DUALITY OF GAUDIN MODELS

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To Katia, Lev, and my parents.

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TABLE OF CONTENTS

Page

Page

Page

ABSTRACT

Uvarov, Filipp PhD, Purdue University, August 2020. Duality of Gaudin models. Major Professor: Vitaly Tarasov.

We consider actions of the current Lie algebras $\mathfrak{gl}_n[t]$ and $\mathfrak{gl}_k[t]$ on the space $\mathfrak{P}_{kn}$ of polynomials in kn anticommuting variables. The actions depend on parameters $\bar{z} = (z_1, \dots, z_k)$ and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, respectively. We show that the images of the Bethe algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle} \subset U(\mathfrak{gl}_n[t])$ and $\mathcal{B}_{\bar{z}}^{\langle k \rangle} \subset U(\mathfrak{gl}_k[t])$ under these actions coincide.

To prove the statement, we use the Bethe ansatz description of eigenvectors of the Bethe algebras via spaces of quasi-exponentials. We establish an explicit correspondence between the spaces of quasi-exponentials describing eigenvectors of $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}$ and the spaces of quasi-exponentials describing eigenvectors of $\mathcal{B}_{\bar{z}}^{\langle k \rangle}$.

One particular aspect of the duality of the Bethe algebras is that the Gaudin Hamiltonians exchange with the Dynamical Hamiltonians. We study a similar relation between the trigonometric Gaudin and Dynamical Hamiltonians. In trigonometric Gaudin model, spaces of quasi-exponentials are replaced by spaces of quasi-polynomials. We establish an explicit correspondence between the spaces of quasi-polynomials describing eigenvectors of the trigonometric Gaudin Hamiltonians and the spaces of quasi-exponentials describing eigenvectors of the trigonometric Dynamical Hamiltonians.

We also establish the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for the rational, trigonometric and difference versions of Knizhnik-Zamolodchikov and Dynamical equations.

1. INTRODUCTION

1.1 Classical $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality

The classical $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality plays an important role in the representation theory and the classical invariant theory, for example, see [1], [2]. It states the following. Let $e^{\langle n\rangle}_{ij}$, $i, j = 1, \dots, n$, and $e^{\langle k\rangle}_{ab}$, $a, b = 1, \dots, k$, be the standard generators of the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$, respectively. Define $\mathfrak{gl}_n$- and $\mathfrak{gl}_k$-actions on the space $P_{kn} = \mathbb{C}[x_{11}, \dots, x_{kn}]$ of polynomials in kn commuting variables:

$$p^{\langle n,k\rangle} : e^{\langle n\rangle}_{ij} \mapsto \sum_{a=1}^{k} x_{ai} \frac{\partial}{\partial x_{aj}},$$

$$p^{\langle k,n\rangle} : e^{\langle k\rangle}_{ab} \mapsto \sum_{i=1}^{n} x_{ai} \frac{\partial}{\partial x_{bi}}.$$

Then the images $p^{\langle n,k\rangle}(U(\mathfrak{gl}_n))$ and $p^{\langle k,n\rangle}(U(\mathfrak{gl}_k))$ of the universal enveloping algebras of $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$, respectively, are mutual centralizers in $\mathrm{End}(P_{kn})$, and there is an isomorphism of $\mathfrak{gl}_n \oplus \mathfrak{gl}_k$-modules

$$P_{kn} \cong \bigoplus_{\lambda} L^{\langle n\rangle}_{\lambda} \otimes L^{\langle k\rangle}_{\lambda},$$

where $L^{\langle n\rangle}_{\lambda}$ and $L^{\langle k\rangle}_{\lambda}$ are the irreducible representations of $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ of highest weight λ, respectively. In particular, the centers of the algebras $p^{\langle n,k\rangle}(U(\mathfrak{gl}_n))$ and $p^{\langle k,n\rangle}(U(\mathfrak{gl}_k))$ coincide.

Instead of P_{kn}, one can consider the space $\mathfrak{P}_{kn}$ of polynomials in kn anticommuting variables $\xi_{11}, \dots, \xi_{kn}$. Define $\mathfrak{gl}_n$- and $\mathfrak{gl}_k$-actions on the space $\mathfrak{P}_{kn}$ by

$$\pi^{\langle n,k\rangle} : e^{\langle n\rangle}_{ij} \mapsto \sum_{a=1}^{k} \xi_{ai} \partial_{aj},$$

$$\pi^{\langle k,n\rangle} : e^{\langle k\rangle}_{ab} \mapsto \sum_{i=1}^{n} \xi_{ai} \partial_{bi},$$

where ∂_{ai}, $a = 1, \ldots, k$, $i = 1, \ldots, n$ are left derivations, see formula (2.10) for the definition. Then, similarly to the case of P_{kn}, the algebras $\pi^{\langle n,k \rangle}(U(\mathfrak{gl}_n))$ and $\pi^{\langle k,n \rangle}(U(\mathfrak{gl}_k))$ are mutual centralizers in $\mathrm{End}(\mathfrak{P}_{kn})$, and there is an isomorphism of $\mathfrak{gl}_n \oplus \mathfrak{gl}_k$-modules

$$\mathfrak{P}_{kn} \cong \bigoplus_{\lambda} L_{\lambda}^{\langle n \rangle} \otimes L_{\lambda'}^{\langle k \rangle},$$

where the sum runs over $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \leqslant k$, and λ' denotes the conjugate of λ, see Section 1.7 for the definition. In particular, the centers of the algebras $\pi^{\langle n,k \rangle}(U(\mathfrak{gl}_n))$ and $\pi^{\langle k,n \rangle}(U(\mathfrak{gl}_k))$ coincide. In this dissertation, we will focus on the space $\mathfrak{P}_{kn}$ rather than P_{kn}.

The pair $(\mathfrak{gl}_k, \mathfrak{gl}_n)$ is an example of a Howe dual pair. A pair of reductive Lie algebras (g_1, g_2) is called a Howe dual pair if both g_1 and g_2 act on P_{kn} or $\mathfrak{P}_{kn}$, and the images of their universal enveloping algebras under the corresponding actions are mutual centralizers in $\mathrm{End}(P_{kn})$ or $\mathrm{End}(\mathfrak{P}_{kn})$, respectively. Other examples of Howe dual pairs include $(\mathfrak{o}_k, \mathfrak{sp}_{2n})$, $(\mathfrak{sp}_k, \mathfrak{o}_{2n})$ for the space P_{kn}, and $(\mathfrak{o}_k, \mathfrak{o}_{2n})$, $(\mathfrak{sp}_k, \mathfrak{sp}_{2n})$ for the space $\mathfrak{P}_{kn}$, see [1] for details. It is expected that the results of this work can be obtained for pairs other than $(\mathfrak{gl}_k, \mathfrak{gl}_n)$. The Howe duality was originally developed as an useful tool in the representation theory of classical Lie groups and algebras and in the classical invariant theory, in particular, it is closely related to the famous Schur-Weyl duality. The Howe duality is also used in the representation theory of Yangians and twisted Yangians, see [3]. The generalization of this duality to the case of Lie superalgebras was systematically studied in [4]. There are also analogs of Howe dual pairs involving some infinite-dimensional Lie algebras acting on Fock spaces, see [5].

1.2 Duality of Bethe algebras

Consider the current Lie algebras $\mathfrak{gl}_n[t]$ and $\mathfrak{gl}_k[t]$, which are the Lie algebras of polynomials of t with coefficients in $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$, respectively, with the pointwise Lie bracket. Fix sequences of pairwise distinct complex numbers $\bar{z} = (z_1, \ldots, z_k)$

and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$. The actions $\pi^{\langle n,k\rangle}$ and $\pi^{\langle k,n\rangle}$ can be extended to $\mathfrak{gl}_n[t]$- and $\mathfrak{gl}_k[t]$-actions, respectively, by the formulas

$$\pi_{\bar{z}}^{\langle n,k\rangle} : e_{ij}^{\langle n\rangle} \otimes t^s \mapsto \sum_{a=1}^{k} z_a^s \, \xi_{ai}\partial_{aj},$$

$$\pi_{-\bar{\alpha}}^{\langle k,n\rangle} : e_{ab}^{\langle k\rangle} \otimes t^s \mapsto \sum_{i=1}^{n} (-\alpha_i)^s \, \xi_{ai}\partial_{bi}.$$

Unlike in the case of finite-dimensional Lie algebras, $\pi_{\bar{z}}^{\langle n,k\rangle}(U(\mathfrak{gl}_n[t]))$ and $\pi_{\bar{\alpha}}^{\langle k,n\rangle}(U(\mathfrak{gl}_k[t]))$ do not commute in $\mathrm{End}(\mathfrak{P}_{kn})$. But the statement about the equality of centers above does have a generalization to the case of current Lie algebras.

It is known that a generating set of the center of $U(\mathfrak{gl}_n)$ can be obtained from the determinant appearing in the Capelli identity. A generalization of such a determinant gives a differential operator

$$\mathcal{D}_{\bar{\alpha}} = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^{n} \sum_{j=0}^{\infty} B_{ij}^{\bar{\alpha}} x^{-j} \left(\frac{d}{dx}\right)^{n-i}, \tag{1.1}$$

depending on the parameters $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where $B_{ij}^{\bar{\alpha}} \in U(\mathfrak{gl}_n[t])$, see Section 2.3 for details. The elements $B_{ij}^{\bar{\alpha}}$ generate a large commutative subalgebra $\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}$ called the Bethe algebra. The definition of the Bethe algebra that we use is due to D. Talalaev, see [6].

The Bethe algebra $\mathcal{B}_{\bar{z}}^{\langle k\rangle} \subset U(\mathfrak{gl}_k[t])$ depending on the parameters $\bar{z} = (z_1, \ldots, z_k)$ is defined in a similar way. One of the main results of this work is Theorem 2.4.2, which states that the images of the Bethe algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle k\rangle}$ in $\mathrm{End}(\mathfrak{P}_{kn})$ under the actions $\pi_{\bar{z}}^{\langle n,k\rangle}$ and $\pi_{-\bar{\alpha}}^{\langle k,n\rangle}$, respectively, coincide:

$$\pi_{\bar{z}}^{\langle n,k\rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}) = \pi_{-\bar{\alpha}}^{\langle k,n\rangle}(\mathcal{B}_{\bar{z}}^{\langle k\rangle}).$$

This result was inspired by the similar duality when the Bethe algebras act on the space P_{kn} of polynomials in kn commuting variables, see [7].

The Bethe algebras are important objects in the theory of quantum integrable models. Namely, the algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle k\rangle}$ are closely related to the rational quantum Gaudin model, see [8]. As shown in Section 2.3.3, the images of the (rational)

Gaudin Hamiltonians $H_a \in U(\mathfrak{gl}_n)^{\otimes k}$, $a = 1, \dots, k$, under certain actions are elements of $\pi_{\bar{z}}^{\langle n,k \rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle})$. Together with the Gaudin Hamiltonians, we are going to consider elements G_i, $i = 1, \dots, n$ of $U(\mathfrak{gl}_n)^{\otimes k}$ called the Dynamical Hamiltonians, whose images also belong to $\pi_{\bar{z}}^{\langle n,k \rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle})$. The elements $H_1, \dots, H_k$, $G_1, \dots, G_n$ pairwise commute. In the proof of Theorem 2.4.2, we used the observation that under the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality, the Gaudin and Dynamical Hamiltonians exchange, see Lemma 2.4.3 for more precise statement.

In Chapter 3, we study the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the trigonometric Gaudin and Dynamical Hamiltonians. Although we are not going to discuss the Bethe algebras for this case below, let us mention them here for the completeness of the picture. In the case of the trigonometric Gaudin Hamiltonians, instead of the algebra $U(\mathfrak{gl}_k[t])$, one should consider the universal enveloping algebra $U(\widehat{\mathfrak{gl}_k})$ of the affine Lie algebra $\widehat{\mathfrak{gl}_k}$. The commutative algebra inside $U(\widehat{\mathfrak{gl}_k})$ playing the role of the Bethe algebra $\mathcal{B}_{\bar{\alpha}}^{\langle k \rangle}$ was introduced recently in [9]. We will call this algebra the trigonometric Gaudin Bethe algebra. On the other hand, the trigonometric Dynamical Hamiltonians are related to the Yangian $Y(\mathfrak{gl}_n)$. The corresponding commutative algebra inside $Y(\mathfrak{gl}_n)$ is generated by the higher transfer matrices of the XXX spin chain model, see for example [10]. We will call this algebra the Yangian Bethe algebra. Therefore, the ultimate goal in studying the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality related to the trigonometric Gaudin model would be to establish the equality of images of the trigonometric Gaudin Bethe algebra and the Yangian Bethe algebra. The results of Chapter 3 may be considered as first steps in achieving this goal.

