ldots, N$, is typical. In this section, we show that $\mathfrak{gl}_{m|n}$ populations associated to typical $\boldsymbol{\lambda}$ are isomorphic to the variety of the full superflags.

### 2.6.1 Polynomials $\pi_{a,b}$

Let $\boldsymbol{M} = (M_1 \le M_2 \le \cdots \le M_r)$, $\boldsymbol{N} = (N_1 \le N_2 \le \cdots \le N_r)$, $M_i, N_i \in \mathbb{Z}$, be two generalized partitions with $r$ parts. We say $\boldsymbol{N}$ *dominates* $\boldsymbol{M}$ if $N_i \ge M_i$ for $i = 1, \ldots, r$. This gives a partial ordering on the set of generalized partitions with $r$ parts.

For a generalized partition with $r$ parts $\boldsymbol{M}$, there exists a unique generalized partition $\bar{\boldsymbol{M}}$ with $r$ parts such that:

1. all parts of $\bar{\boldsymbol{M}}$ are distinct;

2. $\bar{\boldsymbol{M}}$ dominates $\boldsymbol{M}$; and

3. if a generalized partition with $r$ distinct parts $\boldsymbol{M}'$ dominates $\boldsymbol{M}$, then $\boldsymbol{M}'$ dominates $\bar{\boldsymbol{M}}$.

We call $\bar{\boldsymbol{M}}$ the *dominant of* $\boldsymbol{M}$.

We identify multisets of integers with generalized partitions (by putting their elements into weakly increasing order).

**Example 2.6.1.** *Let* $\boldsymbol{M} = \{-3, -3, -3, -1, 0, 5, 5, 6\}$. *Then*

$$\bar{\boldsymbol{M}} = \{-3, -2, -1, 0, 1, 5, 6, 7\}.$$

This definition is motivated by the following observation.

Let $V$ be a $d$-dimensional space of functions of $x$ meromorphic around $x = z$ for some $z \in \mathbb{C}$. Then $M \in \mathbb{Z}$ is an *exponent of* $V$ *at* $z$ if there is a function $f(x) \in V$ such that the order of the zero at $x = z$ is $M$: $f(x) = (x - z)^M (c + o(x - z))$, $c \in \mathbb{C}^\times$. Then $V$ has $d$ distinct exponents. We denote $e(V, z)$ the set of exponents of $V$ at $z$.

Let $V_1, \dots, V_k$ be spaces of functions of $x$ meromorphic around $x = z$, $\dim V_i = d_i$. Let $\boldsymbol{M} = \sqcup_{i=1}^k e(V_i, z)$. Let $V = \oplus_{i=1}^k V_i$. Assume that $\dim V = \sum_{i=1}^k d_i$. Then $e(V, z)$ dominates $\bar{\boldsymbol{M}}$. Moreover, generically, $e(V, z) = \bar{\boldsymbol{M}}$.

Let $T_1, \dots, T_{m+n} \in \mathbb{C}(x)$ be rational functions such that $T_i/T_{i+1} \in \mathbb{C}[x]$ is a polynomial for all $i = 1, \dots, m + n - 1$, $i \neq m$. Let $\tau_i(z)$ be the order of the zero of $T_i$ at $x = z$. Set

$$M_i(z) = \tau_{m-i+1}(z) + i - 1,\ i = 1, \dots, m, \qquad N_i(z) = -\tau_{m+i}(z) + i - 1,\ i = 1, \dots, n.$$

We have $M_1(z) < M_2(z) < \cdots < M_m(z)$, $N_1(z) < N_2(z) < \cdots < N_n(z)$.

Let $a \in \{0, \dots, m\}$, $b \in \{0, \dots, n\}$. Let $\bar{M}_{a,b} = \{c_1(z) < \cdots < c_{a+b}(z)\}$ be the dominant of $\{M_1(z), \dots, M_a(z), N_1(z), \dots, N_b(z)\}$. Define

$$\begin{aligned} d_{a,b}(z) &= ab - \sum_{i=1}^{a+b} c_i(z) + \sum_{i=1}^{a} M_i(z) + \sum_{i=1}^{b} N_i(z) \\ &= \binom{a+b}{2} - \sum_{i=1}^{a+b} c_i(z) + \sum_{i=1}^{a} \tau_{m+1-i}(z) - \sum_{i=1}^{b} \tau_{m+i}(z). \end{aligned}$$

Note that $d_{a,b}(z) \geq 0$. Moreover, for all but finitely many $z$ we have $M_i = i-1$, $N_i = i-1$, $c_i = i-1$, and $d_{a,b}(z) = 0$.

We set

$$\pi_{a,b} = \prod_{z\in\mathbb{C}} (x-z)^{d_{a,b}(z)}. \tag{2.23}$$

Note that $\pi_{a,b} \in \mathbb{C}[x]$ is a polynomial.

Note that for any non-zero rational function $f(x)$, the polynomials $\pi_{a,b}$ computed from $T_i$ and $fT_i$ are the same.

### 2.6.2 Properties of $\pi_{a,b}$

Let $\boldsymbol{\lambda}$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $T_i = T_i^{\boldsymbol{s}_0}$ be the corresponding polynomials, see (2.9). Let $\pi_{a,b}$ be the polynomials given by (2.23).

Let $\boldsymbol{s}$ be a parity sequence. Using $\pi_{a,b}$, the polynomials $T_i^{\boldsymbol{s}}$ can be written in terms of the $T_i$.

**Theorem 2.6.2.** *We have*

$$T_i^{\boldsymbol{s}} = T_{\sigma_{\boldsymbol{s}}(i)} \frac{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^-}}{\pi_{\boldsymbol{s}_i^+ +1,\boldsymbol{s}_i^-}}, \text{ if } s_i = 1 \quad \text{and} \quad T_i^{\boldsymbol{s}} = T_{\sigma_{\boldsymbol{s}}(i)} \frac{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^- +1}}{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^-}}, \text{ if } s_i = -1.$$

*Proof.* Let $\boldsymbol{s}$ be a parity sequence such that $s_i \neq s_{i+1}$ and

$$\widetilde{\boldsymbol{s}} = \boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n}).$$

Let $a = \boldsymbol{s}_i^+$, $b = \boldsymbol{s}_i^- + 1$. By Lemma 2.5.1 it is sufficient to check

$$\frac{\pi_{a+1,b}\,\pi_{a,b-1}}{\pi_{a,b}\,\pi_{a+1,b-1}} = \pi\left(T_{M+b}T_{M-a}\frac{\pi_{a,b}}{\pi_{a+1,b-1}}\right).$$

Since $\lambda^{(k)}$ is a polynomial $\mathfrak{gl}_{m|n}$-weight, the exponent of $\pi_{a,b}$ at $z_k$, $d_{a,b}(z_k)$, is given by

$$d_{a,b}(z_k) = \begin{cases} ab & \text{if } b \leq \lambda_m^{(k)}, \\ (a-1)b + \lambda_m^{(k)} & \text{if } \lambda_m^{(k)} < b \leq \lambda_{m-1}^{(k)}, \\ \dots & \dots \\ \lambda_m^{(k)} + \dots + \lambda_{m-a+1}^{(k)} & \text{if } \lambda_{m-a+1}^{(k)} < b. \end{cases}$$

Thus the exponent of $\pi_{a+1,b}/\pi_{a,b}$ at $z_k$ is given by

$$d_{a+1,b}(z_k) - d_{a,b}(z_k) = \min\{b, \lambda^{(k)}_{m-a}\}.$$

The exponent of $(\pi_{a+1,b}\,\pi_{a,b-1})/(\pi_{a,b}\,\pi_{a+1,b-1})$ at $z_k$ is 1 if $b \leq \lambda^{(k)}_{m-a}$ and it is 0 otherwise.

To compute the exponent of $T_{m+b}T_{m-a}\pi_{a,b}/\pi_{a+1,b-1}$ at $z_k$, introduce two extra parameters $c_1, c_2$: $\lambda^{(k)}_{m-c_1+1} < b-1 = \lambda^{(k)}_{m-c_1} = \cdots = \lambda^{(k)}_{m-c_2+1} < b \leq \lambda^{(k)}_{m-c_2}$. We have

$$d_{a,b} - d_{a+1,b-1} = \begin{cases} 1 + a - b - c_2 & \text{if } a \geq c_2, \\ -\lambda_{m-a} & \text{if } a < c_2. \end{cases}$$

Note that $\lambda^{(k)}_{m-a} < b$ implies $\lambda^{(k)}_{m+b} = 0$. A direct computation gives the proof. $\square$

Let $W = V \oplus U$ be a graded space of rational functions of dimension $m+n$, where $V = W_{\bar{0}}$, $U = W_{\bar{1}}$ and $\dim V = m$, $\dim U = n$. For $z \in \mathbb{C}$, define $M_1(z) < M_2(z) < \cdots < M_m(z)$ and $N_1(z) < N_2(z) < \cdots < N_n(z)$ to be the exponents of $V$ and $U$ at $z$ respectively. Define rational functions

$$T_i^V = \prod_{z\in\mathbb{C}} (x-z)^{M_{m-i+1}-m+i}, i = 1, \ldots, m,$$

and

$$T_{m+i}^U = \prod_{z\in\mathbb{C}} (x-z)^{-N_i+i-1}, i = 1, \ldots, n.$$

Let $\pi^{V,U}_{a,b}$ be polynomials as in (2.23) computed from $T_i^V, T_{m+i}^U$. The following lemma is clear.

**Lemma 2.6.3.** *Let $v_1, \ldots, v_a \in V$, $u_1, \ldots, u_b \in U$. Then*

$$\frac{\mathrm{Wr}\,(v_1, \ldots, v_a, u_1, \ldots, u_b)\,\pi^{V,U}_{a,b}\,T^U_{m+1}T^U_{m+2}\ldots T^U_{m+b}}{T^V_m T^V_{m-1}\cdots T^V_{m-a+1}}$$

*is a polynomial.* $\square$

### 2.6.3 The $\mathfrak{gl}_{m|n}$ spaces

Let $W = V \oplus U$ be a graded space of rational functions of dimension $m+n$, where $V = W_{\bar{0}}$, $U = W_{\bar{1}}$ and $\dim V = m$, $\dim U = n$. For $z \in \mathbb{C}$, let as before $M_1(z) < M_2(z) < \cdots < M_m(z)$ and $N_1(z) < N_2(z) < \cdots < N_n(z)$ be the exponents of $V$ and $U$ at $z$ respectively.

We call $W$ a $\mathfrak{gl}_{m|n}$ *space* if the following conditions are satisfied for all $z \in \mathbb{C}$:

1. $N_n(z) \leq n-1$;
2. if $M_1(z) < 0$, then $M_2(z) \geq 1$, $N_1(z) = M_1(z)$, and $N_i(z) = i-1$, $i = 2, \ldots, n$;
3. if $v \in V$, $u \in U$ are not regular at $z$, then there exists a $c \in \mathbb{C}$ such that $(u+cv)(z) = 0$.

These conditions can be reformulated as follows. Let

$$p^V = \prod_{z,\, M_1(z)<0} (x-z)^{-M_1(z)}, \qquad p^U = \prod_{z,\, N_1(z)<0} (x-z)^{-N_1(z)}$$

be the least common denominators. Then $\bar{V} = p^V V$ and $\bar{U} = p^U U$ are spaces of polynomials.

**Lemma 2.6.4.** *The conditions in the definition of the $\mathfrak{gl}_{m|n}$ space are equivalent to:*

1. *$p^U/p^V$ is a polynomial that is relatively prime with $p^V$;*
2. *$T^{\bar{V}}_{m-1}/p^V$ and $T^U_{m+n}$ are polynomials;*
3. *if $T^{\bar{U}}_{m+i}(z) = 0$ for some $i = 2, \ldots, n$, then $(p^U/p^V)(z) = 0$;*
4. *for any $v \in V, u \in U$, $p^V \mathrm{Wr}\,(v,u)$ is regular at every zero of $p^V$.*

*Proof.* Let $\tau^V_i(z)$, $\tau^{\bar{V}}_i(z)$, $\tau^U_j(z)$, and $\tau^{\bar{U}}_j(z)$ be the orders of the zeroes of $T^V_i$, $T^{\bar{V}}_i$, $T^U_j$, and $T^{\bar{U}}_j$ at $z$. If $\tau^V_m(z) < 0$, then $\tau^{\bar{V}}_i(z) = \tau^V_i(z) - \tau^V_m(z)$. If $\tau^U_{m+1}(z) > 0$, then $\tau^{\bar{U}}_j(z) = \tau^U_j(z) - \tau^U_{m+1}(z)$.

The conditions (1) and (2) in the definition of a $\mathfrak{gl}_{m|n}$ space are equivalent to $\tau^U_{m+n}(z) \geq 0$ and if $\tau^V_m(z) < 0$, then $\tau^V_{m-1}(z) \geq 0$, $-\tau^U_{m+1}(z) = \tau^V_m(z)$, and $\tau^U_{m+2}(z) = \cdots = \tau^U_{m+n}(z) = 0$. This is equivalent to the first three conditions in the lemma.

The condition (3) in the definition is equivalent to the condition *(4)* in the lemma in the presence of the other conditions. $\square$

Let $W = V \oplus U$ be a $\mathfrak{gl}_{m|n}$ space. Define polynomials

$$T_i^W = T_i^V = \frac{T_i^{\bar V}}{p^V},\ i = 1, \dots, m-1, \qquad T_m^W = p^V T_m^V = T_m^{\bar V},$$
$$T_{m+1}^W = \frac{T_{m+1}^U}{p^V} = \frac{p^U}{p^V}, \qquad T_{m+i}^W = T_{m+i}^U = p^U T_{m+i}^{\bar U},\ i = 2, \dots, n.$$

**Remark 2.6.5.** *Note that while $T_i^{\bar V}$, $i = 1, \dots, m$, are the standard polynomials describing the exponents of the space of polynomials $p^V V$, our definition of $T_{m+i}^{\bar U}$ has an extra minus sign. The exponents of the space of polynomials $p^U U$ are described by a sequence of polynomials $(p^U/T_{m+n}^W, \dots, p^U/T_{m+2}^W, 1)$.*

Let $\pi_{a,b}^W$ be as in (2.23) computed from polynomials $T_i^W$.

Further, given $a \in \{0, 1, \dots, m\}$, $b \in \{0, 1, \dots, n\}$, $v_1, \dots, v_a \in V$, $u_1, \dots, u_b \in U$, define

$$y_{a,b} = \frac{\mathrm{Wr}\,(v_1, \dots, v_a, u_1, \dots, u_b)\, \pi_{a,b}^W\, p^V\, T_{m+1}^W \dots T_{m+b}^W}{T_m^W \dots T_{m-a+1}^W}.$$

We have

**Lemma 2.6.6.** *The function $y_{a,b}$ is a polynomial.*

*Proof.* The lemma is proved by considering orders of zeroes at each $z \in \mathbb{C}$. $\square$

Note that Lemma 2.6.6 is stronger than Lemma 2.6.3, since $y_{a,b}$ has $p^V$ and not $(p^V)^2$ in the numerator. Lemma 2.6.6 holds due to the additional assumption that $W$ is a $\mathfrak{gl}_{m|n}$ space. Here, we crucially use the condition (3) in the definition of the $\mathfrak{gl}_{m|n}$ space.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights, $\boldsymbol{z} = (z_1, \dots, z_n)$ a sequence of pairwise distinct complex numbers. Let $\boldsymbol{T} = (T_1, \dots, T_{m+n})$ be the corresponding polynomials given by (2.9). Let $\boldsymbol{y}$ represent a solution of the BAE associated to $\boldsymbol{\lambda}, \boldsymbol{z}$, and the standard parity sequence $\boldsymbol{s}_0$. We have the rational pseudodifferential operator $R(\boldsymbol{y}) = D_{\bar 0}(\boldsymbol{y}) D_{\bar 1}^{-1}(\boldsymbol{y})$. Let $V(\boldsymbol{y}) = \ker D_{\bar 0}(\boldsymbol{y})$, $U(\boldsymbol{y}) = \ker D_{\bar 1}(\boldsymbol{y})$.

**Proposition 2.6.7.** *If $\boldsymbol{\lambda}$ is typical, then $W(\boldsymbol{y}) = V(\boldsymbol{y}) \oplus U(\boldsymbol{y})$ is a $\mathfrak{gl}_{m|n}$ space of rational functions and $T_i^W = T_i$, $i = 1, \dots, m+n$.*

*Proof.* Denote $W(\boldsymbol{y})$, $V(\boldsymbol{y})$, and $U(\boldsymbol{y})$ by $W$, $V$, and $U$ respectively.

Note that $y_1, \dots, y_{m-1}$ represents a solution of the $\mathfrak{gl}_m$ BAE. Therefore, the bosonic reproduction procedures generate a $\mathfrak{gl}_m$ population and $y_m \cdot D_{\bar 0} \cdot (y_m)^{-1}$ is the differential operator associated to this population. It follows by [MV04] that $\bar V = y_m V$ is a space of polynomials. Similarly, $y_{m+1}, \dots, y_{m+n-1}$ represents a solution of the $\mathfrak{gl}_n$ BAE and $\bar U = y_m T_{m+1} U$ is also a space of polynomials.

We have $p^V = y_m$, $p^U = T_{m+1} y_m$.

Since $\boldsymbol{\lambda}$ is typical, there exists $k$ such that $\lambda^{(k)}$ is typical, i.e. $\lambda_m^{(k)} \geq n$. Then $\lambda_i^{(k)} + m - i \geq \lambda_i^{(k)} \geq \lambda_m^{(k)} \geq n > j - 1 \geq -\lambda_{m-j}^{(k)} + j - 1$ for $i = 1, \dots, m$, $j = 1, \dots, n$. Therefore the spaces $V$ and $U$ have no exponents in common and hence $V \cap U = 0$.

The only non-trivial condition in Lemma 2.6.4 is *(4)*. The fermionic reproduction procedure in the $m$-th direction (2.19) can be written as

$$y_m \widetilde{y}_m = \mathrm{Wr}\,(v, u) y_m^2 \pi_m T_{m+1}/T_m.$$

Initially, we have $v(\boldsymbol{y}) = T_m y_{m-1}/y_m$, $u(\boldsymbol{y}) = y_{m+1}/(T_{m+1} y_m)$. Generic $u, v$ can be obtained from $v(\boldsymbol{y}), u(\boldsymbol{y})$ by the bosonic reproduction procedures. Therefore, by Theorem 2.5.2, $\widetilde{y}_m$ is a polynomial for generic $v$, $u$. Since $y_m$ is relatively prime to $\pi_m T_{m+1}/T_m$, we obtain condition *(4)* in Lemma 2.6.4. $\square$

**Remark 2.6.8.** *If $\boldsymbol{\lambda}$ is not typical then cancellations may occur in the rational pseudodifferential operator $R(\boldsymbol{y}) = D_{\bar 0}(\boldsymbol{y}) D_{\bar 1}^{-1}(\boldsymbol{y})$ of* (2.22) *and the spaces $V(\boldsymbol{y}) = \ker D_{\bar 0}(\boldsymbol{y})$, $U(\boldsymbol{y}) = \ker D_{\bar 1}(\boldsymbol{y})$ may intersect non-trivially. Compare Lemma 2.4.7. As an important example, consider the tensor product of $N$ copies of the defining representation, $L(\boldsymbol{\lambda}) = (\mathbb{C}^{m|n})^N$. Then $T_1(x) = \prod_{k=1}^N (x - z_k)$ and $T_i(x) = 1$ for*

$i = 2, \ldots, n+m$. *Thus for the vacuum solution to the BAE, i.e.* $\boldsymbol{y} = (1, \ldots, 1)$, *we have*

$$D_{\bar{0}}(\boldsymbol{y}) = \left( \partial - \prod_{k=1}^{N} \frac{1}{x - z_k} \right) \partial^{m-1}, \quad D_{\bar{1}}(\boldsymbol{y}) = \partial^n.$$

### 2.6.4 The generating map

Given a parity sequence $\boldsymbol{s}$ and a full superflag $\mathcal{F} \in \mathcal{F}^{\boldsymbol{s}}(W)$, we define polynomials $y_i(\mathcal{F})$, $i = 1, \ldots, m+n-1$, by the formula

$$y_i(\mathcal{F}) = \begin{cases} y_{\boldsymbol{s}_i^+, \boldsymbol{s}_i^-} & \text{if } s_i = 1, \\ y_{\boldsymbol{s}_i^+, \boldsymbol{s}_i^- + 1} & \text{if } s_i = -1. \end{cases}$$

That defines the *generating map*

$$\beta^{\boldsymbol{s}} : \mathcal{F}^{\boldsymbol{s}}(W) \to (\mathbb{P}(\mathbb{C}[x]))^{m+n-1}, \quad \mathcal{F} \mapsto \boldsymbol{y}(\mathcal{F}) = (y_1(\mathcal{F}), \ldots, y_{m+n-1}(\mathcal{F})).$$

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N)$ be a typical sequence of polynomial $\mathfrak{gl}_{m|n}$ weights, $\boldsymbol{z} = (z_1, \ldots, z_N)$ a sequence of pairwise distinct complex numbers. Let $\boldsymbol{T} = (T_1, \ldots, T_{m+n})$ be the corresponding polynomials given by (2.9). Let $\boldsymbol{y}$ represent a solution of the BAE associated to $\boldsymbol{\lambda}, \boldsymbol{z}$ and the standard parity sequence $\boldsymbol{s}_0$.

Recall that we have the $\mathfrak{gl}_{m|n}$ population $P = P_{\boldsymbol{y}}$, see Section 2.5.1, the rational pseudodifferential operator of the population $R_P = R(\boldsymbol{y}) = D_{\bar{0}}(\boldsymbol{y})(D_{\bar{1}}(\boldsymbol{y}))^{-1}$, see (2.22) and the $\mathfrak{gl}_{m|n}$ space $W_P = V(\boldsymbol{y}) \oplus U(\boldsymbol{y})$, see Proposition 2.6.7.

The following theorem asserts that the population $P$ is canonically identified with full superflags $\mathcal{F}(W_P)$ and the complete factorizations of the pseudodifferential operator $\mathcal{F}(R_P)$.

**Theorem 2.6.9.** *For any flag* $\mathcal{F} \in \mathcal{F}^{\boldsymbol{s}}(W_P)$, *we have* $\beta^{\boldsymbol{s}}(\mathcal{F}) \in P^{\boldsymbol{s}}$. *Moreover, the generating map* $\beta^{\boldsymbol{s}} : \mathcal{F}^{\boldsymbol{s}}(W_P) \to P^{\boldsymbol{s}}$ *is a bijection. Finally, the complete factorization* $\rho^{\boldsymbol{s}}(\mathcal{F})$ *of* $R_P$ *coincides with* $R^{\boldsymbol{s}}(\beta^{\boldsymbol{s}}(\mathcal{F}))$, *see* (2.4), (2.5), *and* (2.22).

*Proof.* The operator $R_P^{\boldsymbol{s}_0}$ coincides with the unique minimal fractional decomposition of $R_P$. Thus, for the standard parity, the theorem is proved in [MV04].

Let $\boldsymbol{y} = \beta^{\boldsymbol{s}}(\mathcal{F}) = (y_1, \dots, y_{m+n-1})$. Lemma 2.6.6 asserts that $\boldsymbol{y}$ is a sequence of polynomials. By Theorem 2.6.2, we have $R^{\boldsymbol{s}}(\boldsymbol{y}) = \rho^{\boldsymbol{s}}(\mathcal{F})$.

Let $\boldsymbol{s}$ be such that $s_i \neq s_{i+1}$. Let $\widetilde{\boldsymbol{s}} = \boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$. Let $\widetilde{\boldsymbol{y}} = \beta^{\widetilde{\boldsymbol{s}}}(\mathcal{F}) = (\widetilde{y}_1, \dots, \widetilde{y}_{m+n-1})$. A direct computation shows $y_r = \widetilde{y}_r$, $r = 1, \dots, m+n-1$, $r \neq i$, and $y_i$, $\widetilde{y}_i$ satisfy equation (2.19). By Theorem 2.5.3 we have $R^{\widetilde{\boldsymbol{s}}}(\widetilde{\boldsymbol{y}}) = \rho^{\widetilde{\boldsymbol{s}}}(\mathcal{F})$.

That reduces the case of any $\boldsymbol{s}$ to the case of $\boldsymbol{s}_0$. □

**Remark 2.6.10.** *Theorem 2.6.9 shows in particular that if two populations intersect, then they coincide.*

Let $W$ be a $\mathfrak{gl}_{m|n}$ space. Let $\boldsymbol{\lambda}_W$ be a sequence of $\mathfrak{gl}_{m|n}$ weights and $\boldsymbol{z}_W$ a sequence of distinct complex numbers such that $T_i^W$ are associated to $\boldsymbol{s}_0, \boldsymbol{\lambda}_W, \boldsymbol{z}_W$.

Let $\boldsymbol{s}$ be a parity sequence. Consider the set of all sequences $(y_1, \dots, y_{m+n-1}) \in \beta^{\boldsymbol{s}}(\mathcal{F}^{\boldsymbol{s}}(W))$. For $i = 1, \dots, m+n-1$, let $l_i^{(\boldsymbol{s},W)}$ be the minimal possible degree of the $i$th polynomial $y_i(x)$ in this set.

Define

$$\lambda_W^{(\boldsymbol{s},\infty)} = \sum_{k=1}^{N} (\lambda_W^{(k)})^{\boldsymbol{s}} - \sum_{i=1}^{m+n-1} \alpha_i^{\boldsymbol{s}}\, l_i^{(\boldsymbol{s},W)}.$$

## 2.7 Conjectures and examples

It is well known that the Bethe ansatz in the naive form is not complete in general. We conjecture how to overcome this problem. We also give a few examples.

### 2.7.1 Conjecture on Bethe vectors

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ be a typical sequence of polynomial $\mathfrak{gl}_{m|n}$ weights, $\boldsymbol{z} = (z_1, \dots, z_N)$ a sequence of distinct complex numbers. Let $\boldsymbol{T} = (T_1, \dots, T_{m+n})$ be the corresponding polynomials given by (2.9).

Let $L(\boldsymbol{\lambda}) = \otimes_{k=1}^{N} L(\lambda^{(k)})$ be the corresponding $\mathfrak{gl}_{m|n}$ module. It is known that the Gaudin Hamiltonians acting in $L(\boldsymbol{\lambda})$ can be included in a natural commutative algebra

$\mathcal{B}(\boldsymbol{\lambda})$ of higher Gaudin Hamiltonians, see [MR14]. The algebra $\mathcal{B}(\boldsymbol{\lambda})$ commutes with the diagonal action of $\mathfrak{gl}_{m|n}$.

If $n = 0$, it is known that the joint eigenvectors of $\mathcal{B}(\boldsymbol{\lambda})$ in $L(\boldsymbol{\lambda})^{sing}$ (up to multiplication by a non-zero constant) are in bijective correspondence with spaces of polynomials $V$, such that $T_i^V = T_i$, see [MTV09].

Let $\boldsymbol{s}$ be a parity sequence. We have the following conjecture.

**Conjecture 2.7.1.** *The algebra $\mathcal{B}(\boldsymbol{\lambda})$ has a simple joint spectrum in $L(\boldsymbol{\lambda})^{\boldsymbol{s}ing}$. There is a bijective correspondence between eigenvectors of $\mathcal{B}(\boldsymbol{\lambda})$ in $L(\boldsymbol{\lambda})^{\boldsymbol{s}ing}_{\lambda^{(\boldsymbol{s},\infty)}}$ (up to multiplication by a non-zero constant) and the $\mathfrak{gl}_{m|n}$ spaces of rational functions $W$ such that $T_i^W = T_i$ and $\lambda_W^{(\boldsymbol{s},\infty)} = \lambda^{(\boldsymbol{s},\infty)}$. Moreover, this bijection is such that, for all $k = 1, \ldots, N$, the eigenvalue of the Gaudin Hamiltonian $\mathcal{H}_k$ is given by* (4.10), *where $\boldsymbol{t}$ is represented by any $k$-admissible $\boldsymbol{y}$ in $\beta(\mathcal{F}(W))$.*

By simple joint spectrum we mean that if $v_1$,$v_2$ are eigenvectors of $\mathcal{B}(\boldsymbol{\lambda})$ and $v_1 \neq cv_2$, $c \in \mathbb{C}^{\times}$, then there exists $b \in \mathcal{B}(\boldsymbol{\lambda})$ such that the eigenvalues of $b$ on $v_1$ and $v_2$ are different.

**Remark 2.7.2.** *If the sequence of polynomial modules $\boldsymbol{\lambda}$ is not typical we expect that the eigenvectors are also parameterized by pairs of spaces of rational functions $V$ and $U$ of dimensions $M$ and $N$ with similar conditions. However, $V$ and $U$ can have a non-trivial intersection (see Remark 2.6.8). Then some fermionic reproduction procedure becomes undefined and the factorization of the rational pseudodifferential operator* (2.22) *is not minimal. We do not deal with this case here.*

In the case of $\mathfrak{gl}_{1|1}$, Conjecture 2.7.1 simplifies as follows. We follow the notation of Section 2.4.2. Let $\mathcal{N}(T) = \ln'(T_1T_2)\pi(T_1T_2)$.

**Conjecture 2.7.3.** *The Gaudin Hamiltonians $\mathcal{H}_k$, $k = 1, \ldots, N$, have a simple joint spectrum in $L(\boldsymbol{\lambda})^{\boldsymbol{s}ing}$. There exists a one-to-one correspondence between the monic divisors $y$ of the polynomial $\mathcal{N}(T)$ of degree $l$ and the joint eigenvectors $v$ of the Gaudin Hamiltonians of weight $(p - l, q + l)$ (up to multiplication by a non-zero* *constant). Moreover, this bijection is such that* $\mathcal{H}_k v = E_k v$, $k = 1, \ldots, N$, *where* $E_k$ *are given by* (2.16)*.*

Recall our conventions from §2.3.2 about what constitutes a solution to the Bethe ansatz equation. With those conventions, a monic divisor of $\mathcal{N}(T)$ is the same thing as a solution to the Bethe ansatz equation, cf. Lemma 2.4.3, and in that sense Conjecture 2.7.3 asserts that the Bethe ansatz is complete for $\mathfrak{gl}_{1|1}$.

### 2.7.2 A $\mathfrak{gl}_{1|1}$ example of double roots

Suppose all the tensor factors $L(\lambda^{(k)})$, $k = 1, \ldots, N$ are non-trivial. In type $\mathfrak{gl}_{1|1}$ that suffices to make them all typical, cf. Remark 2.4.4. Thus we have $\deg \mathcal{N}(T) = N-1$. For generic $z$, all roots of the polynomial $\mathcal{N}(T)$ are distinct, and there are $2^{N-1}$ different monic divisors of $\mathcal{N}(T)$. In such a case we have a basis of Bethe eigenvectors in $L(\boldsymbol{\lambda})^{sing}$, in accordance with Conjecture 2.7.3. But when the polynomial $\mathcal{N}(T)$ has multiple roots the number of its divisors is smaller. Then, according to Conjecture 2.7.3, we should expect non-trivial Jordan blocks in the action of the Gaudin Hamiltonians. We give an example illustrating this point.

We consider the case when $N = 3$. We work with the standard parity sequence.

The modules $L(\lambda^{(k)})$, $k = 1, 2, 3$ are spanned by $v_+^{(k)}$ and $v_-^{(k)}$, where $v_+^{(k)}$ is the highest weight vector with respect to $\boldsymbol{s}_0$, and $v_-^{(k)} = e_{21} v_+^{(k)}$. Denote the vector $v_i^{(1)} \otimes v_j^{(2)} \otimes v_k^{(3)}$, $i, j, k \in \{\pm\}$ by $v_{(ijk)}$. Let $h_k = p_k + q_k$, $k = 1, 2, 3$. We are supposing that $h_k \neq 0$, $k = 1, 2, 3$. We have

$$\begin{aligned} \mathcal{N}(T) = &(h_1 + h_2 + h_3)x^2 \\ &- (h_1(z_2 + z_3) + h_2(z_1 + z_3) + h_3(z_1 + z_2))x \\ &+ (h_1 z_2 z_3 + h_2 z_1 z_3 + h_3 z_1 z_2). \end{aligned} \tag{2.24}$$

The weights $\lambda^{(i)}$ being polynomial means that $h_i \in \mathbb{Z}_{\geq 1}$.