The $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality is also expected in the case of the Bethe algebras associated with the quantum affine algebras $U_q(\widehat{\mathfrak{gl}_k})$ and $U_q(\widehat{\mathfrak{gl}_n})$. This duality should correspond to the duality of the XXZ spin chain models associated with $\mathfrak{gl}_k$ and $\mathfrak{gl}_n$, respectively.

To distinguish the Bethe algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle k \rangle}$ related to the rational Gaudin model from the Bethe algebras for other integrable models, we will sometimes call them the rational Gaudin Bethe algebras.

For the convenience of a reader, we illustrate the most important relations between objects that we introduce on diagram (1.12).

1.3 Duality for spaces of quasi-exponentials

Consider a space of functions V with a basis

$$\{e^{\alpha_i x} p_{ij}(x) |\ i = 1, \ldots, n,\ j = 1, \ldots, n_i\},$$

where $\alpha_1, \ldots, \alpha_n$ are distinct complex numbers, $p_{ij}(x)$ are polynomials such that $\deg p_{ij} \neq \deg p_{il}$ for $j \neq l$. Denote by D_V the monic linear differential operator of degree $\dim V$ annihilating V. Then D_V has rational coefficients.

To the space V, one can associate the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$, where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$ is a sequence of partitions related to degrees of the polynomials $p_{ij}(x)$, $\bar{z} = (z_1, \ldots, z_k)$ is the set of poles of the coefficients of D_V, and $\bar{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ is related to the local behavior of V around these poles, see Section 2.2 for more details. We will say that V is a space of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. Denote the set of all spaces of quasi-exponentials with the fixed data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ by $\mathcal{E}(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$.

In [11], the authors introduced a bijection $\mathfrak{T}_1 : \mathcal{E}(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z}) \to \mathcal{E}(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$ given by

$$\mathfrak{T}_1 : V \mapsto V^{\ddagger} = \ker(pD_V)^{\ddagger}, \tag{1.2}$$

where p is the least common denominator of coefficients of D_V, and $D \mapsto D^{\ddagger}$ is an antiauthomorphism of the algebra of differential operators with polynomial coefficients such that

$$\left(\frac{d}{dx}\right)^{\ddagger} = x, \qquad (x)^{\ddagger} = \frac{d}{dx}.$$

The bijection $\mathfrak{T}_1$ was introduced in relation to the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality on the space P_{kn} of polynomials in kn commuting variables.

Recall that η' denotes the conjugate of a partition η. We will also write $\bar{\eta}'$ if we conjugate all partitions in the sequence $\bar{\eta}$. In this work, we introduce a bijection $\mathfrak{T}_2 : \mathcal{E}(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z}) \to \mathcal{E}(\bar{\mu}', \bar{\lambda}'; -\bar{\alpha}, \bar{z})$ given by

$$\mathfrak{T}_2 : V \mapsto \check{V}^\dagger = \ker(\check{D}_V^\dagger), \tag{1.3}$$

where $\check{D}_V$ is a differential operator such that

$$\prod_{i=1}^{n} \left(\frac{d}{dx} - \alpha_i \right)^{\max_j(\deg p_{ij})+1} = \check{D}_V D_V,$$

and $D \mapsto D^\dagger$ is an antiautomorphism of the algebra of differential operators such that

$$\left(\frac{d}{dx} \right)^\dagger = -\frac{d}{dx}, \qquad (b(x))^\dagger = b(x)$$

for any function $b(x)$. We will call $\check{D}_V$ the quotient differential operator.

We study the bijection $\mathfrak{T}_2$ because it is closely related to the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the rational Gaudin Bethe algebras acting on the space $\mathfrak{P}_{kn}$. Let us briefly describe this relation now.

Consider the subspace $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \in \mathfrak{P}_{kn}$ of $\mathfrak{gl}_k$-weight $\boldsymbol{l}$ and $\mathfrak{gl}_n$-weight $\boldsymbol{m}$ (for definition of a weight subspace, see Section 1.7). Both Bethe algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle k \rangle}$ preserve the subspace $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$. Denote the set of eigenspaces of $\pi_{\bar{z}}^{\langle n,k \rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle})$ in $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ by $\mathcal{V}\left[\pi_{\bar{z}}^{\langle n,k \rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}), \boldsymbol{l}, \boldsymbol{m}\right]$. Similarly, let $\mathcal{V}\left[\pi_{-\bar{\alpha}}^{\langle k,n \rangle}(\mathcal{B}_{\bar{z}}^{\langle k \rangle}), \boldsymbol{l}, \boldsymbol{m}\right]$ be the set of eigenspaces of $\pi_{-\bar{\alpha}}^{\langle k,n \rangle}(\mathcal{B}_{\bar{z}}^{\langle k \rangle})$ in $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$.

The Bethe ansatz is a method of finding common eigenvectors of some commutative families of operators in the theory of quantum integrable models. In particular, the Bethe ansatz method for the rational quamtum Gaudin model produces common eigenvectors for the Gaudin Hamiltonians, or more generaly, common eigenvectors for all elements of the Bethe algebra. Each eigenvector produced in such a way is associated with a solution of a system of algebraic equations called the Bethe ansatz equations.

A natural question is whether the Bethe ansatz gives all the eigenvectors. The answer to this question is given by E. Mukhin, V. Tarasov and A. Varchenko in [12],

[13], [14], and [15]. In particular, it is shown that there is a bijection between the sets $\mathcal{E}(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ and $\mathcal{V}\left[\pi_{\bar{z}}^{\langle n,k\rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}), \boldsymbol{l}, \boldsymbol{m}\right]$, where $\bar{\lambda}$ and $\bar{\mu}$ are some specific sequences defined using $\boldsymbol{l}$ and $\boldsymbol{m}$, respectively. We denote this bijection as $\mathfrak{L}$. The map $\mathfrak{L}$ can be regarded as an example of the geometric Langlands correspondence, see [16].

The analog of the map $\mathfrak{L}$ for the Bethe algebra associated with $\mathfrak{gl}_k$ gives a bijection $\mathcal{E}(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha}) \to \mathcal{V}\left[\pi_{-\bar{\alpha}}^{\langle k,n\rangle}(\mathcal{B}_{\bar{z}}^{\langle k\rangle}), \boldsymbol{l}, \boldsymbol{m}\right]$, which we denote by $\mathfrak{L}'$. To prove the equality of the images of the Gaudin Bethe algebras, we showed that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{V}\left[\pi_{\bar{z}}^{\langle n,k\rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}), \boldsymbol{l}, \boldsymbol{m}\right] \arrow[r, phantom, "="] & \mathcal{V}\left[\pi_{-\bar{\alpha}}^{\langle k,n\rangle}(\mathcal{B}_{\bar{z}}^{\langle k\rangle}), \boldsymbol{l}, \boldsymbol{m}\right] \\
\mathcal{E}(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z}) \arrow[u, "\mathfrak{L}"] \arrow[r, "\mathfrak{T}_1 \circ \mathfrak{T}_2"'] & \mathcal{E}(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha}) \arrow[u, "\mathfrak{L}'"']
\end{tikzcd},$$

see Theorem 2.4.6 and Corollary 2.4.7. In other words, we obtained the duality on the other side of the geometric Langlands correspondence.

To prove the commutativity of the diagram, we used the fact that the construction of the bijection $\mathfrak{L}$ gives an explicit relation between eigenvalues of the generators $B_{ij}^{\bar{\alpha}}$ of the Bethe algebra and the coefficients of a differential operator annihilating the corresponding space of quasi-exponentials, Namely, if $\boldsymbol{l}$ and $\boldsymbol{m}$ have no zero components, and $\mathfrak{L}(V) = v \in \mathcal{V}\left[\pi_{\bar{z}}^{\langle n,k\rangle}(\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}), \boldsymbol{l}, \boldsymbol{m}\right]$, then D_V is the image of the differential operator (1.1), under the action $\pi_{\bar{z}}^{\langle n,k\rangle}$ and restriction to v. If $\boldsymbol{l}$ and $\boldsymbol{m}$ have zero components, then we have to use slightly modified differential operator D_V^{aug}. This helps us to express the eigenvalues of the Gaudin and Dynamical Hamiltonians in terms of the coefficients of D_V or D_V^{aug} and check that that the relation between the eigenvalues coming from the transformation $\mathfrak{T}_1 \circ \mathfrak{T}_2$ matches the relation coming from the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality, see Lemma 2.2.3.

It is convenient to describe the composition $\mathfrak{T}_1 \circ \mathfrak{T}_2$ using pseudo-differential operators, see Section 2.1. In particular, it explains the relation between the maps $\mathfrak{T}_1 \circ \mathfrak{T}_2$ and $\mathfrak{T}_2 \circ \mathfrak{T}_1$, see Proposition 2.5.10. In this description, the essential transformation in the definition of the map $\mathfrak{T}_1 \circ \mathfrak{T}_2$ is to take the inverse of a pseudo-differential operator. This is consistent with the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for the Bethe algebras associated with Lie superalgebras obtained in [17], which we will discuss more in Section 1.6.

1.4 Duality for difference operators

In Chapter 3, we study the analog of the map $\mathfrak{T}_1 \circ \mathfrak{T}_2$ relevant to the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the trigonometric Gaudin and Dynamical Hamiltonians. We start with a space V, which has a basis of the form

$$\{x^{z_a} q_{ab}(x) \,|\, a = 1, \dots, k, b = 1, \dots, k_a\},$$

where $z_1, \dots, z_k$ are distinct complex numbers, and $q_{ab}(x)$ are polynomials such that $\deg q_{ab}(x) \neq \deg q_{ac}(x)$ if $b \neq c$.

Denote $K = \sum_{a=1}^{k} k_a = \dim V$. Consider the differential operator

$$x^K D_V = \sum_{a=0}^{K} \beta_a(x) \left(x \frac{d}{dx} \right)^{K-a}$$

annihilating V. Then $\beta_0(x), \dots, \beta_K(x)$ are rational functions. One can associate the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$ to the space V in a way similar to what we discussed above. We will say that V is a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$.

Consider an operator T defined by $Tf(x) = f(x+1)$. Define a difference operator S_V^m by

$$S_V^m = \tau(p x^K D_V),$$

where p is the least common denominator of $\beta_0(x), \dots, \beta_K(x)$, and τ is a homomorphism such that

$$\tau(x) = T, \quad \tau(x \frac{d}{dx}) = -x.$$

Using the quotient difference operator, which is analogous to the quotient differential operator, we can obtain from S_V^m another difference operator S_W^m. The kernel W of S_W^m has a basis of the form

$$\{\alpha_i^x r_{ij}(x) |\ i = 1, \dots, n,\ j = 1, \dots, n_i\},$$

where $\alpha_1, \dots, \alpha_n$ are distinct non-zero complex numbers and $r_{ij}(x)$ are polynomials such that $\deg r_{ij}(x) \neq \deg r_{il}(x)$ if $j \neq l$.

We prove some properties of the space W, which allows one to assign the data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$ to W, where $-\bar{z} - \bar{\lambda}'_1 + 1 = (-z_1 - (\lambda^{(1)})'_1 + 1, -z_2 - (\lambda^{(2)})'_1 + 1, \dots, -z_k - (\lambda^{(k)})'_1 + 1)$, and $(\lambda^{(a)})'_1$ is the number of non-zero entries of the partition $\lambda^{(a)}$ in the sequence $\bar{\lambda}$. The assignment of the data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$ to the space W is different comparing to the previous cases. In particular, the set $-\bar{z} - \bar{\lambda}'_1 + 1$ is smaller than the set of poles of coefficients of S^m_W, and the sequence $\bar{\lambda}'$ is not related to the local behavior of W anymore, but rather to its behavior associated to strings of points related by integer shifts. We say that W is a space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$.

The map $V \mapsto W$ is described in a more elegant way using pseudo-difference operators S_V and S_W associated to S^m_V and S^m_W, respectively. We have

$$S_W = S_V^{-1}. \tag{1.4}$$

We relate the map $V \mapsto W$ to the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the trigonometric Gaudin and Dynamical Hamiltonians. Eigenvectors of the trigonometric Gaudin Hamiltonians can be produced using the Bethe ansatz for the trigonometric quantum Gaudin model. Since the trigonometric Dynamical Hamiltonians are related to the Yangian Bethe algebra and to the transfer matrices generating it, their eigenvectors can be obtained by the Bethe ansatz for the XXX spin chain model.

The Bethe ansatz method for the trigonometric Gaudin and XXX spin chain models is less developed than for the rational Gaudin case. There are statements that relate the spaces V and W with solutions of the Bethe ansatz equations for the trigonometric Gaudin and XXX spin chain models, respectively, see [18]. Also, there are explicit formulas for eigenvalues of the Hamiltonians in terms of these solutions. But unlike in the rational Gaudin case, explicit relations between eigenvalues and coefficients of difference operators were not written down. In Sections 3.5.2 - 3.5.5, we obtain such relations collecting necessary results from papers [18] and [10]. Then we check that the relation between eigenvalues coming from (1.4) matches the relation coming from the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality, see Theorem 3.5.10. Again, we have to treat the case when the weights $\boldsymbol{l}$ or $\boldsymbol{m}$ have zero components in a slightly different way, see Section 3.5.7.

Similarly to the rational Gaudin case, in the construction of the map $V \mapsto W$, we used a transformation relevant to the duality of the trigonometric Gaudin and Dynamival Hamiltonians acting on the space P_{kn} of polynomials in kn commuting variables. This transformation was introduced in [19], and it defines the so-called bispectral duality. One can first apply bispectral duality and then consider the quotient difference operator, or first consider the quotient differential operator and then apply the bispectral duality. In Section 3.4.7, we show that the result is the same for both choices.

1.5 Duality for Knizhnik-Zamolodchikov and Dynamical equations

The differential equations

$$\left(\kappa \frac{\partial}{\partial z_a} - H_a(\bar{z}, \bar{\alpha})\right) f = 0, \quad a = 1, \ldots, k, \tag{1.5}$$

where f is a $\mathfrak{P}_{kn}$-valued function of $z_1, \ldots, z_k$, $\alpha_1, \ldots, \alpha_n$, and $H_1(\bar{z}, \bar{\alpha}), \ldots, H_k(\bar{z}, \bar{\alpha})$ are the Gaudin Hamiltonians, are called the rational Knizhnik-Zamolodchikov (KZ) equations. They were first introduced as differential equations for the correlation functions in Wess-Zumino-Novikov-Witten (WZNW) conformal field theory, see [20]. The rational KZ equations along with their trigonometric and difference analogs play an important role in the representation theory of Lie algebras and quantum groups, see [21].

The differential equations

$$\left(\kappa \frac{\partial}{\partial \alpha_i} - G_i(\bar{\alpha}, \bar{z})\right) f = 0, \quad i = 1, \ldots, n, \tag{1.6}$$

where f is again a $\mathfrak{P}_{kn}$-valued function of $z_1, \ldots, z_k$, $\alpha_1, \ldots, \alpha_n$, and $G_1(\bar{z}, \bar{\alpha}), \ldots,$ $G_n(\bar{z}, \bar{\alpha})$ are the Dynamical Hamiltonians, are called the rational differential Dynamical (DD) equations. It was proved in [22], that the rational KZ equations and rational DD equations are compatible, see Theorem 4.1.1 for a more precise statement.

We will also consider the trigonometric (trigKZ) and quantized (qKZ) Knizhnik-Zamolodchikov equations. The trigKZ equations are related to the trigonometric Gaudin Hamiltonians similarly to the rational case. The qKZ equations are difference equations of the form

$$K_a f(z_a + \kappa) = f(z_a), \quad a = 1, \ldots, k, \tag{1.7}$$

where f is a $\mathfrak{P}_{kn}$-valued function of $z_1, \ldots, z_k$, $\alpha_1, \ldots, \alpha_n$, and $K_a \in \mathrm{End}(\mathfrak{P}_{kn})$.

In [23] and [24], the authors introduced the trigonometric differential Dynamical (trigDD) equations compatible with the qKZ equations and the difference Dynamical (qDD) equations compatible with the trigKZ equations, respectively. The trigDD equations are related to the trigonometric Dynamical Hamiltonians similarly to the rational case. The qDD equations are difference equations of the form

$$X_i f(\alpha_i + \kappa) = f(\alpha_i), \quad i = 1, \ldots, n, \tag{1.8}$$

where f is a $\mathfrak{P}_{kn}$-valued function of $z_1, \ldots, z_k$, $\alpha_1, \ldots, \alpha_n$, and $X_i \in \mathrm{End}(\mathfrak{P}_{kn})$.

All mentioned equations can be defined using either the Lie algebra $\mathfrak{gl}_n$ or the Lie algebra $\mathfrak{gl}_k$. In Chapter 4, we show that the Knizhnik-Zamolodchikov and Dynamical equations exchange under the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality:

$$\mathrm{KZ} \longleftrightarrow \mathrm{DD}, \tag{1.9}$$

$$\mathrm{trigKZ} \longleftrightarrow \mathrm{trigDD}, \tag{1.10}$$

$$\mathrm{qKZ} \longleftrightarrow \mathrm{qDD}, \tag{1.11}$$

see Theorem 4.2.1 for the precise statement. Similar dualities for the space P_{kn} were observed in [25].

The duality for the rational (resp., trigonometric) KZ and DD equations is equivalent to the duality for the rational (resp, trigonometric) Gaudin and Dynamical

Hamiltonians. Therefore, correspondence (1.9) is related to the duality of the rational Gaudin Bethe algebras, and correspondence (1.10) is related to the duality of the trigonometric Gaudin and Yangian Bethe algebras.

Correspondence (1.11) is not equivalent to any results in the previous chapters, but it is also related to the duality of the trigonometric Gaudin and Yangian Bethe algebras. This is because the space V from the previous section gives eigenvectors for the trigonometric Gaudin Hamiltonians, which are also eigenvectors of the $\mathfrak{gl}_k$-versions of operators $X_1, \dots, X_n$ in formula (1.8), see [26] and [27]. Similarly, the space W from the previous section gives eigenvectors for the trigonometric Dynamical Hamiltonians, which are also eigenvectors of the operators $K_1, \dots, K_k$ in formula (1.7), see [28]. Again, we refer a reader to diagram (1.12).

$U(\mathfrak{gl}_k[t])$ Rational Gaudin model KZ and DD equations $V = \langle e^{z_a x} p_a(x),\, a = 1, \dots, k\rangle$ $D_V = \sum_{a=0}^{k} b_a(x) \left(\frac{d}{dx}\right)^{k-a}$	$\longleftrightarrow$	$U(\mathfrak{gl}_n[t])$ Rational Gaudin model DD and KZ equations $\tilde V = \langle e^{\alpha_i x} \tilde p_i(x),\, i = 1, \dots, n\rangle$ $D_{\tilde V} = \sum_{i=0}^{n} \tilde b_i(x) \left(\frac{d}{dx}\right)^{n-i}$
$U(\widehat{\mathfrak{gl}}_k)$ Trigonometric Gaudin model trigKZ and qDD equations $V = \langle x^{z_a} q_a(x),\, a = 1, \dots, k\rangle$ $x^k D_V = \sum_{a=0}^{k} \beta_a(x) \left(x\frac{d}{dx}\right)^{k-a}$	$\longleftrightarrow$	$Y(\mathfrak{gl}_n)$ XXX spin chain model trigDD and qKZ equations $W = \langle \alpha_i^x r_i(x),\, i = 1, \dots, n\rangle$ $S_W = \sum_{i=0}^{n} b_i(x) T^{n-i}$
$U_q(\widehat{\mathfrak{gl}}_k)$ XXZ spin chain model	$\longleftrightarrow$	$U_q(\widehat{\mathfrak{gl}}_n)$ XXZ spin chain model

(1.12)

1.6 Other related dualities

In this section, we highlight some other results known in the literature, which are related to the dualities we study.

A lot of work was done for the space P_{kn} of polynomials in kn commuting variables. The equality of images of the rational Gaudin Bethe algebras acting on P_{kn} was proved in [7] using a generalization of the Capelli identity. The corresponding transforms of the spaces of quasi-exponentials and quasi-polynomials were introduced in [11] for the rational Gaudin model and in [19] for the trigonometric Gaudin model.

The $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-dualities of the Bethe algebras $\mathcal{B}_{\bar{z}}^{\langle k\rangle}$ and $\mathcal{B}_{\bar{\alpha}}^{\langle n\rangle}$ acting on the spaces P_{kn} and $\mathfrak{P}_{kn}$ are special cases of the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the Bethe algebras $\mathcal{B}_{\bar{z}}^{\langle k\rangle}$ and $\mathcal{B}_{\bar{\alpha}}^{\langle n|m\rangle}$ acting on the space $P_{k,n|m}$, where $\mathcal{B}_{\bar{\alpha}}^{\langle n|m\rangle}$ is the Bethe algebra associated with the Lie superalgebra $\mathfrak{gl}_{n|m}$, and $P_{k,n|m}$ is the space of polynomials in variables x_{ai}, $a = 1, \dots, k$, $i = 1, \dots, m+n$, such that

$$x_{ai}x_{bj} = (-1)^{|i||j|}x_{bj}x_{ai},$$

where $|i| = 0$ if $i \leqslant n$, and $|i| = 1$ otherwise.

The duality between the algebras $\mathcal{B}_{\bar{\alpha}}^{\langle k\rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle n|m\rangle}$ acting on $P_{k,n|m}$ was established in [17]. It was also conjectured in [29] that the eigenvectors and eigenvalues of $\mathcal{B}_{\bar{z}}^{\langle n|m\rangle}$ are described by ratios of differential operators, which are elements of the algebra of pseudo-differential operators. The $(\mathfrak{gl}_k, \mathfrak{gl}_{n|m})$-duality for the Bethe algebras suggests that there exists a correspondence between differential operators and ratios of differential operators. This correspondence is well understood when $m = 0$ ("the even case") and when $n = 0$ ("the odd case"). For the even case, the correspondence is the map $\mathfrak{T}_1$ above. For the odd case, the correspondence is the map $\mathfrak{T}_1 \circ \mathfrak{T}_2$ that we study in this work, which sends the kernel of a differential operator to the kernel of the denominator in the ratio of differential operators mentioned above. This justifies the appearance of the inverse of a pseudo-differential operator in our construction of the map $\mathfrak{T}_1 \circ \mathfrak{T}_2$. The even case and the odd case are linked in a non-trivial way when both m and n are not zero. It is therefore a challenging and interesting problem to establish and study dualities for pseudo-differential operators describing the eigenvectors and eigenvalues of the Bethe algebras in "super-case"

In [30], the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality was studied for another generalization of the algebra $\pi_{\bar z}^{\langle n,k\rangle}(\mathcal{B}_{\bar\alpha}^{\langle n\rangle})$. The image of the determinant used in the definition of $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$ under the map $\pi_{\bar z}^{\langle n,k\rangle}$ is the determinant of a matrix whose entries are rational functions with at most simple poles at $z_1, \dots, z_k$. Also, the numbers $\alpha_1, \dots, \alpha_n$ can be regarded as entries of a diagonal matrix. The authors of [30] consider higher order poles at $z_1, \dots,$ z_k, which in the dual picture corresponds to assigning a Jordan block to each α_i, $i = 1,$ $\dots, n$. One can also consider this generalization as a limit of the algebra $\pi_{\bar z}^{\langle n,k\rangle}(\mathcal{B}_{\bar\alpha}^{\langle n\rangle})$ when $\alpha_i \to \alpha_j$ and $z_a \to z_b$ for some i, j, a, and b. This limit is very interesting to study, in particular, using this limit, it was shown in [31] that the monodromy of the Bethe vectors is related to crystal bases. In our work, the results for the spaces of quasi-exponentials and quasi-polynomials are much more general than what we need for the duality of $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$ and $\mathcal{B}_{\bar z}^{\langle k\rangle}$. We expect that these results are relevant to the generalization of the Bethe algebras to higher order poles.