The subspace $L(\boldsymbol{\lambda})^{\mathrm{sing}}_{(p-1,q+1)}$ is spanned by $w_1 = -h_2 v_{(-++)} + h_1 v_{(+-+)}$ and $w_2 = -h_3 v_{(+-+)} + h_2 v_{(++-)}$. The action of the Gaudin Hamiltonians in this subspace is explicitly given by

$$\mathcal{H}_1 = \Big(\frac{p_1p_2 - q_1q_2}{z_1 - z_2} + \frac{p_1p_3 - q_1q_3}{z_1 - z_3}\Big) I + \begin{pmatrix} -\frac{h_1+h_2}{z_1-z_2} & -\frac{h_3}{z_1-z_2} \\ -\frac{h_2}{z_1-z_3} & -\frac{h_1+h_3}{z_1-z_3} \end{pmatrix},$$

$$\mathcal{H}_2 = \Big(\frac{p_2p_1 - q_2q_1}{z_2 - z_1} + \frac{p_2p_3 - q_2q_3}{z_2 - z_3}\Big) I + \begin{pmatrix} -\frac{h_1+h_2}{z_2-z_1} & \frac{h_1}{z_2-z_3} \\ \frac{h_3}{z_2-z_1} & -\frac{h_2+h_3}{z_2-z_3} \end{pmatrix}.$$

The discriminants of the characteristic polynomials of both of the above $2\times 2$ matrices coincide with the right-hand side of (2.24) up to multiplication by nonzero factors. Therefore the polynomial $\mathcal{N}(T)$ has distinct roots if and only if $\mathcal{H}_1, \mathcal{H}_2$ have distinct eigenvalues, that is, if and only if the Gaudin Hamiltonians are diagonalizable.

We note that in the case of double roots of $y(x)$, the corresponding Bethe vector is zero. Therefore an actual eigenvector should be obtained via an appropriate derivative. It can be done in the case of $\mathfrak{gl}_{1|1}$ without difficulties, but in general the algebraic procedure is not known.

### 2.7.3 A $\mathfrak{gl}_{1|1}$ example with non polynomial modules

Conjecture 2.7.3 may be true for arbitrary modules, not only polynomial ones if we make the following modification. Let $\boldsymbol{\lambda}$ be a sequence of arbitrary $\mathfrak{gl}_{1|1}$ weights.

In general $L(\boldsymbol{\lambda})$ need not be completely reducible. That is, there may exist a nonzero singular vector $v \in L(\boldsymbol{\lambda})^{sing}$ such that $v = e^{\boldsymbol{s}}_{21} w$ for some $w \in L(\boldsymbol{\lambda})$. If $v$ and $w$ are eigenvectors then the eigenvalues of $v$ and $w$ are the same and we do not expect to obtain a new divisor of $\mathcal{N}(T)$ for $v$.

**Conjecture 2.7.4.** *Consider the subspace of $L(\boldsymbol{\lambda})^{sing}$ spanned by the joint eigenvectors of the Gaudin Hamiltonians $\mathcal{H}_k$, $k = 1, \dots, N$. Quotient it by its intersection with the image of $e^{\boldsymbol{s}}_{21}$. On this subquotient, the Gaudin Hamiltonians $\mathcal{H}_k$, $k = 1, \dots, N$ have a simple joint spectrum and their joint eigenvectors of weight $(p-l, q+l)$ (up to multiplication by a non-zero constant) are in one-to-one correspondence with the* *monic divisors $y$ of the polynomial $\mathcal{N}(T)$ of degree $l$. Moreover, this bijection is such that $\mathcal{H}_k v = E_k v$, $k = 1, \dots, N$, where $E_k$ are given by* (2.16).

Here we give an example of a such a phenomenon. We consider the case when $N = 3$. Suppose $h_1 + h_2 + h_3 = 0$, that is $p + q = 0$. Then the polynomial $\mathcal{N}(T)$ given by (2.24) is linear. In particular, we have only two divisors instead of the four which we had in a generic situation. We denote the only root of $\mathcal{N}(T)$ by $t$.

The subspace $L(\boldsymbol{\lambda})_{(p-1,-p+1)}$ is three dimensional. It has a basis $\{w, e_{21}v_{(+++)}, v\}$ where $w$ is any vector such that $e_{12}w = v_{(+++)}$, and the two other vectors $e_{21}v_{(+++)} = v_{(-++)} + v_{(+-+)} + v_{(++-)}$ and $v = (t-z_1)^{-1}v_{(-++)} + (t-z_2)^{-1}v_{(+-+)} + (t-z_3)^{-1}v_{(++-)}$ are singular.

The subspace $L(\boldsymbol{\lambda})_{(p-2,-p+2)}$ is also three dimensional. It has a basis $\{u, e_{21}w, e_{21}v\}$ where $u$ is any vector such that $e_{21}u = v_{(---)}$. One can check that $e_{21}v$ is proportional to $e_{12}v_{(---)}$, and is therefore singular since $e_{12}^2 = 0$.

The structure of the module can be pictured as follows:

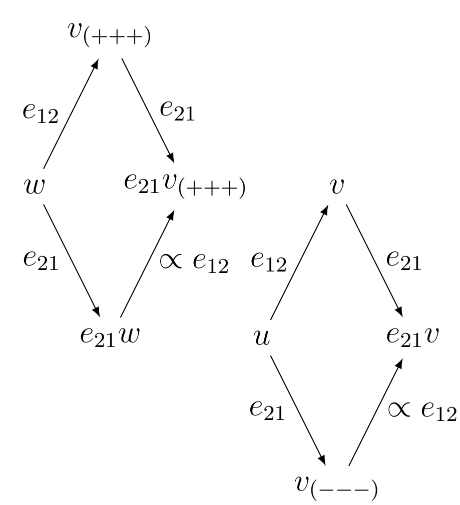



Fig. 2.2. Structure of the module

While the singular space $L(\boldsymbol{\lambda})^{\text{sing}}$ is four dimensional, its quotient by the image of $e_{21}$ is two dimensional and generated by the images of $v_{(+++)}$ and $v$, in accordance with the Conjecture 2.7.4.

Let $\boldsymbol{s}_1 = (-1, 1)$ be the only non-standard parity sequence. The subspace of $\boldsymbol{s}_1$-singular vectors has a basis $\{v_{(---)}, e_{21}v, e_{21}w, e_{21}v_{(+++)}\}$. Its quotient by the image of $e_{21}^{\boldsymbol{s}_1}$ is generated by images of $v_{(---)}$ and $e_{21}w$.

The reproduction procedure connects $v_{(+++)}$ with $e_{21}w$ and $v$ with $v_{(---)}$. In particular, it connects vectors with the same eigenvalues, see Lemma 2.4.5; however the weight now changes by $2\alpha$ and Corollary 2.4.6 is not true in this situation.

# 3. DUALITY OF SUPERSYMMETRIC GAUDIN MODELS

## 3.1 Preliminaries

### 3.1.1 Superspaces and superalgebras

A *superalgebra* is a vector superspace with an even, bilinear, associative, unital product operation. Given superalgebras $\mathcal{A}, \mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is a superalgebra. For any homogeneous elements $x, x' \in \mathcal{A}$, $y, y' \in \mathcal{B}$, the product in the superalgebra $\mathcal{A} \otimes \mathcal{B}$ is

$$(x \otimes y)(x' \otimes y') = (-1)^{|x'||y|}(xx' \otimes yy').$$

For $x \in \mathcal{A}$, $a \in \{1, \dots, k\}$, denote $1^{\otimes(a-1)} \otimes x \otimes 1^{\otimes(k-a)} \in \mathcal{A}^{\otimes k}$ by $x^{(a)}$.

### 3.1.2 The $\mathfrak{gl}_{m|n}$ current algebra and the evaluation modules

Let $t$ be an even variable. Let $\mathfrak{gl}_{m|n}[t] = \mathfrak{gl}_{m|n} \otimes \mathbb{C}[t]$ be the Lie superalgebra of $\mathfrak{gl}_{m|n}$ valued polynomials with pointwise superbracket. We call $\mathfrak{gl}_{m|n}[t]$ the *current algebra*. Denote by $U\mathfrak{gl}_{m|n}[t]$ the universal enveloping algebra of $\mathfrak{gl}_{m|n}[t]$.

We identify the Lie superalgebra $\mathfrak{gl}_{m|n}$ with the subalgebra $\mathfrak{gl}_{m|n} \otimes 1$ of constant polynomials in $\mathfrak{gl}_{m|n}[t]$. Therefore any $\mathfrak{gl}_{m|n}[t]$-module has the canonical structure of a $\mathfrak{gl}_{m|n}$-module.

The standard generators of $\mathfrak{gl}_{m|n}[t]$ are $e_{i,j} \otimes t^r$, $i, j = 1, \dots, m+n$, $r \in \mathbb{Z}_{\geq 0}$. The superbracket is given by

$$(u - v)[e_{i,j}(u), e_{p,q}(v)] = -[e_{i,j}, e_{p,q}](u) + [e_{i,j}, e_{p,q}](v), \tag{3.1}$$

where

$$e_{i,j}(v) = \sum_{r=0}^{\infty} (e_{i,j} \otimes t^r) v^{-r-1} \tag{3.2}$$

are the formal power series.

For each $z \in \mathbb{C}$, there exists a *shift of spectral parameter automorphism* $\rho_z$ of $\mathfrak{gl}_{m|n}[t]$ sending $g(v)$ to $g(v-z)$ for all $g \in \mathfrak{gl}_{m|n}$. Given a $\mathfrak{gl}_{m|n}[t]$-module $V$, denote by $V_z$ the pull-back of $V$ through the automorphism $\rho_z$. As $\mathfrak{gl}_{m|n}$-modules, $V$ and $V_z$ are isomorphic by the identify map.

We have the *evaluation homomorphism*, $ev : \mathfrak{gl}_{m|n}[t] \to \mathfrak{gl}_{m|n}$, $ev : g(v) \mapsto gv^{-1}$. For any $\mathfrak{gl}_{m|n}$-module $V$, denote by the same letter the $\mathfrak{gl}_{m|n}[t]$-module, obtained by pull-back of $V$ through the evaluation homomorphism $ev$. Given a $\mathfrak{gl}_{m|n}$-module $V$ and $z \in \mathbb{C}$, the $\mathfrak{gl}_{m|n}[t]$-module $V_z$ is called an *evaluation module.* The action of $\mathfrak{gl}_{m|n}[t]$ in $V_z$ is given by

$$e_{i,j}(v)w = \frac{e_{i,j}w}{v-z}, \tag{3.3}$$

for any $w \in V$, $i,j = 1,\dots,m+n$.

Note that if $\lambda^{(1)},\dots,\lambda^{(k)}$ are polynomial weights and $z_1,\dots,z_k$ are pairwise distinct complex numbers, then the module $\otimes_{a=1}^{k} L(\lambda^{(a)})_{z_a}$ is irreducible.

## 3.2 Berezinians of affine Manin matrices

In this section, we recall some facts about Berezinians, following [MR14]. We give a definition of Berezinians of affine Manin matrices to arbitrary parities and study its properties.

Let $\mathcal{A}$ be a superalgebra. Given a matrix $A = \left(a_{i,j}\right)_{i,j=1}^{m+n}$, $a_{i,j} \in \mathcal{A}$, with a two sided inverse $A^{-1}$, we denote the $(i,j)$ entry of $A^{-1}$ by $\widetilde{a}_{i,j}$.

### 3.2.1 Berezinian of standard parity

Let $A = \big(a_{i,j}\big)_{i,j=1}^{m+n}$ be a matrix with a two sided inverse. The *Berezinian of standard parity* of $A$, see [MR14], is

$$\operatorname{Ber} A = \Big( \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \ a_{\sigma(1),1} \dots a_{\sigma(m),m} \Big) \times \Big( \sum_{\tau \in \mathfrak{S}_n} \operatorname{sgn} \tau \ \widetilde{a}_{m+1,m+\tau(1)} \dots \widetilde{a}_{m+n,m+\tau(n)} \Big), \tag{3.4}$$

where $\mathfrak{S}_r$ is the symmetric group on $r$ letters. In the case of $n = 0$, the above formula is the *column determinant* which we denote by $\operatorname{cdet} A$. In the case of $m = 0$, the above formula is the *row determinant* of the inverse matrix which we denote by $\operatorname{rdet} A^{-1}$.

We call $A = \big(a_{i,j}\big)_{i,j=1}^{m+n}$, $a_{i,j} \in \mathcal{A}$, a *matrix of standard parity over* $\mathcal{A}$, if $|a_{i,j}| = |i| + |j|$.

We call $A$ a *Manin matrix of standard parity*, if $A$ is of standard parity and

$$[a_{i,j}, a_{p,q}] = (-1)^{|i||j|+|i||p|+|j||p|}[a_{p,j}, a_{i,q}], \qquad i, j, p, q = 1, \dots, m+n.$$

Many properties of even Manin matrices are known, see [CFR09]. Similar properties can be proved in the supersymmetric case, but we need here only a couple of facts which we extract from [MR14].

Let $w$ be an even formal variable. We call $A(w) = \big(a_{i,j}(w)\big)_{i,j=1}^{m+n}$ an *affine matrix*, if

$$a_{i,j}(w) = \sum_{r=0}^{\infty} a_{i,j,r} w^r, \ a_{i,j,r} \in \mathcal{A}, \qquad a_{i,j,0} = \delta_{i,j}, \ i, j = 1, \dots, m+n.$$

In other words, an affine matrix is a matrix whose entries $a_{i,j}(w) \in \mathcal{A}[[w]]$ are formal power series in variable $w$ and such that $A(0) = I$. In particular, every affine matrix has a two sided inverse.

Given a Manin matrix $A$ of standard parity, the matrix $(1 + wA)$ is an affine Manin matrix of standard parity.

**Lemma 3.2.1** ( [MR14]). *Let $A(w)$ be an affine Manin matrix of standard parity. Then the inverse matrix $A^{-1}(w)$ is an affine Manin matrix of standard parity.* $\square$

For an arbitrary $(m+n)\times(m+n)$ matrix $A$ with a two sided inverse, the $(i,j)$ *quasideterminant of* $A$ is $\widetilde{a}_{j,i}^{-1}$. If $\widetilde{a}_{j,i}^{-1}$ does not exist in $\mathcal{A}$, then the $(i,j)$ quasideterminant of $A$ is not defined. We write

$$\widetilde{a}_{j,i}^{-1} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j} & \dots & a_{1,m+n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i,1} & \dots & \boxed{a_{i,j}} & \dots & a_{i,m+n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m+n,1} & \dots & a_{m+n,j} & \dots & a_{m+n,m+n} \end{pmatrix}.$$

For $i=1,\dots,m+n$, define the *principal quasi-minors of* $A$ by

$$d_i(A) = \begin{pmatrix} a_{1,1} & \dots & a_{1,i} \\ \dots & \dots & \dots \\ a_{i,1} & \dots & \boxed{a_{i,i}} \end{pmatrix}. \tag{3.5}$$

If $A(w)$ is an affine matrix, then the principal quasi-minors $d_i(A(w))$ are well defined for all $i$.

The Berezinian of Manin matrices of standard parity is computed in terms of quasi-minors.

**Theorem 3.2.2** ( [MR14]). *Let $A(w)$ be an affine Manin matrix of standard parity. The Berezinian* $\operatorname{Ber} A(w)$ *admits the quasideterminant factorization:*

$$\operatorname{Ber} A(w) = d_1(A(w))\dots d_m(A(w)) \times d_{m+1}^{-1}(A(w))\dots d_{m+n}^{-1}(A(w)).$$

□

### 3.2.2 Berezinian of general parity

Fix a parity sequence $\boldsymbol{s}\in S_{m|n}$, see Section 2.1.1. Set $i^{\boldsymbol{s}} = \sigma_{\boldsymbol{s}}(i)$, see (2.1).

We call $A = \left(a_{i,j}\right)_{i,j=1}^{m+n}$, $a_{i,j}\in\mathcal{A}$, a *matrix of parity* $\boldsymbol{s}$, if $|a_{i,j}| = |i^{\boldsymbol{s}}| + |j^{\boldsymbol{s}}|$. Note that 0 is both odd and even, in particular, the zero and the identity matrices are matrices of arbitrary parity $\boldsymbol{s}$.

We call $A$ a *Manin matrix of parity* $\boldsymbol{s}$ if $A$ is of parity $\boldsymbol{s}$ and

$$[a_{i,j}, a_{p,q}] = (-1)^{|i^{\boldsymbol{s}}||j^{\boldsymbol{s}}|+|i^{\boldsymbol{s}}||p^{\boldsymbol{s}}|+|j^{\boldsymbol{s}}||p^{\boldsymbol{s}}|}[a_{p,j}, a_{i,q}], \qquad i,j,p,q = 1,\dots,m+n.$$

The symmetric groups $\mathfrak{S}_{m+n}$ acts on matrices and parities by the following rule. For $\sigma \in \mathfrak{S}_{m+n}$, we set

$$\sigma(A) = \sigma A \sigma^{-1} = \big(a_{\sigma^{-1}(i),\sigma^{-1}(j)}\big)_{i,j=1}^{m+n}$$

and $\sigma(\boldsymbol{s}) = (s_{\sigma^{-1}(1)},\dots,s_{\sigma^{-1}(m+n)})$.

The following lemma is straightforward.

**Lemma 3.2.3.** *Let $A$ be a Manin matrix of parity $\boldsymbol{s}$. Then $\sigma(A)$ is a Manin matrix of parity $\sigma(\boldsymbol{s})$.* □

Lemma 3.2.1 is extended to affine Manin matrices of arbitrary parities.

**Lemma 3.2.4.** *Let $A(w)$ be an affine Manin matrix of parity $\boldsymbol{s}$. Then $A^{-1}(w)$ is an affine Manin matrix of parity $\boldsymbol{s}$.*

*Proof.* There exists $\sigma \in \mathfrak{S}_{m+n}$ such that $\sigma(\boldsymbol{s}) = \boldsymbol{s}_0$. By Lemma 3.2.3, $\sigma(A(w))$ is an affine matrix of standard parity. By Lemma 3.2.1, the matrix $(\sigma(A(w)))^{-1}$ is an affine Manin matrix of standard parity. We have $(\sigma(A(w)))^{-1} = \sigma(A^{-1}(w))$. Therefore by Lemma 3.2.3, the matrix $A^{-1}(w) = \sigma^{-1}((\sigma(A(w)))^{-1})$ is an affine Manin matrix of parity $\boldsymbol{s}$. □

Let $A(w)$ be an affine Manin matrix of parity $\boldsymbol{s}$. We define the *Berezinian of parity* $\boldsymbol{s}$ of $A(w)$ by

$$\operatorname{Ber}^{\boldsymbol{s}} A(w) = d_1^{s_1}(A(w))\dots d_{m+n}^{s_{m+n}}(A(w)). \tag{3.6}$$

By Theorem 3.2.2, definition (3.6) coincides with definition (3.4) in the case of standard parity.

Let $A(w)$ be an affine Manin matrix of parity $\boldsymbol{s}$. Fix $r \in \{1,\dots,m+n\}$ and consider the corresponding blocks. Namely, let $W(w), X(w), Y(w), Z(w)$ be submatrices of $A(w) = \begin{pmatrix} W(w) & X(w) \\ Y(w) & Z(w) \end{pmatrix}$ of size $r \times r$, $r \times (m+n-r)$, $(m+n-r) \times r$, and $(m+n-r) \times (m+n-r)$ respectively.

Then $W(w)$ and $Z(w)$ are affine Manin matrices of parities $\boldsymbol{s}|^r$ and $\boldsymbol{s}|_{m+n-r}$, where $\boldsymbol{s}|^r = (s_1, \ldots, s_r)$ and $\boldsymbol{s}|_{m+n-r} = (s_{r+1}, \ldots, s_{m+n})$.

We have the Gauss decomposition:

$$\begin{aligned} A(w) &= \begin{pmatrix} W(w) & X(w) \\ Y(w) & Z(w) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ Y(w)W^{-1}(w) & 1 \end{pmatrix} \begin{pmatrix} W(w) & X(w) \\ 0 & Z(w) - Y(w)W^{-1}(w)X(w) \end{pmatrix}. \end{aligned} \tag{3.7}$$

The next proposition claims that the Gauss decomposition is compatible with the definition of Berezinian.

**Proposition 3.2.5.** *The matrices $W(w)$ and $Z(w) - Y(w)W^{-1}(w)X(w)$ are affine Manin matrices. We have*

$$\mathrm{Ber}^{\boldsymbol{s}} A(w) = \mathrm{Ber}^{\boldsymbol{s}|^r} W(w) \times \mathrm{Ber}^{\boldsymbol{s}|_{m+n-r}} \left( Z(w) - Y(w)W^{-1}(w)X(w) \right). \tag{3.8}$$

*Proof.* The matrix $\big(Z(w) - Y(w)W^{-1}(w)X(w)\big)^{-1}$ is a submatrix of $A^{-1}(w)$, see (3.7). Therefore, by Lemma 3.2.4, the matrix $\big(Z(w) - Y(w)W^{-1}(w)X(w)\big)^{-1}$ is an affine Manin matrix of parity $\boldsymbol{s}|_{m+n-r}$, which implies in turn that $Z(w) - Y(w)W^{-1}(w)X(w)$ is an affine Manin matrix of parity $\boldsymbol{s}|_{m+n-r}$.

For $i = r+1, \ldots, m+n$, denote by $X(w)|_i$ the submatrix of size $r \times (i-r)$ formed by the first $(i-r)$ columns of $X(w)$, denote by $Y(w)|^i$ the submatrix of size $(i-r) \times r$ formed by the first $(i-r)$ rows of $Y(w)$, and denote by $Z(w)|_i^i$ the top left $(i-r) \times (i-r)$ submatrix of $Z(w)$. Similar to (3.7), we have

$$\begin{aligned} &\begin{pmatrix} W(w) & X(w)|_i \\ Y(w)|^i & Z(w)|_i^i \end{pmatrix}^{-1} \\ &\quad = \begin{pmatrix} W(w) & X(w)|_i \\ 0 & Z(w)|_i^i - Y(w)|^i W^{-1}(w) X(w)|_i \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -Y(w)|^i W^{-1}(w) & 1 \end{pmatrix}. \end{aligned} \tag{3.9}$$

From the definition of principal quasi-minors, we have $d_i(A(w)) = d_i(W(w))$, $i = 1, \dots, r$. From (3.9), we have

$$d_i(A(w)) = d_{i-r}(Z(w) - Y(w)W^{-1}(w)X(w)),\ i = r+1, \dots, m+n. \tag{3.10}$$

□

Now we can prove that the action of $\mathfrak{S}_{m+n}$ does not change the Berezinian.

**Proposition 3.2.6.** *Let $A(w)$ be an affine Manin matrix of parity $\boldsymbol{s}$. Let $\sigma \in \mathfrak{S}_{m+n}$. We have*

$$\mathrm{Ber}^{\boldsymbol{s}} A(w) = \mathrm{Ber}^{\sigma(\boldsymbol{s})} \sigma(A(w)). \tag{3.11}$$

*Proof.* It suffices to consider $\sigma = (i, i+1)$, $i = 1, \dots, m+n-1$. Moreover, it is sufficient to show

$$d_i^{s_i}(A(w)) d_{i+1}^{s_{i+1}}(A(w)) = d_i^{s_{i+1}}(\sigma(A(w))) d_{i+1}^{s_i}(\sigma(A(w))).$$

Without losing generality we treat the case $i = m+n-1$.

Consider the block decomposition of $A(w)$ with $r = m+n-2$. In particular, $Z(w)$ is a $2 \times 2$ matrix. By (3.10) with $i = m+n-1, m+n$,

$$d_{m+n-1}(A(w)) = d_1(Z(w) - Y(w)W^{-1}(w)X(w)),$$
$$d_{m+n}(A(w)) = d_2(Z(w) - Y(w)W^{-1}(w)X(w)),$$

and

$$d_{m+n-1}(\sigma(A(w))) = d_1(\bar{\sigma}(Z(w) - Y(w)W^{-1}(w)X(w))),$$
$$d_{m+n}(\sigma(A(w))) = d_2(\bar{\sigma}(Z(w) - Y(w)W^{-1}(w)X(w))),$$

where $\bar{\sigma} = (1, 2) \in \mathfrak{S}_2$.

Thus, the proposition is reduced to the case of $2 \times 2$ affine Manin matrices. This is proved by a direct computation. □

### 3.2.3 Affine-like Manin matrices

We extend the results on Berezinians of affine matrices to another class of matrices which we call affine-like matrices.

Denote $\mathcal{A}((w))$ the superalgebra of formal Laurent series in $w$ with coefficients in $\mathcal{A}$,

$$\mathcal{A}((w)) = \{ \sum_{r=-N}^{\infty} b_r w^r,\ N \in \mathbb{Z},\ b_r \in \mathcal{A}\}.$$

Let $A = \big(a_{i,j}\big)_{i,j=1}^{m+n}$ be a matrix of parity $\boldsymbol{s}$ with entries $a_{i,j}$ in $\mathcal{A}$. We call $A$ an *affine-like matrix of parity* $\boldsymbol{s}$ if the following two conditions are met:

- for any subset $\mathfrak{a} \subset \{1, \ldots, m+n\}$, the matrix $A_{\mathfrak{a}} = \big(a_{i,j}\big)_{i,j\in\mathfrak{a}}$ has a two sided inverse with entries in $\mathcal{A}$ and the diagonal entries of $A_{\mathfrak{a}}^{-1}$ are invertible in $\mathcal{A}$.
- there exists an injective homomorphism of superalgebras $\Phi_A : \mathcal{A} \to \mathcal{A}((w))$ such that $a_{i,j} \mapsto a_{i,j} + \delta_{i,j} w^{-1}$.

If $A$ is an affine-like matrix, then the principal quasi-minors $d_i(A)$ are well-defined. If $A$ is an affine-like matrix then $\sigma(A)$ is affine-like for any $\sigma \in \mathfrak{S}_{m+n}$.

Our definition is motivated by the following simple observation.

**Lemma 3.2.7.** *If $A$ is an affine-like matrix, then $w\Phi_A(A) = 1 + wA$ is an affine matrix. Moreover, we have $\Phi_A(A^{-1}) = (\Phi_A(A))^{-1}$ and $\Phi_A(d_i(A)) = d_i(\Phi_A(A))$, $i = 1, \ldots, m+n$.*

*If $A$ is an affine-like Manin matrix of parity $\boldsymbol{s}$, then $w\Phi_A(A)$ is an affine Manin matrix of parity $\boldsymbol{s}$ and $A^{-1}$ is also an affine-like Manin matrix of parity $s$.* $\square$

Now we can extend the definition of the Berezinian and its properties to affine-like matrices.

Let $A$ be an affine-like Manin matrix of parity $\boldsymbol{s}$. Define Berezinian $\operatorname{Ber}^{\boldsymbol{s}} A$ by formula (3.6).

**Proposition 3.2.8.** *Propositions 3.2.5 and 3.2.6 hold for affine-like Manin matrices of parity $\boldsymbol{s}$.* $\square$

### 3.3 Bethe algebra $\mathfrak{B}_{m|n}(\mathbf{\Lambda})$

In this section we discuss Bethe subalgebras $\mathfrak{B}_{m|n}(\mathbf{\Lambda}) \subset U\mathfrak{gl}_{m|n}[t]$. The Bethe subalgebras $\mathfrak{B}_{m|n}(\mathbf{\Lambda})$ are commutative and depend on parameters $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{m+n}) \in \mathbb{C}^{m+n}$. The algebra $\mathcal{B}(\boldsymbol{\lambda})$ of higher Gaudin Hamiltonians in Section 2.7 is the image of $\mathfrak{B}_{m|n}(\mathbf{0})$ acting in $L(\boldsymbol{\lambda})$.

#### 3.3.1 Algebra of pseudodifferential operators

Let $\mathcal{A}$ be a differential superalgebra with an even derivation $\partial : \mathcal{A} \to \mathcal{A}$. For $r \in \mathbb{Z}_{\geq 0}$, denote the $r$-th derivative of $a \in \mathcal{A}$ by $a_{(r)}$.

Let $\mathcal{A}((\partial^{-1}))$ be the *algebra of pseudodifferential operators.* The elements of $\mathcal{A}((\partial^{-1}))$ are Laurent series in $\partial^{-1}$ with coefficients in $\mathcal{A}$, and the product follows from the relations

$$\partial\partial^{-1} = \partial^{-1}\partial = 1, \quad \partial^r a = \sum_{s=0}^{\infty} \binom{r}{s} a_{(s)} \partial^{r-s},\ r \in \mathbb{Z},\ a \in \mathcal{A},$$

where

$$\binom{r}{s} = \frac{r(r-1)\dots(r-s+1)}{s!}.$$

Let $\mathcal{A}[\partial] \subset \mathcal{A}((\partial^{-1}))$ be the *subalgebra of differential operators*,

$$\mathcal{A}[\partial] = \{\sum_{r=0}^{M} a_r \partial^r, M \in \mathbb{Z}_{\geq 0}, a_r \in \mathcal{A}\}.$$

Consider a linear map $\Phi : \mathcal{A}((\partial^{-1})) \to \mathcal{A}[\partial]((w))$,

$$\Phi : \sum_{r=-\infty}^{N} a_r \partial^r \mapsto \sum_{r=-\infty}^{N} a_r (w^{-1} + \partial)^r, \tag{3.12}$$

where the right hand side is expanded by the rule $(w^{-1} + \partial)^r = \sum_{s=0}^{\infty} \binom{r}{s} \partial^s w^{-r+s}$.

**Lemma 3.3.1.** *The map $\Phi$ is an injective homomorphism of superalgebras.*

*Proof.* For any $r$, the coefficient of $w^r$ in the right hand side of (3.12) is a summation of finitely many terms.

The coefficient of $w^{-N}$ in $\Phi(\sum_{r=-\infty}^{N} a_r\partial^r)$ is $a_N$. Therefore, $\Phi$ is injective.

For any $a \in \mathcal{A}$, we have

$$\Phi(\partial^r a) = \Phi\left(\sum_{s=0}^{\infty}\binom{r}{s}a_{(s)}\partial^{r-s}\right) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\binom{r}{s}\binom{r-s}{t}a_{(s)}\partial^t w^{-r+s+t}.$$

Then, changing the summation indices we obtain

$$\Phi(\partial^r)\Phi(a) = \Phi(\partial^r)a = \sum_{s=0}^{\infty}\binom{r}{s}\partial^s w^{-r+s}a = \sum_{s=0}^{\infty}\sum_{t=0}^{s}\binom{r}{s}\binom{s}{t}a_{(t)}\partial^{s-t}u^{-r+s} = \Phi(\partial^r a).$$

Therefore, the map $\Phi$ is a homomorphism of superalgebras. $\square$

### 3.3.2 Bethe subalgebra

Let

$$\mathcal{A}_v^{m|n} = U\mathfrak{gl}_{m|n}[t]((v^{-1})) = \left\{ \sum_{r=-\infty}^{N} g_r v^r,\ N \in \mathbb{Z},\ g_r \in U\mathfrak{gl}_{m|n}[t] \right\}$$

be the superalgebra of Laurent series in $v^{-1}$ with coefficients in $U\mathfrak{gl}_{m|n}[t]$. The algebra $\mathcal{A}_v^{m|n}$ is a differential superalgebra with derivation $\partial_v$.

Let $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_{m+n})$ be a sequence of complex numbers. Consider the matrix $B(\mathbf{\Lambda})$ with entries in the algebra of pseudodifferential operators $\mathcal{A}_v^{m|n}((\partial_v^{-1}))$ given by

$$B(\mathbf{\Lambda}) = \Big(\delta_{i,j}(\partial_v - \Lambda_i) - (-1)^{|i|}e_{i,j}(v)\Big)_{i,j=1}^{m+n}. \tag{3.13}$$

The following lemma is checked by a straightforward computation.