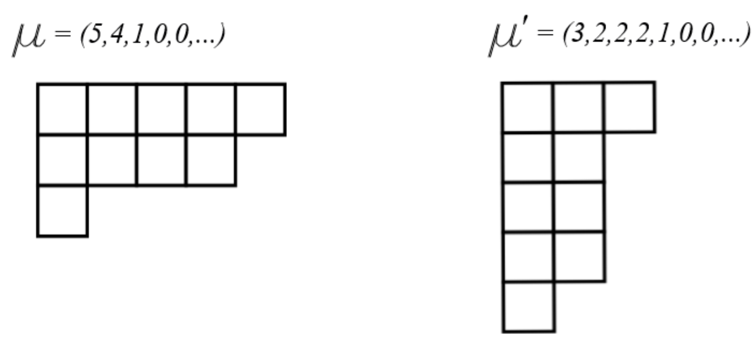
The $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality was also studied in the context of quantum toroidal algebras, see [32], where the authors proved that the corresponding Bethe algebras commute.

The recently established connection between quantum integrable models and Nakajima quiver varieties became a quickly developing research area nowadays, see [33], [34], [35], [36], and [37]. A natural question is whether the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality is somehow reflected in the context of this connection. A possible answer to this question involves the $3d$ mirror symmetry for algebraic varieties introduced in [38], [39]. In [38], the authors constructed the $3d$ mirror dual X' for the variety $X = T^*Gr(l, n)$, where $T^*Gr(l, n)$ is the cotangent bundle of the Grassmanian $Gr(l, n)$ of l-planes in an n-dimensional space. The hypergeometric solutions of the trigonometric qKZ equation associated with the quantum affine algebra $U_q(\widehat{\mathfrak{gl}_n})$ can be constructed using the K-theory of the cotangent bundles of partial flag varieties, see [40], [37]. Then the pair (X, X') above gives a correspondence between the hypergeometric solutions of the trigonometric KZ equations associated with $U_q(\widehat{\mathfrak{gl}_2})$ and $U_q(\widehat{\mathfrak{gl}_n})$, which is believed to correspond to the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for quantum affine algebras (the last duality on diagram (1.12)).

1.7 Basic notation and conventions

In this dissertation, a partition $\mu = (\mu_1, \mu_2, \dots)$ is an infinite nonincreasing sequence of nonnegative integers stabilizing at zero. Let $\mu' = (\mu'_1, \mu'_2, \dots)$ denote the conjugate partition, that is, $\mu'_i = \#\{j \mid \mu_j \geqslant i\}$. In particular, μ'_1 equals the number of nonzero entries in μ.

The Young diagram corresponding to a partition $\mu = (\mu_1, \mu_2, \dots)$ consists of rows of boxes aligned by their left side, such that the top row has μ_1 boxes, the next row has μ_2 boxes, and so on. An example of the Young diagram for a partition and its conjugate is given below. Columns of the Young diagram of the conjugate partition correspond to rows of the Young diagram of the original partition.



The general linear Lie algebra $\mathfrak{gl}_n$ is a Lie algebra spanned by the elements e_{ij}, $i, j = 1, \dots, n$ with the Lie bracket $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$. The elements e_{ij}, $i, j = 1, \dots, n$ are called the standard generators of $\mathfrak{gl}_n$. A $\mathfrak{gl}_n$-module (or representation of $\mathfrak{gl}_n$) is a vector space L endowed with a $\mathfrak{gl}_n$-action. A representation of $\mathfrak{gl}_n$ is irreducible if it does not contain an invariant subspace.

A $\mathfrak{gl}_n$-weight is a sequence of n complex numbers. Notice that a partition with at most n non-zero entries defines a $\mathfrak{gl}_n$-weight. We denote by $(L)_\lambda$ the weight subspace of a $\mathfrak{gl}_n$-module L of weight $\lambda = (\lambda_1, \dots, \lambda_n)$, which is the subspace of all $v \in L$ such that $e_{ii}v = \lambda_i v$, $i = 1, \dots, n$. We denote by L_λ the irreducible highest weight $\mathfrak{gl}_n$-module of highest weight λ, which is a unique up to an isomorhism irreducible $\mathfrak{gl}_n$-module with a vector v such that $e_{ij}v = 0$ for all $i < j$, and $e_{ii}v = \lambda_i v$, $i = 1, \dots,$ n.

We will often consider the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ together. We will write the superscripts $\langle n\rangle$ and $\langle k\rangle$ to distinguish objects associated with algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$, respectively. For example, $e_{ij}^{\langle n\rangle}$, $i, j = 1, \dots, n$, are the generators of $\mathfrak{gl}_n$, and $e_{ab}^{\langle k\rangle}$, $a, b = 1, \dots, k$, are the generators of $\mathfrak{gl}_k$.

All vector spaces are over the field of complex numbers if not specified otherwise.

Here are some other notations that we used throughout the text:

D_V - the fundamental monic differential operator of the space V of quasi-exponentials or quasi-polynomials.

$\bar{D}_V$ - the fundamental regularized differential operator of the space of quasi-polynomials V.

S_W^m - the fundamental monic difference operator of the space of quasi-exponentials W.

S_V and S_W - the fundamental pseudo-difference operators of the space of quasi-polynomials V and of the space of quasi-exponentials W, respectively.

$\bar{S}_W$ - the fundamental regularized difference operator of the space of quasi-exponentials W.

$\mathrm{Wr}(f_1, \dots, f_n)$ (resp., $\mathcal{W}r(f_1, \dots, f_n)$) - differential (resp., difference) Wronskian of $f_1, \dots, f_n$.

1.8 Other notes

Chapter 2 is based on the joint paper [41] with V. Tarasov.

Chapter 4 is based on the joint paper [42] with V. Tarasov.

The results of Chapter 3 are not published yet.

2. DUALITY OF RATIONAL GAUDIN BETHE ALGEBRAS

2.1 Algebra of pseudo-differential operators

The algebra of pseudo-differential operators $\Psi\mathfrak{D}$ consists of all formal series of the form

$$\sum_{m=-\infty}^{M}\sum_{k=-\infty}^{K} C_{km}x^k\left(\frac{d}{dx}\right)^m,$$

where integers M and K can differ for different series, and C_{km} are complex numbers. One can check that the rule

$$\left(\frac{d}{dx}\right)^m x^k = \sum_{j=0}^{\infty}\frac{(m)_j(k)_j}{j!}\,x^{k-j}\left(\frac{d}{dx}\right)^{m-j}, \qquad m,k\in\mathbb{Z}\,, \tag{2.1}$$

where $(a)_i = a(a-1)(a-2)\dots(a-i+1)$, yields a well-defined multiplication on $\Psi\mathfrak{D}$. The verification of associativity is straightforward using the Chu-Vandermonde identity:

$$\sum_{j=1}^{i}\frac{(m-n)_j}{j!}\cdot\frac{(n)_{i-j}}{(i-j)!} = \frac{(m)_i}{i!}\,.$$

Lemma 2.1.1 *If* $D=\sum_{m=-\infty}^{M}\sum_{k=-\infty}^{K}C_{km}\,x^k(d/dx)^m$, *with* $C_{KM}\neq 0$, *then* D *is invertible in* $\Psi\mathfrak{D}$.

Proof Define $\acute{D}$ by the rule $1+\acute{D}=C_{KM}^{-1}\,x^{-K}D\,(d/dx)^{-M}$. Then $\sum_{j=0}^{\infty}(-1)^j\acute{D}^j$ is a well-defined element of $\Psi\mathfrak{D}$ and the inverse of D is given by the formula:

$$D^{-1}=C_{KM}^{-1}\left(\frac{d}{dx}\right)^{-M}\left(\sum_{j=0}^{\infty}(-1)^j\acute{D}^j\right)x^{-K}.$$

■

We consider a formal series $\sum_{m=-\infty}^{M}f_m(x)(d/dx)^m$, where all $f_m(x)$ are rational functions, as an element of $\Psi\mathfrak{D}$ replacing each $f_m(x)$ by its Laurent series at infinity.

In particular, we identify the algebra of linear differential operators with rational coefficients and the corresponding subalgebra of $\Psi\mathfrak{D}$. Next corollary follows immediately from the Lemma 2.1.1.

Corollary 2.1.2 *Let $D = \sum_{m=-\infty}^{M} f_m(x)(d/dx)^m$, where all $f_m(x)$ are rational functions regular at infinity. Then D is invertible in $\Psi\mathfrak{D}$.*

Using (2.1), one can check that for any complex numbers C_{km}, the series

$$\sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km} \left(-\frac{d}{dx}\right)^m x^k$$

is a well-defined element of $\Psi\mathfrak{D}$. We define a map $(\cdot)^\dagger : \Psi\mathfrak{D} \to \Psi\mathfrak{D}$ by the rule

$$\left(\sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km}\, x^k \left(\frac{d}{dx}\right)^m\right)^\dagger = \sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km} \left(-\frac{d}{dx}\right)^m x^k.$$

Lemma 2.1.3 *The map $(\cdot)^\dagger$ is an involutive antiautomorphism of $\Psi\mathfrak{D}$.*

Proof To check that $(\cdot)^\dagger$ is involutive, we need to verify that $\left(\left(x^k(d/dx)^m\right)^\dagger\right)^\dagger = x^k(d/dx)^m$. By (2.1), it reads as

$$\sum_{j=0}^{\infty} (-1)^j \frac{(k)_j (m)_j}{j!} \left(\frac{d}{dx}\right)^{m-j} x^{k-j} = x^k \left(\frac{d}{dx}\right)^m . \tag{2.2}$$

The equality holds since $$\sum_{j=0}^{i} \frac{(-1)^j}{j!(i-j)!} = \delta_{i0}\,.$$

Using (2.1) and (2.2), one can check that $(\cdot)^\dagger$ is an antiautomorphism as well. ■

We also define the following involutive antiautomorphism on $\Psi\mathfrak{D}$:

$$\left(\sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km}\, x^k \left(\frac{d}{dx}\right)^m\right)^\ddagger = \sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km}\, x^m \left(\frac{d}{dx}\right)^k .$$

For $D \in \Psi\mathfrak{D}$, we say that $D^\dagger$ is the *formal conjugate* to D and $D^\ddagger$ is the *bispectral dual* to D. Let $D^\# = \left(D^\dagger\right)^\ddagger$.

Lemma 2.1.4 *The map $(\cdot)^\#$ is an automorphism on $\Psi\mathfrak{D}$ of order 4*

Proof The map $(\cdot)^{\#}$ is an automorphism because it is a composition of two anti-automorphisms. Since $\left(\left(x^k(d/dx)^m\right)^{\#}\right)^{\#} = (-x)^k\,(-d/dx)^m$, the map $(\cdot)^{\#}$ has order 4.

■

2.2 Spaces of quasi-exponentials

In this dissertation, a partition $\mu = (\mu_1, \mu_2, \dots)$ is an infinite nonincreasing sequence of nonnegative integers stabilizing at zero. Let $\mu' = (\mu'_1, \mu'_2, \dots)$ denote the conjugate partition, that is, $\mu'_i = \#\{j \mid \mu_j \geqslant i\}$. In particular, μ'_1 equals the number of nonzero entries in μ.

Fix complex numbers $\alpha_1, \dots, \alpha_n$ and nonzero partitions $\mu^{(1)}, \dots, \mu^{(n)}$. Assume that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let V be a vector space of functions in one variable with a basis $\{q_{ij}(x)e^{\alpha_i x} \mid i = 1, \dots, n, j = 1, \dots, (\mu^{(i)})'_1\}$, where $q_{ij}(x)$ are polynomials and $\deg q_{ij} = (\mu^{(i)})'_1 + \mu^{(i)}_j - j$.

Denote $M' = \sum_{i=1}^{n} (\mu^{(i)})'_1 = \dim V$. For $z \in \mathbb{C}$, define *the sequence of exponents of V at z* as a unique sequence of integers $\boldsymbol{e} = \{e_1 > \dots > e_{M'}\}$, with the property: for each $i = 1, \dots, M'$, there exists $f \in V$ such that $f(x) = (x - z)^{e_i}\big(1 + o(1)\big)$ as $x \to z$.

We say that $z \in \mathbb{C}$ is a singular point of V if the set of exponents of V at z differs from the set $\{0, \dots, M'-1\}$. A space of quasi-exponentials has finitely many singular points.