**Lemma 3.3.2.** *The matrix $B(\mathbf{\Lambda})$ is an affine-like Manin matrix of standard parity with the map $\Phi_{B(\mathbf{\Lambda})} = \Phi$, see* (3.12)*.* $\square$

Consider the expansion of the Berezinian of the affine Manin matrix $w\Phi(B(\mathbf{\Lambda})) = 1 + wB(\mathbf{\Lambda})$:

$$\mathrm{Ber}\,(1 + wB(\mathbf{\Lambda})) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} B_{r,s}^{\mathbf{\Lambda}}(v)\partial_v^{r-s}w^r, \tag{3.14}$$

where $B_{r,s}^{\mathbf{\Lambda}}(v) \in \mathcal{A}_v^{m|n}$. The following fundamental result is known.

**Theorem 3.3.3** ( [MR14])**.** *The series $B^{\mathbf{\Lambda}}_{r,s}(v)$ pairwise commute,*

$$[B^{\mathbf{\Lambda}}_{r_1,s_1}(v_1), B^{\mathbf{\Lambda}}_{r_2,s_2}(v_2)] = 0,$$

*for all $r_1, s_1, r_2, s_2$.*

*The series $B^{\mathbf{\Lambda}}_{r,s}(v)$ commute with the Cartan subalgebra $\mathfrak{h} \subset U\mathfrak{gl}_{m|n}$,*

$$[B^{\mathbf{\Lambda}}_{r,s}(v), e_{i,i}] = 0,$$

*for all $r, s, i$.* □

We call the commutative subalgebra generated by coefficients of series $B^{\mathbf{\Lambda}}_{r,s}(v)$, $r, s \in \mathbb{Z}_{\geq 0}$, $s \leq r$, the *Bethe subalgebra of $U\mathfrak{gl}_{m|n}[t]$* and denote it by $\mathfrak{B}_{m|n}(\mathbf{\Lambda})$.

Alternatively, we can expand $\mathrm{Ber}\, B(\mathbf{\Lambda})$ directly

$$\mathrm{Ber}\, B(\mathbf{\Lambda}) = \sum_{r=-\infty}^{m-n} B^{\mathbf{\Lambda}}_{r}(v)\partial_v^r, \tag{3.15}$$

where $B^{\mathbf{\Lambda}}_{r}(v) \in \mathcal{A}_v^{m|n}$.

**Proposition 3.3.4.** *The coefficients of the series $B^{\mathbf{\Lambda}}_{r}(v)$, $r \in \mathbb{Z}_{\leq m-n}$ generate the Bethe algebra $\mathfrak{B}_{m|n}(\mathbf{\Lambda})$.*

*Proof.* We have

$$w^{m-n}\Phi(\mathrm{Ber}\, B(\mathbf{\Lambda})) = w^{m-n}\mathrm{Ber}\, \Phi(B(\mathbf{\Lambda})) = \mathrm{Ber}\,(1 + wB(\mathbf{\Lambda})),$$

since $\Phi$ is a homomorphism of superalgebras by Lemma 3.3.1. Moreover, $\Phi(a) = a$, for $a \in \mathcal{A}_v^{m|n}$. The proposition follows. □

## 3.4 Duality between $\mathfrak{B}_{m|n}$ and $\mathfrak{B}_k$

In this section we show the duality between $\mathfrak{B}_{m|n}(\mathbf{\Lambda})$ and $\mathfrak{B}_k(\boldsymbol{z})$ acting in the space of supersymmetric polynomials. The duality in the case of $n = 0$ is given in [MTV09b].

### 3.4.1 The duality between $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$

We start with the standard duality between $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$.

Let $\mathcal{D}$ be the superalgebra generated by $x_{i,a}, \partial_{i,a}, i = 1, \dots, m+n, a = 1, \dots, k$, with parity given by $|x_{i,a}| = |\partial_{i,a}| = |i|$ and the relations given by supercommutators

$$[x_{i,a}, x_{j,b}] = [\partial_{i,a}, \partial_{j,b}] = 0, \qquad [\partial_{i,a}, x_{j,b}] = \delta_{i,j}\delta_{a,b}, \text{ for all } i, j, a, b.$$

Let $V \subset \mathcal{D}$ be the subalgebra generated by $x_{i,a},\ i = 1, \dots, m+n,\ a = 1, \dots, k$. Then

$$V = \mathbb{C}[x_{i,a},\ i = 1, \dots, m,\ a = 1, \dots, k] \otimes \Lambda(x_{j,a},\ j = m+1, \dots, m+n,\ a = 1, \dots, k)$$

is the product of a polynomial algebra and a Grassmann algebra. We call $V$ the *space of supersymmetric polynomials* or *bosonic-fermionic space*. The algebra $\mathcal{D}$ acts on $V$ in the obvious way.

We have a homomorphism of superalgebras $\pi_{m|n} : \mathfrak{gl}_{m|n} \to \mathcal{D}$ given by

$$\pi_{m|n}(e^{[m|n]}_{i,j}) = \sum_{a=1}^{k} x_{i,a}\partial_{j,a},\ i, j = 1, \dots, m+n,$$

where we write the suffix in $e^{[m|n]}_{i,j}$ to indicate that these are elements of $\mathfrak{gl}_{m|n}$. In particular, $\mathfrak{gl}_{m|n}$ acts on $V$.

For $a \in \{1, \dots, k\}$, let $V^{(a)}_{m|n} \subset V$ be the subalgebra generated by $x_{1,a}, \dots, x_{m+n,a}$. Then we have isomorphisms of $\mathfrak{gl}_{m|n}$-modules:

$$V^{(a)}_{m|n} = \bigoplus_{d=0}^{\infty} L^{(a)}_{m|n}(d\epsilon_1), \qquad V = \bigotimes_{a=1}^{k} V^{(a)}_{m|n},$$

where $L^{(a)}_{m|n}(d\epsilon_1)$ is the the irreducible $\mathfrak{gl}_{m|n}$-module with highest weight $(d, 0, \dots, 0)$ and highest weight vector $x^d_{1,a}$. The submodule $L^{(a)}_{m|n}(d\epsilon_1)$ is spanned by all monomials of total degree $d$ in $V^{(a)}_{m|n}$.

We also have the homomorphism of superalgebras $\pi_k : \mathfrak{gl}_k \to \mathcal{D}$ given by

$$\pi_k(e^{[k]}_{a,b}) = \sum_{i=1}^{m+n} x_{i,a}\partial_{i,b}, \qquad a, b = 1, \dots, k.$$

In particular, $\mathfrak{gl}_k$ also acts on $V$.

For $i \in \{1, \dots, m+n\}$, let $V_k^{(i)} \subset V$ be the subalgebra generated by $x_{i,1}, \dots, x_{i,k}$. If $i \le m$, the space $V_k^{(i)}$ is the polynomial ring of $k$ variables, otherwise the space $V_k^{(i)}$ is the Grassmann algebra of $k$ variables. Then we have isomorphisms of $\mathfrak{gl}_k$-modules:

$$V_k^{(i)} = \bigoplus_{d=0}^{\infty} L_k^{(i)}(d\epsilon_1),\ i \le m, \qquad V_k^{(i)} = \bigoplus_{a=0}^{k} L_k^{(i)}(\omega_a),\ i > m, \qquad V = \bigotimes_{i=1}^{m+n} V_k^{(i)}.$$

Here, $L_k^{(i)}(d\epsilon_1)$, $i \le m$, is the irreducible $\mathfrak{gl}_k$-module with highest weight $(d, 0, \dots, 0)$ and highest weight vector $x_{i,1}^d$. The submodule $L_k^{(i)}(d\epsilon_1)$ is spanned by all monomials of total degree $d$ in $V_k^{(i)}$. The module $L_k^{(i)}(\omega_a)$, $i > m$, is the irreducible $\mathfrak{gl}_k$-module with highest weight $(\underbrace{1, \dots, 1}_{a}, 0 \dots, 0)$ and highest weight vector $x_{i,1} \dots x_{i,a}$. This submodule is spanned by all monomials of total degree $a$ in $V_k^{(i)}$.

In particular we have the canonical identification of weight spaces:

$$\begin{aligned}
&\left(L_{m|n}^{(1)}(\lambda_1\epsilon_1) \otimes \dots \otimes L_{m|n}^{(k)}(\lambda_k\epsilon_1)\right)[(\mu_1, \dots, \mu_{m+n})] \\
&= \Big(L_k^{(1)}(\mu_1\epsilon_1) \otimes \dots \otimes L_k^{(m)}(\mu_m\epsilon_1) \\
&\qquad\qquad \otimes L_k^{(m+1)}(\omega_{\mu_{m+1}}) \otimes \dots \otimes L_k^{(m+n)}(\omega_{\mu_{m+n}})\Big)[(\lambda_1, \dots, \lambda_k)]. \qquad (3.16)
\end{aligned}$$

These weight spaces are spanned by monomials in $V$ which have total degree $\lambda_a$ with respect to variables $x_{1,a}, \dots, x_{m+n,a}$ and total degree $\mu_i$ with respect to variables $x_{i,1}, \dots, x_{i,k}$.

The standard duality between $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ is the following well-known statement.

**Lemma 3.4.1.** *The actions of $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ on $V$ commute. We have the isomorphism of $\mathfrak{gl}_{m|n} \oplus \mathfrak{gl}_k$ modules*

$$V = \bigoplus_{\mu \in P_{m,n;k}} L_{m|n}(\mu^\natural) \otimes L_k(\mu),$$

*where $P_{m,n;k}$ is the set of all $(m|n)$-hook partition with length at most $k$.* $\square$

### 3.4.2 The duality of Bethe algebras $\mathfrak{B}_{m|n}(\boldsymbol{\Lambda})$ and $\mathfrak{B}_k(\boldsymbol{z})$

Let $\boldsymbol{z} = (z_1, \dots, z_k)$ and $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_{m+n})$ be two sequences of complex numbers. We extend actions of $\mathfrak{gl}_{m|n}$ and $\mathfrak{gl}_k$ on $V$ to the actions of the current algebras $\mathfrak{gl}_{m|n}[t]$ and $\mathfrak{gl}_k[t]$ as follows.

Let $\hat{\pi}_{m|n} : U\mathfrak{gl}_{m|n}[t] \to \mathcal{D}$ and $\hat{\pi}_k : U\mathfrak{gl}_k[t] \to \mathcal{D}$ be homomorphisms of superalgebras given by

$$\hat{\pi}_{m|n} : e^{[m|n]}_{i,j}(v) \mapsto \sum_{a=1}^{k} \frac{x_{i,a}\partial_{j,a}}{v - z_a}, \qquad i, j = 1, \dots, m+n, \tag{3.17}$$

$$\hat{\pi}_k : e^{[k]}_{a,b}(u) \mapsto \sum_{i=1}^{m+n} \frac{x_{i,a}\partial_{i,b}}{u - \Lambda_i}, \qquad a, b = 1, \dots, k. \tag{3.18}$$

Then the $\mathfrak{gl}_{m|n}$-module $V^{(a)}_{m|n}$ becomes evaluation $\mathfrak{gl}_{m|n}[t]$-module $(V^{(a)}_{m|n})_{z_a}$ and the $\mathfrak{gl}_k$-module $V^{(i)}_k$ becomes evaluation $\mathfrak{gl}_k[t]$-module $(V^{(i)}_k)_{\Lambda_i}$, see (3.3).

The actions of $\mathfrak{gl}_{m|n}[t]$ and $\mathfrak{gl}_k[t]$ on $V$ do not commute anymore. However, we prove the theorem saying that the actions of Bethe algebras $\mathfrak{B}_{m|n}(\boldsymbol{\Lambda}) \subset U\mathfrak{gl}_{m|n}[t]$ and $\mathfrak{B}_k(\boldsymbol{z}) \subset U\mathfrak{gl}_k[t]$ on $V$ coincide.

Recall that the Bethe algebra $\mathfrak{B}_{m|n}(\boldsymbol{\Lambda})$ is generated by the coefficients of the Berezinian of the matrix

$$B(\boldsymbol{\Lambda}) = \Big(\delta_{i,j}(\partial_v - \Lambda_i) - (-1)^{|i|} e^{[m|n]}_{i,j}(v)\Big)_{i,j=1}^{m+n} .$$

Similarly, the Bethe algebra $\mathfrak{B}_k(\boldsymbol{z})$ is generated by the coefficients of the column determinant of the matrix

$$G(\boldsymbol{z}) = \Big(\delta_{a,b}(\partial_u - z_a) - e^{[k]}_{a,b}(u)\Big)_{a,b=1}^{k} .$$

**Theorem 3.4.2.** *The Bethe algebras $\hat{\pi}_k \mathfrak{B}_k(\boldsymbol{z})$ and $\hat{\pi}_{m|n}\mathfrak{B}_{m|n}(\boldsymbol{\Lambda})$ coincide.*

*Moreover, we have the following identification of generators. Suppose $b_{r,s}(\boldsymbol{z}, \boldsymbol{\Lambda})$, $g_{r,s}(\boldsymbol{\Lambda}, \boldsymbol{z}) \in \mathcal{D}$ do not depend on $v$, $\partial_v$, $u$, $\partial_u$, and*

$$(v - z_1) \dots (v - z_k)\ \hat{\pi}_{m|n} \operatorname{Ber} B(\boldsymbol{\Lambda}) = \sum_{r=0}^{k} \sum_{s=-\infty}^{m-n} b_{r,s}(\boldsymbol{z}, \boldsymbol{\Lambda}) v^r \partial_v^s\ ,$$

$$\frac{(u - \Lambda_1) \dots (u - \Lambda_m)}{(u - \Lambda_{m+1}) \dots (u - \Lambda_{m+n})}\ \hat{\pi}_k \operatorname{cdet} G(\boldsymbol{z}) = \sum_{r=-\infty}^{m-n} \sum_{s=0}^{k} g_{r,s}(\boldsymbol{\Lambda}, \boldsymbol{z}) u^r \partial_u^s\ ,$$

*then*

$$b_{r,s}(\boldsymbol{z},\boldsymbol{\Lambda}) = g_{s,r}(\boldsymbol{\Lambda},\boldsymbol{z}). \tag{3.19}$$

*Proof.* The proof of this theorem is given in Sections 3.4.3 and 3.4.4. $\square$

By Theorem 3.3.3, Bethe algebras preserve weight spaces. In particular, Theorem 3.4.2 gives an identification of action of Bethe algebras $\mathfrak{B}_k(\boldsymbol{z})$ and $\mathfrak{B}_{m|n}(\boldsymbol{\Lambda})$ on the weight spaces (3.16). In particular we can now translate the known properties from the even to supersymmetric case.

Denote the right-hand side of (3.16) by $V(\boldsymbol{\lambda},\boldsymbol{\mu})$. Let $\boldsymbol{z}=(z_1,\dots,z_k),\ z_a\neq z_b$ and $\boldsymbol{\Lambda}=(\Lambda_1,\dots,\Lambda_{m+n}),\ \Lambda_i\neq\Lambda_j$. Denote $\mathfrak{B}_{m|n}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\mu})$ the image of the Bethe algebra $\mathfrak{B}_{m|n}(\Lambda)$ in $\mathrm{End}(V(\boldsymbol{\lambda},\boldsymbol{\mu}))$ with evaluation parameters $z_1,\dots,z_k$.

**Corollary 3.4.3.** *We have:*

1. *The algebra* $\mathfrak{B}_{m|n}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\mu})$ *is a Frobenius algebra of dimension* $\dim(V(\boldsymbol{\lambda},\boldsymbol{\mu}))$.
2. *The space* $V(\boldsymbol{\lambda},\boldsymbol{\mu})$ *is a regular representation of* $\mathfrak{B}_{m|n}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\mu})$.
3. *All eigenspaces of* $\mathfrak{B}_{m|n}(\boldsymbol{z},\boldsymbol{\lambda},\boldsymbol{\mu})$ *in* $V(\boldsymbol{\lambda},\boldsymbol{\mu})$ *are one dimensional.*

*Proof.* The corollary follows from the corresponding statements for $\mathfrak{B}_k(\boldsymbol{z})$, see Theorem 7.1 in [MTV08b]. $\square$

### 3.4.3 An identity of Capelli type

In this section we give an explicit expansion of $\hat{\pi}_k\mathrm{cdet}\,G(\boldsymbol{z})$.

Let $\mathcal{D}_u=\mathcal{D}((u^{-1}))$ be the superalgebra of Laurent series in $u^{-1}$ with values in $\mathcal{D}$. The algebra $\mathcal{D}_u$ has a derivation $\partial_u$ and $\mathcal{D}_u((\partial_u^{-1}))$ is the superalgebra of pseudodifferential operators.

Let $G(\boldsymbol{\Lambda},\boldsymbol{z})$ be a $k\times k$ matrix with entries in $\mathcal{D}_u[\partial_u]\subset\mathcal{D}_u((\partial_u^{-1}))$ given by

$$G(\boldsymbol{\Lambda},\boldsymbol{z}) = \hat{\pi}_k G(\boldsymbol{z}) = \left(\delta_{a,b}(\partial_u - z_a) - \sum_{i=1}^{m+n}\frac{x_{i,a}\partial_{i,b}}{u-\Lambda_i}\right)_{a,b=1}^{k}.$$

The matrix $G(\boldsymbol{\Lambda}, \boldsymbol{z})$ is a Manin matrix of parity $(1, \dots, 1)$. We want to expand $\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z})$. In order to do that, we introduce some notation.

The superalgebra $\mathcal{D}_u((\partial_u^{-1}))$ is topologically generated by $x_{i,a}, \partial_{i,a}, u^{\pm 1}, \partial_u^{\pm 1}$. Define an ordering on the generators such that $x_{i,a} < \partial_{j,b} < u^{\pm 1} < \partial_u^{\pm 1}$, $i, j = 1, \dots, m+n$, $a, b = 1, \dots, k$, and $x_{i,a} < x_{j,b}$, $\partial_{i,a} < \partial_{j,b}$, if either $a < b$ or $a = b$ and $i < j$.

Let $m$ be a monomial in the generators. Denote by $:m:$ the new monomial where all participating generators are multiplied in the increasing order and the sign is changed by the usual supercommutativity rule. For example,

$$:\partial_u^{-1} u^{-1} \partial_{1,1} x_{1,1} \partial_{m+1,2} x_{m+1,1} := -x_{1,1} x_{m+1,1} \partial_{1,1} \partial_{m+1,2} u^{-1} \partial_u^{-1}.$$

We call $:m:$ the *normal ordered monomial.*

Let

$$F^i_{a,b} = \begin{cases} -x_{i,a}\partial_{i,b}(u - \Lambda_i)^{-1}, & i = 1, \dots, m+n,\ a, b = 1, \dots, k, \\ \partial_u - z_a, & a = 1, \dots, k,\ b = a,\ i = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.20}$$

Note that in all cases $F^i_{a,b}$ is even and normal ordered. Every term will be given as a product of $F^i_{a,b}$ in the expansion of $\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z})$.

Denote by $|S|$ the cardinality of a set $S$.

Let $\mathfrak{a} = \{1 \le a_1 < \cdots < a_l \le k\}$ be a subset of $\{1, \dots, k\}$, where $l = \#\mathfrak{a}$. Let $J(\mathfrak{a})$ be the set of function $j : \{1, \dots, k\} \to \{0, 1, \dots, m+n\}$ such that $j(a) = 0$ if and only if $a \notin \mathfrak{a}$ and such that for any $i \in \{1, \dots, m\}$, $\#j^{-1}(i) \le 1$.

We have

$$|J(\mathfrak{a})| = \sum_{s=0}^{\#\mathfrak{a}} \binom{l}{s} \binom{m}{s}\ s!\ n^{l-s}.$$

For $j_1, j_2 \in J(\mathfrak{a})$, we write $j_1 \sim j_2$ if $\#j_1^{-1}(i) = \#j_2^{-1}(i)$ for all $i$. Clearly, $\sim$ is an equivalence relation in $J(\mathfrak{a})$. The cardinality of the equivalence class of $j \in J(\mathfrak{a})$ is $l!/(\prod_{i=m+1}^{m+n}(\#j^{-1}(i))!)$.

For $j \in J(\mathfrak{a})$, $j^{-1}(\{1, \dots, m+n\}) = \mathfrak{a}$. Therefore the symmetric group $\mathfrak{S}_{\#\mathfrak{a}}$ acts on the preimage $j^{-1}(\{1, \dots, m+n\})$. Given $j_1 \sim j_2$, there exists a unique permutation $\sigma_{j_1,j_2} \in \mathfrak{S}_{\#\mathfrak{a}}$ such that $\sigma_{j_1,j_2} : j_2^{-1}(i) \to j_1^{-1}(i)$ is an increasing function for all $i = 1, \dots, m+n$. Note that $j_1 \circ \sigma_{j_1,j_2} = j_2$ on $\mathfrak{a}$.

We also define

$$\operatorname{sgn}(j_1, j_2) = (-1)^{\mathcal{N}},$$
$$\mathcal{N} = \#\{(s,s') \mid 1 \le s < s' \le l,\ \sigma_{j_1,j_2}(s) < \sigma_{j_1,j_2}(s'),\ j_2(a_s) > m,\ j_2(a_{s'}) > m\}.$$

Given $j_1, j_2 \in J(\mathfrak{a})$, $j_1 \sim j_2$, define the sign

$$c(j_1, j_2) = \operatorname{sgn}(j_1, j_2)\ \operatorname{sgn}(\sigma_{j_1,j_2})\,(-1)^l.$$

For $j \in J(\mathfrak{a})$, set

$$\boldsymbol{x}_j = x_{j(a_1),a_1} x_{j(a_2),a_2} \dots x_{j(a_l),a_l}, \qquad \boldsymbol{\partial}_j = \partial_{j(a_1),a_1} \partial_{j(a_2),a_2} \dots \partial_{j(a_l),a_l}.$$

Note that monomials $\boldsymbol{x}_j$ and $\boldsymbol{\partial}_j$ are normal ordered.

Now we are ready to state the main result of this section.

**Proposition 3.4.4.** *The normal ordered expansion of the column determinant of $G(\boldsymbol{\Lambda}, \boldsymbol{z})$ is given by*

$$\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z}) = \sum_{\mathfrak{a} \subset \{1,\dots,k\}} \sum_{\substack{j_1, j_2 \in J(\mathfrak{a}) \\ j_1 \sim j_2}} c(j_1, j_2) \prod_{i=m+1}^{m+n} (\# j_2^{-1}(i))!\ \boldsymbol{x}_{j_1} \boldsymbol{\partial}_{j_2} \tag{3.21}$$
$$\times \prod_{i \in j_2(\mathfrak{a})} (u - \Lambda_i)^{-1} \prod_{a \notin \mathfrak{a}} (\partial_u - z_a).$$

*Proof.* We first assume all generators are supercommutative and show equation (3.21) holds. Then we show that the additional terms created by non-trivial supercommutation relations cancel in pairs and do not contribute to the expansion.

Recall even elements $F^i_{a,b}$ given in (3.20). We have the expansion

$$\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z}) = \sum_{\sigma \in \mathfrak{S}_k} \sum_{i_1, \dots, i_k = 0}^{m+n} \operatorname{sgn}(\sigma) F^{i_1}_{\sigma(1),1} \dots F^{i_k}_{\sigma(k),k}.$$

Now we want to normal order it.

Assume the supercommutators are all zero, $[u, \partial_u] = [x_{i,a}, \partial_{i,a}] = 0$.

For a nonzero term $\operatorname{sgn}(\sigma) F^{i_1}_{\sigma(1),1} \dots F^{i_k}_{\sigma(k),k}$, let $\mathfrak{a} = \{a, i_a \neq 0\} \subset \{1, \dots, k\}$. We write the set $\mathfrak{a} = \{a_1 < \dots < a_l\}$. Then we can rewrite our sum as follows

$$\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z}) = \sum_{l=0}^{k} \sum_{1 \le a_1 < \dots < a_l \le k} \sum_{\sigma \in \mathfrak{S}_l} \sum_{i_1, \dots, i_l = 1}^{m+n} \operatorname{sgn}(\sigma) F^{i_1}_{a_{\sigma(1)}, a_1} \dots F^{i_l}_{a_{\sigma(l)}, a_l} \prod_{a,\ a \neq a_1, \dots, a_l} (\partial - z_a).$$

We normal order the term corresponding to $a_1 < \dots < a_l, \sigma \in \mathfrak{S}_l, i_1, \dots, i_l$. Let $i_{\bar{1}}$ be the number of upper indices greater than $m$, $i_{\bar{1}} = \#\{i_s > m, s = 1, \dots, l\}$. We have

$$F^{i_1}_{a_{\sigma(1)}, a_1} \dots F^{i_l}_{a_{\sigma(l)}, a_l} = (-1)^{l + i_{\bar{1}}(i_{\bar{1}} - 1)/2} \, \frac{x_{i_1, a_{\sigma(1)}} \dots x_{i_l, a_{\sigma(l)}} \partial_{i_1, a_1} \dots \partial_{i_l, a_l}}{(u - \Lambda_{i_1}) \dots (u - \Lambda_{i_l})}.$$

Note that monomial $\partial_{i_1, a_1} \dots \partial_{i_l, a_l}$ is normal ordered. We now observe some simplifications before ordering $x_{i_1, a_{\sigma(1)}} \dots x_{i_l, a_{\sigma(l)}}$.

Consider a term corresponding to $a_1 < \dots < a_l$, $\sigma$, $i_1, \dots, i_l$.

Fix an $i \in \{1, \dots, m+n\}$. Let $\mathfrak{b} = \{s,\ i_s = i\} \subset \{1, \dots, l\}$. If $\#\mathfrak{b} = r > 1$, then we have $r!$ terms which correspond to the same $a_1 < \dots < a_l$, $i_1, \dots, i_l$, and permutations of the form $\tau\sigma$, where $\tau \in \mathfrak{S}_l$ permutes elements of $a_s,\ s \in \mathfrak{b}$, and leaves others preserved.

If $i \le m$, then after normal ordering all these $r!$ terms will produce the same monomial with different signs and cancel out. On the other hand, if $i > m$, then after normal ordering, all these $r!$ terms will produce the same monomial with the same sign and therefore can be combined.

Therefore the summands in the expansion can be reparametrized by $\mathfrak{a} \subset \{1, \dots, k\}$ and $j_1, j_2 \in J(\mathfrak{a})$, $j_1 \sim j_2$. The correspondence is given by

$$\mathfrak{a} = \{a_1 < \dots < a_l\}, \qquad j_1(a_{\sigma(s)}) = j_2(a_s) = i_s, \qquad s = 1, \dots, l. \tag{3.22}$$

Note that $\sigma$ is not recovered from $\mathfrak{a}, j_1, j_2$. In fact, we have $(\#\{s,\ i_s = m+1\})! \dots (\#\{s,\ i_s = m+n\})!$ choices for $\sigma$, which all correspond to equal summands. We choose one permutation, namely $\sigma_{j_1, j_2}$, and multiply the corresponding term by $(\# j_1^{-1}(m+1))! \dots (\# j_1^{-1}(m+n))!$.

So we have

$$\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z}) = \sum_{\mathfrak{a}\subset\{1,\dots,k\}} \sum_{\substack{j_1,j_2\in J(\mathfrak{a})\\ j_1\sim j_2}} \operatorname{sgn}(\sigma_{j_1,j_2})\, (-1)^{\#_{\bar 1} j_1(\#_{\bar 1} j_1 - 1)/2 + \#\mathfrak{a}} \prod_{i=m+1}^{m+n} |j_2^{-1}(i)|!$$
$$\times \frac{x_{j_2(a_1),a_{\sigma_{j_1,j_2}(1)}} \dots x_{j_2(a_{\#\mathfrak{a}}),a_{\sigma_{j_1,j_2}(\#\mathfrak{a})}} \partial_{j_2(a_1),a_1} \dots \partial_{j_2(a_{\#\mathfrak{a}}),a_{\#\mathfrak{a}}}}{(u-\Lambda_{i_1})\dots(u-\Lambda_{i_{\#\mathfrak{a}}})} \prod_{a\notin\mathfrak{a}} (\partial_u - z_a)\ ,$$

where we denoted by $\#_{\bar 1} j_1$ the cardinality of $j_1^{-1}(\{m+1,\dots,m+n\})$. We rewrite the first indices of $x_{i,a}$ variables through $j_1$, using (3.22), and then we normal order them, getting the additional sign and arriving at (3.21).

Now we proceed to the non-commutative setting. We call the additional terms "quantum corrections" and show that they cancel in pairs.

We normal order monomials from right to left. The induction is based on the number of $F^i_{a,b}$ on the right which have been normal ordered. Namely we prove

$$\operatorname{cdet} G(\boldsymbol{\Lambda}, \boldsymbol{z}) = \sum_{\sigma\in\mathfrak{S}_k} \sum_{i_1,\dots,i_k=0}^{m+n} \operatorname{sgn}(\sigma)\, F^{i_1}_{\sigma(1),1} \cdots : F^{i_a}_{\sigma(a),a} \dots F^{i_k}_{\sigma(k),k} : \tag{3.23}$$

by induction on $a$.

The basis $a = k$ of induction is a tautology. We show the step of induction from $a = a_0$ to $a = a_0 - 1$.

We use the following simple formula:

$$F^{i_1}_{a_1,b_1} F^{i_2}_{a_2,b_2} =: F^{i_1}_{a_1,b_1} F^{i_2}_{a_2,b_2} : - \begin{cases} F^{i_2}_{a_2,b_2}(u-\Lambda_{i_2})^{-1}, & i_1 = 0,\ a_1 = b_1,\ i_2 \neq 0,\\ \delta_{i_1,i_2}\delta_{b_1,a_2} F^{i_1}_{a_1,b_2}(u-\Lambda_{i_2})^{-1}, & i_1 \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Consider a nonzero term $\operatorname{sgn}(\sigma) F^{i_{a_0-1}}_{\sigma(a_0-1),a_0-1} : F^{i_{a_0}}_{\sigma(a_0),a_0} \dots F^{i_k}_{\sigma(k),k} :$. Then $i_a = 0$ implies $\sigma(a) = a$.

We have two cases: $i_{a_0-1} \neq 0$ and $i_{a_0-1} = 0$.

Let $i_{a_0-1} \neq 0$. Then $F^{i_{a_0-1}}_{\sigma(a_0-1),a_0-1}$ creates at most one quantum correction. Namely, if there exists $b \in \{a_0, \ldots, k\}$ such that $i_{a_0-1} = i_b$, and $a_0 - 1 = \sigma(b)$, then such $b$ is unique and

$$\begin{aligned}\operatorname{sgn}(\sigma)\, F^{i_{a_0-1}}_{\sigma(a_0-1),a_0-1} :& F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_b}_{\sigma(b),b} \ldots F^{i_k}_{\sigma(k),k} : \\ &= \operatorname{sgn}(\sigma) : F^{i_{a_0-1}}_{\sigma(a_0-1),a_0-1} F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_b}_{\sigma(b),b} \ldots F^{i_k}_{\sigma(k),k} : \\ &\quad - \operatorname{sgn}(\sigma) \frac{1}{u - \Lambda_{i_b}} : F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_b}_{\sigma(a_0-1),b} \ldots F^{i_k}_{\sigma(k),k} : . \end{aligned} \tag{3.24}$$

If such $b$ does not exist then there is no quantum correction (the second term on the right hand side is absent).

Let $i_{a_0-1} = 0$. Then we possibly have many quantum corrections:

$$\begin{aligned}\operatorname{sgn}(\sigma) F^{0}_{a_0-1,a_0-1} : F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_k}_{\sigma(k),k} :&= \operatorname{sgn}(\sigma) : F^{0}_{a_0-1,a_0-1} F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_k}_{\sigma(k),k} : \\ &- \operatorname{sgn}(\sigma) \sum_{\substack{a=a_0 \\ i_a \neq 0}}^{k} \frac{1}{u - \Lambda_{i_a}} : F^{i_{a_0}}_{\sigma(a_0),a_0} \ldots F^{i_k}_{\sigma(k),k} : . \end{aligned} \tag{3.25}$$

The quantum correction in (3.24) corresponding to the term labeled by $\sigma, i_1, \ldots, i_k$ in (3.23) cancels with the quantum correction corresponding to $a = b$ summand in (3.25) applied to the term in (3.23) labeled by $\sigma(a_0-1, b)$, $\{i_1, \ldots, i_{a_0-2}, 0, i_{a_0}, \ldots, i_k\}$. This proves the induction step.