Let $z_1, \dots, z_k$ be all singular points of V and let $\boldsymbol{e}^{(a)} = \{e^{(a)}_1 > \dots > e^{(a)}_{M'}\}$ be the set of exponents of V at z_a. For each $a = 1, \dots, k$, define a partition $\lambda^{(a)} = (\lambda^{(a)}_1, \lambda^{(a)}_2, \dots)$ as follows: $e^{(a)}_i = M' + \lambda^{(a)}_i - i$ for $i = 1, \dots, M'$, and $\lambda^{(a)}_i = 0$ for $i > M'$. Clearly, all partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$ are nonzero.

Denote the sequences $(\mu^{(1)}, \dots, \mu^{(n)})$, $(\lambda^{(1)}, \dots, \lambda^{(k)})$, $(\alpha_1, \dots, \alpha_n)$, $(z_1, \dots, z_k)$ as $\bar{\mu}$, $\bar{\lambda}$, $\bar{\alpha}$, $\bar{z}$, respectively. We will say that V is a *space of quasi-exponentials with the data* $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$.

For arbitrary sequences of partitions $\bar\mu = (\mu^{(1)}, \dots, \mu^{(n)})$, $\bar\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$, and sequences of complex numbers $\bar\alpha = (\alpha_1, \dots, \alpha_n)$, $\bar z = (z_1, \dots, z_k)$, define the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)^{\mathrm{red}}$ by removing all zero partitions from the sequences $\bar\mu\,, \bar\lambda$ and the corresponding numbers from the sequences $\bar\alpha\,, \bar z$. We will call the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$ *reduced* if $(\bar\mu, \bar\lambda; \bar\alpha, \bar z) = (\bar\mu, \bar\lambda; \bar\alpha, \bar z)^{\mathrm{red}}$.

We will say that V is a *space of quasi-exponentials with the data* $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$ if V is a space of quasi-exponentials with the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)^{\mathrm{red}}$.

The fundamental differential operator of V is a unique monic linear differential operator of order M' annihilating V. Denote the fundamental differential operator of V by D_V.

Define $D_V^{\mathrm{aug}} = D_V \prod_{i=1,\, \mu^{(i)}=0}^{n} (d/dx - \alpha_i)$. We will say that the space $V^{\mathrm{aug}} = \ker D_V^{\mathrm{aug}}$ is the *augmentation of* V *with the data* $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$, and the space V is the *reduction of* V^{aug}. Clearly, $V = \prod_{i=1,\, \mu^{(i)}=0}^{n} (d/dx - \alpha_i)\, V^{\mathrm{aug}}$.

Lemma 2.2.1 *The coefficients of D_V and D_V^{aug} are rational functions in x regular at infinity.*

The lemma will be proved in Section 2.5.5.

Recall that we identify the algebra of linear differential operators with rational coefficients and the corresponding subalgebra of $\Psi\mathfrak{D}$.

Let V be a *space of quasi-exponentials with the data* $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$. By Lemma 2.2.1 and Corollary 2.1.2, the operator D_V is an invertible element of $\Psi\mathfrak{D}$. Consider the following pseudo-differential operator:

$$\widetilde{D}_V = (-1)^{M'} \prod_{i=1}^{n} (x + \alpha_i)^{(\mu^{(i)})'_1} \left(D_V^{-1}\right)^{\#} \prod_{a=1}^{k} \left(\frac{d}{dx} - z_a\right)^{\lambda_1^{(a)}} . \tag{2.3}$$

Clearly, $\widetilde{D}_V$ depends only on the reduced data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)^{\mathrm{red}}$.

Theorem 2.2.2 *The following holds:*

1. *$\widetilde{D}_V$ is a monic differential operator of order $L = \sum_{a=1}^{k} \lambda_1^{(a)}$.*

2. *The vector space* $\tilde{V} = \ker \widetilde{D}_V$ *is a space of quasi-exponentials with the data* $(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha})$, *where* $\bar{\mu}' = ((\mu^{(1)})', \dots, (\mu^{(n)})')$, $\bar{\lambda}' = ((\lambda^{(1)})', \dots, (\lambda^{(k)})')$ *and* $-\bar{\alpha} = (-\alpha_1, \dots, -\alpha_n)$.

3. *Let* b_{ij} *and* $\tilde{b}_{st}$ *be the coefficients in the following expansions of* D_V^{aug} *and* $\widetilde{D}_V^{\mathrm{aug}} = \widetilde{D}_V \prod_{a=1,\,\lambda^{(a)}=0}^{k} (d/dx - z_a)$*:*
$$D_V^{\mathrm{aug}} = \sum_{i=0}^{M'_{\mathrm{aug}}} \sum_{j=0}^{\infty} b_{ij} x^{-j} \left(\frac{d}{dx}\right)^{M'_{\mathrm{aug}}-i}, \qquad \widetilde{D}_V^{\mathrm{aug}} = \sum_{s=0}^{L_{\mathrm{aug}}} \sum_{t=0}^{\infty} \tilde{b}_{st} x^{-t} \left(\frac{d}{dx}\right)^{L_{\mathrm{aug}}-s}.$$
Then there are polynomials P_{st} *in variables* r_{ij}, $i = 0, \dots, M'_{\mathrm{aug}}$, $j \geqslant 0$, *depending only on the data* $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$, *such that* $\tilde{b}_{st}$ *equals the value of* P_{st} *under the substitution* $r_{ij} = b_{ij}$ *for all* i, j. *Moreover, the coefficients of* P_{st} *are polynomials in* $\bar{\alpha}, \bar{z}$.

The theorem will be proved in Section 2.5.

Let $b_i(x)$ and $\tilde{b}_i(x)$ be the coefficients of D_V^{aug} and $\widetilde{D}_V^{\mathrm{aug}}$:
$$D_V^{\mathrm{aug}} = \left(\frac{d}{dx}\right)^{M'_{\mathrm{aug}}} + \sum_{i=1}^{M'_{\mathrm{aug}}} b_i(x) \left(\frac{d}{dx}\right)^{M'_{\mathrm{aug}}-i},$$
$$\widetilde{D}_V^{\mathrm{aug}} = \left(\frac{d}{dx}\right)^{L_{\mathrm{aug}}} + \sum_{a=1}^{L_{\mathrm{aug}}} \tilde{b}_a(x) \left(\frac{d}{dx}\right)^{L_{\mathrm{aug}}-a}.$$
By Lemma 2.2.1, $b_i(x)$ and $\tilde{b}_i(x)$ are rational functions of x. Define functions $\tilde{c}_i(u)$, $i \in \mathbb{Z}_{\geqslant 0}$, by the rule:
$$\prod_{\substack{a=1 \\ \lambda_1^{(a)}=0}}^{k} (u - z_a) \prod_{a=1}^{k} (u - z_a)^{\lambda_1^{(a)}} \sum_{i=0}^{\infty} \tilde{c}_i(u) x^{-i} = u^{L_{\mathrm{aug}}} + \sum_{a=1}^{L_{\mathrm{aug}}} \tilde{b}_a(x) u^{L_{\mathrm{aug}}-a}. \tag{2.4}$$
Set
$$h_a = \mathrm{Res}_{x=z_a}\Big(\frac{b_1^2(x)}{2} - b_2(x)\Big), \qquad \tilde{g}_a = \mathrm{Res}_{u=z_a}\Big(\frac{\tilde{c}_1^2(u)}{2} - \tilde{c}_2(u)\Big). \tag{2.5}$$
We will need the following lemma:

Lemma 2.2.3 *For each* $a = 1, \dots, k$, *we have*
$$\tilde{g}_a = -h_a.$$

The Lemma will be proved in Section 2.5.6.

2.3 Bethe algebra

2.3.1 Universal differential operator

The current algebra $\mathfrak{gl}_n[t] = \mathfrak{gl}_n \otimes \mathbb{C}[t]$ is the Lie algebra of $\mathfrak{gl}_n$-valued polynomials with pointwise commutator. We identify the Lie algebra $\mathfrak{gl}_n$ with the subalgebra $\mathfrak{gl}_n \otimes 1$ of constant polynomials in $\mathfrak{gl}_n[t]$.

For each $g \in \mathfrak{gl}_n$, let $g(x) = \sum_{s=0}^{\infty}(g \otimes t^s)\, x^{-s-1}$. It is a formal power series in x^{-1} with coefficients in $\mathfrak{gl}_n[t]$.

For an $n\times n$ matrix A with possibly noncommuting entries a_{ij}, its row determinant is

$$\operatorname{rdet} A = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}.$$

Let e_{ij}, $i, j = 1, \dots, n$, be the standard generators of the Lie algebra $\mathfrak{gl}_n$ satisfying the relations $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$. Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{gl}_n$ spanned by the generators $e_{11}, \dots, e_{nn}$.

Fix $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, a sequence of pairwise distinct complex numbers. Define the *universal differential operator* $\mathcal{D}_{\bar{\alpha}}$ by the formula

$$\mathcal{D}_{\bar{\alpha}} = \operatorname{rdet} \left(\left(\frac{d}{dx} - \alpha_i \right) \delta_{ij} - e_{ji}(x) \right)_{i,j=1}^{n}.$$

It is a differential operator in the variable x whose coefficients are formal power series in x^{-1} with coefficients in $U(\mathfrak{gl}_n[t])$,

$$\mathcal{D}_{\bar{\alpha}} = \left(\frac{d}{dx} \right)^{n} + \sum_{i=1}^{n} B_i(x) \left(\frac{d}{dx} \right)^{n-i}, \tag{2.6}$$

where

$$B_i(x) = \sum_{j=0}^{\infty} B_{ij} x^{-j}. \tag{2.7}$$

and $B_{ij} \in U(\mathfrak{gl}_n[t])$ for $i = 1, \dots, n$, $j \geqslant 0$. Notice that $\sum_{i=0}^{n} B_{i0} u^{n-i} = \prod_{j=1}^{n} (u - \alpha_j)$.

Definition 2.3.1 *The subalgebra $\mathcal{B}_{\bar{\alpha}}$ of $U(\mathfrak{gl}_n[t])$ generated by B_{ij}, $i = 1, \dots, n$, $j \geqslant 1$, is called the Bethe algebra.*

The proof of the following theorem can be found in [6].

Theorem 2.3.2 *The algebra $\mathcal{B}_{\bar\alpha}$ is commutative. The algebra $\mathcal{B}_{\bar\alpha}$ commutes with the subalgebra $U(\mathfrak{h}) \subset U(\mathfrak{gl}_n[t])$.*

2.3.2 Action of Bethe algebra in a tensor product of evaluation modules.

For $a \in \mathbb{C}$, let ρ_a be the automorphism of $\mathfrak{gl}_n[t]$ such that $\rho_a : g(x) \mapsto g(x - a)$. Given a $\mathfrak{gl}_n[t]$-module M, we denote by $M(a)$ the pullback of M through the automorphism ρ_a.

Let $ev : \mathfrak{gl}_n[t] \to \mathfrak{gl}_n$ be the evaluation homomorphism, $ev : g(x) \mapsto gx^{-1}$. For any $\mathfrak{gl}_n$-module M, we denote by the same letter the $\mathfrak{gl}_n[t]$-module, obtained by pulling M back through the evaluation homomorphism. For each $a \in \mathbb{C}$ and $\mathfrak{gl}_n$-module M, the $\mathfrak{gl}_n[t]$- module $M(a)$ is called an *evaluation module*.

For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and an $\mathfrak{h}$-module M, we denote by $(M)_\lambda$ the weight subspace of M of weight λ. Note that any partition λ with $\lambda_{n+1} = 0$ can be considered as an element of $\mathbb{C}^n$.

Let M be a $\mathfrak{gl}_n[t]$-module. As a subalgebra of $U(\mathfrak{gl}_n[t])$, the algebra $\mathcal{B}_{\bar\alpha}$ acts on M. Since $\mathcal{B}_{\bar\alpha}$ commutes with $U(\mathfrak{h})$, it preserves the weight subspaces $(M)_\lambda$.

Given a $\mathcal{B}_{\bar\alpha}$-module M, a subspace $H \subset M$ is called an eigenspace of $\mathcal{B}_{\bar\alpha}$-action on M if there is a homomorphism $\xi : \mathcal{B}_{\bar\alpha} \to \mathbb{C}$ such that $H = \bigcap_{F \in \mathcal{B}_{\bar\alpha}} \ker\big(F - \xi(F)\big)$.

Denote by L_λ the irreducible finite-dimensional $\mathfrak{gl}_n$-module with highest weight λ. Fix $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n_{\geqslant 0}$, $\bar\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$, $\bar z = (z_1, \dots, z_k) \in \mathbb{C}^k$ such that $z_a \neq z_b$ for $a \neq b$, and a sequence of partitions $\bar\lambda = (\lambda^{(1)}, \dots, \lambda^{(k)})$. Define the sequence of partitions $\bar\mu = (\mu^{(1)}, \dots, \mu^{(n)})$ setting $\mu^{(i)} = (\mu_i, 0, 0, \dots)$. The next theorem states the results from [43] that we need.

Theorem 2.3.3 *Consider a tensor product $L_{\bar\lambda}(\bar z) = L_{\lambda^{(1)}}(z_1) \otimes \ldots \otimes L_{\lambda^{(k)}}(z_k)$ of evaluation $\mathfrak{gl}_n[t]$-modules. Then the following holds.*

1. *Each eigenspace of the action of $\mathcal{B}_{\bar\alpha}$ on $\big(L_{\bar\lambda}(\bar z)\big)_\mu$ is one-dimensional.*

2. *For generic $\bar\alpha$ and $\bar z$, the action of $\mathcal{B}_{\bar\alpha}$ on $\big(L_{\bar\lambda}(\bar z)\big)_\mu$ is diagonalizable.*

3. *Let $v \in \big(L_{\bar\lambda}(\bar z)\big)_\mu$ be an eigenvector of the action of $\mathcal{B}_{\bar\alpha}$. Then there exist rational functions $b_1(x), \dots, b_n(x)$ with Laurent series at infinity $\hat b_1(x), \dots, \hat b_n(x)$, respectively, such that $B_i(x)v = \hat b_i(x)v$ for all $i = 1, \dots, n$, and the kernel of the differential operator $D = (d/dx)^n + \sum_{i=1}^n b_i(x)(d/dx)^{n-i}$ is the augmentation of a space of quasi-exponentials with the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$.*

4. *The correspondence between eigenspaces of the action of $\mathcal{B}_{\bar\alpha}$ on $\big(L_{\bar\lambda}(\bar z)\big)_\mu$ and spaces of quasi-exponentials with the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$ given in part (3) is bijective.*

2.3.3 Gaudin and Dynamical Hamiltonians

For $g \in \mathfrak{gl}_n$, define $g_{(a)} = 1^{\otimes(a-1)} \otimes g \otimes 1^{\otimes(k-a)} \in U(\mathfrak{gl}_n)^{\otimes k}$. We will use the same notation for an element of $U(\mathfrak{gl}_n)$ and its image under the diagonal embedding $g \mapsto \sum_{a=1}^k (g)_{(a)} \in U(\mathfrak{gl}_n)^{\otimes k}$. Let $\Omega_{(ab)} = \sum_{i,j=1}^n (e_{ij})_{(a)} \, (e_{ji})_{(b)}$.

For sequences of pairwise distinct numbers $\bar\alpha = (\alpha_1, \dots, \alpha_n)$ and $\bar z = (z_1, \dots, z_k)$, define the following elements of $U(\mathfrak{gl}_n)^{\otimes k}$:

$$H_a(\bar z, \bar\alpha) = \sum_{i=1}^n \alpha_i (e_{ii})_{(a)} + \sum_{\substack{b=1\\ b\neq a}}^k \frac{\Omega_{(ab)}}{z_a - z_b}\,, \qquad G_i(\bar z, \bar\alpha) = \sum_{a=1}^k z_a (e_{ii})_{(a)} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{e_{ij}e_{ji} - e_{ii}}{\alpha_i - \alpha_j}\,.$$

The elements $H_1(\bar z, \bar\alpha), \dots, H_k(\bar z, \bar\alpha)$ are called the Gaudin Hamiltonians. The elements $G_1(\bar z, \bar\alpha), \dots, G_n(\bar z, \bar\alpha)$ are called the Dynamical Hamiltonians.

Consider an algebra homomorphism $ev_{\bar z} : U(\mathfrak{gl}_n[t]) \to U(\mathfrak{gl}_n)^{\otimes k}$, given by

$$ev_{\bar z} : g \otimes t^s \;\mapsto\; \sum_{a=1}^k g_{(a)} z_a^s\,.$$

For each $i = 1, \dots, n$, let $\widehat B_i(x)$ be the image of the series $B_i(x)$, see (2.6), under the map $ev_{\bar z}$. The series $\widehat B_i(x)$ is a formal power series in x^{-1} with coefficients in $U(\mathfrak{gl}_n)^{\otimes k}$. There exists a rational function of the form $\sum_{a=1}^k \sum_{j=0}^i \widehat B_{ija}(x - z_a)^{-j}$, where $\widehat{B}_{ija} \in U(\mathfrak{gl}_n)^{\otimes k}$, such that $\widehat{B}_i(x)$ is the Laurent series of this function as $x \to \infty$. We will identify the series $\widehat{B}_i(x)$ and this rational function.

Let $\widehat{C}_j(u),\ j \in \mathbb{Z}_{\geqslant 0}$, be rational functions in u defined by the formula

$$\prod_{i=1}^{n}(u-\alpha_i)\sum_{j=0}^{\infty}\widehat{C}_j(u)x^{-j} \,=\, u^n + \sum_{i=1}^{n}\widehat{B}_i(x)u^{n-i}\,. \tag{2.8}$$

Lemma 2.3.4 *The following holds:*

$$H_a(\bar{z},\bar{\alpha}) \,=\, \mathrm{Res}_{x=z_a}\Big(\frac{\widehat{B}_1^2(x)}{2} - \widehat{B}_2(x)\Big)\,, \qquad G_i(\bar{z},\bar{\alpha}) = \mathrm{Res}_{u=\alpha_i}\Big(\frac{\widehat{C}_1^2(u)}{2} - \widehat{C}_2(u)\Big)\,. \tag{2.9}$$

Proof The proof is straightforward. ■

2.4 $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality

2.4.1 $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for Bethe algebras

Let $\mathfrak{X}_n$ be the space of polynomials in anticommuting variables $\xi_1, \ldots, \xi_n$. Since $\xi_i\xi_j = -\xi_j\xi_i$ for any i,j, in particular, $\xi_i^2 = 0$ for any i, the monomials $\xi_{i_1}\ldots\xi_{i_l}$, $1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant n$, form a basis of $\mathfrak{X}_n$.

The left derivations $\partial_1, \ldots, \partial_n$ on $\mathfrak{X}_n$ are linear maps such that

$$\begin{aligned} &\partial_i\,(\xi_{j_1}\ldots\xi_{j_l}) = (-1)^{s-1}\xi_{j_1}\ldots\xi_{j_{s-1}}\,\xi_{j_{s+1}}\ldots\xi_{j_l}\,, && \text{if } \ i = j_s \text{ for some } \ s\,, \\ &\partial_i\,(\xi_{j_1}\ldots\xi_{j_l}) = 0\,, && \text{otherwise}\,. \end{aligned} \tag{2.10}$$

It is easy to check that $\partial_i\partial_j = -\partial_j\partial_i$ for any i,j, in particular, $\partial_i^2 = 0$ for any i, and $\partial_i\xi_j + \xi_j\partial_i = \delta_{ij}$ for any i,j.

Define a $\mathfrak{gl}_n$-action on $\mathfrak{X}_n$ by the rule $e_{ij} \mapsto \xi_i\partial_j$. As a $\mathfrak{gl}_n$-module, $\mathfrak{X}_n$ is isomorphic to $\bigoplus_{l=0}^{n} L_{\omega_l}$, where

$$\omega_l = (\underbrace{1,\ldots,1}_{l},0,\ldots,0)\,, \tag{2.11}$$

and the component L_{ω_l} is spanned by the monomials of degree l.

Notice that $\mathfrak{X}_n$ as an algebra coincides with the exterior algebra of $\mathbb{C}^n$. The operators of left multiplication by $\xi_1, \ldots, \xi_n$ and the left derivations $\partial_1, \ldots, \partial_n$ give on $\mathfrak{X}_n$ the irreducible representation of the Clifford algebra Cliff_n.

From now on, we will consider the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ together. We will write superscripts $\langle n\rangle$ and $\langle k\rangle$ to distinguish objects associated with algebras $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$, respectively. For example, $e_{ij}^{\langle n\rangle}$, $i, j = 1, \ldots, n$, are the generators of $\mathfrak{gl}_n$, and $e_{ab}^{\langle k\rangle}$, $a, b = 1, \ldots, k$, are the generators of $\mathfrak{gl}_k$.

Let $\mathfrak{P}_{kn}$ be the vector space of polynomials in kn pairwise anticommuting variables ξ_{ai}, $a = 1, \ldots, k$, $i = 1, \ldots, n$. We have two vector space isomorphisms $\psi_1 : (\mathfrak{X}_n)^{\otimes k} \to \mathfrak{P}_{kn}$ and $\psi_2 : (\mathfrak{X}_k)^{\otimes n} \to \mathfrak{P}_{kn}$, given by:

$$\psi_1 : (p_1 \otimes \ldots \otimes p_k) \mapsto p_1(\xi_{11}, \ldots, \xi_{1n}) p_2(\xi_{21}, \ldots, \xi_{2n}) \ldots p_k(\xi_{k1}, \ldots, \xi_{kn}) \,, \tag{2.12}$$

$$\psi_2 : (p_1 \otimes \ldots \otimes p_n) \mapsto p_1(\xi_{11}, \ldots, \xi_{k1}) p_2(\xi_{12}, \ldots, \xi_{k2}) \ldots p_n(\xi_{1n}, \ldots, \xi_{kn}) \,. \tag{2.13}$$

Let ∂_{ai}, $a = 1, \ldots, k$, $i = 1, \ldots, n$, be the left derivations on $\mathfrak{P}_{kn}$ defined similarly to the left derivations on $\mathfrak{X}_\mathfrak{n}$, see (2.10). Define actions of $\mathfrak{gl}_n$ and $\mathfrak{gl}_k$ on $\mathfrak{P}_{kn}$ by the formulas

$$e_{ij}^{\langle n\rangle} \mapsto \sum_{a=1}^{k} \xi_{ai}\partial_{aj} \,, \qquad e_{ab}^{\langle k\rangle} \mapsto \sum_{i=1}^{n} \xi_{ai}\partial_{bi}.$$

Then ψ_1 and ψ_2 are isomorphisms of $\mathfrak{gl}_n$- and $\mathfrak{gl}_k$-modules, respectively.

It is easy to check that $\mathfrak{gl}_n$- and $\mathfrak{gl}_k$-actions on $\mathfrak{P}_{kn}$ commute. For the next theorem, see for example [4]:

Theorem 2.4.1 *The $\mathfrak{gl}_n \oplus \mathfrak{gl}_k$-module $\mathfrak{P}_{kn}$ has the decomposition $\mathfrak{P}_{kn} = \bigoplus_\lambda V_\lambda^{\langle n\rangle} \otimes V_{\lambda'}^{\langle k\rangle}$, where the sum runs over $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \leqslant k$.*

The $\mathfrak{gl}_n$- and $\mathfrak{gl}_k$-actions on $\mathfrak{P}_{kn}$ can be extended to the actions of corresponding current Lie algebras by the formulas

$$e_{ij}^{\langle n\rangle} \otimes t^s \mapsto \sum_{a=1}^{k} \, z_a^s \, \xi_{ai}\partial_{aj}, \tag{2.14}$$

$$e_{ab}^{\langle k\rangle} \otimes t^s \mapsto \sum_{i=1}^{n} (-\alpha_i)^s \, \xi_{ai}\partial_{bi}. \tag{2.15}$$

Then ψ_1 and ψ_2 are respective isomorphisms of the following $\mathfrak{gl}_n[t]$- and $\mathfrak{gl}_k[t]$-modules:

$$\psi_1 : \mathfrak{X}_n(z_1) \otimes \mathfrak{X}_n(z_2) \otimes \ldots \otimes \mathfrak{X}_n(z_k) \to \mathfrak{P}_{kn}\,, \tag{2.16}$$

$$\psi_2 : \mathfrak{X}_k(-\alpha_1) \otimes \mathfrak{X}_k(-\alpha_2) \otimes \ldots \otimes \mathfrak{X}_k(-\alpha_n) \to \mathfrak{P}_{kn}\,. \tag{2.17}$$

The actions (2.14) and (2.15) do not commute. Nevertheless, it turns out that the images of the subalgebras $\mathcal{B}^{\langle n\rangle}_{\bar\alpha}$ and $\mathcal{B}^{\langle k\rangle}_{\bar z}$ in $\mathrm{End}(\mathfrak{P}_{kn})$ given by these actions coincide. We will use Theorems 2.2.2 and 2.3.3 to show the following.