The statement of induction with $a = 1$ proves the proposition. □

### 3.4.4 Another identity of Capelli type

Let $\mathcal{D}_v = \mathcal{D}((v^{-1}))$, be the superalgebra of Laurent series in $v^{-1}$ with coefficients in $\mathcal{D}$. The superalgebra $\mathcal{D}_v$ has a derivation $\partial_v$ and we consider the superalgebra of pseudodifferential operators $\mathcal{D}_v((\partial_v^{-1}))$.

Let $B(\boldsymbol{z}, \boldsymbol{\Lambda})$ be a $(m+n) \times (m+n)$ matrix with entries in $\mathcal{D}_v[\partial_v] \subset \mathcal{D}_v((\partial_v^{-1}))$ given by

$$B(\boldsymbol{z}, \boldsymbol{\Lambda}) = \hat{\pi}_{m|n} B(\boldsymbol{\Lambda}) = \left( \delta_{ij}(\partial_v - \Lambda_i) - \sum_{a=1}^{k} \frac{(-1)^{|i|} x_{i,a} \partial_{j,a}}{v - z_a} \right)_{i,j=1}^{m+n} .$$

The matrix $B(\boldsymbol{z},\boldsymbol{\Lambda})$ is a Manin matrix of standard parity.

Let $\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})$ be a $(m+n+k)\times(m+n+k)$ matrix given by

$$\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}) = \begin{pmatrix} v-Z & D^t \\ SX & \partial_v - \Lambda \end{pmatrix} \tag{3.26}$$

where the submatrices are $Z = \operatorname{diag}(z_1,\dots,z_k)$, $\Lambda = \operatorname{diag}(\Lambda_1,\dots,\Lambda_{m+n})$, $D = \big(\partial_{i,a}\big)_{i=1,\dots,m+n}^{a=1,\dots,k}$, $X = \big(x_{i,a}\big)_{i=1,\dots,m+n}^{a=1,\dots,k}$, $S = \operatorname{diag}(\underbrace{1,\dots,1}_{m},\underbrace{-1,\dots,-1}_{n})$, and $D^t$ is the transpose of $D$. In particular, $SX = \big((-1)^{|i|}x_{i,a}\big)_{i=1,\dots,m+n}^{a=1,\dots,k}$.

Let $\mathcal{D}_v((\partial_v^{-1}))((w))$ be the superalgebra of Laurent series in $w$ with coefficients in $\mathcal{D}_v((\partial_v^{-1}))$. Define the homomorphism of superalgebras

$$\begin{aligned} &\hat{\Phi} : \mathcal{D}_v((\partial_v^{-1})) \to \mathcal{D}_v((\partial_v^{-1}))((w)), \\ &v \mapsto v + w^{-1},\ \partial_v \mapsto \partial_v + w^{-1},\ \text{and } g \mapsto g,\ g \in \mathcal{D}. \end{aligned} \tag{3.27}$$

Note that in our convention we first expand in positive powers of $w$ then in powers of $\partial_v^{-1}$ and then in powers of $v^{-1}$, cf. (3.12). As a result, if a series is in the image of $\hat{\Phi}$, then it belongs to $\mathcal{D}[v,\partial_v]((w))$, in other words, a coefficient of $w^k$ is always a polynomial in $\partial_v$ and $v$ for any $k \in \mathbb{Z}$.

The map $\hat{\Phi}$ is a composition of map $\Phi$, see (3.12) and of the shift homomorphism $v \to v + w^{-1}$. Therefore, $\hat{\Phi}$ is a well-defined injective homomorphism.

Then, it is straightforward to check the following statement.

**Lemma 3.4.5.** *The matrix $\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})$ is an affine-like Manin matrix of parity $\hat{\boldsymbol{s}}_0 = (\underbrace{1,\dots,1}_{k+m},\underbrace{-1,\dots,-1}_{n})$ with the map $\hat{\Phi}$.* □

We would like to expand and normal order the Berezinian of $B(z,\boldsymbol{\Lambda})$. However, it is sufficient to expand and normal order Berezinian of $\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})$. Indeed, by Proposition 3.2.8, we have

$$\operatorname{Ber}^{\hat{\boldsymbol{s}}_0} \hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}) = (v-z_1)\dots(v-z_k)\operatorname{Ber} B(\boldsymbol{z},\boldsymbol{\Lambda}), \tag{3.28}$$

cf. Corollary 2.2 of [MTV09b].

The expansion of the Berezinian of $\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})$ is given by the following proposition.

**Proposition 3.4.6.** *We have*

$$\operatorname{Ber}^{\hat{\boldsymbol{s}}_0} \hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}) = \sum_{\mathfrak{a}\subset\{1,\dots,k\}} \sum_{\substack{j_1,j_2\in J(\mathfrak{a})\\ j_1\sim j_2}} c(j_1,j_2) \prod_{i=m+1}^{m+n} (\# j_2^{-1}(i))!\, \boldsymbol{x}_{j_1}\boldsymbol{\partial}_{j_2} \tag{3.29}$$

$$\times \prod_{a\notin\mathfrak{a}} (v-z_a) \prod_{i\in j_1(\mathfrak{a})} (\partial_v-\Lambda_i)^{-1} \frac{(\partial_v-\Lambda_1)\dots(\partial_v-\Lambda_m)}{(\partial_v-\Lambda_{m+1})\dots(\partial_v-\Lambda_{m+n})} \ .$$

*Proof.* Let $\sigma\in\mathfrak{S}_{m+n+k}$ be defined by $\sigma^{-1}(a)=m+n+a$, $a=1,\dots,k$, and $\sigma^{-1}(k+i)=i$, $i=1,\dots,m+n$. Then

$$\sigma(\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})) = \begin{pmatrix} \partial_v-\Lambda & SX \\ D^t & v-Z \end{pmatrix}.$$

The matrix $\sigma(\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}))$ is an affine-like Manin matrix of parity

$$\boldsymbol{s} = (\underbrace{1,\dots,1}_{m},\underbrace{-1,\dots,-1}_{n},\underbrace{1,\dots,1}_{k})$$

with the map $\hat{\Phi}$. By Proposition 3.2.8, we have

$$\operatorname{Ber}^{\hat{\boldsymbol{s}}_0}\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}) = \operatorname{Ber}^{\boldsymbol{s}}\sigma(\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})).$$

Using Proposition 3.2.8 once again (we use $r=m+n$), we further see

$$\operatorname{Ber}^{\boldsymbol{s}}\sigma(\hat{B}(\boldsymbol{z},\boldsymbol{\Lambda})) = \frac{(\partial_v-\Lambda_1)\dots(\partial_v-\Lambda_m)}{(\partial_v-\Lambda_{m+1})\dots(\partial_v-\Lambda_{m+n})}\ \operatorname{cdet} B'(\boldsymbol{z},\boldsymbol{\Lambda})\ ,$$

where $B'(\boldsymbol{z},\boldsymbol{\Lambda})$ is an even matrix given by

$$B'(\boldsymbol{z},\boldsymbol{\Lambda}) = \Big(\delta_{a,b}(v-z_a) - \sum_{i=1}^{m+n}\frac{(-1)^{|i|}\partial_{i,a}x_{i,b}}{\partial_v-\Lambda_i}\Big)_{a,b=1}^{k}\ .$$

Next we move the factor $\frac{(\partial_v-\Lambda_1)\dots(\partial_v-\Lambda_m)}{(\partial_v-\Lambda_{m+1})\dots(\partial_v-\Lambda_{m+n})}$ to the right of the column determinant. Note that for $i\in\{1,\dots,m\}$, $a\in\{1,\dots,k\}$, we have

$$(\partial_v-\Lambda_i)(v-z_a-\frac{\partial_{i,a}x_{i,a}}{\partial_v-\Lambda_i}) = (v-z_a-\frac{x_{i,a}\partial_{i,a}}{\partial_v-\Lambda_i})(\partial_v-\Lambda_i)\ .$$

Similarly, for $i\in\{m+1,\dots,m+n\}$, $a\in\{1,\dots,k\}$,

$$\frac{1}{(\partial_v-\Lambda_i)}(v-z_a+\frac{\partial_{i,a}x_{i,a}}{\partial_v-\Lambda_i}) = (v-z_a-\frac{x_{i,a}\partial_{i,a}}{\partial_v-\Lambda_i})\frac{1}{(\partial_v-\Lambda_i)}\ .$$

Therefore, we have

$$\mathrm{Ber}^{\hat{\boldsymbol{s}}_0}\ \hat{B}(\boldsymbol{z},\boldsymbol{\Lambda}) = \mathrm{cdet}\left(\delta_{a,b}(v-z_a) - \sum_{i=1}^{m+n} \frac{x_{i,b}\partial_{i,a}}{\partial_v - \Lambda_i}\right)_{a,b=1}^{k} \times \frac{(\partial_v-\Lambda_1)\dots(\partial_v-\Lambda_m)}{(\partial_v-\Lambda_{m+1})\dots(\partial_v-\Lambda_{m+n})}\ .$$

Finally, the expansion of the above column determinant is done by a computation similar to the one in Proposition 3.4.4. □

Theorem 3.4.2 follows from Propositions 3.4.4 and 3.4.6.

We remark that the $k\times k$ column determinant $\mathrm{cdet}\, G(\boldsymbol{\Lambda},\boldsymbol{z})$ in Proposition 3.4.4 is also essentially a Berezinian of an $(m+n+k)\times(m+n+k)$ matrix. Namely, let $\hat{G}(\boldsymbol{\Lambda},\boldsymbol{z})$ be a $(m+n+k)\times(m+n+k)$ matrix given by

$$\hat{G}(\boldsymbol{\Lambda},\boldsymbol{z}) = \begin{pmatrix} u-\Lambda & D \\ X^t & \partial_u - Z \end{pmatrix}.$$

Then $\hat{G}(\boldsymbol{\Lambda},\boldsymbol{z})$ is an affine-like Manin matrix of parity

$$\boldsymbol{s} = (\underbrace{1,\dots,1}_{m},\underbrace{-1,\dots,-1}_{n},\underbrace{1,\dots,1}_{k})$$

with the same homomorphism of superalgebras $\hat{\Phi}$, see (3.27). By Proposition 3.2.8, the Berezinian of parity $\boldsymbol{s}$ of $\hat{G}(\boldsymbol{\Lambda},\boldsymbol{z})$ is given by

$$\mathrm{Ber}^{\boldsymbol{s}}\, \hat{G}(\boldsymbol{\Lambda},\boldsymbol{z}) = \frac{(u-\Lambda_1)\dots(u-\Lambda_m)}{(u-\Lambda_{m+1})\dots(u-\Lambda_{m+n})}\ \mathrm{cdet}\, G(\boldsymbol{\Lambda},\boldsymbol{z}). \tag{3.30}$$

# 4. BETHE ANSATZ EQUATION AND RATIONAL DIFFERENCE OPERATORS

## 4.1 Rational difference operators and their factorizations

We study properties of ratios of difference operators, following the treatment of ratios of differential operators in [CDSK12]. We also describe the relation between the complete factorizations and the superflag varieties.

### 4.1.1 Rational difference operators

Fix a non-zero number $h \in \mathbb{C}^\times$. Let $\mathbb{K}$ be the field of complex valued rational functions $\mathbb{K} = \mathbb{C}(x)$, with an automorphism $\tau : \mathbb{K} \to \mathbb{K}$, $(\tau f)(x) \mapsto f(x-h)$.

Consider the algebra $\mathbb{K}[\tau]$ of *difference operators* where the shift operator $\tau$ satisfies

$$\tau \cdot f = f(x-h) \cdot \tau$$

for all $f \in \mathbb{K}$. By definition, an element $\mathcal{D} \in \mathbb{K}[\tau]$ has the form

$$\mathcal{D} = \sum_{j=0}^{r} a_j \tau^j, \quad a_j \in \mathbb{K}, \quad r \in \mathbb{Z}_{\geqslant 0}. \tag{4.1}$$

The difference operator $\mathcal{D}$ has *order* $r$, ord $\mathcal{D} = r$, if $a_r \neq 0$. One says that $\mathcal{D}$ is *monic* if $a_r = 1$. We call $a_0$ the *constant term* of $\mathcal{D}$.

Let $\mathcal{D} \in \mathbb{K}[\tau]$ be a difference operator of order $r$ as in (4.1). We say a difference operator $\mathcal{D}$ of order $r$ is *completely factorable over* $\mathbb{K}$ if there exist $f_i \in \mathbb{K}$, $i = 1, \dots, r$, such that $\mathcal{D} = a_r\, d_1 \dots d_r$, where $d_i = \tau - f_i$. We focus on completely factorable difference operators with non-zero constant terms $a_0$. In this case, we consider factorizations of the form $\mathcal{D} = a_0 d_1 \cdots d_r$, where $d_i = 1 - \tilde{f}_i \tau$, $\tilde{f}_i \in \mathbb{K}$, $i = 1, \dots, r$.

Let $\ker \mathcal{D} = \{u \in \mathbb{K} \mid \mathcal{D}u = 0\}$ be the kernel of $\mathcal{D}$. It is clear that if $\dim(\ker \mathcal{D}) = \operatorname{ord} \mathcal{D}$, then $\mathcal{D}$ is completely factorable over $\mathbb{K}$.

Let $\mathbb{K}(\tau)$ be the division ring generated by $\mathbb{K}[\tau]$. The division ring $\mathbb{K}(\tau)$ is called the *ring of rational difference operators*. Elements in $\mathbb{K}(\tau)$ are called *rational difference operators*.

A *fractional factorization* of a rational difference operator $\mathcal{R}$ is the equality $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$, where $\mathcal{D}_{\bar{0}}, \mathcal{D}_{\bar{1}} \in \mathbb{K}[\tau]$. A fractional factorization $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$ is called *minimal* if $\mathcal{D}_{\bar{1}}$ is monic and has the minimal possible order.

**Proposition 4.1.1.** *Any rational difference operator $\mathcal{R} \in \mathbb{K}(\tau)$ has the following properties.*

1. *There exists a unique minimal fractional factorization of $\mathcal{R}$.*
2. *Let $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$ be the minimal fractional factorization. If $\mathcal{R} = \widetilde{\mathcal{D}}_{\bar{0}}\widetilde{\mathcal{D}}_{\bar{1}}^{-1}$ is a fractional factorization, then there exists $\mathcal{D} \in \mathbb{K}[\tau]$ such that $\widetilde{\mathcal{D}}_{\bar{0}} = \mathcal{D}_{\bar{0}}\mathcal{D}$ and $\widetilde{\mathcal{D}}_{\bar{1}} = \mathcal{D}_{\bar{1}}\mathcal{D}$.*
3. *Let $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$ be a fractional factorization such that* $\dim(\ker \mathcal{D}_{\bar{0}}) = \operatorname{ord} \mathcal{D}_{\bar{0}}$ *and* $\dim(\ker \mathcal{D}_{\bar{1}}) = \operatorname{ord} \mathcal{D}_{\bar{1}}$. *Then $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$ is the minimal fractional factorization of $\mathcal{R}$ if and only if* $\ker \mathcal{D}_{\bar{0}} \cap \ker \mathcal{D}_{\bar{1}} = 0$.

*Proof.* We have the analogs of [CDSK12, Proposition 2.1, Corollary 2.2, Lemma 3.2] for difference operators. Namely, the algebra $\mathbb{K}[\tau]$ is right Euclidean, therefore $\mathbb{K}[\tau]$ satisfies the right Ore condition and every right ideal of $\mathbb{K}[\tau]$ is principal. This statement is proved similarly as [CDSK12, Proposition 3.4]. □

We call $\mathcal{R}$ an $(m|n)$*-rational difference operator* if in the minimal fractional factorization $\mathcal{R} = \mathcal{D}_{\bar{0}}\mathcal{D}_{\bar{1}}^{-1}$, $\mathcal{D}_{\bar{0}}, \mathcal{D}_{\bar{1}}$ are completely factorable over $\mathbb{K}$, and $\operatorname{ord}(\mathcal{D}_{\bar{0}}) = m$, $\operatorname{ord}(\mathcal{D}_{\bar{1}}) = n$, and $\mathcal{D}_{\bar{0}}, \mathcal{D}_{\bar{1}}$ have the same non-zero constant term.

Let $\mathcal{R}$ be an $(m|n)$-rational difference operator. Note that $\mathcal{R}$ can also be written in the form $\mathcal{R} = \widetilde{\mathcal{D}}_{\bar{1}}^{-1}\widetilde{\mathcal{D}}_{\bar{0}}$, where $\widetilde{\mathcal{D}}_{\bar{1}}, \widetilde{\mathcal{D}}_{\bar{0}} \in \mathbb{K}[\tau]$, $\operatorname{ord}(\widetilde{\mathcal{D}}_{\bar{0}}) = m$, and $\operatorname{ord}(\widetilde{\mathcal{D}}_{\bar{1}}) = n$. More generally, let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. Then we call the form $\mathcal{R} = d_1^{s_1} \dots d_{m+n}^{s_{m+n}}$, where $d_i = 1 - f_i\tau$, $f_i \in \mathbb{K}$, $i = 1, \dots, m+n$, a *complete factorization with the parity sequence* $\boldsymbol{s}$. Let $\mathfrak{F}^{\boldsymbol{s}}(\mathcal{R})$ be the set of all complete factorizations of $\mathcal{R}$ with parity sequence $\boldsymbol{s}$ and $\mathfrak{F}(\mathcal{R}) = \bigsqcup_{\boldsymbol{s} \in S_{m|n}} \mathfrak{F}^{\boldsymbol{s}}(\mathcal{R})$ the set of all complete factorizations of $\mathcal{R}$.

Throughout the paper, we use the following useful notation: for any $i \in \mathbb{Z}$ and $f \in \mathbb{K}$,

$$f[i] := \tau^i(f) = f(x - ih).$$

Define the *discrete logarithmic derivative* of a function $f(x)$ by $\ln'(f) = f/f[1]$.

Consider two $(1|1)$-rational difference operators

$$\mathcal{R}_1 = (1 - a\,\tau)(1 - b\,\tau)^{-1} \quad \text{and} \quad \mathcal{R}_2 = (1 - c\,\tau)^{-1}(1 - d\,\tau),$$

where $a, b, c, d \in \mathbb{K}$, $a \neq b$, and $c \neq d$.

**Lemma 4.1.2.** *We have $\mathcal{R}_1 = \mathcal{R}_2$ if and only if*

$$\begin{cases} c = b[1]\ln'(a-b), \\ d = a[1]\ln'(a-b), \end{cases} \quad \textit{or equivalently} \quad \begin{cases} a[1] = d/\ln'(c-d), \\ b[1] = c/\ln'(c-d). \end{cases} \qquad \square$$

Let $\mathcal{R}$ be an $(m|n)$-rational difference operator with a complete factorization $\mathcal{R} = d_1^{s_1} \cdots d_{m+n}^{s_{m+n}}$, where $d_i = 1 - f_i\tau$. Suppose $s_i \neq s_{i+1}$ and $d_i \neq d_{i+1}$. Using Lemma 4.1.2, one constructs $\tilde{d}_i$ and $\tilde{d}_{i+1}$ such that $d_i^{s_i} d_{i+1}^{s_{i+1}} = \tilde{d}_i^{s_{i+1}} \tilde{d}_{i+1}^{s_i}$. This induces a new complete factorization of $\mathcal{R} = d_1^{s_1} \cdots \tilde{d}_i^{s_{i+1}} \tilde{d}_{i+1}^{s_i} \cdots d_{m+n}^{s_{m+n}}$ with the new parity sequence $\tilde{\boldsymbol{s}} = \boldsymbol{s}^{[i]} = (s_1, \dots, s_{i+1}, s_i, \dots, s_{m+n})$.

Repeating this procedure, we see that there exists a canonical bijection between the sets of complete factorizations with respect to any two parity sequences.

### 4.1.2 Complete factorizations and superflag varieties

Let $W = W_{\bar{0}} \oplus W_{\bar{1}}$ be a vector superspace with $\dim(W_{\bar{0}}) = m$ and $\dim(W_{\bar{1}}) = n$. Consider a *full flag* $\mathscr{F}$ of $W$, $\mathscr{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{m+n} = W\}$ such that $\dim(F_i) = i$. A basis $\{w_1, \dots, w_{m+n}\}$ of $W$ *generates the full flag* $\mathscr{F}$ if $F_i$ is spanned by $w_1, \ldots, w_i$. A full flag is called a *full superflag* if it is generated by a homogeneous basis. We denote by $\mathscr{F}(W)$ the set of all full superflags.

To a homogeneous basis $\{w_1, \ldots, w_{m+n}\}$ of $W$, we associate the unique parity sequence $\boldsymbol{s} \in S_{m|n}$ such that $s_i = (-1)^{|w_i|}$. We say a full superflag $\mathscr{F}$ *has parity sequence* $\boldsymbol{s}$ if it is generated by a homogeneous basis whose parity sequence is $\boldsymbol{s}$. We denote by $\mathscr{F}^{\boldsymbol{s}}(W)$ the set of all full superflags of parity $\boldsymbol{s}$.

Clearly, we have

$$\mathscr{F}(W) = \bigsqcup_{\boldsymbol{s} \in S_{m|n}} \mathscr{F}^{\boldsymbol{s}}(W), \qquad \mathscr{F}^{\boldsymbol{s}}(W) \cong \mathscr{F}\left(W_{\bar{0}}\right) \times \mathscr{F}\left(W_{\bar{1}}\right).$$

Given a basis $\{v_1, \ldots, v_m\}$ of $W_{\bar{0}}$, a basis $\{u_1, \ldots, u_n\}$ of $W_{\bar{1}}$, and a parity sequence $\boldsymbol{s} \in S_{m|n}$, define a homogeneous basis $\{w_1, \ldots, w_{m+n}\}$ of $W$ by the rule $w_i = v_{\boldsymbol{s}_i^+ + 1}$ if $s_i = 1$ and $w_i = u_{\boldsymbol{s}_i^- + 1}$ if $s_i = -1$. Conversely, any homogeneous basis of $W$ gives a basis of $W_{\bar{0}}$, a basis of $W_{\bar{1}}$, and a parity sequence $\boldsymbol{s}$. We say that the basis $\{w_1, \ldots, w_{m+n}\}$ is *associated to* $\{v_1, \ldots, v_m\}$, $\{u_1, \ldots, u_n\}$, *and* $\boldsymbol{s}$.

Define the *discrete Wronskian* $\mathrm{Wr}$ (or Casorati determinant) of $g_1, \ldots, g_r$ by

$$\mathrm{Wr}^{\pm}(g_1, \ldots, g_r) = \det\left(g_j[\mp(i-1)]\right)_{i,j=1}^r = \det\left(g_j(x \pm (i-1)h)\right)_{i,j=1}^r.$$

We simply write $\mathrm{Wr}$ for $\mathrm{Wr}^-$.

Let $\mathcal{R}$ be an $(m|n)$-rational difference operator over $\mathbb{K}$. Let $\mathcal{R} = \mathcal{D}_{\bar{0}} \mathcal{D}_{\bar{1}}^{-1}$ be a fractional factorization such that $\operatorname{ord} \mathcal{D}_{\bar{1}} = n$ and the constant term of $\mathcal{D}_{\bar{1}}$ is 1. By Proposition 4.1.1, such a fractional factorization of $\mathcal{R}$ is unique.

Let $V = W_{\bar{0}} = \ker \mathcal{D}_{\bar{0}}$, $U = W_{\bar{1}} = \ker \mathcal{D}_{\bar{1}}$, $W = W_{\bar{0}} \oplus W_{\bar{1}}$.

Given a basis $\{v_1, \ldots, v_m\}$ of $V$, a basis $\{u_1, \ldots, u_n\}$ of $U$, and a parity sequence $\boldsymbol{s} \in S_{m|n}$, define $d_i = 1 - f_i \tau$, where

$$\begin{aligned} f_i &= \ln' \frac{\mathrm{Wr}\left(v_1, v_2, \ldots, v_{\boldsymbol{s}_i^+ + 1}, u_1, u_2, \ldots, u_{\boldsymbol{s}_i^-}\right)}{\mathrm{Wr}\left(v_1, v_2, \ldots, v_{\boldsymbol{s}_i^+}, u_1, u_2, \ldots, u_{\boldsymbol{s}_i^-}\right)[1]}, \qquad \text{if } s_i = 1, \\ f_i &= \ln' \frac{\mathrm{Wr}\left(v_1, v_2, \ldots, v_{\boldsymbol{s}_i^+}, u_1, u_2, \ldots, u_{\boldsymbol{s}_i^- + 1}\right)}{\mathrm{Wr}\left(v_1, v_2, \ldots, v_{\boldsymbol{s}_i^+}, u_1, u_2, \ldots, u_{\boldsymbol{s}_i^-}\right)[1]}, \qquad \text{if } s_i = -1. \end{aligned} \tag{4.2}$$

Note that if two bases $\{v_1,\dots,v_m\}$, $\{\tilde v_1,\dots,\tilde v_m\}$ generate the same full flag of $V$ and two bases $\{u_1,\dots,u_n\}$, $\{\tilde u_1,\dots,\tilde u_n\}$ generate the same full flag of $U$, then the coefficients $f_i$ computed from $v_j,u_j$ and from $\tilde v_j,\tilde u_j$ are the same.

**Proposition 4.1.3.** *We have a complete factorization of $\mathcal{R}$ with parity $\boldsymbol{s}$: $\mathcal{R} = d_1^{s_1}\cdots d_{m+n}^{s_{m+n}}$.*

*Proof.* The statement for the case of $\boldsymbol{s}=\boldsymbol{s}_0$ follows from [MV03].

Let $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ be two parity sequences which differ only in positions $i,i+1$. Explicitly, $s_j=\tilde s_j$ for $j\neq i,i+1$ and $s_i=-s_{i+1}=-\tilde s_i=\tilde s_{i+1}$. It is clear that $d_j=\tilde d_j$ for $j\neq i,i+1$. In addition, the equality $d_i^{s_i}d_{i+1}^{s_{i+1}}=\tilde d_i^{\tilde s_i}\tilde d_{i+1}^{\tilde s_{i+1}}$ follows from the discrete Wronskian identity, see [MV03, Lemma 9.5],

$$\begin{aligned}&\mathrm{Wr}\left(\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+ +1},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-}),\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^- +1})\right)\\ =&\,\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+ +1},u_1,u_2,\dots,u_{\boldsymbol{s}_i^- +1})\mathrm{Wr}\,(v_1,v_2,\dots,v_{\boldsymbol{s}_i^+},u_1,u_2,\dots,u_{\boldsymbol{s}_i^-})[1].\end{aligned}$$ □

By Proposition 4.1.3, we have maps $\varpi:\mathscr{F}(W)\to\mathfrak{F}(\mathcal{R})$ and $\varpi^{\boldsymbol{s}}:\mathscr{F}^{\boldsymbol{s}}(W)\to\mathfrak{F}^{\boldsymbol{s}}(\mathcal{R})$.

**Corollary 4.1.4.** *The maps $\varpi$ and $\varpi^{\boldsymbol{s}}$ are bijections.* □

Thus the set of complete factorizations of $\mathcal{R}$ is canonically identified with the variety of full superflags of $W$.

## 4.2 XXX model

In this section we recall the definition of the super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and some facts about the XXX model associated with $\mathrm{Y}(\mathfrak{gl}_{m|n})$. Our main source is [BR08].

### 4.2.1 Super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ and transfer matrix

Let $\mathbb{C}^{m|n}$ be the complex vector superspace with $\dim(\mathbb{C}^{m|n}_{\bar 0})=m$ and $\dim(\mathbb{C}^{m|n}_{\bar 1})=n$. We choose a homogeneous basis $e_1,\dots,e_{m+n}$ of $\mathbb{C}^{m|n}$ such that $|e_i|=0$ for $1\leqslant$

$i \leqslant m$ and $|e_j| = 1$ for $m+1 \leqslant j \leqslant m+n$. Denote by $E_{ij} \in \mathrm{End}(\mathbb{C}^{m|n})$ the linear operator of parity $|i|+|j|$ such that $E_{ij}e_k = \delta_{jk}e_i$ for $1 \leqslant i,j,k \leqslant m+n$.

The *super Yangian* $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a unital associative algebra with generators $\mathcal{L}_{ij}^{(k)}$ of parity $|i|+|j|$, $i,j = 1,\dots,m+n$, $k \in \mathbb{Z}_{>0}$.

Consider the generating series

$$\mathcal{L}_{ij}(x) = \sum_{k=0}^{\infty} \mathcal{L}_{ij}^{(k)} x^{-k}, \qquad \mathcal{L}_{ij}^{(0)} = \delta_{ij},$$

and combine the series into a linear operator

$$\mathcal{L}(x) = \sum_{i,j=1}^{m+n} E_{ij} \otimes \mathcal{L}_{ij}(x) \in \mathrm{End}(\mathbb{C}^{m|n}) \otimes \mathrm{Y}(\mathfrak{gl}_{m|n})[[x^{-1}]].$$

The defining relations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ are given by

$$R^{(12)}(x_1-x_2)\mathcal{L}^{(13)}(x_1)\mathcal{L}^{(23)}(x_2) = \mathcal{L}^{(23)}(x_2)\mathcal{L}^{(13)}(x_1)R^{(12)}(x_1-x_2), \tag{4.3}$$

where $R(x) \in \mathrm{End}(\mathbb{C}^{m|n} \otimes \mathbb{C}^{m|n})$ is the super R-matrix defined by

$$x\,R(x) = x\,\mathrm{id} + h\sum_{i,j=1}^{m+n} (-1)^{|j|} E_{ij} \otimes E_{ji}.$$

**Remark 4.2.1.** *Note that, for any non-zero $z \in \mathbb{C}^{\times}$, the map $\mathcal{L}_{ij}(x) \mapsto \mathcal{L}_{ij}(x/z)$ induces an automorphism of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, therefore the super Yangians $\mathrm{Y}(\mathfrak{gl}_{m|n})$ defined by different non-zero $h$ are actually isomorphic. In particular, we can always rescale $h$ to* 1. □

The R-matrix $R(x)$ satisfies the graded Yang-Baxter equation,

$$R^{(12)}(x_1-x_2)R^{(13)}(x_1)R^{(23)}(x_2) = R^{(23)}(x_2)R^{(13)}(x_1)R^{(12)}(x_1-x_2).$$

The super commutator relations obtained from (4.3) are explicitly given by

$$\begin{aligned}(x_1-x_2)[\mathcal{L}_{ij}(x_1), \mathcal{L}_{k\ell}(x_2)] &= (-1)^{|i||k|+|\ell||i|+|\ell||k|} h\big(\mathcal{L}_{kj}(x_2)\mathcal{L}_{i\ell}(x_1) - \mathcal{L}_{kj}(x_1)\mathcal{L}_{i\ell}(x_2)\big) \\ &= (-1)^{|i||j|+|\ell||i|+|\ell||j|} h\big(\mathcal{L}_{i\ell}(x_1)\mathcal{L}_{kj}(x_2) - \mathcal{L}_{i\ell}(x_2)\mathcal{L}_{kj}(x_1)\big).\end{aligned} \tag{4.4}$$

In particular, one has

$$[\mathcal{L}_{ij}^{(1)}, \mathcal{L}_{k\ell}(x)] = (-1)^{|i||k|+|\ell||i|+|\ell||k|} h\big(\delta_{i\ell}\mathcal{L}_{kj}(x) - \delta_{kj}\mathcal{L}_{i\ell}(x)\big). \tag{4.5}$$

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ is a Hopf algebra with the coproduct

$$\Delta : \mathcal{L}_{ij}(x) \mapsto \sum_{k=1}^{m+n} (-1)^{(|k|+|i|)(|k|+|j|)} \mathcal{L}_{ik}(x) \otimes \mathcal{L}_{kj}(x), \qquad i, j = 1, \ldots, m+n.$$

The super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$ contains the algebra $\mathrm{U}(\mathfrak{gl}_{m|n})$ as a Hopf subalgebra. The embedding is given by the map $e_{ij} \mapsto (-1)^{|i|}\mathcal{L}_{ji}^{(1)}/h$ for $1 \leqslant i, j \leqslant m+n$. We identify $\mathrm{U}(\mathfrak{gl}_{m|n})$ with the image of this map.