Theorem 2.4.2 *Let* $\pi^{\langle n\rangle}_{\bar z} : U(\mathfrak{gl}_n[t]) \to \mathrm{End}(\mathfrak{P}_{kn})$ *and* $\pi^{\langle k\rangle}_{-\bar\alpha} : U(\mathfrak{gl}_k[t]) \to \mathrm{End}(\mathfrak{P}_{kn})$ *be the homomorphisms defined by formulas* (2.14) *and* (2.15)*, respectively. Then*

$$\pi^{\langle n\rangle}_{\bar z}(\mathcal{B}^{\langle n\rangle}_{\bar\alpha}) = \pi^{\langle k\rangle}_{-\bar\alpha}(\mathcal{B}^{\langle k\rangle}_{\bar z}). \tag{2.18}$$

Theorem 2.4.2 is proved in Section 2.4.5.

Remark 1 *Let* $B^{\langle n\rangle}_{ij,\bar\alpha}$ *be the generators of the algebra* $\mathcal{B}^{\langle n\rangle}_{\bar\alpha}$*, cf* (2.7)*. Here, we indicated the dependence on* $\bar\alpha$ *explicitly. Then we have* $\pi^{\langle n\rangle}_{-\bar z}(B^{\langle n\rangle}_{ij,-\bar\alpha}) = (-1)^{n-i-j}\pi^{\langle n\rangle}_{\bar z}(B^{\langle n\rangle}_{ij,\bar\alpha})$*. Therefore* $\pi^{\langle n\rangle}_{-\bar z}(\mathcal{B}^{\langle n\rangle}_{-\bar\alpha}) = \pi^{\langle n\rangle}_{\bar z}(\mathcal{B}^{\langle n\rangle}_{\bar\alpha})$*.*

2.4.2 $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for Gaudin and Dynamical Hamiltonians

Define $U(\mathfrak{gl}_n)^{\otimes k}$ and $U(\mathfrak{gl}_k)^{\otimes n}$-actions on $\mathfrak{P}_{kn}$ by

$$(e^{\langle n\rangle}_{ij})_{(a)} \mapsto \xi_{ai}\partial_{aj}, \tag{2.19}$$

$$(e^{\langle k\rangle}_{ab})_{(i)} \mapsto \xi_{ai}\partial_{bi}. \tag{2.20}$$

Then ψ_1 and ψ_2 are isomorphisms of $U(\mathfrak{gl}_n)^{\otimes k}$- and $U(\mathfrak{gl}_k)^{\otimes n}$-modules, respectively.

In Section 2.3.3, we introduced elements $H_a(\bar z, \bar\alpha)$ and $G_i(\bar z, \bar\alpha)$ of $U(\mathfrak{gl}_n)^{\otimes k}$. We will write them now as $H^{\langle n,k\rangle}_a(\bar z, \bar\alpha)$, $G^{\langle n,k\rangle}_i(\bar z, \bar\alpha)$. We will also consider analogous elements $H^{\langle k,n\rangle}_i(\bar\alpha, \bar z)$, $G^{\langle k,n\rangle}_a(\bar\alpha, \bar z)$ of $U(\mathfrak{gl}_k)^{\otimes n}$. The following result can be found in [41]:

Lemma 2.4.3 *Let* $\rho^{\langle n,k\rangle} : U(\mathfrak{gl}_n)^{\otimes k} \to \mathrm{End}(\mathfrak{P}_{kn})$ *and* $\rho^{\langle k,n\rangle} : U(\mathfrak{gl}_k)^{\otimes n} \to \mathrm{End}(\mathfrak{P}_{kn})$ *be the homomorphisms defined by* (2.19) *and* (2.20) *respectively. Then for any* $i = 1, \dots, n$, *and* $a = 1, \dots, k$ *we have:*

$$\rho^{\langle n,k\rangle}\big(H_a^{\langle n,k\rangle}(\bar z,\bar\alpha)\big) = -\rho^{\langle k,n\rangle}\big(G_a^{\langle k,n\rangle}(-\bar\alpha,\bar z)\big)\,, \tag{2.21}$$

$$\rho^{\langle n,k\rangle}\big(G_i^{\langle n,k\rangle}(\bar z,\bar\alpha)\big) = \rho^{\langle k,n\rangle}\big(H_i^{\langle k,n\rangle}(-\bar\alpha,\bar z)\big)\,. \tag{2.22}$$

Proof The proof is straightforward. ■

2.4.3 Restriction to the subspaces $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$.

Let $\mathcal{Z}_{kn}$ be the subset of all pairs $(\boldsymbol{l},\boldsymbol{m}) \in \mathbb{Z}^k_{\geqslant 0} \times \mathbb{Z}^n_{\geqslant 0}\,,\ \boldsymbol{l} = (l_1,\dots,l_k),\ \boldsymbol{m} = (m_1, \dots, m_n)$, such that $l_a \leqslant n$ for all a, $m_i \leqslant k$ for all i, and $\sum_{a=1}^k l_a = \sum_{i=1}^n m_i$. For each $(\boldsymbol{l},\boldsymbol{m}) \in \mathcal{Z}_{kn}$, denote by $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}] \subset \mathfrak{P}_{kn}$ the span of all monomials $\xi_{11}^{d_{11}}\dots\xi_{k1}^{d_{k1}}\dots\xi_{1n}^{d_{1n}}\dots\xi_{kn}^{d_{kn}}$ such that $\sum_{a=1}^k d_{ai} = m_i$ and $\sum_{i=1}^n d_{ai} = l_a$. Note that $d_{ai} \in \{0,1\}$ for all a,i. Clearly, we have a vector space decomposition:

$$\mathfrak{P}_{kn} = \bigoplus_{(\boldsymbol{l},\boldsymbol{m})\in\mathcal{Z}_{kn}} \mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]\,.$$

Lemma 2.4.4 *For any* $(\boldsymbol{l},\boldsymbol{m}) \in \mathcal{Z}_{kn}$, *the subspace* $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ *is invariant under the actions of the algebras* $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$ *and* $\mathcal{B}_{\bar z}^{\langle k\rangle}$.

Proof Recall $\mathfrak{X}_n = \bigoplus_{l=0}^n L_{\omega_l}$ as a $\mathfrak{gl}_n$-module. Then by the isomorphism ψ_1, see (2.16), the $\mathfrak{gl}_n[t]$-module $\mathfrak{P}_{kn}$ is the direct sum of tensor products $L^{\langle n\rangle}_{\omega_{l_1}}(z_1) \otimes \dots \otimes L^{\langle n\rangle}_{\omega_{l_k}}(z_k)$, and

$$\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}] = \psi_1\big(\big(L^{\langle n\rangle}_{\omega_{l_1}}(z_1) \otimes \dots \otimes L^{\langle n\rangle}_{\omega_{l_k}}(z_k)\big)_{\boldsymbol{m}}\big)\,. \tag{2.23}$$

Hence, $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ is invariant under the action of $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$, see Section 2.3.2.

Similarly, $\mathfrak{X}_k = \bigoplus_{m=0}^k L_{\omega_m}$ as a $\mathfrak{gl}_k$-module. Then by the isomorphism ψ_2, see (2.17), the $\mathfrak{gl}_k[t]$-module $\mathfrak{P}_{kn}$ is the direct sum of tensor products $L^{\langle k\rangle}_{\omega_{m_1}}(-\alpha_1) \otimes \dots \otimes L^{\langle k\rangle}_{\omega_{m_n}}(-\alpha_n)$, and

$$\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}] = \psi_2\big(\big(L^{\langle k\rangle}_{\omega_{m_1}}(-\alpha_1) \otimes \dots \otimes L^{\langle k\rangle}_{\omega_{m_n}}(-\alpha_n)\big)_{\boldsymbol{l}}\big)\,. \tag{2.24}$$

Thus $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ is invariant under the action of $\mathcal{B}^{\langle k\rangle}_{\bar z}$. ■

We will prove Theorem 2.4.2 by showing that the restrictions of $\pi^{\langle n\rangle}_{\bar z}(\mathcal{B}^{\langle n\rangle}_{\bar\alpha})$ and $\pi^{\langle k\rangle}_{-\bar\alpha}(\mathcal{B}^{\langle k\rangle}_{\bar z})$ to each subspace $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ coincide. We will also need the following lemma.

Lemma 2.4.5 *Fix $(\boldsymbol{l},\boldsymbol{m})\in\mathcal{Z}_{kn}$. For generic $\bar\alpha,\bar z$, the common eigenspaces of the operators $\rho^{\langle n,k\rangle}(H^{\langle n,k\rangle}_a(\bar z,\bar\alpha))$, $a=1,\dots,k$, restricted to $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ are one-dimensional. Similarly, for generic $\bar\alpha,\bar z$, the common eigenspaces of the operators $\rho^{\langle k,n\rangle}(H^{\langle k,n\rangle}_i(-\bar\alpha,\bar z))$, $i=1,\dots,n$, restricted to $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ are one-dimensional.*

Proof For every monomial $p\in\mathfrak{P}_{kn}$, we have $(e^{\langle n\rangle}_{ii})_{(a)}p=m^a_i(p)p$ and $m^a_i(p)\in\mathbb{Z}$. Moreover, if $p\neq p'$, there exist i,a such that $m^a_i(p)\neq m^a_i(p')$. Take $\bar\alpha$ such that $\alpha_1,\dots,\alpha_n$ are linearly independent over $\mathbb{Z}$. Then for the operators $K_a=\rho^{\langle n,k\rangle}(\sum_{i=1}^n\alpha_i(e^{\langle n\rangle}_{ii})_{(a)})$, $a=1,\dots,k$, the common eigenspaces are one-dimensional. Therefore, the common eigenspaces of the operators $\rho^{\langle n,k\rangle}(H^{\langle n,k\rangle}_a(\bar z,\bar\alpha))=K_a+\sum_{b\neq a}\Omega_{(ab)}(z_a-z_b)^{-1}$, $a=1,\dots,k$, restricted to a finite-dimensional submodule of $\mathfrak{P}_{kn}$ are one-dimensional provided all the differences $|z_a-z_b|$ are sufficiently large. Hence, for generic $\bar\alpha$ and $\bar z$, the common eigenspaces of the operators $\rho^{\langle n,k\rangle}(H^{\langle n,k\rangle}_a(\bar z,\bar\alpha))$, $a=1,\dots,k$, restricted to a $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ are one-dimensional.

The proof of the second claim is similar. ■

2.4.4 Spaces of quasi-exponentials and $(\mathfrak{gl}_k,\mathfrak{gl}_n)$-duality

Fix $(\boldsymbol{l},\boldsymbol{m})\in\mathcal{Z}_{kn}$, and define $\bar\mu=(\mu^{(1)},\dots,\mu^{(n)})$, $\bar\lambda=(\lambda^{(1)},\dots,\lambda^{(k)})$ as follows. If $\boldsymbol{l}=(l_1,\dots,l_k)$ and $\boldsymbol{m}=(m_1,\dots,m_n)$, then $\mu^{(i)}=(m_i,0,\dots)$, $i=1,\dots,n$, and $\lambda^{(a)}=\omega_{l_a}$, $a=1,\dots,k$, see (2.11).

By Theorem 2.3.3 and formulas (2.23), (2.24), a space of quasi-exponentials with the data $(\bar\mu,\bar\lambda;\bar\alpha,\bar z)$ defined above gives rise to an eigenvector of the action $\pi^{\langle n\rangle}_{\bar z}$ of $B^{\langle n\rangle}_{\bar\alpha}$ on $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$. Similarly, a space of quasi-exponentials with the data $(\bar\lambda',\bar\mu';\bar z,-\bar\alpha)$ gives rise to an eigenvector of the action $\pi^{\langle k\rangle}_{-\bar\alpha}$ of $B^{\langle k\rangle}_{\bar z}$ on $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$. We have the following theorem.