The *transfer matrix* $\mathcal{T}(x)$ is defined as the supertrace of $\mathcal{L}(x)$,

$$\mathcal{T}(x) = \mathrm{str}(\mathcal{L}(x)) = \sum_{i=1}^{m+n} (-1)^{|i|} \mathcal{L}_{ii}(x).$$

It is known that the transfer matrices commute, $[\mathcal{T}(x_1), \mathcal{T}(x_2)] = 0$. Moreover, the transfer matrix $\mathcal{T}(x)$ commutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_{m|n})$.

Since the transfer matrices commute, the transfer matrix can be considered as a generating function of integrals of motion of an integrable system.

For any given complex number $z \in \mathbb{C}$, there is an automorphism

$$\zeta_z : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{Y}(\mathfrak{gl}_{m|n}), \qquad \mathcal{L}_{ij}(x) \to \mathcal{L}_{ij}(x-z),$$

where $(x-z)^{-1}$ is expanded as a power series in $x^{-1}$. The *evaluation homomorphism* $\mathrm{ev} : \mathrm{Y}(\mathfrak{gl}_{m|n}) \to \mathrm{U}(\mathfrak{gl}_{m|n})$ is defined by the rule:

$$\mathcal{L}_{ji}^{(a)} \mapsto (-1)^{|i|}\delta_{1a} h e_{ij},$$

for $a \in \mathbb{Z}_{>0}$.

For any $\mathfrak{gl}_{m|n}$-module $V$ denote by $V(z)$ the $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module obtained by pulling back of $V$ through the homomorphism $\mathrm{ev}\circ\zeta_z$. The module $V(z)$ is called the *evaluation module with the evaluation point* $z$.

Let $V$ be a $\mathrm{Y}(\mathfrak{gl}_{m|n})$-module. Given a parity sequence $\boldsymbol{s} \in S_{m|n}$, a non-zero vector $v \in V$ is called an $\boldsymbol{s}$*-singular vector* if

$$\mathcal{L}^{\boldsymbol{s}}_{ii}(x)v = \Lambda_i(x)v, \qquad \mathcal{L}^{\boldsymbol{s}}_{ij}(x)v = 0, \quad i > j,$$

where $\Lambda_i(x) \in \mathbb{C}[[x^{-1}]]$ and $\mathcal{L}^{\boldsymbol{s}}_{a,b}(x) = \mathcal{L}_{\sigma_{\boldsymbol{s}}(a),\sigma_{\boldsymbol{s}}(b)}(x)$.

**Example 4.2.2.** *Let $L_\lambda$ be an irreducible polynomial $\mathfrak{gl}_{m|n}$-module of highest weight $\lambda$ with highest weight vector $v_\lambda$. Let $z$ be a complex number. Then the $\mathfrak{gl}_{m|n}$ $\boldsymbol{s}$-singular vector $v^{\boldsymbol{s}}_\lambda \in L_\lambda(z)$ is a $\mathrm{Y}(\mathfrak{gl}_{m|n})$ $\boldsymbol{s}$-singular vector. Moreover, we have*

$$\mathcal{L}^{\boldsymbol{s}}_{ii}(x)v^{\boldsymbol{s}}_\lambda = \Big(1 + \frac{s_i\,\lambda^{\boldsymbol{s}}(e^{\boldsymbol{s}}_{ii})h}{x-z}\Big)v^{\boldsymbol{s}}_\lambda = \frac{x - z + s_i\,\lambda^{\boldsymbol{s}}(e^{\boldsymbol{s}}_{ii})h}{x-z}v^{\boldsymbol{s}}_\lambda, \quad i = 1, 2, \dots, m+n. \quad \square$$

### 4.2.2 Bethe ansatz equation

We fix a parity sequence $\boldsymbol{s} \in S_{m|n}$, a sequence $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(p)})$ of polynomial $\mathfrak{gl}_{m|n}$ weights, and a sequence $\boldsymbol{z} = (z_1, \dots, z_p)$ of complex numbers. We call $\lambda^{(\boldsymbol{s},k)}$, see Section 2.3.2, the *weight at point $z_k$ with respect to $\boldsymbol{s}$*.

Let $\boldsymbol{l} = (l_1, \dots, l_{m+n-1})$ be a sequence of non-negative integers. Define $l = \sum_{i=1}^{m+n-1} l_i$. Let $\boldsymbol{t} = (t^{(1)}_1, \dots, t^{(1)}_{l_1}; \dots; t^{(m+n-1)}_1, \dots, t^{(m+n-1)}_{l_{m+n-1}})$ be a collection of variables. We say that $t^{(i)}_j$ has *color* $i$. Define the $\mathfrak{gl}_{m|n}$ *weight at $\infty$ with respect to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$* by

$$\lambda^{(\boldsymbol{s},\infty)} = \sum_{k=1}^{p} \lambda^{(\boldsymbol{s},k)} - \sum_{i=1}^{m+n-1} l_i \alpha^{\boldsymbol{s}}_i.$$

The *Bethe ansatz equation* (BAE) *associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$*, is a system of algebraic equations in variables $\boldsymbol{t}$:

$$\prod_{k=1}^{p} \frac{t^{(i)}_j - z_k + s_i\lambda^{(\boldsymbol{s},k)}_i h}{t^{(i)}_j - z_k + s_{i+1}\lambda^{(\boldsymbol{s},k)}_{i+1} h} \prod_{r=1}^{l_{i-1}} \frac{t^{(i)}_j - t^{(i-1)}_r + s_i h}{t^{(i)}_j - t^{(i-1)}_r} \prod_{\substack{r=1\\ r\neq j}}^{l_i} \frac{t^{(i)}_j - t^{(i)}_r - s_i h}{t^{(i)}_j - t^{(i)}_r + s_{i+1} h} \prod_{r=1}^{l_{i+1}} \frac{t^{(i)}_j - t^{(i+1)}_r}{t^{(i)}_j - t^{(i+1)}_r - s_{i+1} h} = 1, \tag{4.6}$$

where $i = 1, \dots, m+n-1$, $j = 1, \dots, l_i$. We call the single equation (4.6) the *BAE for $\boldsymbol{t}$ related to $t^{(i)}_j$*.

We allow the following cancellations in the BAE,

$$\frac{t_j^{(i)} - z_k + s_i\lambda_i^{(\boldsymbol{s},k)}h}{t_j^{(i)} - z_k + s_{i+1}\lambda_{i+1}^{(\boldsymbol{s},k)}h} = 1, \text{ if } s_i\lambda_i^{(\boldsymbol{s},k)} = s_{i+1}\lambda_{i+1}^{(\boldsymbol{s},k)};$$

$$\frac{t_j^{(i)} - t_r^{(i)} - s_i h}{t_j^{(i)} - t_r^{(i)} + s_{i+1}h} = 1, \text{ if } s_i = -s_{i+1}. \tag{4.7}$$

After these cancellations, we consider only the solutions that do not make the remaining denominators in (4.6) vanish.

In addition, we impose the following condition. Suppose $(\alpha_i^{\boldsymbol{s}}, \alpha_i^{\boldsymbol{s}}) = 0$ for some $i$. Consider the BAE for $\boldsymbol{t}$ related to $t_j^{(i)}$ with all $t_b^{(a)}$ fixed, where $a \neq i$ and $1 \leqslant b \leqslant l_a$, this equation does not depend on $j$. Let $t_0^{(i)}$ be a solution of this equation with multiplicity $r$. Then we require that the number of $j$ such that $t_j^{(i)} = t_0^{(i)}$ is at most $r$, c.f. Lemma 4.16, Theorem 4.4.1.

The group $\mathfrak{S}_{\boldsymbol{l}} = \mathfrak{S}_{l_1} \times \cdots \times \mathfrak{S}_{l_{m+n-1}}$ acts on $\boldsymbol{t}$ by permuting the variables of the same color.

We do not distinguish between solutions of the BAE in the same $\mathfrak{S}_{\boldsymbol{l}}$-orbit.

**Remark 4.2.3.** *Note that in the quasiclassical limit $h \to 0$, system* (4.6) *becomes system (4.2) of [MVY14], which is the Bethe ansatz equation of Gaudin model associated to $\mathfrak{gl}_{m|n}$.* □

### 4.2.3 Bethe vector

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $v_k^{\boldsymbol{s}} = v_{\lambda^{(\boldsymbol{s},k)}}^{\boldsymbol{s}}$ be an $\boldsymbol{s}$-singular vector in the irreducible $\mathfrak{gl}_{m|n}$-module $L_{\lambda^{(k)}}$. Consider the tensor product of evaluation modules $L(\boldsymbol{\lambda}, \boldsymbol{z}) = \bigotimes_{k=1}^{p} L_{\lambda^{(k)}}(z_k)$. We also denote by $L(\boldsymbol{\lambda})$ the corresponding $\mathfrak{gl}_{m|n}$-module.

Let $\boldsymbol{l} = (l_1, \ldots, l_{m+n-1})$ be a collection of non-negative integers. The *weight function* is a vector $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z})$ in $L(\boldsymbol{\lambda}, \boldsymbol{z})$ depending on variables

$$\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{l_1}^{(1)}; \ldots; t_1^{(m+n-1)}, \ldots, t_{l_{m+n-1}}^{(m+n-1)})$$

and parameters $\boldsymbol{z} = (z_1, \ldots, z_p)$. The weight function $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z})$ is constructed as follows, see [BR08, Section 5.2].

Set $l^{<a} = l_1 + \cdots + l_{a-1}$, $a = 1, \ldots, m+n$. Note that $l = l^{<m+n}$. Consider a series in $l$ variables $\boldsymbol{t}$ with coefficients in $\mathrm{Y}(\mathfrak{gl}_{m|n})$:

$$\mathbb{B}^{\boldsymbol{s}}_l(\boldsymbol{t}) = (\mathrm{str}_{12\cdots l} \otimes \mathrm{id})\Big(\mathcal{L}^{(1,l+1)}(t^{(1)}_1) \cdots \mathcal{L}^{(l,l+1)}(t^{(m+n-1)}_{l_{m+n-1}}) \\ \times \mathfrak{R}^{(1\ldots l)}(\boldsymbol{t}) E^{\boldsymbol{s}}_{m+n,m+n-1}{}^{\otimes l_{m+n-1}} \otimes \cdots \otimes E^{\boldsymbol{s}}_{21}{}^{\otimes l_1} \otimes 1\Big),$$

where

$$\mathfrak{R}^{(1\ldots l)}(\boldsymbol{t}) = \prod_{a<b} \overrightarrow{\prod_{1\leqslant j\leqslant l_b}} \overleftarrow{\prod_{1\leqslant i\leqslant l_a}} \frac{t^{(b)}_j - t^{(a)}_i}{t^{(b)}_j - t^{(a)}_i + s_b h} R^{(l^{<b}+j, l^{<a}+i)}(t^{(b)}_j - t^{(a)}_i) \tag{4.8}$$

and the first product in (4.8) runs over $1 \leqslant a < b \leqslant m+n-1$.

The weight function $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) \in L(\boldsymbol{\lambda}, \boldsymbol{z})$ is given by

$$w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) = \mathbb{B}^{\boldsymbol{s}}_l(\boldsymbol{t})\big(v^{\boldsymbol{s}}_1 \otimes \cdots \otimes v^{\boldsymbol{s}}_p\big)\,.$$

**Example 4.2.4.** *Let $m+n=2$ and $\boldsymbol{t} = (t_1, \ldots, t_l)$, then*

$$w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) = (-1)^{l|2|} \mathcal{L}^{\boldsymbol{s}}_{12}(t_1) \cdots \mathcal{L}^{\boldsymbol{s}}_{12}(t_l)\big(v^{\boldsymbol{s}}_1 \otimes \cdots \otimes v^{\boldsymbol{s}}_p\big) \tag{4.9}$$

*is an example of the weight function.* □

The following theorem is known.

**Theorem 4.2.5** ( [BR08])**.** *Suppose that $\boldsymbol{\lambda}$ is a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights and $\boldsymbol{t}$ a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$. If the vector $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) \in L(\boldsymbol{\lambda}, \boldsymbol{z})$ is well-defined and non-zero, then $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) \in L(\boldsymbol{\lambda}, \boldsymbol{z})$ is an eigenvector of the transfer matrix $\mathcal{T}(x)$, $\mathcal{T}(x) w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) = \mathcal{E}(x) w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z})$, where the eigenvalue $\mathcal{E}(x)$ is given by*

$$\mathcal{E}(x) = \sum_{a=1}^{m+n} s_a \prod_{k=1}^{p} \frac{x - z_k + s_a \lambda^{(\boldsymbol{s},k)}_a h}{x - z_k} \prod_{j=1}^{l_{a-1}} \frac{x - t^{(a-1)}_j + s_a h}{x - t^{(a-1)}_j} \prod_{j=1}^{l_a} \frac{x - t^{(a)}_j - s_a h}{x - t^{(a)}_j}\ . \tag{4.10}$$

Note that the eigenvalue $\mathcal{E}(x)$ depends on the parameters $\boldsymbol{t}$, $\boldsymbol{s}$, $\boldsymbol{z}$, and $\boldsymbol{\lambda}$. We drop this dependence for our notation.

If $\boldsymbol{t}$ is a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then the value of the weight function $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z})$ is called the *Bethe vector.*

We have the following standard statement regarding to Bethe vectors, c.f. [MTV06, Proposition 6.2] and [MVY14, Theorem 4.3].

**Proposition 4.2.6.** *The Bethe vector $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z})$ is a $\mathfrak{gl}_{m|n}$ $\boldsymbol{s}$-singular vector of weight $\lambda^{(\boldsymbol{s},\infty)}$.*

*Proof.* Clearly, the Bethe vector $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z})$ is a vector of weight $\lambda^{(\boldsymbol{s},\infty)}$. We then show that $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z})$ is $\mathfrak{gl}_{m|n}$ $\boldsymbol{s}$-singular.

We show it for the case of $m=n=1$ with the standard parity $\boldsymbol{s}_0$ in Section 6. The general case follows from a similar computation using a combination of nested Bethe ansatz, as in [BR08, Section 4], and induction on $m+n$, see e.g. [MTV06, Proposition 6.2]. □

### 4.2.4 Sequences of polynomials

We use the following convenient notation. We say that a sequence $\boldsymbol{z}=(z_1,\dots,z_p)$ of complex numbers is *$h$-generic* if $z_i-z_j\notin h\mathbb{Z}$ for all $1\leqslant i<j\leqslant p$.

Let $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z}=(z_1,\dots,z_p)$ be an $h$-generic sequence of complex numbers. Fix a parity sequence $\boldsymbol{s}\in S_{m|n}$.

Define a sequence of polynomials $\boldsymbol{T}^{\boldsymbol{s}}=(T_1^{\boldsymbol{s}},\dots,T_{m+n}^{\boldsymbol{s}})$ associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$ and $\boldsymbol{z}$,

$$T_i^{\boldsymbol{s}}(x)=\prod_{k=1}^{p}\prod_{j=1}^{\lambda_i^{(\boldsymbol{s},k)}}(x-z_k+s_ijh),\qquad i=1,\dots,m+n. \tag{4.11}$$

Note that $T_i^{\boldsymbol{s}}(T_{i+1}^{\boldsymbol{s}})^{-s_is_{i+1}}$ is a polynomial for all $i=1,\dots,m+n-1$.

Let $\boldsymbol{l}=(l_1,\dots,l_{m+n-1})$ be a sequence of non-negative integers.

Let $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{l_1}^{(1)}; \dots; t_1^{(m+n-1)}, \dots, t_{l_{m+n-1}}^{(m+n-1)})$ be a sequence of complex numbers. Define a sequence of polynomials $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ by

$$y_i(x) = \prod_{j=1}^{l_i} (x - t_j^{(i)}), \qquad i = 1, \dots, m+n-1. \tag{4.12}$$

We say the *sequence of polynomials* $\boldsymbol{y}$ *represents* $\boldsymbol{t}$. We have $\deg y_i = l_i$.

We also set $y_0(x) = y_{m+n}(x) = 1$.

If $\boldsymbol{t}$ is a solution of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\boldsymbol{l}$, then the eigenvalue $\mathcal{E}(x)$ of the transfer matrix $\mathcal{T}(x)$ acting on the Bethe vector $w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z})$, see (2.8), can be written in terms of $\boldsymbol{y}$ and $\boldsymbol{T}^{\boldsymbol{s}}$. Namely, we have

$$\mathcal{E}(x) = \mathcal{E}_{\boldsymbol{y}}(x) = \sum_{a=1}^{m+n} s_a \, \frac{T_a^{\boldsymbol{s}}}{T_a^{\boldsymbol{s}}[s_a]} \, \frac{y_{a-1}[-s_a]}{y_{a-1}} \, \frac{y_a[s_a]}{y_a} \, . \tag{4.13}$$

We do not consider zero polynomials $y_i(x)$ and do not distinguish between polynomials $y_i(x)$ and $cy_i(x)$, $c \in \mathbb{C}^\times$. Hence, a sequence $\boldsymbol{y}$ defines a point in $\big(\mathbb{P}(\mathbb{C}[x])\big)^{m+n-1}$, the direct product of $m+n-1$ copies of the projective space associated to the vector space of polynomials.

We say that a sequence of polynomials $\boldsymbol{y}$ is *generic with respect to* $\boldsymbol{s}$*,* $\boldsymbol{\lambda}$*, and* $\boldsymbol{z}$ if it satisfies the following conditions:

1. if $s_i s_{i+1} = 1$, then $y_i$ has only simple roots and $y_i$ has no common roots with the polynomial $y_i[1]$;

2. the polynomial $y_i$ has no common roots with polynomials $y_{i-1}$, $y_{i-1}[-s_i]$, and $y_{i+1}[s_{i+1}]$;

3. all roots of $y_i$ are different from the roots of polynomial $T_i^{\boldsymbol{s}}(T_{i+1}^{\boldsymbol{s}})^{-s_i s_{i+1}}$,

for $i = 1, \dots, m+n-1$.

Not all solutions of the BAE correspond to generic sequences of polynomials. For instance, if $m = 2$, $n = p = 0$, and $l$ is even, then $t_1 = \dots = t_l = 0$ is a solution of the BAE.

## 4.3 Reproduction procedures for $\mathfrak{gl}_2$ and $\mathfrak{gl}_{1|1}$

In this section, we recall the reproduction procedure for the XXX model associated to $\mathfrak{gl}_2$ from [MV03, Section 2] and define its analogue for $\mathfrak{gl}_{1|1}$. We define a rational difference operator associated to a solution of BAE. We also show that the reproduction procedure does not alter the rational difference operator and the corresponding eigenvalues obtained from Theorem 4.2.5.

### 4.3.1 Reproduction procedure for $\mathfrak{gl}_2$.

Set $m = 2$ and $n = 0$. We have the following identifications $\mathrm{Y}(\mathfrak{gl}_{2|0}) \cong \mathrm{Y}(\mathfrak{gl}_{0|2}) \cong \mathrm{Y}(\mathfrak{gl}_2)$. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(p)}) = ((a_1, b_1), \dots, (a_p, b_p))$ be a sequence of polynomial $\mathfrak{gl}_2$ weights. We have $a_k, b_k \in \mathbb{Z}$, $a_k \geqslant b_k \geqslant 0$, $k = 1, \dots, p$. Let $\boldsymbol{z} = (z_1, \dots, z_p)$ be an $h$-generic sequence of complex numbers. We have

$$T_1(x) = \prod_{k=1}^{p} \prod_{j=1}^{a_k} (x - z_k + jh), \qquad T_2(x) = \prod_{k=1}^{p} \prod_{j=1}^{b_k} (x - z_k + jh).$$

Let $a = \deg T_1$ and $b = \deg T_2$.

Give a non-negative integer $l$ and variables $\boldsymbol{t} = (t_1, \dots, t_l)$. The BAE associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $l$ is simplified to

$$\prod_{k=1}^{p} \frac{t_j - z_k + a_k h}{t_j - z_k + b_k h} \prod_{i=1, i\neq j}^{l} \frac{t_j - t_i - h}{t_j - t_i + h} = 1, \qquad j = 1, \dots, l. \tag{4.14}$$

It is known that the BAE (4.14) can be reformulated in terms of discrete Wronskian. Moreover, starting from a generic solution of BAE, one can construct a family of new solutions of the BAE in the following way.

**Lemma 4.3.1** ( [MV03]). *Let $y$ be a polynomial of degree $l$ which is generic with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{z}$.*

1. *The polynomial $y \in \mathbb{C}[x]$ represents a solution of the BAE* (4.14) *associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$ and $l$, if and only if there exists a polynomial $\tilde{y} \in \mathbb{C}[x]$, such that*

$$\mathrm{Wr}^{+}(y, \tilde{y}) = T_1 T_2^{-1}. \tag{4.15}$$

2. *If $\tilde{y}$ is generic, then $\tilde{y}$ represents a solution of the BAE associated to $\boldsymbol{\lambda}$, $\boldsymbol{z}$ and $\tilde{l}$, where $\tilde{l} = \deg \tilde{y}$.* □

Almost all $\tilde{y}$ are generic with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{z}$, and therefore by Lemma 4.3.1 represent solutions of the BAE (4.14). Thus, from one solution of the BAE, we obtain a family of new solutions. Following the terminology of [MV03], we call this construction the $\mathfrak{gl}_2$ *reproduction procedure*.

Let $P_y$ be the closure of the set containing $y$ and all $\tilde{y}$ as in Lemma 4.3.1 in $\mathbb{P}(\mathbb{C}[x])$. We call $P_y$ the $\mathfrak{gl}_2$ *population originated at* $y$. The population $P_y$ can be identified with the projective line $\mathbb{CP}^1$ through the correspondence $c_1 y + c_2 \tilde{y} \mapsto (c_1 : c_2)$.

The weight at infinity associated to the data $\boldsymbol{\lambda}$ and $l$ is given by $\lambda^{(\infty)} = (a-l, b+l)$. Suppose that the weight $\lambda^{(\infty)}$ is dominant, namely $2l \leqslant a-b$. If $\tilde{l} \neq l$, then the weight at infinity associated to $\boldsymbol{\lambda}$ and $\tilde{l}$ is

$$\tilde{\lambda}^{(\infty)} = (a-\tilde{l}, b+\tilde{l}) = (b+l-1, a-l+1) = s \cdot \lambda^{(\infty)},$$

where $s \in \mathfrak{S}_2$ is the non-trivial element in the Weyl group of $\mathfrak{gl}_2$, and the dot denotes the shifted action.

Let $\tilde{y} = \prod_{r=1}^{\tilde{l}} (x - \tilde{t}_r)$ and $\tilde{\boldsymbol{t}} = (\tilde{t}_1, \ldots, \tilde{t}_{\tilde{l}})$. If $y$ is generic, then by Lemma 4.3.1, $\tilde{\boldsymbol{t}}$ is a solution of the BAE (4.14) with $l$ replaced by $\tilde{l}$. By Proposition 4.2.6, the value of the weight function $w(\tilde{\boldsymbol{t}}, \boldsymbol{z})$ is a singular vector. At the same time, $\tilde{\lambda}^{(\infty)}$ is not dominant and therefore $w(\tilde{\boldsymbol{t}}, \boldsymbol{z}) = 0$ in $L(\boldsymbol{\lambda})$. So, in a $\mathfrak{gl}_2$ population only the unique polynomial (the one of the smallest degree) corresponds to an actual eigenvector in $L(\boldsymbol{\lambda})$.

The eigenvalues corresponding to the solutions $y$ and $\tilde{y}$, see (4.13), are given by

$$\mathcal{E}(x) = \frac{T_1 y[1]}{T_1[1] y} + \frac{T_2 y[-1]}{T_2[1] y}, \qquad \tilde{\mathcal{E}}(x) = \frac{T_1 \tilde{y}[1]}{T_1[1] \tilde{y}} + \frac{T_2 \tilde{y}[-1]}{T_2[1] \tilde{y}}.$$

**Lemma 4.3.2.** *The eigenvalues $\mathcal{E}(x)$ and $\tilde{\mathcal{E}}(x)$ are the same.*

*Proof.* Note that

$$\tilde{\mathcal{E}}(x) - \mathcal{E}(x) = \frac{\mathrm{Wr}^+(y, \tilde{y})[1]}{y \tilde{y}} \frac{T_1}{T_1[1]} - \frac{\mathrm{Wr}^+(y, \tilde{y})}{y \tilde{y}} \frac{T_2}{T_2[1]}.$$

By (4.15), we have

$$\frac{\mathrm{Wr}^{+}(y,\tilde{y})}{\mathrm{Wr}^{+}(y,\tilde{y})[1]} = \frac{T_1T_2[1]}{T_2T_1[1]}.$$

Therefore the lemma follows. $\square$

This fact can be reformulated in the following form.

Define a difference operator

$$\mathcal{D}(y) = \Big(1 - \frac{T_1 y[1]}{T_1[1]y}\,\tau\Big)\Big(1 - \frac{T_2 y[-1]}{T_2[1]y}\,\tau\Big).$$

The operator $\mathcal{D}(y)$ does not depend on a choice of polynomial $y$ in a population, $\mathcal{D}(y) = \mathcal{D}(\tilde{y})$.

### 4.3.2 Reproduction procedure for $\mathfrak{gl}_{1|1}$.

Set $m = n = 1$. We have $S_{1|1} = \{(1,-1),(-1,1)\}$. Let $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}} = \boldsymbol{s}^{[1]}$ be two different parity sequences in $S_{1|1}$. Let $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{1|1}$ weights. For each $k = 1,\dots,p$, let us write $(\lambda^{(k)})^{\boldsymbol{s}}_{[\boldsymbol{s}]} = (a_k, b_k)$, where $a_k, b_k \in \mathbb{Z}_{\geqslant 0}$ and if $a_k = 0$ then $b_k = 0$. Note that $\lambda^{(k)}$ is atypical if and only if $a_k + b_k = 0$. Let $\boldsymbol{z} = (z_1,\dots,z_p)$ be an $h$-generic sequence of complex numbers.

Let

$$\tilde{a}_k = \begin{cases} b_k + 1 & \text{if } \ a_k + b_k \neq 0, \\ 0 & \text{if } \ a_k + b_k = 0, \end{cases} \qquad \tilde{b}_k = \begin{cases} a_k - 1 & \text{if } \ a_k + b_k \neq 0, \\ 0 & \text{if } \ a_k + b_k = 0. \end{cases}$$

Equation (4.11) becomes

$$T_1^{\boldsymbol{s}} = \prod_{k=1}^{p}\prod_{j=1}^{a_k}(x - z_k + s_1 jh), \quad T_2^{\boldsymbol{s}} = \prod_{k=1}^{p}\prod_{j=1}^{b_k}(x - z_k + s_2 jh),$$

$$T_1^{\tilde{\boldsymbol{s}}} = \prod_{k=1}^{p}\prod_{j=1}^{\tilde{a}_k}(x - z_k + \tilde{s}_1 jh), \quad T_2^{\tilde{\boldsymbol{s}}} = \prod_{k=1}^{p}\prod_{j=1}^{\tilde{b}_k}(x - z_k + \tilde{s}_2 jh).$$

Let $a = \deg T_1^{\boldsymbol{s}}$, $b = \deg T_2^{\boldsymbol{s}}$. Similarly, let $\tilde{a} = \deg T_1^{\tilde{\boldsymbol{s}}}$, $\tilde{b} = \deg T_2^{\tilde{\boldsymbol{s}}}$. Suppose the number of typical weights in $\boldsymbol{\lambda}$ is $r$, then $\tilde{a} = b + r$ and $\tilde{b} = a - r$.

Let $l$ be a non-negative integer. Let $\boldsymbol{t} = (t_1, \dots, t_l)$ be a collection of variables. The Bethe ansatz equation associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $l$, is given as follows,

$$\prod_{\substack{k=1 \\ a_k+b_k\neq 0}}^{p} \frac{t_j - z_k + s_1 a_k h}{t_j - z_k + s_2 b_k h} = 1, \qquad j = 1, \dots, l. \tag{4.16}$$

The Bethe ansatz equation (4.16) can be rewritten in the form

$$\varphi^{\boldsymbol{s}}(t_j) - \psi^{\boldsymbol{s}}(t_j) = 0,$$

where

$$\varphi^{\boldsymbol{s}} = \prod_{\substack{k=1 \\ a_k+b_k\neq 0}}^{p} (x - z_k + s_1 a_k h), \qquad \psi^{\boldsymbol{s}} = \prod_{\substack{k=1 \\ a_k+b_k\neq 0}}^{p} (x - z_k + s_2 b_k h).$$

Note that $\varphi^{\boldsymbol{s}} = \psi^{\tilde{\boldsymbol{s}}}[-s_1]$ and $\psi^{\boldsymbol{s}} = \varphi^{\tilde{\boldsymbol{s}}}[-s_1]$. Thus, in the case of $\mathfrak{gl}_{1|1}$, the BAEs (4.16) associated to $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ coincide up to a shift.

We call a sequence of polynomial $\mathfrak{gl}_{1|1}$ weights $\boldsymbol{\lambda}$ *typical* if at least one of the weights $\lambda^{(k)}$ is typical. Note that $\boldsymbol{\lambda}$ is typical if and only if $a + b \neq 0$. In other words, $\boldsymbol{\lambda}$ is typical if and only if $T_1^{\boldsymbol{s}} T_2^{\boldsymbol{s}} \neq 1$.

The BAE (4.16) is reformulated as follows, c.f. [GLM18, equation (A.12)].

**Lemma 4.3.3.** *Let $y$ be a polynomial of degree $l$. Let $\boldsymbol{\lambda}$ be typical.*

1. *The polynomial $y$ represents a solution of the BAE* (4.16) *associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $l$, if and only if there exists a polynomial $\tilde{y}$, such that*

$$y \cdot \tilde{y}[-s_1] = \varphi^{\boldsymbol{s}} - \psi^{\boldsymbol{s}}. \tag{4.17}$$

2. *The polynomial $\tilde{y}$ represents a solution of the BAE* (4.16) *associated to $\tilde{\boldsymbol{s}}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, and $\tilde{l}$, where $\tilde{l} = \deg \tilde{y} = r - 1 - l$.* □

For each solution $y$, we can construct exactly one solution $\tilde{y}$. We call this construction the $\mathfrak{gl}_{1|1}$ *reproduction procedure.*

The set $P_y$ consisting of $y$ and $\tilde{y}$ is called the $\mathfrak{gl}_{1|1}$ *population originated at $y$.*

The weight at infinity associated to $\boldsymbol{s}, \boldsymbol{\lambda}$, and $l$ is $\lambda^{(\boldsymbol{s},\infty)}_{[\boldsymbol{s}]} = (a-l, b+l)$. The weight at infinity associated to $\tilde{\boldsymbol{s}}, \boldsymbol{\lambda}$ and $\tilde{l}$ is $\tilde{\lambda}^{(\tilde{\boldsymbol{s}},\infty)}_{[\tilde{\boldsymbol{s}}]} = (\tilde{a}-\tilde{l}, \tilde{b}+\tilde{l}) = (b+l+1, a-l-1)$. Thus we have $\lambda^{(\boldsymbol{s},\infty)} = \tilde{\lambda}^{(\tilde{\boldsymbol{s}},\infty)} + \alpha^{\boldsymbol{s}}$. In particular, in contrast to the case of $\mathfrak{gl}_2$, both $y$ and $\tilde{y}$ correspond to actual eigenvectors of the transfer matrix.

If $\boldsymbol{\lambda}$ is not typical, then all participating representations are one-dimensional, where the situation is trivial. In particular, we have $y(x) = 1$. We do not discuss this case.

### 4.3.3 Motivation for $\mathfrak{gl}_{1|1}$ reproduction procedure

Suppose $y$ and $\tilde{y}$ are in the same $\mathfrak{gl}_{1|1}$ population as in Section 4.3.2. Parallel to the $\mathfrak{gl}_2$ reproduction procedure, we show that the eigenvalues of transfer matrix corresponding to the Bethe vectors obtained from polynomials $y$ and $\tilde{y}$ coincide.

Let $y = \prod_{r=1}^{l}(x - t_r)$, $\tilde{y} = \prod_{r=1}^{\tilde{l}}(x - \tilde{t}_r)$. Let $\boldsymbol{t} = (t_1, \dots, t_l)$, $\tilde{\boldsymbol{t}} = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{l}})$. By Theorem 4.2.5 and (4.13), we have $\mathcal{T}(x)w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z}) = \mathcal{E}(x)w^{\boldsymbol{s}}(\boldsymbol{t}, \boldsymbol{z})$ and $\mathcal{T}(x)w^{\tilde{\boldsymbol{s}}}(\tilde{\boldsymbol{t}}, \boldsymbol{z}) = \tilde{\mathcal{E}}(x)w^{\tilde{\boldsymbol{s}}}(\tilde{\boldsymbol{t}}, \boldsymbol{z})$, where

$$\mathcal{E}(x) = s_1 \frac{T_1^{\boldsymbol{s}} y[s_1]}{T_1^{\boldsymbol{s}}[s_1] y} + s_2 \frac{T_2^{\boldsymbol{s}} y[-s_2]}{T_2^{\boldsymbol{s}}[s_2] y}, \qquad \tilde{\mathcal{E}}(x) = \tilde{s}_1 \frac{T_1^{\tilde{\boldsymbol{s}}} y[\tilde{s}_1]}{T_1^{\tilde{\boldsymbol{s}}}[\tilde{s}_1] y} + \tilde{s}_2 \frac{T_2^{\tilde{\boldsymbol{s}}} y[-\tilde{s}_2]}{T_2^{\tilde{\boldsymbol{s}}}[\tilde{s}_2] y}. \tag{4.18}$$

**Lemma 4.3.4.** *The eigenvalues $\mathcal{E}(x)$ and $\tilde{\mathcal{E}}(x)$ of transfer matrix are the same.*

*Proof.* By (4.17), we have

$$\mathcal{E}(x) = s_1 \frac{y[s_1]}{y}(\varphi^{\boldsymbol{s}} - \psi^{\boldsymbol{s}}) \prod_{\substack{k=1 \\ a_k+b_k \neq 0}}^{p} (x - z_k)^{-1} = s_1 y[s_1]\tilde{y}[-s_1] \prod_{\substack{k=1 \\ a_k+b_k \neq 0}}^{p} (x - z_k)^{-1},$$

and

$$\tilde{\mathcal{E}}(x) = s_1 \frac{\tilde{y}[-s_1]}{\tilde{y}}(\varphi^{\boldsymbol{s}}[s_1] - \psi^{\boldsymbol{s}}[s_1]) \prod_{\substack{k=1 \\ a_k+b_k \neq 0}}^{p} (x - z_k)^{-1} = s_1 y[s_1]\tilde{y}[-s_1] \prod_{\substack{k=1 \\ a_k+b_k \neq 0}}^{p} (x - z_k)^{-1}.$$

Therefore the lemma follows. $\square$

Define a rational difference operator:

$$\mathcal{R}^{\boldsymbol{s}}(y) = \Big(1 - \frac{T_1^{\boldsymbol{s}} y[s_1]}{T_1^{\boldsymbol{s}}[s_1] y}\,\tau\Big)^{s_1} \Big(1 - \frac{T_2^{\boldsymbol{s}} y[-s_2]}{T_2^{\boldsymbol{s}}[s_2] y}\,\tau\Big)^{s_2}.$$

It is clear that $\mathcal{R}^{\boldsymbol{s}}(y)=1$ if $\boldsymbol{\lambda}$ is not typical.

We have the following lemma.

**Lemma 4.3.5.** *If $\boldsymbol{\lambda}$ is typical, then $\mathcal{R}^{\boldsymbol{s}}(y)$ is a $(1|1)$-rational difference operator. Moreover, this $(1|1)$-rational difference operator is independent of a choice of a polynomial in a population, $\mathcal{R}^{\boldsymbol{s}}(y)=\mathcal{R}^{\tilde{\boldsymbol{s}}}(\tilde{y})$.*

*Proof.* The lemma is proved by a direct computation using Lemma 4.1.2 and Equation (4.17). □

## 4.4 Reproduction procedure for $\mathfrak{gl}_{m|n}$

We define the reproduction procedure and the populations in the general case.

### 4.4.1 Reproduction procedure

Let $\boldsymbol{s}\in S_{m|n}$ be a parity sequence. Let $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z}=(z_1,\dots,z_p)$ be an $h$-generic sequence of complex numbers. Let $\boldsymbol{T}^{\boldsymbol{s}}$ be a sequence of polynomials associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, see (4.11). If $s_i\neq s_{i+1}$, we also set

$$\varphi_i^{\boldsymbol{s}}=\prod_{\substack{k=1\\ \lambda_i^{(\boldsymbol{s},k)}+\lambda_{i+1}^{(\boldsymbol{s},k)}\neq 0}}^{p}(x-z_k+s_i\lambda_i^{(\boldsymbol{s},k)}h),\qquad \psi_i^{\boldsymbol{s}}=\prod_{\substack{k=1\\ \lambda_i^{(\boldsymbol{s},k)}+\lambda_{i+1}^{(\boldsymbol{s},k)}\neq 0}}^{p}(x-z_k+s_{i+1}\lambda_{i+1}^{(\boldsymbol{s},k)}h).$$

Let $\boldsymbol{l}=(l_1,\dots,l_{m+n-1})$ be a sequence of non-negative integers.

For $i\in\{1,\dots,m+n-1\}$, set $\boldsymbol{s}^{[i]}=(s_1,\dots,s_{i+1},s_i,\dots,s_{m+n})$. Set $y_0=y_{m+n}=1$.

For $g_1,g_2\in\mathbb{K}$, we also use the notation

$$\mathrm{Wr}^{\,s_i}(g_1,g_2)=g_1g_2[-s_i]-g_2g_1[-s_i].$$

We now reformulate the BAE (4.6) which allows us to construct a family of new solutions.

**Theorem 4.4.1.** *Let $\boldsymbol{y}=(y_1,\dots,y_{m+n-1})$ be a sequence of polynomials generic with respect to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, such that $\deg y_k=l_k$, $k=1,\dots,m+n-1$.*

1. *The sequence* $\boldsymbol{y}$ *represents a solution of the BAE* (4.6) *associated to* $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, *and* $\boldsymbol{l}$, *if and only if for each* $i = 1, \dots, m+n-1$, *there exists a polynomial* $\tilde{y}_i$, *such that*

$$\mathrm{Wr}^{\,s_i}(y_i, \tilde{y}_i) = T_i^{\boldsymbol{s}} \left(T_{i+1}^{\boldsymbol{s}}\right)^{-1} y_{i-1}[-s_i] y_{i+1}, \qquad \text{if } s_i = s_{i+1}, \tag{4.19}$$

$$y_i\, \tilde{y}_i[-s_i] = \varphi_i^{\boldsymbol{s}} y_{i-1}[-s_i] y_{i+1} - \psi_i^{\boldsymbol{s}} y_{i-1} y_{i+1}[-s_i], \qquad \text{if } s_i \neq s_{i+1}. \tag{4.20}$$

2. *Let* $i \in \{1, \dots, m+n-1\}$ *be such that* $\tilde{y}_i \neq 0$. *If* $\boldsymbol{y}^{[i]} = (y_1, \dots, \tilde{y}_i, \dots, y_{m+n-1})$ *is generic with respect to* $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, *and* $\boldsymbol{z}$, *then* $\boldsymbol{y}^{[i]}$ *represents a solution of the BAE associated to* $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, *and* $\boldsymbol{l}^{[i]}$, *where* $\boldsymbol{l}^{[i]} = (l_1, \dots, \tilde{l}_i, \dots, l_{m+n-1})$, $\tilde{l}_i = \deg \tilde{y}_i$.

*Proof.* Part (i) follows from Lemmas 4.3.1 and 4.3.3.

Now we consider Part (ii). Let $y_r = \prod_{j=1}^{l_r}(x - t_j^{(r)})$ and $\tilde{y}_r = \prod_{j=1}^{\tilde{l}_r}(x - \tilde{t}_j^{(r)})$, $r = 1, \dots, m+n-1$. Let $\boldsymbol{t} = (t_j^{(r)})_{r=1,\dots,m+n-1}^{j=1,\dots,l_r}$ and $\tilde{\boldsymbol{t}} = (\tilde{t}_j^{(r)})_{r=1,\dots,m+n-1}^{j=1,\dots,\tilde{l}_r}$, where we set $l_r = \tilde{l}_r, t_j^{(r)} = \tilde{t}_j^{(r)}$ if $r \neq i$.

The sequence $\boldsymbol{t}$ satisfies the BAE associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $\boldsymbol{l}$. We prove that $\tilde{\boldsymbol{t}}$ satisfies the BAE associated to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $\boldsymbol{l}^{[i]}$. Clearly, the BAEs for $\tilde{\boldsymbol{t}}$ and $\boldsymbol{t}$ related to $t_j^{(r)}$ with $|r-i| > 1$ are the same. On the other hand, the BAE for $\tilde{\boldsymbol{t}}$ related to $\tilde{t}_j^{(i)}$ holds by Lemmas 4.3.1 and 4.3.3. We only need to establish the BAE for $\tilde{\boldsymbol{t}}$ related to $t_j^{(i-1)}$ and $t_j^{(i+1)}$. We have two main cases depending on the sign of $s_i s_{i+1}$.

Suppose $s_i = s_{i+1}$. Dividing (4.19) by $y_i[-s_i]\tilde{y}_i[-s_i]$ and evaluating at $x = t_j^{(i-1)} - s_i h$ and $x = t_j^{(i+1)}$, we obtain

$$\prod_{a=1}^{l_i} \frac{t_j^{(i-1)} - t_a^{(i)}}{t_j^{(i-1)} - t_a^{(i)} - s_i h} = \prod_{a=1}^{\tilde{l}_i} \frac{t_j^{(i-1)} - \tilde{t}_a^{(i)}}{t_j^{(i-1)} - \tilde{t}_a^{(i)} - s_i h},$$

$$\prod_{a=1}^{l_i} \frac{t_j^{(i+1)} - t_a^{(i)} + s_i h}{t_j^{(i+1)} - t_a^{(i)}} = \prod_{a=1}^{\tilde{l}_i} \frac{t_j^{(i+1)} - \tilde{t}_a^{(i)} + s_i h}{t_j^{(i+1)} - \tilde{t}_a^{(i)}}.$$

Thus, the BAE for $\tilde{\boldsymbol{t}}$ related to $t_j^{(i\pm1)}$ follows from the BAE for $\boldsymbol{t}$ related to $t_j^{(i\pm1)}$.

If $s_i = -s_{i+1}$, then the argument depends on $s_{i-1}$, $s_{i+2}$. Here we only treat the case of $s_{i-1} = -s_i$. All other cases are similar, we omit further details.

We prove the BAE for $\tilde{\boldsymbol{t}}$ related to $t_j^{(i-1)}$, which has the form

$$\frac{\varphi_{i-1}^{\boldsymbol{s}^{[i]}}(t_j^{(i-1)})}{\psi_{i-1}^{\boldsymbol{s}^{[i]}}(t_j^{(i-1)})} \cdot \frac{y_{i-2}(t_j^{(i-1)} + s_{i-1}h)}{y_{i-2}(t_j^{(i-1)})} \cdot \frac{y_{i-1}(t_j^{(i-1)} - s_{i-1}h)}{y_{i-1}(t_j^{(i-1)} + s_{i+1}h)} \cdot \frac{\tilde{y}_i(t_j^{(i-1)})}{\tilde{y}_i(t_j^{(i-1)} + s_i h)} = -1\,. \quad (4.21)$$

Substituting $x = t_j^{(i-1)} - s_i h$ and $x = t_j^{(i-1)}$ to (4.20) and dividing, we get

$$\frac{\tilde{y}_i(t_j^{(i-1)})}{\tilde{y}_i(t_j^{(i-1)} + s_i h)} = -\frac{\psi_i^{\boldsymbol{s}}(t_j^{(i-1)} - s_i h) y_{i-1}(t_j^{(i-1)} + s_{i+1}h) y_i(t_j^{(i-1)})}{\varphi_i^{\boldsymbol{s}}(t_j^{(i-1)}) y_{i-1}(t_j^{(i-1)} - s_{i-1}h) y_i(t_j^{(i-1)} - s_i h)}\,. \quad (4.22)$$

Changing $i$ in (4.20) to $i-1$ (recall $s_{i-1} = -s_i$) and substituting $x = t_j^{(i-1)}$, we have

$$\frac{\varphi_{i-1}^{\boldsymbol{s}}(t_j^{(i-1)}) y_{i-2}(t_j^{(i-1)} + s_{i-1}h) y_i(t_j^{(i-1)})}{\psi_{i-1}^{\boldsymbol{s}}(t_j^{(i-1)}) y_{i-2}(t_j^{(i-1)}) y_i(t_j^{(i-1)} - s_i h)} = 1\,. \quad (4.23)$$

Equation (4.21) follows from (4.22), (4.23), and the equality

$$\frac{\varphi_{i-1}^{\boldsymbol{s}^{[i]}}(t_j^{(i-1)})}{\psi_{i-1}^{\boldsymbol{s}^{[i]}}(t_j^{(i-1)})} = \frac{\varphi_{i-1}^{\boldsymbol{s}}(t_j^{(i-1)}) \varphi_i^{\boldsymbol{s}}(t_j^{(i-1)})}{\psi_{i-1}^{\boldsymbol{s}}(t_j^{(i-1)}) \psi_i^{\boldsymbol{s}}(t_j^{(i-1)} - s_i h)}\,. \qquad \square$$

**Remark 4.4.2.** *Suppose $s_i \neq s_{i+1}$. It is not hard to see that if $\varphi_i^{\boldsymbol{s}} y_{i-1}[-s_i] y_{i+1}$ and $\psi_i^{\boldsymbol{s}} y_{i-1} y_{i+1}[-s_i]$ in (4.20) have common roots, then $\boldsymbol{y}^{[i]}$ is not generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$.*

If $s_i = s_{i+1}$, then starting from a solution of the BAE we construct a family of new solutions represented by sequences $\boldsymbol{y}^{[i]}$. Here we use (4.19) and the parity sequence remains unchanged. We call this construction the *bosonic reproduction procedure in i-th direction.*

If $\varphi_i^{\boldsymbol{s}} y_{i-1}[-s_i] y_{i+1} \neq \psi_i^{\boldsymbol{s}} y_{i-1} y_{i+1}[-s_i]$, then starting from a solution of the BAE we construct a single new solution represented by $\boldsymbol{y}^{[i]}$. We use (4.20) and the parity sequence changes from $\boldsymbol{s}$ to $\boldsymbol{s}^{[i]}$. We call this construction the *fermionic reproduction procedure in i-th direction.*

From the very definition of the fermionic reproduction procedure, $(\boldsymbol{y}^{[i]})^{[i]} = \boldsymbol{y}$.

If $\boldsymbol{y}^{[i]}$ is generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, then by Theorem 4.4.1 we can apply the reproduction procedure again.

Let

$$P_{(\boldsymbol{y},\boldsymbol{s})} \subset \big(\mathbb{P}(\mathbb{C}[x])\big)^{m+n-1} \times S_{m|n} \quad (4.24)$$

be the closure of the set of all pairs $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{s}})$ obtained from the initial pair $(\boldsymbol{y}, \boldsymbol{s})$ by repeatedly applying all possible reproductions. We call $P_{(\boldsymbol{y},\boldsymbol{s})}$ the $\mathfrak{gl}_{m|n}$ *population of solutions of the BAE associated to* $\boldsymbol{s}$*,* $\boldsymbol{z}$*, and* $\boldsymbol{\lambda}$ *, originated at* $\boldsymbol{y}$. By definition, $P_{(\boldsymbol{y},\boldsymbol{s})}$ is a disjoint union over parity sequences,

$$P_{(\boldsymbol{y},\boldsymbol{s})} = \bigsqcup_{\tilde{\boldsymbol{s}} \in S_{m|n}} P^{\tilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})}, \qquad P^{\tilde{\boldsymbol{s}}}_{(\boldsymbol{y},\boldsymbol{s})} = P_{(\boldsymbol{y},\boldsymbol{s})} \cap \Big( \big(\mathbb{P}(\mathbb{C}[x])\big)^{m+n-1} \times \{\tilde{\boldsymbol{s}}\} \Big).$$

### 4.4.2 Rational difference operator associated to population

We define a rational difference operator which does not change under the reproduction procedure.

Let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. Let $\boldsymbol{z} = (z_1, \dots, z_p)$ be an $h$-generic sequence of complex numbers. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. The sequence $\boldsymbol{T}^{\boldsymbol{s}} = (T^{\boldsymbol{s}}_1, \dots, T^{\boldsymbol{s}}_{m+n})$ is given by (4.11).

Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials. Recall our convention that $y_0 = y_{m+n} = 1$. Define a rational difference operator $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y})$ over $\mathbb{K} = \mathbb{C}(x)$,

$$\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y}) = \overrightarrow{\prod_{1 \leqslant i \leqslant m+n}} \Big(1 - \frac{T^{\boldsymbol{s}}_i y_{i-1}[-s_i] y_i[s_i]}{T^{\boldsymbol{s}}_i[s_i] y_{i-1} y_i}\, \tau\Big)^{s_i}. \tag{4.25}$$

The following theorem is the main result of this section.

**Theorem 4.4.3.** *Let* $P$ *be a* $\mathfrak{gl}_{m|n}$ *population. Then the rational difference operator* $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y})$ *does not depend on the choice of* $\boldsymbol{y}$ *in* $P$*.*

*Proof.* We want to show

$$\Big(1 - \frac{T^{\boldsymbol{s}}_i y_{i-1}[-s_i] y_i[s_i]}{T^{\boldsymbol{s}}_i[s_i] y_{i-1} y_i}\, \tau\Big)^{s_i} \Big(1 - \frac{T^{\boldsymbol{s}}_{i+1} y_i[-s_{i+1}] y_{i+1}[s_{i+1}]}{T^{\boldsymbol{s}}_{i+1}[s_{i+1}] y_i y_{i+1}}\, \tau\Big)^{s_{i+1}}$$
$$= \Big(1 - \frac{T^{\boldsymbol{s}^{[i]}}_i y_{i-1}[-s_{i+1}] \tilde{y}_i[s_{i+1}]}{T^{\boldsymbol{s}^{[i]}}_i[s_{i+1}] y_{i-1} \tilde{y}_i}\, \tau\Big)^{s_{i+1}} \Big(1 - \frac{T^{\boldsymbol{s}^{[i]}}_{i+1} \tilde{y}_i[-s_i] y_{i+1}[s_i]}{T^{\boldsymbol{s}^{[i]}}_{i+1}[s_i] \tilde{y}_i y_{i+1}}\, \tau\Big)^{s_i}.$$

We have four cases, $(s_i, s_{i+1}) = (\pm 1, \pm 1)$. The cases of $s_i = s_{i+1}$ are proved similarly to Lemma 4.3.2.

The case of $s_i = -s_{i+1} = 1$ is similar to Lemma 4.3.5. Namely, we want to show

$$\Big(1 - \frac{T_i^{\boldsymbol{s}} y_{i-1}[-1] y_i[1]}{T_i^{\boldsymbol{s}}[1] y_{i-1} y_i}\,\tau\Big)\Big(1 - \frac{T_{i+1}^{\boldsymbol{s}} y_i[1] y_{i+1}[-1]}{T_{i+1}^{\boldsymbol{s}}[-1] y_i y_{i+1}}\,\tau\Big)^{-1}$$
$$= \Big(1 - \frac{T_i^{\boldsymbol{s}^{[i]}} y_{i-1}[1] \tilde{y}_i[-1]}{T_i^{\boldsymbol{s}^{[i]}}[-1] y_{i-1} \tilde{y}_i}\,\tau\Big)^{-1}\Big(1 - \frac{T_{i+1}^{\boldsymbol{s}^{[i]}} \tilde{y}_i[-1] y_{i+1}[1]}{T_{i+1}^{\boldsymbol{s}^{[i]}}[1] \tilde{y}_i y_{i+1}}\,\tau\Big).$$

This equation is proved by a direct computation using Lemma 4.1.2 and (4.20). We only note that the following identities

$$\frac{T_i^{\boldsymbol{s}^{[i]}}}{T_i^{\boldsymbol{s}^{[i]}}[-1]}\,\frac{T_{i+1}^{\boldsymbol{s}}}{T_{i+1}^{\boldsymbol{s}}[1]} = \frac{T_{i+1}^{\boldsymbol{s}^{[i]}}}{T_{i+1}^{\boldsymbol{s}^{[i]}}[1]}\,\frac{T_i^{\boldsymbol{s}}[2]}{T_i^{\boldsymbol{s}}[1]} = \prod_{\substack{k=1 \\ \lambda_i^{(\boldsymbol{s},k)} + \lambda_{i+1}^{(\boldsymbol{s},k)} \neq 0}}^{p} \frac{x - z_k - h}{x - z_k}$$

are used.

The case of $s_i = -s_{i+1} = -1$ is similar. $\square$

We denote the rational difference operator corresponding to a population $P$ by $\mathcal{R}_P$.

**Remark 4.4.4.** *Taking the quasiclassical limit $h \to 0$, a solution $\boldsymbol{t_h}$ of BAE* (4.6) *tends to a solution of BAE for the Gaudin model associated to $\mathfrak{gl}_{m|n}$ represented by a tuple $\boldsymbol{\mathcal{Y}} = (\mathcal{Y}_1, \dots, \mathcal{Y}_{m+n-1})$, see Remark 4.2.3. Note that $\tau = e^{-h\partial_x}$, we have*

$$1 - \frac{T_i^{\boldsymbol{s}} y_{i-1}[-s_i] y_i[s_i]}{T_i^{\boldsymbol{s}}[s_i] y_{i-1} y_i}\,\tau = h\left(\partial_x - s_i\Big(\ln \frac{\mathscr{T}_i^{\boldsymbol{s}} \mathcal{Y}_{i-1}}{\mathcal{Y}_i}\Big)'\right) + \mathcal{O}(h^2),$$

*where $\mathscr{T}_i^{\boldsymbol{s}} = \prod_{k=1}^{p} (x - z_k)^{\lambda_i^{(\boldsymbol{s},k)}}$, $\mathcal{Y}_0 = \mathcal{Y}_{m+n} = 1$. Ignoring the terms in $\mathcal{O}(h^2)$ for each factor, one gets from $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y})$ the rational pseudo-differential operator $R^{\boldsymbol{s}}(\boldsymbol{\mathcal{Y}})$ defined in [HMVY19, equation (6.5)].* $\square$

The transfer matrix $\mathcal{T}(x)$ (associated to the vector representation) can be included in a natural commutative algebra $\mathcal{B}$ generated by transfer matrices associated to other finite dimensional representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$, c.f. [KSZ08], [TZZ15]. We expect that similar to the even case, the rational difference operator $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y})$ encodes eigenvalues of algebra $\mathcal{B}$ acting on the Bethe vector corresponding to $\boldsymbol{y}$, c.f [T06]. Then, Theorem 4.4.3 would assert that formulas for eigenvalues of $\mathcal{B}$ acting on $L(\boldsymbol{\lambda}, \boldsymbol{z})$ do not depend on a choice of $\boldsymbol{y}$ in the population.

Similar to Lemmas 4.3.2 and 4.3.4, we show that formula for eigenvalue (2.8) or (4.13) does not change under $\mathfrak{gl}_{m|n}$ reproduction procedure.

**Lemma 4.4.5.** *Let* $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ *be a sequence of polynomials such that there exists a polynomial* $\tilde{y}_i$ *satisfying* (4.19) *if* $s_i = s_{i+1}$ *or* (4.20) *if* $s_i = -s_{i+1}$. *Then* $\mathcal{E}_{\boldsymbol{y}}(x) = \mathcal{E}_{\boldsymbol{y}^{[i]}}(x)$, *where* $\boldsymbol{y}^{[i]} = (y_1, \dots, \tilde{y}_i, \dots, y_{m+n-1})$.

*Proof.* The proof is similar to proofs of Lemmas 4.3.2 and 4.3.4. □

### 4.4.3 Example of a $\mathfrak{gl}_{2|1}$ population

In this section, we give an example of a population for the case of $\mathfrak{gl}_{2|1}$.

Set $m = 2$, $n = 1$, and $p = 3$. There are three parity sequences in $S_{2|1}$, namely, $\boldsymbol{s}_0 = (1, 1, -1)$, $\boldsymbol{s}_1 = (1, -1, 1)$, and $\boldsymbol{s}_2 = (-1, 1, 1)$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, where $\lambda^{(i)} = (1, 1, 0)$, for $i = 1, 2, 3$, in standard parity sequence $\boldsymbol{s}_0$. Let $\boldsymbol{l} = (0, 0)$ and $\boldsymbol{y} = (y_1, y_2) = (1, 1)$. We also set $h = 1$.

Let $\boldsymbol{z} = (0, \sqrt{2}, -\sqrt{2})$. Our choice of $\boldsymbol{z}$ is such that $z_i - z_j \notin h\mathbb{Z}$ for $i \neq j$. We have $\boldsymbol{T} = \boldsymbol{T}^{\boldsymbol{s}_0} = (x^3 + 3x^2 + x - 1, x^3 + 3x^2 + x - 1, 1)$.We consider the population $P_{(1,1)}$ of solutions of the BAE associated to $\boldsymbol{s}_0$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, originated at $\boldsymbol{y}$.

1. Applying bosonic reproduction procedure in the first direction to $\boldsymbol{y}$, we have $\boldsymbol{s}_0^{[1]} = \boldsymbol{s}_0$, $\boldsymbol{T}^{\boldsymbol{s}_0} = \boldsymbol{T}$, and $\boldsymbol{y}_c^{[1]} = (y_1^{[1]}, y_2^{[1]}) = (x - c, 1)$, where $c \in \mathbb{CP}^1$. Note that $\boldsymbol{y}_\infty^{[1]} = (1, 1) = \boldsymbol{y}$.

2. We then apply fermionic reproduction procedure in the second direction to $\boldsymbol{y}_c^{[1]}$. We have $(\boldsymbol{s}_0)^{[2]} = \boldsymbol{s}_1$ and $\boldsymbol{T}^{\boldsymbol{s}_1} = (x^3 + 3x^2 + x - 1, x^3 - 3x^2 + x + 1, 1)$. We have
$$(\boldsymbol{y}_c^{[1]})^{[2]} = (x - c, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1).$$

3. Finally, apply fermionic reproduction procedure in the first direction to $(\boldsymbol{y}_c^{[1]})^{[2]}$. We have $(\boldsymbol{s}_1)^{[1]} = \boldsymbol{s}_2$ and $\boldsymbol{T}^{\boldsymbol{s}_2} = ((x - 1)(x - 2)(x^2 - 2x - 1)(x^2 - 4x + 2), 1, 1)$. We have
$$((\boldsymbol{y}_c^{[1]})^{[2]})^{[1]} = \big(6(x - 1)^4 - 9(x - 1)^2 + 1, 4x^3 - (6 + 3c)x^2 + 3cx + c + 1\big).$$

It is easy to check that all further reproduction procedures cannot create a new pair of polynomials. Therefore the $\mathfrak{gl}_{2|1}$ population $P_{(1,1)}$ is the union of three $\mathbb{CP}^1$, $P^{\boldsymbol{s}_0}_{(1,1)} = \{(x-c,1) \mid c \in \mathbb{CP}^1\}$, $P^{\boldsymbol{s}_1}_{(1,1)} = \{(x-c, 4x^3-(6+3c)x^2+3cx+c+1) \mid c \in \mathbb{CP}^1\}$, and $P^{\boldsymbol{s}_2}_{(1,1)} = \{(6(x-1)^4 - 9(x-1)^2 + 1, 4x^3 - (6+3c)x^2 + 3cx + c + 1) \mid c \in \mathbb{CP}^1\}$.

## 4.5 Populations and superflag varieties

In this section, we show that $\mathfrak{gl}_{m|n}$ populations associated to typical $\boldsymbol{\lambda}$ are isomorphic to the variety of the full superflags.

### 4.5.1 Discrete exponents and dominants

Following [HMVY19], we introduce the following partial ordering on the set of partitions with $r$ parts. Let $\boldsymbol{a} = (a_1 \leqslant a_2 \leqslant \ldots \leqslant a_r)$ and $\boldsymbol{b} = (b_1 \leqslant b_2 \leqslant \ldots \leqslant b_r)$, $a_i, b_i \in \mathbb{Z}_{\geqslant 0}$, be two partitions with $r$ parts. If $b_i \geqslant a_i$ for all $i = 1, \ldots, r$, we say that $\boldsymbol{b}$ *dominates* $\boldsymbol{a}$.

For a partition $\boldsymbol{a}$ with $r$ parts, we call the smallest partition with $r$ distinct parts that dominates $\boldsymbol{a}$ the *dominant* of $\boldsymbol{a}$ and denote it by $\bar{\boldsymbol{a}} = (\bar{a}_1 < \bar{a}_2 < \cdots < \bar{a}_r)$. Namely, the partition $\bar{\boldsymbol{a}}$ is such that $\bar{\boldsymbol{a}}$ dominates $\boldsymbol{a}$ and if a partition $\boldsymbol{a}'$ with $r$ distinct parts dominates $\boldsymbol{a}$ then $\boldsymbol{a}'$ dominates $\bar{\boldsymbol{a}}$. The partition $\bar{\boldsymbol{a}}$ is unique.

We identify a set of non-negative integers with a partition by rearranging their elements into weakly increasing order.

This definition is motivated by the relation of exponents for a sum of spaces of functions to exponents of the summands. We describe this phenomenon for the discrete exponents of spaces of functions.

Let $V$ be an $r$-dimensional space of functions. Let $z \in \mathbb{C}$ be such that all functions in $V$ are well-defined at $z - h\mathbb{Z}$. Then there exists a partition with $r$ distinct parts $\boldsymbol{c} = (c_1 < \cdots < c_r)$ and a basis of $\{v_1, \cdots, v_r\}$ of $V$ such that for $i = 1, \ldots, r$, we have $v_i(z - jh) = 0$ for $j = 1, \ldots, c_i$ and $v_i(z - (c_i + 1)h) \neq 0$. This sequence of integers is defined uniquely and will be called the *sequence of discrete exponents of* $V$ *at* $z$. We denote the set $\boldsymbol{c}$ by $\boldsymbol{E}_z(V)$.

Let $V_1, \dots, V_k$ be spaces of functions such that the sum $V = \sum_{i=1}^k V_i$ is a direct sum. Let $\boldsymbol{a}_z = \sqcup_{i=1}^k \boldsymbol{E}_z(V_i)$, then $\boldsymbol{E}_z(V)$ dominates $\bar{\boldsymbol{a}}_z$. Moreover, for generic spaces of functions $V_i$, we have the equality $\boldsymbol{E}_z(V) = \bar{\boldsymbol{a}}_z$.

### 4.5.2 Space of rational functions associated to a solution of BAE

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z} = (z_1, \dots, z_p)$ be an $h$-generic sequence of complex numbers.

Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ represent a solution of the BAE associated to $\boldsymbol{\lambda}, \boldsymbol{z}$, and the standard parity sequence $\boldsymbol{s}_0$. Suppose further that $\boldsymbol{y}$ is generic with respect to $\boldsymbol{\lambda}, \boldsymbol{z}, \boldsymbol{s}_0$. Recall the rational difference operator $\mathcal{R}^{\boldsymbol{s}_0}(\boldsymbol{y}) = \mathcal{D}_{\bar{0}}(\boldsymbol{y})\mathcal{D}_{\bar{1}}^{-1}(\boldsymbol{y})$ associated to the population $P_{(\boldsymbol{y},\boldsymbol{s}_0)}$ generated by $\boldsymbol{y}$, see (4.25). Let $V_{\boldsymbol{y}} = \ker \mathcal{D}_{\bar{0}}(\boldsymbol{y})$ and $U_{\boldsymbol{y}} = \ker \mathcal{D}_{\bar{1}}(\boldsymbol{y})$.

Note that the sequence $(y_1, \dots, y_{m-1})$ represents a solution of the BAE associated to the Lie algebra $\mathfrak{gl}_m$. It follows from [MV03] that one can generate a $\mathfrak{gl}_m$ population starting from $(y_1, \dots, y_{m-1})$ using bosonic reproduction procedures. Moreover, the corresponding difference operator to this population is given by $y_m \cdot \mathcal{D}_{\bar{0}}(\boldsymbol{y}) \cdot (y_m)^{-1}$. Therefore, by [MV03, Proposition 4.7], the space $y_m \cdot V_{\boldsymbol{y}}$ is an $m$-dimensional space of polynomials. Similarly, since $(y_{m+1}, \dots, y_{m+n-1})$ represents a solution of the BAE associated to the Lie algebra $\mathfrak{gl}_n$, the space $T_{m+1}[-1]y_m \cdot U_{\boldsymbol{y}}$ is an $n$-dimensional space of polynomials. In particular, $V_{\boldsymbol{y}}$ and $U_{\boldsymbol{y}}$ are spaces of rational functions.

In the remainder of Section 4.5, we impose the condition that $y_m(z_i + kh) \neq 0$ for $i = 1, \dots, p$ and $k \in \mathbb{Z}$.

Since $\boldsymbol{z}$ is $h$-generic and $y_m(z_i + kh) \neq 0$ for $1 \leqslant i \leqslant p$ and $k \in \mathbb{Z}$, it follows from [MTV07, Corollary 7.5] that the sequence of discrete exponents $\boldsymbol{E}_{z_i}(y_m \cdot V_{\boldsymbol{y}})$ is given by

$$\big(\lambda_m^{(i)} < \lambda_{m-1}^{(i)} + 1 < \cdots < \lambda_{m-k+1}^{(i)} + k - 1 < \cdots < \lambda_1^{(i)} + m - 1\big).$$

Therefore the sequence of discrete exponents $\boldsymbol{E}_{z_i+\lambda^{(i)}_{m+1}h}(T_{m+1}[-1]y_m \cdot V_{\boldsymbol{y}})$ is given by

$$\big(\lambda^{(i)}_m + \lambda^{(i)}_{m+1} < \lambda^{(i)}_{m-1} + \lambda^{(i)}_{m+1} + 1 < \cdots < \\ \lambda^{(i)}_{m-k+1} + \lambda^{(i)}_{m+1} + k - 1 < \cdots < \lambda^{(i)}_1 + \lambda^{(i)}_{m+1} + m - 1\big).$$

Similarly, the sequence of discrete exponents $\boldsymbol{E}_{z_i+\lambda^{(i)}_{m+1}h}(T_{m+1}[-1]y_m \cdot U_{\boldsymbol{y}})$ is given by

$$\big(0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+k} + k - 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+n} + n - 1\big).$$

**Lemma 4.5.1.** *If $\boldsymbol{\lambda}$ is typical, then $V_{\boldsymbol{y}} \cap U_{\boldsymbol{y}} = 0$.*

*Proof.* Since $\boldsymbol{\lambda}$ is typical, there exists some $i_0 \in \{1, \ldots, p\}$ such that $\lambda^{(i_0)}_m \geqslant n$. Therefore the largest discrete exponent of $T_{m+1}[-1]y_m \cdot U_{\boldsymbol{y}}$ at $z_{i_0} + \lambda^{(i_0)}_{m+1}h$ is strictly less than the smallest discrete exponent of $T_{m+1}[-1]y_m \cdot V_{\boldsymbol{y}}$ at $z_{i_0} + \lambda^{(i_0)}_{m+1}h$, namely,

$$\lambda^{(i_0)}_{m+1} - \lambda^{(i_0)}_{m+n} + n - 1 < n + \lambda^{(i_0)}_{m+1} \leqslant \lambda^{(i_0)}_m + \lambda^{(i_0)}_{m+1}.$$

Therefore, by the definition of discrete exponents, we have $(T_{m+1}[-1]y_m \cdot U_{\boldsymbol{y}}) \cap (T_{m+1}[-1]y_m \cdot V_{\boldsymbol{y}}) = 0$, which completes the proof. $\square$

Therefore, by Proposition 4.1.1, the operator $\mathcal{R}^{\boldsymbol{s}_0}(\boldsymbol{y})$ is an $(m|n)$-rational difference operator.

**Remark 4.5.2.** *If $\boldsymbol{\lambda}$ is not typical, then the intersection $V_{\boldsymbol{y}} \cap U_{\boldsymbol{y}}$ may be non-trivial. For example, consider the tensor product of the vector representations, namely $L(\boldsymbol{\lambda}) = (\mathbb{C}^{m|n})^{\otimes p}$, and the sequence of polynomials $\boldsymbol{y} = (1, \ldots, 1)$. Then we have $T_1(x) = (x - z_1 + h) \cdots (x - z_p + h)$ and $T_i(x) = 1$ for $i = 2, \ldots, m+n$. Therefore for the rational difference operator $\mathcal{R}^{\boldsymbol{s}_0}(\boldsymbol{y}) = \mathcal{D}_{\bar 0}(\boldsymbol{y})\mathcal{D}_{\bar 1}^{-1}(\boldsymbol{y})$, we have*

$$\mathcal{D}_{\bar 0}(\boldsymbol{y}) = \Big(1 - \frac{(x - z_1 + h) \cdots (x - z_p + h)}{(x - z_1) \cdots (x - z_p)}\tau\Big)(1 - \tau)^{m-1}, \qquad \mathcal{D}_{\bar 1}(\boldsymbol{y}) = (1 - \tau)^n. \quad \square$$

Fix $a \in \{0, 1, \ldots, m\}$ and $b \in \{0, 1, \ldots, n\}$. For each $1 \leqslant i \leqslant p$, set

$$\boldsymbol{A}_i = \big(\lambda^{(i)}_m + \lambda^{(i)}_{m+1} < \lambda^{(i)}_{m-1} + \lambda^{(i)}_{m+1} + 1 < \cdots < \lambda^{(i)}_{m-a+1} + \lambda^{(i)}_{m+1} + a - 1\big),$$

$$\boldsymbol{B}_i = \big(0 < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+2} + 1 < \cdots < \lambda^{(i)}_{m+1} - \lambda^{(i)}_{m+b} + b - 1\big).$$

**Lemma 4.5.3.** *If $b \leqslant \lambda_m^{(i)}$, then the dominant of $\boldsymbol{A}_i \sqcup \boldsymbol{B}_i$ is given by*

$$(0 < \lambda_{m+1}^{(i)} - \lambda_{m+2}^{(i)} + 1 < \ldots < \lambda_{m+1}^{(i)} - \lambda_{m+b}^{(i)} + b - 1 < \\ \lambda_m^{(i)} + \lambda_{m+1}^{(i)} < \cdots < \lambda_{m-a+1}^{(i)} + \lambda_{m+1}^{(i)} + a - 1).$$

*If $\lambda_{m-j+1}^{(i)} < b \leqslant \lambda_{m-j}^{(i)}$ for some $1 \leqslant j \leqslant a-1$, then the dominant of $\boldsymbol{A}_i \sqcup \boldsymbol{B}_i$ is given by*

$$(0 < \lambda_{m+1}^{(i)} - \lambda_{m+2}^{(i)} + 1 < \cdots < \\ \lambda_{m+1}^{(i)} - \lambda_{m+b}^{(i)} + b - 1 < \lambda_{m+1}^{(i)} + b < \lambda_{m+1}^{(i)} + b + 1 < \cdots < \\ \lambda_{m+1}^{(i)} + b + j - 1 < \lambda_{m-j}^{(i)} + \lambda_{m+1}^{(i)} + j < \cdots < \lambda_{m-a+1}^{(i)} + \lambda_{m+1}^{(i)} + a - 1).$$

*If $\lambda_{m-a+1}^{(i)} < b$, then the dominant of $\boldsymbol{A}_i \sqcup \boldsymbol{B}_i$ is given by*

$$(0 < \lambda_{m+1}^{(i)} - \lambda_{m+2}^{(i)} + 1 < \ldots < \lambda_{m+1}^{(i)} - \lambda_{m+b}^{(i)} + b - 1 < \\ \lambda_{m+1}^{(i)} + b < \lambda_{m+1}^{(i)} + b + 1 < \cdots < \lambda_{m+1}^{(i)} + b + a - 1).$$

*Proof.* If $b \leqslant \lambda_m^{(i)}$, the statement is clear. If $\lambda_{m-j+1}^{(i)} < b \leqslant \lambda_{m-j}^{(i)}$ for some $1 \leqslant j \leqslant a-1$. Let $\lambda_m^{(i)} = \ell$. Since $\lambda^{(i)}$ is a polynomial $\mathfrak{gl}_{m|n}$ weight, we have $\lambda_{m+\ell+k}^{(i)} = 0$ for $k = 1, \ldots, b-\ell$. In particular, the last $b-\ell$ numbers in $\boldsymbol{B}_i$ are consecutive integers from $\lambda_{m+1}^{(i)} + \ell$ to $\lambda_{m+1}^{(i)} + b - 1$. Adding $\lambda_m^{(i)} + \lambda_{m+1}^{(i)}$ into $\boldsymbol{B}_i$, the dominant of the new set is obtained by changing $\lambda_m^{(i)} + \lambda_{m+1}^{(i)}$ to $\lambda_{m+1}^{(i)} + b$. We add the numbers of $\boldsymbol{A}_i$ one by one (from left to right) into $\boldsymbol{B}_i$. Inductively, adding $\lambda_{m+1}^{(i)} + \lambda_{m-k+1}^{(i)} + k - 1$, if $\lambda_{m-k+1}^{(i)} < b$, then the dominant is obtained by changing $\lambda_{m+1}^{(i)} + \lambda_{m-k+1}^{(i)} + k - 1$ to $\lambda_{m+1}^{(i)} + b + k - 1$. Therefore the lemma follows. $\square$

### 4.5.3 Polynomials $\pi_{a,b}$

Let $\boldsymbol{s} \in S_{m|n}$ be a parity sequence. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ be a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights. Let $\boldsymbol{z} = (z_1, \ldots, z_p)$ be an $h$-generic sequence of complex numbers. Let $\boldsymbol{T}^{\boldsymbol{s}}$ be a sequence of polynomials associated to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, see (4.11). We set $T_i = T_i^{\boldsymbol{s}_0}$ the polynomials corresponding to the standard parity $\boldsymbol{s}_0$.

Define polynomials $\pi_{a,b}^{\boldsymbol{\lambda},\boldsymbol{z}}$ by

$$\pi_{a,b}^{\boldsymbol{\lambda},\boldsymbol{z}}(x) = \prod_{k=1}^{p}\prod_{i=1}^{a}\prod_{j=1}^{\min\{b,\lambda_{m-i+1}^{(k)}\}}(x - z_k + (i+j-a-b-1)h). \tag{4.26}$$

We often abbreviate $\pi_{a,b}^{\boldsymbol{\lambda},\boldsymbol{z}}$ to $\pi_{a,b}$.

The polynomials $T_i^{\boldsymbol{s}}$ can be expressed in terms of $T_i$ and $\pi_{a,b}$. Recall that we have

$$\boldsymbol{s}_i^+ = \begin{cases} m - \sigma_{\boldsymbol{s}}(i), & \text{if } s_i = 1, \\ \sigma_{\boldsymbol{s}}(i) - i, & \text{if } s_i = -1, \end{cases} \qquad \boldsymbol{s}_i^- = \begin{cases} i - \sigma_{\boldsymbol{s}}(i), & \text{if } s_i = 1, \\ \sigma_{\boldsymbol{s}}(i) - m - 1, & \text{if } s_i = -1. \end{cases} \tag{4.27}$$

**Theorem 4.5.4.** *We have*

$$T_i^{\boldsymbol{s}} = T_{\sigma_{\boldsymbol{s}}(i)}[\boldsymbol{s}_i^-]\,\frac{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^-}}{\pi_{\boldsymbol{s}_i^++1,\boldsymbol{s}_i^-}[-1]},\ \text{if } s_i = 1; \qquad T_i^{\boldsymbol{s}} = T_{\sigma_{\boldsymbol{s}}(i)}[\boldsymbol{s}_i^+]\,\frac{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^-+1}}{\pi_{\boldsymbol{s}_i^+,\boldsymbol{s}_i^-}[1]},\ \text{if } s_i = -1.$$

*Proof.* It is not hard to see that

$$\lambda_i^{(\boldsymbol{s},k)} = \begin{cases} \lambda_{\sigma_{\boldsymbol{s}}(i)}^{(k)} - \min\{\boldsymbol{s}_i^-, \lambda_{\sigma_{\boldsymbol{s}}(i)}^{(k)}\}, & \text{if } s_i = 1, \\ \lambda_{\sigma_{\boldsymbol{s}}(i)}^{(k)} + \#\{j \mid \lambda_{m-j+1}^{(k)} > \boldsymbol{s}_i^-, j = 1,2,\dots,\boldsymbol{s}_i^+\}, & \text{if } s_i = -1. \end{cases}$$

The theorem follows from a direct computation. $\square$

Note that polynomials $\pi_{a,b}$ are discrete versions of $\pi_{a,b}$ in [HMVY19, equation (7.1)], even though our definition here is more explicit. In particular, Theorem 4.5.4 is the counterpart of [HMVY19, Theorem 7.2].

The polynomial $\pi_{a,b}$ is related to the dominants of $\boldsymbol{A}_i \sqcup \boldsymbol{B}_i$ for all $1 \leqslant i \leqslant p$. Write the dominant $\overline{\boldsymbol{A}_i \sqcup \boldsymbol{B}_i}$ of $\boldsymbol{A}_i \sqcup \boldsymbol{B}_i$ as

$$0 = c_{a+b}^{(i)} < c_{a+b-1}^{(i)} + 1 < \cdots < c_{a+b-j}^{(i)} + j < \cdots < c_1^{(i)} + a + b - 1,$$

where $c_j^{(i)}$ are computed explicitly from Lemma 4.5.3. Let $\tilde{z}_i = z_i + \lambda_{m+1}^{(i)} h$ and set

$$\mathscr{T}_i(x) = \prod_{k=1}^{p}\prod_{j=1}^{c_i^{(k)}}(x - \tilde{z}_k + jh). \tag{4.28}$$

**Proposition 4.5.5.** *We have*

$$\pi_{a,b}\prod_{j=1}^{a}\mathscr{T}_j[j] = \prod_{i=1}^{a}\big(T_{m-a+i}[b+i]T_{m+1}[i-1]\big).$$

*Proof.* The lemma is obtained from Lemma 4.5.3 by a direct computation. $\square$

### 4.5.4 Generating map

Recall the notation from the beginning of Section 4.5.2, where $V_{\boldsymbol{y}} = \ker \mathcal{D}_{\bar{0}}(\boldsymbol{y})$ and $U_{\boldsymbol{y}} = \ker \mathcal{D}_{\bar{1}}(\boldsymbol{y})$.

For $a \in \{0, 1, \ldots, m\}$, $b \in \{0, 1, \ldots, n\}$, $v_1, \ldots, v_a \in V_{\boldsymbol{y}}$, $u_1, \ldots, u_b \in U_{\boldsymbol{y}}$, we define the function

$$y_{a,b} = \mathrm{Wr}\,(v_1, \ldots, v_a, u_1, \ldots, u_b)[1]\pi_{a,b} y_m[a+b]\frac{T_{m+1}[a+b-1]\cdots T_{m+b}[a]}{T_m[a+b]\cdots T_{m-a+1}[b+1]}.$$

We impose the technical condition that $y_m$ has only simple roots and is relatively prime to $y_m[k]$ for all non-zero integers $k$.

**Proposition 4.5.6.** *The function $y_{a,b}$ is a polynomial.*

*Proof.* This proposition is proved in Section 4.5.5. $\square$

In the following ,we assume that $\boldsymbol{\lambda}$ is typical. Set $W_{\boldsymbol{y}} = V_{\boldsymbol{y}} \oplus U_{\boldsymbol{y}}$. Given a parity sequence $\boldsymbol{s}$ and a full superflag $\mathscr{F} \in \mathscr{F}^{\boldsymbol{s}}(W_{\boldsymbol{y}})$ generated by a homogeneous basis $\{w_1, \ldots, w_{m+n}\}$, we define polynomials $y_i(\mathscr{F})$, $i = 1, \ldots, m+n-1$, by the formula

$$y_i(\mathscr{F}) = \begin{cases} y_{\boldsymbol{s}_i^+, \boldsymbol{s}_i^-}, & \text{if } s_i = 1, \\ y_{\boldsymbol{s}_i^+, \boldsymbol{s}_i^- + 1}, & \text{if } s_i = -1, \end{cases}$$

where we choose $\{v_1, \ldots, v_m\}$ and $\{u_1, \ldots, u_n\}$ such that the basis $\{w_1, \ldots, w_{m+n}\}$ is associated to $\{v_1, \ldots, v_m\}$, $\{u_1, \ldots, u_n\}$, and $\boldsymbol{s}$, see Section 4.1.2.

Define the *generating map* by

$$\beta^{\boldsymbol{s}} : \mathscr{F}^{\boldsymbol{s}}(W_{\boldsymbol{y}}) \to \big(\mathbb{P}(\mathbb{C}[x])\big)^{m+n-1}, \quad \mathscr{F} \mapsto \boldsymbol{y}(\mathscr{F}) = (y_1(\mathscr{F}), \ldots, y_{m+n-1}(\mathscr{F})).$$

The following theorem is our main result of this section.

**Theorem 4.5.7.** *For any superflag $\mathscr{F} \in \mathscr{F}^{\boldsymbol{s}}(W_{\boldsymbol{y}})$, we have $\beta^{\boldsymbol{s}}(\mathscr{F}) \in P^{\boldsymbol{s}}_{(\boldsymbol{y}, \boldsymbol{s}_0)}$. Moreover, the generating map $\beta^{\boldsymbol{s}} : \mathscr{F}^{\boldsymbol{s}}(W_{\boldsymbol{y}}) \to P^{\boldsymbol{s}}_{(\boldsymbol{y}, \boldsymbol{s}_0)}$ is a bijection and the complete factorization $\varpi^{\boldsymbol{s}}(\mathscr{F})$ of $\mathcal{R}^{\boldsymbol{s}_0}(\boldsymbol{y})$ given by* (4.2) *coincides with $\mathcal{R}^{\boldsymbol{s}}(\beta^{\boldsymbol{s}}(\mathscr{F}))$ given by* (4.25).

*Proof.* Note that the even case of this theorem is proved in [MV03, Theorem 4.16]. Due to Theorem 4.5.4 and Proposition 4.5.6, the proof is parallel to that of [HMVY19, Theorem 7.9]. □

This theorem does not rely on the technical condition imposed above Proposition 4.5.6, see Remark 4.5.10.

### 4.5.5 Proof of Proposition 4.5.6

We prepare several lemmas which will be used in the proof.

**Lemma 4.5.8.** *For any $v \in V_{\boldsymbol{y}}, u \in U_{\boldsymbol{y}}$, the function $T_{m+1}y_m[1]\mathrm{Wr}\,(v,u)$ is a polynomial. In particular, if $v \in V_{\boldsymbol{y}}, u \in U_{\boldsymbol{y}}$ are not regular at $z$, then there exists a $c \in \mathbb{C}$ such that $(u+cv)(z-h)=0$.*

*Proof.* The case of $\mathfrak{gl}_{1|1}$ is clear. Now we assume that either $m \geqslant 2$ or $n \geqslant 2$.

If the fermionic reproduction in the $m$-th direction is not applicable, then we can slightly change $y_{m-1}$ or $y_{m+1}$ using bosonic reproduction procedure such that the fermionic reproduction in the $m$-th direction can be applied to the new tuple of polynomials $\tilde{\boldsymbol{y}}$. Therefore we can assume that the fermionic reproduction in the $m$-th direction is applicable to $\boldsymbol{y}$ at the beginning.

It follows from (4.2) and Theorem 4.4.1 that

$$T_{m+1}y_m[1]\mathrm{Wr}\,(v,u) = T_{m+1}^{\boldsymbol{s}_0^{[m]}}\tilde{y}_m[-1].$$

Here $\tilde{y}_m$ depends on $u$ and $v$.

Initially, we have $v(\boldsymbol{y}) = T_m y_{m-1}[-1]/y_m$ and $u(\boldsymbol{y}) = y_{m+1}[-1]/(T_{m+1}[-1]y_m)$. Generic $u$ and $v$ can be obtained from $\boldsymbol{y}$ using only bosonic reproduction procedures. Moreover, the polynomial $y_m$ never changes. Note that, by Theorem 4.4.1, $\tilde{y}_m$ is a polynomial for generic $u$ and $v$. Therefore the first part of the lemma follows.

Recall that $y_m$ has only simple zeros and $y_m$ is relatively prime to $y_m[1]$. In addition, none of zeros of $y_m$ belongs to the sets $z_k+h\mathbb{Z}$, $k=1,\dots,p$. If $v \in V_{\boldsymbol{y}}, u \in U_{\boldsymbol{y}}$

are not regular at $z$, then $z$ is a root of $y_m$. Moreover, $v$ and $u$ have simple pole at $x = z$. The second statement follows directly from the first statement. $\square$

Suppose $V$ is an $r$-dimensional space of polynomials with the sequence of discrete exponents at $z$ given by $c_r < c_{r-1} + 1 < \cdots < c_{r-i} + i < \cdots < c_1 + r - 1$. Let $\mathscr{T}_i(x) = (x - z + h) \cdots (x - z + c_i h)$, $i = 1, \ldots, r$.

The following lemma is well-known, see e.g. [MTV08, Theorem 3.3].

**Lemma 4.5.9.** *Let* $f_1, \ldots, f_i \in V$, *then* $\mathrm{Wr}\,(f_1, \ldots, f_i)$ *is divisible by* $\prod_{j=1}^{i} \mathscr{T}_{r+1-j}[i-j]$. $\square$

*Proof of Proposition 4.5.6.* It is clear that we only need to consider the case when $v_1, \ldots, v_a, u_1, \ldots, u_b$ are linearly independent. The rational function $y_{a,b}$ can only have poles at $z_i + h\mathbb{Z}$, $1 \leqslant i \leqslant p$, and at zeros of the product of polynomials $\prod_{j=1}^{a+b} y_m[j]$.

Denote by $W_{a,b}$ the space of polynomials spanned by $\tilde{v}_j := T_{m+1}[-1] y_m v_j, \tilde{u}_k := T_{m+1}[-1] y_m u_k$, $1 \leqslant j \leqslant a$ and $1 \leqslant k \leqslant b$, then $\boldsymbol{E}_{\tilde{z}_i}(W_{a,b})$ dominates $\overline{\boldsymbol{A}_i \sqcup \boldsymbol{B}_i}$, where $\tilde{z}_i = z_i + \lambda_{m+1}^{(i)} h$. Therefore it follows from Lemma 4.5.9 that $\mathrm{Wr}\,(\tilde{v}_1, \ldots, \tilde{v}_a, \tilde{u}_1, \ldots, \tilde{u}_b)$ is divisible by $\prod_{j=1}^{a+b} \mathscr{T}_j[j-1]$, where $\mathscr{T}_j$ are defined in (4.28). It follows from Proposition 4.5.5 that the function $y_{a,b}$ is regular at $z_i + h\mathbb{Z}$, $1 \leqslant i \leqslant p$.

Write $y_m = \prod_{i=1}^{r} (x - z_i' + h)$, then by assumption $z_i' - z_j' \notin h\mathbb{Z}$ for $1 \leqslant i < j \leqslant r$. It follows from [MTV07, Corollary 7.5] that $\boldsymbol{E}_{z_i'}(\mathrm{span}\langle \tilde{v}_1, \ldots, \tilde{v}_a \rangle)$ dominates the partition $(0 < 2 < 3 < \cdots < a)$ with $a$ parts and $\boldsymbol{E}_{z_i'}(\mathrm{span}\langle \tilde{u}_1, \ldots, \tilde{u}_b \rangle)$ dominates the partition $(0 < 2 < 3 < \cdots < b)$ with $b$ parts. Therefore it follows from Lemma 4.5.8 that $\boldsymbol{E}_{z_i'}(W_{a,b})$ dominates the partition $(0 < 2 < 3 < \cdots < a + b)$ with $a + b$ parts. Hence, by Lemma 4.5.9, $\mathrm{Wr}\,(\tilde{v}_1, \ldots, \tilde{v}_a, \tilde{u}_1, \ldots, \tilde{u}_b)$ is divisible by $\prod_{j=2}^{a+b} y_m[j-2]$. In particular, $\mathrm{Wr}\,(v_1, \ldots, v_a, u_1, \ldots, u_b) y_m[a+b-1]$ is regular at zeros of the product of polynomials $\prod_{j=1}^{a+b} y_m[j-1]$. $\square$

**Remark 4.5.10.** *If* $\boldsymbol{\lambda}$ *is typical, the proof of Proposition 4.5.6 can be simplified as follows. Since* $\boldsymbol{\lambda}$ *is typical, generically the reproduction procedure is applicable for all parity sequences and all directions. Therefore, it follows from Theorem 4.4.1 that* $y_{a,b}$

*is a polynomial for generic* $v_1, \ldots, v_a$*,* $u_1, \ldots, u_b$*. Hence* $y_{a,b}$ *is a polynomial for all* $v_1, \ldots, v_a$*,* $u_1, \ldots, u_b$*.* $\square$

### 4.6 Quasi-periodic Case

In this section, we generalize our results to the quasi-periodic case.

#### 4.6.1 Twisted transfer matrix and Bethe ansatz

We follow the notation in Section 4.2.2.

Let $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_{m+n})$ be a sequence of complex numbers such that $e^{h\kappa_i} \neq e^{h\kappa_j}$ for $1 \leqslant i < j \leqslant m+n$. Let $Q_{\boldsymbol{\kappa}}$ be the diagonal matrix $\mathrm{diag}(e^{h\kappa_1}, \ldots, e^{h\kappa_{m+n}})$. Define the twisted transfer matrix $\mathcal{T}_{\boldsymbol{\kappa}}(x)$ by

$$\mathcal{T}_{\boldsymbol{\kappa}}(x) = \mathrm{str}(Q_{\boldsymbol{\kappa}}\mathcal{L}(x)) = \sum_{i=1}^{m+n} (-1)^{|i|} e^{h\kappa_i} \mathcal{L}_{ii}(x).$$

It is known that the twisted transfer matrices commute, $[\mathcal{T}_{\boldsymbol{\kappa}}(x_1), \mathcal{T}_{\boldsymbol{\kappa}}(x_2)] = 0$. Moreover, $\mathcal{T}_{\boldsymbol{\kappa}}(x)$ commutes with the subalgebra $\mathrm{U}(\mathfrak{h})$.

The Bethe ansatz equation associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$, and $\boldsymbol{l}$ is a system of algebraic equations in variables $\boldsymbol{t}$:

$$\begin{aligned} e^{h(\kappa_i - \kappa_{i+1})} \prod_{k=1}^{p} \frac{t_j^{(i)} - z_k + s_i \lambda_i^{(\boldsymbol{s},k)} h}{t_j^{(i)} - z_k + s_{i+1} \lambda_{i+1}^{(\boldsymbol{s},k)} h} \prod_{r=1}^{l_{i-1}} \frac{t_j^{(i)} - t_r^{(i-1)} + s_i h}{t_j^{(i)} - t_r^{(i-1)}} & \\ \times \prod_{\substack{r=1 \\ r \neq j}}^{l_i} \frac{t_j^{(i)} - t_r^{(i)} - s_i h}{t_j^{(i)} - t_r^{(i)} + s_{i+1} h} \prod_{r=1}^{l_{i+1}} \frac{t_j^{(i)} - t_r^{(i+1)}}{t_j^{(i)} - t_r^{(i+1)} - s_{i+1} h} &= 1, \end{aligned} \tag{4.29}$$

where $i = 1, \ldots, m+n-1$, $j = 1, \ldots, l_i$.

After making cancellations as in (4.7), we require the solutions do not make the remaining denominators in (4.29) vanish.

We also impose the same condition, see Section 4.2.2, for variables which correspond to a simple odd root of the same color. Suppose $(\alpha_i^{\boldsymbol{s}}, \alpha_i^{\boldsymbol{s}}) = 0$ for some $i$.

Consider the BAE for $\boldsymbol{t}$ related to $t_j^{(i)}$ with all $t_b^{(a)}$ fixed, where $a \neq i$ and $1 \leqslant b \leqslant l_a$, this equation does not depend on $j$. Let $t_0^{(i)}$ be a solution of this equation with multiplicity $r$. Then we require that the number of $j$ such that $t_j^{(i)} = t_0^{(i)}$ is at most $r$, c.f. Theorem 4.6.1.

Suppose that $\boldsymbol{\lambda}$ is a sequence of polynomial $\mathfrak{gl}_{m|n}$ weights and $\boldsymbol{t}$ a solution of the BAE (4.29) associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$, and $\boldsymbol{l}$. Similar to Theorem 4.2.5, see [BR09], if the vector $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z}) \in L(\boldsymbol{\lambda},\boldsymbol{z})$ is well-defined and non-zero, then $w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z}) \in L(\boldsymbol{\lambda},\boldsymbol{z})$ is an eigenvector of twisted transfer matrix, $\mathcal{T}_{\boldsymbol{\kappa}}(x)w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z}) = \mathcal{E}_{\boldsymbol{\kappa}}(x)w^{\boldsymbol{s}}(\boldsymbol{t},\boldsymbol{z})$, where the eigenvalue $\mathcal{E}_{\boldsymbol{\kappa}}(x)$ is given by

$$\mathcal{E}_{\boldsymbol{\kappa}}(x) = \sum_{a=1}^{m+n} s_a\, e^{h\kappa_a} \prod_{k=1}^{p} \frac{x - z_k + s_a \lambda_a^{(\boldsymbol{s},k)} h}{x - z_k} \prod_{j=1}^{l_{a-1}} \frac{x - t_j^{(a-1)} + s_a h}{x - t_j^{(a-1)}} \prod_{j=1}^{l_a} \frac{x - t_j^{(a)} - s_a h}{x - t_j^{(a)}} \,. \tag{4.30}$$

Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials representing the solution $\boldsymbol{t}$, then

$$\mathcal{E}_{\boldsymbol{\kappa}}(x) = \mathcal{E}_{(\boldsymbol{y},\boldsymbol{\kappa})}(x) = \sum_{a=1}^{m+n} s_a\, e^{h\kappa_a}\, \frac{T_a^{\boldsymbol{s}}}{T_a^{\boldsymbol{s}}[s_a]} \frac{y_{a-1}[-s_a]}{y_{a-1}} \frac{y_a[s_a]}{y_a} \,.$$

### 4.6.2 Reproduction procedure and rational difference operators

Recall the notation given at the beginning of Section 4.4.1.

Set $\boldsymbol{\kappa}^{[i]} = (\kappa_1, \dots, \kappa_{i+1}, \kappa_i, \dots, \kappa_{m+n})$.

**Theorem 4.6.1.** *Let $\boldsymbol{y} = (y_1, \dots, y_{m+n-1})$ be a sequence of polynomials generic with respect to $\boldsymbol{s}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, such that $\deg y_k = l_k$, $k = 1, \dots, m+n-1$.*

1. *The sequence $\boldsymbol{y}$ represents a solution of the BAE (4.29) associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}$, and $\boldsymbol{l}$, if and only if for each $i = 1, \ldots, m+n-1$, there exists a unique polynomial $\tilde{y}_i$, such that*

$$\mathrm{Wr}^{\,s_i}\left(y_i, e^{(\kappa_i-\kappa_{i+1})x}\tilde{y}_i\right) = e^{(\kappa_i-\kappa_{i+1})x}T_i^{\boldsymbol{s}}\left(T_{i+1}^{\boldsymbol{s}}\right)^{-1} y_{i-1}[-s_i]y_{i+1}, \quad \text{if } s_i = s_{i+1}, \tag{4.31}$$

$$y_i\,\tilde{y}_i[-s_i] = e^{h\kappa_i}\varphi_i^{\boldsymbol{s}} y_{i-1}[-s_i]y_{i+1} - e^{h\kappa_{i+1}}\psi_i^{\boldsymbol{s}} y_{i-1}y_{i+1}[-s_i], \qquad \text{if } s_i \neq s_{i+1}. \tag{4.32}$$

2. *If $\boldsymbol{y}^{[i]} = (y_1, \ldots, \tilde{y}_i, \ldots, y_{m+n-1})$ is generic with respect to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, and $\boldsymbol{z}$, then $\boldsymbol{y}^{[i]}$ represents a solution of the BAE (4.29) associated to $\boldsymbol{s}^{[i]}$, $\boldsymbol{\lambda}$, $\boldsymbol{\kappa}^{[i]}$, $\boldsymbol{z}$, and $\boldsymbol{l}^{[i]}$, where $\boldsymbol{l}^{[i]} = (l_1, \ldots, \tilde{l}_i, \ldots, l_{m+n-1})$, $\tilde{l}_i = \deg \tilde{y}_i$.*

*Proof.* For part (i), the case of (4.31) is proved in [MV08, Theorem 7.4]. The proofs of (4.32) in part (i) and part (ii) are similar to that of Theorem 4.4.1. □

Thanks to Theorem 4.6.1, we define similarly the twisted bosonic and fermionic reproduction procedures in $i$-th direction, the twisted $\mathfrak{gl}_{m|n}$ population $P(\boldsymbol{y},\boldsymbol{\kappa})$ of solutions of the BAE associated to $\boldsymbol{s}$, $\boldsymbol{z}$, $\boldsymbol{\lambda}$, originated at $(\boldsymbol{y},\boldsymbol{\kappa})$. Here the reproduction procedure in $i$-th direction sends $(\boldsymbol{y},\boldsymbol{\kappa})$ to $(\boldsymbol{y}^{[i]},\boldsymbol{\kappa}^{[i]})$. Note that for both twisted bosonic and fermionic reproduction procedures, the sequence $\boldsymbol{\kappa}$ is changed to $\boldsymbol{\kappa}^{[i]}$.

Define a rational difference operator $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y},\boldsymbol{\kappa})$ over $\mathbb{K} = \mathbb{C}(x)$,

$$\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y},\boldsymbol{\kappa}) = \overrightarrow{\prod_{1\leqslant i\leqslant m+n}}\left(1 - e^{h\kappa_i}\,\frac{T_i^{\boldsymbol{s}} y_{i-1}[-s_i]y_i[s_i]}{T_i^{\boldsymbol{s}}[s_i]y_{i-1}y_i}\,\tau\right)^{s_i}. \tag{4.33}$$

**Theorem 4.6.2.** *Let $P$ be a twisted $\mathfrak{gl}_{m|n}$ population. Then the rational difference operator $\mathcal{R}^{\boldsymbol{s}}(\boldsymbol{y},\boldsymbol{\kappa})$ does not depend on a choice of $(\boldsymbol{y},\boldsymbol{\kappa})$ in $P$.*

*Proof.* The proof is similar to that of Theorem 4.4.3. □

**Proposition 4.6.3.** *Let $\boldsymbol{y} = (y_1, \ldots, y_{m+n-1})$ be a sequence of polynomials such that there exists a sequence of polynomials $\boldsymbol{y}^{[i]} = (y_1, \ldots, \tilde{y}_i, \ldots, y_{m+n-1})$ satisfying (4.31) if $s_i = s_{i+1}$ or (4.32) if $s_i = -s_{i+1}$. Then $\mathcal{E}_{(\boldsymbol{y},\boldsymbol{\kappa})}(x) = \mathcal{E}_{(\boldsymbol{y}^{[i]},\boldsymbol{\kappa}^{[i]})}(x)$.*

*Proof.* The proof is similar to proofs of Lemmas 4.3.2 and 4.3.4. $\square$

Let $\sigma_i$ be the permutation $(i, i+1)$ in the symmetric group $\mathfrak{S}_{m+n}$. There is a natural action of $\mathfrak{S}_{m+n}$ on the set of sequences of $m+n$ complex numbers. Namely, for a sequence $\boldsymbol{\kappa}$, we have $\sigma_i \boldsymbol{\kappa} = \boldsymbol{\kappa}^{[i]}$.

**Theorem 4.6.4.** *The map $P(\boldsymbol{y}, \boldsymbol{\kappa}) \to \mathfrak{S}_{m+n}\boldsymbol{\kappa}$ given by $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\kappa}}) \mapsto \tilde{\boldsymbol{\kappa}}$ is a bijection between the twisted population $P(\boldsymbol{y}, \boldsymbol{\kappa})$ and the orbit of $\boldsymbol{\kappa}$ under the action of symmetric group $\mathfrak{S}_{m+n}$. In particular, it gives a bijection between the twisted population $P(\boldsymbol{y}, \boldsymbol{\kappa})$ and the symmetric group $\mathfrak{S}_{m+n}$.*

*Proof.* The proof is similar to that of [MV08, Corollary 4.12]. $\square$

# 5. SUMMARY

The reproduction procedure produces a family $P$ of other solutions called the population with a given solutions of the $\mathfrak{gl}_{m|n}$ Gaudin Bethe ansatz equation associated to a tensor product of polynomial modules. We associate a rational pseudodifferential operator $R$ and a superspace $W$ of rational functions to a population.

If at least one module is typical then the population $P$ is canonically identified with the set of minimal factorizations of $R$ and with the space of full superflags in $W$.

We also establish a duality of the non-periodic Gaudin model associated with superalgebra $\mathfrak{gl}_{m|n}$ and the non-periodic Gaudin model associated with algebra $\mathfrak{gl}_k$.

We conjecture that the singular eigenvectors (up to rescaling) of all $\mathfrak{gl}_{m|n}$ Gaudin Hamiltonians are in a bijective correspondence with certain superspaces of rational functions.

The reproduction procedure produces a family $P$ of other solutions called the population with a given solutions of the Bethe ansatz equations of the non-homogeneous periodic XXX model associated to super Yangian $\mathrm{Y}(\mathfrak{gl}_{m|n})$.

We associate a rational difference operator $\mathcal{D}$ and a superspace of rational functions $W$ to a population. We show that the set of complete factorizations of $\mathcal{D}$ is in canonical bijection with the variety of superflags in $W$ and that each generic superflag defines a solution of the Bethe ansatz equation. We also give the analogous statements for the quasi-periodic supersymmetric spin chains.

# 6. RECOMMENDATIONS

Here are some possible future directions of research.

Conjecture 2.7.1 says the eigenvalues of the Bethe algebra $\mathcal{B}_{m|n}$ acting on the Bethe vector $v(\boldsymbol{s}, \boldsymbol{y})$ can be found by expanding the corresponding operator $R_P$, see (2.22). In the $\mathfrak{gl}_k$ case, this conjecture is first proved in [MTV06b] by an explicit computation. Later, this conjecture is proved again in [MM17] by the affine Harish-Chandra map and a theorem in [FFR94]. The theorem in [FFR94] relates the eigenvalues of the Bethe algebra with certain Cartan algebra-valued rational functions.

The affine Harish-Chandra map in the $\mathfrak{gl}_{m|n}$ case is known. It seems that a theorem similar to the one in [FFR94] should hold in the $\mathfrak{gl}_{m|n}$ case. In order to prove the theorem, one need to show some properties in the super Wakimoto module, see, e.g., [IK02]. In [FFR94], the image of $\mathcal{B}_k$ under affine Harish-Chandra map is isomorphic to the classical $\mathcal{W}$-algebra. Recently, the supersymmetric $\mathcal{W}$-algebra is given in [MRS19], where a finite set of free generators in the case of $A(n, n \pm 1)$ is provided. The classical supersymmetric $\mathcal{W}$-algebra should be obtained by taking a certain limit. It is interesting to see how the Bethe algebra $\mathcal{B}_{m|n}$ is related with the classical supersymmetric $\mathcal{W}$-algebra. It is also interesting to construct the classical $\mathcal{W}$-algebra in the case of $A(m, n)$ for arbitrary $m, n$.

Conjecture 2.7.1 in the case of $n = 0$ is proved in [MTV07], which requires to interpret the $\mathfrak{gl}_m$ space as an intersection of Schubert cells. The obstacles in the $\mathfrak{gl}_{m|n}$ case is obvious: there is no such Grassmannian. From some examples we computed, it seems that we need to consider a space $G$: each point in $G$ is $(U, V, f)$, where $U, V$ are subspaces of $\mathbb{C}[x]$, $f \in \mathbb{C}[x]$, such that for any $u \in U$, $v \in V$, we have $f | \mathrm{Wr}\,(u, v)$. Conjecture 2.7.1 seems doable when $V$ is one-dimensional, namely in the $\mathfrak{gl}_{m|1}$ case.

In Theorem 3.4.2, we discover the duality between the images of $\mathcal{B}_{m|n}$ and $\mathcal{B}_k$, which suggests there should be a duality between the differential operators associated to the $\mathfrak{gl}_k$ populations and the rational pseudodifferential operators associated to the $\mathfrak{gl}_{m|n}$ populations. In the case of $m = 0$, this duality is given in [TU19]. In some sense, [TU19] manages to expand the inverses of differential operators. Then in the case of $m \neq 0$ case, we expect that a similar approach should work.

We were informed recently that certain order one pseudodifferential operators are classified by Wilson's Adelic Grassmannian, see [W93]. This connection may provide another geometric object we are trying to find for the $\mathfrak{gl}_{m|n}$ spaces.

The Weyl module associated to $\mathfrak{sl}_{m|n}[t]$ was introduced in [CLS19] with certain restrictions on $m, n$. The examples we computed for Conjecture 2.7.1 involve computing the graded characters of some Weyl modules $\mathcal{M}$. The $\mathfrak{gl}_{m|n}[t]$-module $\mathcal{M}$, roughly speaking, is a finite dimensional highest weight module generated by the highest weight vectors $v$ with highest $\mathfrak{gl}_{m|n}$ weight $k\omega_1$, where $\omega_1$ is the first fundamental $\mathfrak{gl}_{m|n}$ weight.

In the $n = 0$ case, the graded character of $\mathcal{M}_\lambda^{\text{sing}}$, the $\mathfrak{gl}_m$ singular subspaces of weights $\lambda$, is given by a certain **Kostka polynomial**, see, e.g., [CL05]. It seems that in the $\mathfrak{gl}_{m|n}$ case, the graded character of $\mathcal{M}_\lambda^{\text{sing}}$ is still given by a Kostka polynomial.

The reproduction procedure for other types of lie superalgebras should be developed. In the $\mathfrak{osp}(1|2)$ case, the reproduction procedure has some interesting phenomenon: the reproduction procedure is almost the same as the one in the $\mathfrak{sl}_2$ case. We are still trying to associated rational pseudodifferential operators to such populations.

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# APPENDIX

# APPENDIX A: THE BETHE ANSATZ FOR $\mathrm{Y}(\mathfrak{gl}_{1|1})$

In this section, we give the basics of Bethe ansatz for $\mathfrak{gl}_{1|1}$ XXX model (supersymmetric spin chains associated to $\mathfrak{gl}_{1|1}$). We follow the notation of Section 4.3.2. We also set $h = 1$.

### Super Yangian $\mathrm{Y}(\mathfrak{gl}_{1|1})$ and its representations

Recall that for $\mathrm{Y}(\mathfrak{gl}_{1|1})$ we have

$$[\mathcal{L}_{ii}(x_1), \mathcal{L}_{ii}(x_2)] = 0, \quad \mathcal{L}_{ij}(x_1)\mathcal{L}_{ij}(x_2) = \frac{x_1 - x_2 - (-1)^{|i|}}{x_2 - x_1 - (-1)^{|i|}} \mathcal{L}_{ij}(x_2)\mathcal{L}_{ij}(x_1), \tag{1}$$

$$\mathcal{L}_{kk}(x_1)\mathcal{L}_{ij}(x_2) = \frac{x_1 - x_2 - (-1)^{|i|}}{x_1 - x_2}\mathcal{L}_{ij}(x_2)\mathcal{L}_{kk}(x_1) + \frac{(-1)^{|i|}}{x_1 - x_2}\mathcal{L}_{ij}(x_1)\mathcal{L}_{kk}(x_2), \tag{2}$$

where $i \neq j$ and $i, j, k \in \{1, 2\}$.

In what follows we work with the standard parity sequence $\boldsymbol{s}_0$.

The description of finite dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_{1|1})$ is well known.

Let $\lambda = (\lambda_1, \lambda_2)$ be a $\mathfrak{gl}_{1|1}$ weight, we say that $\lambda$ is *non-degenerate* if $\lambda_1 + \lambda_2 \neq 0$. Clearly, $L_\lambda$ is two-dimensional if $\lambda$ is non-degenerate and one-dimensional otherwise. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ be a sequence of non-degenerate $\mathfrak{gl}_{1|1}$ weights, $\boldsymbol{z}$ a sequence of complex numbers. Let $\lambda^{(k)} = (a_k, b_k)$, $a_k, b_k \in \mathbb{C}$,

$$a = \sum_{k=1}^{p} a_k, \quad b = \sum_{k=1}^{p} b_k, \quad \varphi(x) = \prod_{k=1}^{p}(x - z_k + a_k), \quad \psi(x) = \prod_{k=1}^{p}(x - z_k - b_k).$$

**Theorem .0.1** ( [Zha95]). *Every finite dimensional irreducible representation of the algebra $\mathrm{Y}(\mathfrak{gl}_{1|1})$ is a tensor product of evaluation $\mathrm{Y}(\mathfrak{gl}_{1|1})$-modules up to twisting by a one-dimensional $\mathrm{Y}(\mathfrak{gl}_{1|1})$-module. Moreover, $L(\boldsymbol{\lambda}, \boldsymbol{z})$ is irreducible if and only if $\varphi(x)$ and $\psi(x)$ are relatively prime.*

Clearly, the $\mathrm{Y}(\mathfrak{gl}_{1|1})$-module $L(\boldsymbol{\lambda},\boldsymbol{z})$ is irreducible if and only if $z_i - z_j - a_i - b_j \neq 0$ for all $i \neq j$. Moreover, it satisfies the binary property. Namely, $L(\boldsymbol{\lambda},\boldsymbol{z})$ is irreducible if and only if $L_{\lambda^{(i)}}(z_i) \otimes L_{\lambda^{(j)}}(z_j)$ is irreducible for all $1 \leqslant i < j \leqslant p$. Furthermore, every finite dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_{1|1})$ has dimension $2^r$ for some non-negative integer $r$.

Let $v_1^{(k)}$ be the highest weight vector of $L_{\lambda^{(k)}}$ with respect to the standard root system, and $v_2^{(k)} = e_{21}v_1^{(k)}$. Then $v_1^{(k)}$, $v_2^{(k)}$ is a basis of $L_{\lambda^{(k)}}$. We use the shorthand notation $|0\rangle$ for $v_1^{(1)} \otimes \cdots \otimes v_1^{(p)}$.

Let $E_{ij}$, $i,j = 1,2$, be the linear operator in $\mathrm{End}(L_{\lambda^{(k)}})$ of parity $|i|+|j|$ such that $E_{ij}v_r^{(k)} = \delta_{jr}v_i^{(k)}$ for $r = 1,2$.

The R-matrix $R(x) \in \mathrm{End}(L_{\lambda^{(i)}}) \otimes \mathrm{End}(L_{\lambda^{(j)}})$ is given by

$$\begin{aligned} R(x) = E_{11}\otimes E_{11} &- \frac{b_i + a_j + x}{a_i + b_j - x} E_{22}\otimes E_{22} + \frac{b_j - b_i - x}{a_i + b_j - x} E_{11}\otimes E_{22} \\ &+ \frac{a_i - a_j - x}{a_i + b_j - x} E_{22}\otimes E_{11} - \frac{a_i + b_i}{a_i + b_j - x} E_{12}\otimes E_{21} + \frac{a_j + b_j}{a_i + b_j - x} E_{21}\otimes E_{12}. \end{aligned}$$

Clearly, $L_{\lambda^{(i)}}(z_i) \otimes L_{\lambda^{(j)}}(z_j)$ is irreducible if and only if $R(z_i - z_j)$ is well-defined and invertible.

Define an anti-automorphism $\iota : \mathrm{Y}(\mathfrak{gl}_{1|1}) \to \mathrm{Y}(\mathfrak{gl}_{1|1})$ by the rule, $\iota(\mathcal{L}_{ij}(x)) = (-1)^{|i||j|+|i|}\mathcal{L}_{ji}(x)$, $i,j = 1$. One has $\iota(X_1X_2) = (-1)^{|X_1||X_2|}\iota(X_2)\iota(X_1)$ for $X_1, X_2 \in \mathrm{Y}(\mathfrak{gl}_{1|1})$. Recall that $\mathcal{T}(x) = \mathcal{L}_{11}(x) - \mathcal{L}_{22}(x)$, therefore $\iota(\mathcal{T}(x)) = \mathcal{T}(x)$.

The *Shapovalov form* $B_{\lambda^{(i)}}$ on $L_{\lambda^{(i)}}$ is a bilinear form such that

$$B_{\lambda^{(i)}}(e_{ij}w_1, w_2) = (-1)^{(|i|+|j|)|w_1|} B_{\lambda^{(i)}}(w_1, (-1)^{|i||j|+|i|} e_{ji} w_2),$$

for all $i,j$ and $w_1, w_2 \in L_{\lambda^{(i)}}$, and $B_{\lambda^{(i)}}(v_1^{(i)}, v_1^{(i)}) = 1$. Explicitly, it is given by

$$B_{\lambda^{(i)}}(v_1^{(i)}, v_1^{(i)}) = 1,\ B_{\lambda^{(i)}}(v_1^{(i)}, v_2^{(i)}) = B_{\lambda^{(i)}}(v_2^{(i)}, v_1^{(i)}) = 0,\ B_{\lambda^{(i)}}(v_2^{(i)}, v_2^{(i)}) = -(a_i + b_i).$$

The Shapovalov forms $B_{\lambda^{(i)}}$ on $L_{\lambda^{(i)}}$ induce a bilinear form $B_{\boldsymbol{\lambda}} = \bigotimes_{k=1}^{p} B_{\lambda^{(k)}}$ (following the usual sign convention) on $L(\boldsymbol{\lambda})$.

Let $R_{\boldsymbol{\lambda},\boldsymbol{z}} \in \mathrm{End}(L(\boldsymbol{\lambda}))$ be the product of R-matrices,

$$R_{\boldsymbol{\lambda},\boldsymbol{z}} = \prod_{1\leqslant i\leqslant p}^{\longrightarrow} \prod_{i<j\leqslant p}^{\longrightarrow} R^{(i,j)}(z_i - z_j).$$

Define a bilinear form $B_{\boldsymbol{\lambda},\boldsymbol{z}}$ on $L(\boldsymbol{\lambda},\boldsymbol{z})$ by

$$B_{\boldsymbol{\lambda},\boldsymbol{z}}(w_1,w_2)=B_{\boldsymbol{\lambda}}(w_1,R_{\boldsymbol{\lambda},\boldsymbol{z}}w_2),$$

for all $w_1,w_2\in L(\boldsymbol{\lambda},\boldsymbol{z})$.

One shows that, c.f. [MTV06, Section 7],

$$B_{\boldsymbol{\lambda},\boldsymbol{z}}(|0\rangle,|0\rangle)=1,\qquad B_{\boldsymbol{\lambda},\boldsymbol{z}}(Xw_1,w_2)=(-1)^{|X||w_1|}B_{\boldsymbol{\lambda},\boldsymbol{z}}(w_1,\iota(X)w_2),$$

for all $X\in \mathrm{Y}(\mathfrak{gl}_{1|1})$, $w_1,w_2\in L(\boldsymbol{\lambda},\boldsymbol{z})$. In addition, if $L(\boldsymbol{\lambda},\boldsymbol{z})$ is irreducible, then $B_{\boldsymbol{\lambda},\boldsymbol{z}}$ is non-degenerate.

### Bethe ansatz for $\mathfrak{gl}_{1|1}$ XXX model

In this section, we study the spectrum of the transfer matrix $\mathcal{T}(x)=\mathcal{L}_{11}(x)-\mathcal{L}_{22}(x)$.

Let $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(p)})$ be a sequence of non-degenerate $\mathfrak{gl}_{1|1}$ weights. Recall from Section 4.3.2 that if $y=(x-t_1)\cdots(x-t_l)$ is a divisor of $\varphi(x)-\psi(x)$, then $\boldsymbol{t}=(t_1,\dots,t_l)$ is a solution of the BAE associated to $\boldsymbol{s}_0$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $l$.

It is convenient to renormalize the Bethe vector $w(\boldsymbol{t},\boldsymbol{z})$ associated to $\boldsymbol{t}$, see (4.9),:

$$\tilde{w}(\boldsymbol{t},\boldsymbol{z})=c_0w(\boldsymbol{t},\boldsymbol{z}),\qquad c_0=\prod_{i=1}^{l}\prod_{k=1}^{p}(t_i-z_k).$$

The factor $c_0$ clears up the denominators and the Bethe vector $\tilde{w}(\boldsymbol{t},\boldsymbol{z})$ is well-defined for all $\boldsymbol{z},\boldsymbol{t}$.

The following theorem is well known, see e.g. [BR08].

**Theorem .0.2.** *If the Bethe vector $\tilde{w}(\boldsymbol{t},\boldsymbol{z})$ is non-zero, then $\tilde{w}(\boldsymbol{t},\boldsymbol{z})$ is an eigenvector of the transfer matrix $\mathcal{T}(x)$ with the corresponding eigenvalue*

$$\mathcal{E}(x)=\frac{y[1]}{y}(\varphi-\psi)\prod_{k=1}^{p}(x-z_k)^{-1}. \tag{3}$$

*Proof.* For $j = 1, 2$, one has the following relation,

$$\mathcal{L}_{jj}(x)\mathcal{L}_{12}(t_1)\cdots\mathcal{L}_{12}(t_l) = \xi(x;\boldsymbol{t})\mathcal{L}_{12}(t_1)\cdots\mathcal{L}_{12}(t_l)\mathcal{L}_{jj}(x) + \sum_{i=1}^{l}\xi_i(x;\boldsymbol{t})\mathcal{L}_{12}(x)\mathcal{L}_{12}(t_1)\cdots\widehat{\mathcal{L}_{12}(t_i)}\cdots\mathcal{L}_{12}(t_l)\mathcal{L}_{jj}(t_i). \tag{4}$$

Here the symbol $\widehat{\mathcal{L}_{12}(t_i)}$ means the factor $\mathcal{L}_{12}(t_i)$ is skipped and the functions $\xi(x;\boldsymbol{t})$ and $\xi_i(x;\boldsymbol{t})$ are given by

$$\xi(x;\boldsymbol{t}) = \prod_{1\leqslant r\leqslant l}\frac{x-t_r-1}{x-t_r} = \frac{y[1]}{y},$$

$$\xi_i(x;\boldsymbol{t}) = (-1)^{i-1}\frac{1}{x-t_i}\prod_{1\leqslant r<i}\frac{t_i-t_r+1}{t_i-t_r}\prod_{i<r\leqslant l}\frac{t_i-t_r-1}{t_i-t_r}.$$

We have

$$\mathcal{T}(x)|0\rangle = (\varphi-\psi)\prod_{k=1}^{p}(x-z_k)^{-1}|0\rangle.$$

Since $\boldsymbol{t}$ is a solution of the BAE, we have $c_0\mathcal{T}(t_i)|0\rangle = 0$ for $i = 1, \ldots, l$. Therefore it follows from (4) that

$$\mathcal{T}(x)\tilde{w}(\boldsymbol{z},\boldsymbol{t}) = c_0(\mathcal{L}_{11}(x)-\mathcal{L}_{22}(x))\mathcal{L}_{12}(t_1)\cdots\mathcal{L}_{12}(t_l)|0\rangle = \frac{y[1]}{y}(\varphi-\psi)\prod_{k=1}^{p}(x-z_k)^{-1}\tilde{w}(\boldsymbol{z},\boldsymbol{t}).$$

□

Recall that the transfer matrix $\mathcal{T}(x)$ commutes with the subalgebra $\mathrm{U}(\mathfrak{gl}_{1|1})$ of $\mathrm{Y}(\mathfrak{gl}_{1|1})$.

**Proposition .0.3.** *The Bethe vector $\tilde{w}(\boldsymbol{t},\boldsymbol{z})$ is $\mathfrak{gl}_{1|1}$ singular.*

*Proof.* By (4.5), one has the following relation,

$$[\mathcal{L}_{21}^{(1)}, \mathcal{L}_{12}(t_1)\cdots\mathcal{L}_{12}(t_l)] = \sum_{i=1}^{l}\nu_i(\boldsymbol{t})\mathcal{L}_{12}(t_1)\cdots\widehat{\mathcal{L}_{12}(t_i)}\cdots\mathcal{L}_{12}(t_l)\mathcal{T}(t_i).$$

The functions $\nu_k(\boldsymbol{t})$ are given by

$$\nu_i(\boldsymbol{t}) = (-1)^i \prod_{1\leqslant r<i} \frac{t_i - t_r + 1}{t_i - t_r} \prod_{i<r\leqslant l} \frac{t_i - t_r - 1}{t_i - t_r}.$$

Note that $\mathcal{L}_{21}^{(1)}|0\rangle = 0$ and $c_0\mathcal{T}(t_i)|0\rangle = 0$ for $i = 1, \dots, l$, therefore the statement follows. □

**Proposition .0.4.** *Suppose $\varphi \neq \psi$. Let $\boldsymbol{t}$ and $\tilde{\boldsymbol{t}}$ be two different solutions of Bethe ansatz equation associated to $\boldsymbol{s}_0$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, then the Bethe vectors $\tilde{w}(\boldsymbol{t}, \boldsymbol{z})$ and $\tilde{w}(\tilde{\boldsymbol{t}}, \boldsymbol{z})$ are orthogonal with respect to the form $B_{\boldsymbol{\lambda},\boldsymbol{z}}$.*

*Proof.* Let $y$ and $\tilde{y}$ represent $\boldsymbol{t}$ and $\tilde{\boldsymbol{t}}$ respectively. Note that we have

$$B_{\boldsymbol{\lambda},\boldsymbol{z}}(\mathcal{T}(x)\tilde{w}(\boldsymbol{t}, \boldsymbol{z}), \tilde{w}(\tilde{\boldsymbol{t}}, \boldsymbol{z})) = B_{\boldsymbol{\lambda},\boldsymbol{z}}(\tilde{w}(\boldsymbol{t}, \boldsymbol{z}), \mathcal{T}(x)\tilde{w}(\tilde{\boldsymbol{t}}, \boldsymbol{z})).$$

It follows from Theorem .0.2 that

$$\Big(\frac{y[1]}{y} - \frac{\tilde{y}[1]}{\tilde{y}}\Big)(\varphi - \psi) \prod_{k=1}^{p} (x - z_k)^{-1} B_{\boldsymbol{\lambda},\boldsymbol{z}}(\tilde{w}(\boldsymbol{t}, \boldsymbol{z}), \tilde{w}(\tilde{\boldsymbol{t}}, \boldsymbol{z})) = 0.$$

Since $y$ and $\tilde{y}$ are linearly independent and $\varphi \neq \psi$, the statement follows. □

The following theorem is a particular case of [HLPRS18, Theorem 4.1] which asserts that the square of the norm of the Bethe vector is essentially given by the Jacobian of the BAE.

**Theorem .0.5** ( [HLPRS18])**.** *The square of the norm of the Bethe vector $\tilde{w}(\boldsymbol{t}, \boldsymbol{z})$ is given by*

$$\begin{aligned} B_{\boldsymbol{\lambda},\boldsymbol{z}}(\tilde{w}(\boldsymbol{t}, \boldsymbol{z}), \tilde{w}(\boldsymbol{t}, \boldsymbol{z})) = {} & (-1)^{l(l-1)/2} \prod_{1\leqslant i<j\leqslant l} \Big(\frac{t_i - t_j - 1}{t_i - t_j}\Big)^2 \\ & \times \prod_{i=1}^{l}\prod_{k=1}^{p} \big((t_i - z_k + a_k)(t_i - z_k - b_k)\big) \prod_{i=1}^{l} \Big(\sum_{k=1}^{p} \frac{a_k + b_k}{(t_i - z_k + a_k)(t_i - z_k - b_k)}\Big). \end{aligned}$$

□

**Theorem .0.6.** *Suppose $a + b \neq 0$. For generic $\boldsymbol{z}$, the Bethe ansatz is complete. In other words, there are exactly $2^{p-1}$ solutions $\boldsymbol{t}_i$, $i = 1, \dots, 2^{p-1}$, to the BAE associated to $\boldsymbol{s}_0$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, and $l$ such that the corresponding Bethe vectors $\tilde{w}(\boldsymbol{t}_i, \boldsymbol{z})$, $i = 1, \dots, 2^{p-1}$, form a basis of $L(\boldsymbol{\lambda}, \boldsymbol{z})^{\mathrm{sing}}$.*

*Proof.* Since $a+b \neq 0$, we have $\deg(\varphi-\psi) = p-1$. It is not difficult to see that $\dim L(\boldsymbol{\lambda})^{\mathrm{sing}} = 2^{p-1}$ and for generic $\boldsymbol{z}$ there are exactly $2^{p-1}$ distinct monic divisors of the polynomial $\varphi-\psi$. Each monic divisor of $\varphi-\psi$ corresponds to a solution $\boldsymbol{t}_i$, $i=1,\dots,2^{p-1}$, of BAE associated to $\boldsymbol{s}_0$, $\boldsymbol{\lambda}$, $\boldsymbol{z}$, with possibly different $l$. Due to Proposition .0.3 and Theorem .0.5, the Bethe vectors $\tilde{w}(\boldsymbol{t}_i,\boldsymbol{z})$ are singular and non-zero. Moreover, it follows from Proposition .0.4 that $\tilde{w}(\boldsymbol{t}_i,\boldsymbol{z})$, $i=1,\dots,2^{p-1}$, are linearly independent and hence form a basis of $L(\boldsymbol{\lambda},\boldsymbol{z})^{\mathrm{sing}}$. □

Let $\lambda^{(k)} = (1,0)$ and $z_k = 0$ for all $k=1,\dots,p$. This case is the *homogeneous super* XXX *model.* We obtain the completeness of homogeneous super XXX model.

Let $\theta$ be a primitive $p$-th root of unity. Set $\vartheta_i = 1/(\theta^i - 1)$, $i=1,\dots,p-1$.

**Corollary .0.7.** *The Bethe ansatz is complete for super homogeneous XXX model. Explicitly, the Bethe vectors form a basis of* $\big((\mathbb{C}^{1|1})^{\otimes p}\big)^{\mathrm{sing}}$ *and the transfer matrix* $\mathcal{T}(x)$ *acts on* $\big((\mathbb{C}^{1|1}(0))^{\otimes p}\big)^{\mathrm{sing}}$ *diagonally with simple spectrum. Moreover, the spectrum of* $\mathcal{T}(x)$ *acting on* $\big((\mathbb{C}^{1|1}(0))^{\otimes p}\big)^{\mathrm{sing}}$ *is given by*

$$\Big\{ \frac{(x-\vartheta_{i_1}-1)\cdots(x-\vartheta_{i_l}-1)}{(x-\vartheta_{i_1})\cdots(x-\vartheta_{i_l})}\cdot\frac{(x+1)^p - x^p}{x^p}, \qquad 1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant p-1, l=0,\dots,p-1\Big\}.$$

*Proof.* Note that $\varphi(x) = (x+1)^p$ and $\psi(x) = x^p$. Clearly, we have $\varphi-\psi = p(x-\vartheta_1)\cdots(x-\vartheta_{p-1})$. It is easy to see that $\vartheta_i - \vartheta_j \neq 0,1$ for $i\neq j$ and $\vartheta_i \notin \mathbb{Z}$. Therefore we have exactly $2^{p-1}$ distinct monic divisors

$$(x-\vartheta_{i_1})\cdots(x-\vartheta_{i_l}),\ 1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant p-1,\ l=0,\dots,p-1,$$

of the polynomial $\varphi-\psi$ and hence $2^{p-1}$ different solutions $\boldsymbol{t}_i$, $i=1,\dots,2^{p-1}$, of BAE. Therefore, as in Theorem .0.6, the Bethe wectors $\tilde{w}(\boldsymbol{t}_i,\boldsymbol{z})$, $i=1,\dots,2^{p-1}$, form a basis of $\big((\mathbb{C}^{1|1}(0))^{\otimes p}\big)^{\mathrm{sing}}$. □

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## Presentations

*The solutions of $\mathfrak{gl}(m|n)$ Gaudin Bethe ansatz equation, rational pseudodifferential operators, and the $\mathfrak{gl}(m|n)$ spaces*, Algebra and Number Theory Seminar, LSU, Oct. 2019

*The solutions of $\mathfrak{gl}_{m|n}$ Bethe ansatz equation and rational pseudodifferential operators*, Representation Theory and Integrable Systems, ETH Zurich, Aug. 2019