Theorem 2.4.6 *Let V be a space of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$, and $v \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ be the eigenvector of the action $\pi_{\bar{z}}^{\langle n \rangle}$ of $B_{\bar{\alpha}}^{\langle n \rangle}$ corresponding to V. For the fundamental differential operator D_V of the space V, define the operator $\widetilde{D}_V$ by formula (2.3), and set $\tilde{V} = \ker(\widetilde{D}_V)$. Then, for generic $\bar{\alpha}, \bar{z}$, the vector v is the eigenvector of the action $\pi_{-\bar{\alpha}}^{\langle k \rangle}$ of $B_{\bar{z}}^{\langle k \rangle}$ corresponding to $\tilde{V}$.*

Proof For each $a = 1, \dots, k$, let h_a and $\tilde{g}_a$ be the numbers defined in formula (2.5). Comparing formulae (2.4), (2.5), (2.8), and (2.9), and using that $\lambda_1^{(a)} = 0$ or 1, we see that the vector v is an eigenvector of $\rho^{\langle n,k \rangle}(H_a^{\langle n,k \rangle}(\bar{z}, \bar{\alpha}))$ with eigenvalue h_a. Similarly, an eigenvector $\tilde{v} \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ of the action $\pi_{-\bar{\alpha}}^{\langle k \rangle}$ of $B_{\bar{z}}^{\langle k \rangle}$ corresponding to $\tilde{V}$ is an eigenvector of $\rho^{\langle k,n \rangle}(G_a^{\langle k,n \rangle}(-\bar{\alpha}, \bar{z}))$ with eigenvalue $\tilde{g}_a$. Therefore, by formula (2.21) and Lemma 2.2.3, for each $a = 1, \dots, k$, the vector $\tilde{v}$ is an eigenvector of $\rho^{\langle n,k \rangle}(H_a^{\langle n,k \rangle}(\bar{z}, \bar{\alpha}))$ with eigenvalue h_a, the same as for v. Hence, by Lemma 2.4.5, the vector $\tilde{v}$ is proportional to v. ■

2.4.5 Proof of Theorem 2.4.2

Let $B_{ij,\bar{\alpha}}^{\langle n \rangle}$, $i = 1, \dots, n$, $j \in \mathbb{Z}_{\geqslant 0}$, and $B_{st,\bar{z}}^{\langle k \rangle}$, $s = 1, \dots, k$, $t \in \mathbb{Z}_{\geqslant 0}$, be the generators of the algebras $\mathcal{B}_{\bar{\alpha}}^{\langle n \rangle}$ and $\mathcal{B}_{\bar{z}}^{\langle k \rangle}$, respectively, see (2.7).

Assume first that $\bar{\alpha}$ and $\bar{z}$ are generic. Take a common eigenvector v of $\pi_{\bar{z}}^{\langle n \rangle}(B_{ij,\bar{\alpha}}^{\langle n \rangle})$, $i = 1, \dots, n$, $j \in \mathbb{Z}_{\geqslant 0}$, corresponding to a space V of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ as in Theorem 2.4.6. The eigenvalue of $\pi_{\bar{z}}^{\langle n \rangle}(B_{ij,\bar{\alpha}}^{\langle n \rangle})$ associated to v is the coefficient b_{ij} in the expansion $D_V^{\mathrm{aug}} = \sum_{i=0}^{n} \sum_{j=0}^{\infty} b_{ij} x^{-j} (d/dx)^{n-i}$. By Theorem 2.4.6, v is also a common eigenvector of $\pi_{-\bar{\alpha}}^{\langle k \rangle}(B_{st,\bar{z}}^{\langle k \rangle})$, $s = 1, \dots, k$, $t \in \mathbb{Z}_{\geqslant 0}$, and the corresponding eigenvalue of $\pi_{-\bar{\alpha}}^{\langle k \rangle}(B_{st,\bar{z}}^{\langle k \rangle})$ is the coefficient $\tilde{b}_{st}$ in the expansion $\widetilde{D}_V^{\mathrm{aug}} = \sum_{s=0}^{k} \sum_{t=0}^{\infty} \tilde{b}_{st} x^{-t} (d/dx)^{k-s}$, where $\widetilde{D}_V$ is given by the formula (2.3). Due to Theorem 2.2.2, part (3), there exist polynomials P_{st} in variables r_{ij}, $i = 0, \dots, M'$, $j \geqslant 0$, independent of the eigenvector v, such that $\tilde{b}_{st}$ are obtained by the substitution $r_{ij} = b_{ij}$ for all i, j, into the polynomial P_{st},

$$\tilde{b}_{st} = P_{st}(\{b_{ij}\}). \tag{2.25}$$

By Theorem 2.3.3, part (2), the subspace $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ has a basis consisting of common eigenvectors of the operators $\pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})$, $i=1,\dots,n$, $j\in\mathbb{Z}_{\geqslant 0}$. Since the operator $\pi_{-\bar\alpha}^{\langle k\rangle}(B_{st,\bar z}^{\langle k\rangle})$ is diagonal in such a basis, relation (2.25) for eigenvalues implies the analogous relation for the operators:

$$\pi_{-\bar\alpha}^{\langle k\rangle}(B_{st,\bar z}^{\langle k\rangle}) = P_{st}(\{\pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})\}). \tag{2.26}$$

Since the operators $\pi_{-\bar\alpha}^{\langle k\rangle}(B_{ab,\bar z}^{\langle k\rangle})$, $\pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})$, and the coefficients of P_{st} depend polynomially on $\bar\alpha$ and $\bar z$, relation (2.26) holds for any $\bar\alpha$ and $\bar z$, and $\pi_{-\bar\alpha}^{\langle k\rangle}(\mathcal{B}_{\bar z}^{\langle k\rangle})\subset\pi_{\bar z}^{\langle n\rangle}(\mathcal{B}_{\bar\alpha}^{\langle n\rangle})$.

Exchanging the roles of $\mathfrak{gl}_k$ and $\mathfrak{gl}_n$, we obtain that $\pi_{-\bar z}^{\langle n\rangle}(\mathcal{B}_{-\bar\alpha}^{\langle n\rangle})\subset\pi_{-\bar\alpha}^{\langle k\rangle}(\mathcal{B}_{\bar z}^{\langle k\rangle})$ as well. Since $\pi_{-\bar z}^{\langle n\rangle}(\mathcal{B}_{-\bar\alpha}^{\langle n\rangle})=\pi_{\bar z}^{\langle n\rangle}(\mathcal{B}_{\bar\alpha}^{\langle n\rangle})$, see the remark at the end of Section 2.4.1, Theorem 2.4.2 is proved.

Corollary 2.4.7 *Theorem 2.4.6 holds for any $\bar\alpha=(\alpha_1,\dots,\alpha_n)$ and $\bar z=(z_1,\dots,z_k)$ such that $\alpha_i\neq\alpha_j$ if $i\neq j$ and $z_a\neq z_b$ if $a\neq b$.*

Proof Let V be a space of quasi-exponentials with the data $(\bar\mu,\bar\lambda;\bar\alpha,\bar z)$, where $\bar\mu$ and $\bar\lambda$ are defined by $\boldsymbol{m}$ and $\boldsymbol{l}$, like in Section 2.4.4. Let $v\in\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ be the eigenvector of the action $\pi_{\bar z}^{\langle n\rangle}$ of $B_{\bar\alpha}^{\langle n\rangle}$ corresponding to V. By Theorem 2.4.2, the vector v is also an eigenvector of the action $\pi_{-\bar\alpha}^{\langle k\rangle}$ of $B_{\bar z}^{\langle k\rangle}$. Denote by $\widetilde V'$ the space of quasi-exponentials with the data $(\bar\lambda',\bar\mu';\bar z,-\bar\alpha)$ corresponding to v as an eigenvector of $B_{\bar z}^{\langle k\rangle}$. By Theorem 2.4.6, for generic $\bar\alpha$ and $\bar z$, we have $\widetilde V'=\widetilde V$. We need to prove that $\widetilde V'=\widetilde V$ for any $\bar\alpha=(\alpha_1,\dots,\alpha_n)$ and $\bar z=(z_1,\dots,z_k)$ such that $\alpha_i\neq\alpha_j$ if $i\neq j$ and $z_a\neq z_b$ if $a\neq b$.

Let $B_{ij,\bar\alpha}^{\langle n\rangle}$, $i=1,\dots,n$, $j\in\mathbb{Z}_{\geqslant 0}$, and $B_{st,\bar z}^{\langle k\rangle}$, $s=1,\dots,k$, $t\in\mathbb{Z}_{\geqslant 0}$, be the generators of the algebras $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$ and $\mathcal{B}_{\bar z}^{\langle k\rangle}$, respectively, see (2.7). Then the eigenvalue of $\pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})$ associated to v is the coefficient b_{ij} in the expansion $D_V^{\mathrm{aug}}=\sum_{i=0}^{n}\sum_{j=0}^{\infty}b_{ij}x^{-j}(d/dx)^{n-i}$, and the eigenvalue of $\pi_{-\bar\alpha}^{\langle k\rangle}(B_{st,\bar z}^{\langle k\rangle})$ associated to v is the coefficient $\widetilde b'_{st}$ in the expansion $D_{\widetilde V'}^{\mathrm{aug}}=\sum_{s=0}^{k}\sum_{t=0}^{\infty}\widetilde b'_{st}x^{-t}(d/dx)^{k-s}$.

Consider the differential operator $\widetilde D_V$ defined by formula (2.3), and let $\widetilde b_{st}$ be the coefficient in the expansion $\widetilde D_V^{\mathrm{aug}}=\sum_{s=0}^{k}\sum_{t=0}^{\infty}\widetilde b_{st}x^{-t}(d/dx)^{k-s}$. By Theorem 2.2.2, part 3, there exist polynomials P_{st} in variables r_{ij}, $i = 0, \ldots, n$, $j \geqslant 0$, such that $\widetilde{b}_{st}$ are obtained by the substitution $r_{ij} = b_{ij}$ for all i, j, into the polynomial P_{st},

$$\widetilde{b}_{st} = P_{st}(\{b_{ij}\}).$$

As noted in the proof of Theorem 2.4.2, the relation

$$\pi^{\langle k\rangle}_{-\bar{\alpha}}(B^{\langle k\rangle}_{st,\bar{z}}) = P_{st}(\{\pi^{\langle n\rangle}_{\bar{z}}(B^{\langle n\rangle}_{ij,\bar{\alpha}})\}).$$

holds for any $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\bar{z} = (z_1, \ldots, z_k)$ such that $\alpha_i \neq \alpha_j$ if $i \neq j$ and $z_a \neq z_b$ if $a \neq b$. Therefore, we have

$$\widetilde{b}'_{st} = P_{st}(\{b_{ij}\}),$$

which gives $\widetilde{b}'_{st} = \widetilde{b}_{st}$. Hence, we have $\widetilde{V}' = \widetilde{V}$, and the corollary is proved. ■

2.5 Quotient differential operator

2.5.1 Factorization of a differential operator

For any functions $g_1, \ldots, g_n$, let

$$\mathrm{Wr}(g_1, \ldots, g_n) = \det((g_i^{(j-1)})_{i,j=1}^n)$$

be their Wronski determinant. Let $\mathrm{Wr}_i(g_1, \ldots, g_n)$ be the determinant of the $n \times n$ matrix whose j-th row is $g_j, g'_j, \ldots, g_j^{(n-i-1)}, g_j^{(n-i+1)}, \ldots, g_j^{(n)}$.

Consider a monic differential operator D of order n with coefficients $a_i(x)$, $i = 1, \ldots, n$:

$$D = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^n a_i(x) \left(\frac{d}{dx}\right)^{n-i}, \tag{2.27}$$

and let $f_1, f_2, \ldots, f_n$ be linearly independent solutions of the differential equation $Df = 0$.

Lemma 2.5.1 *The coefficients $a_1(x), \ldots, a_n(x)$ of the differential operator D are given by the formulas*

$$a_i(x) = (-1)^i \frac{\mathrm{Wr}_i(f_1, \ldots, f_n)}{\mathrm{Wr}(f_1, \ldots, f_n)}, \qquad i = 1, \ldots, n, \tag{2.28}$$

Moreover, for any function g, we have

$$Dg = \frac{\mathrm{Wr}(f_1, \ldots, f_n, g)}{\mathrm{Wr}(f_1, \ldots, f_n)} \,. \tag{2.29}$$

Proof The equations $Df_1 = 0, \ldots, Df_n = 0$ give a linear system of equations for the coefficients $a_1(x), \ldots, a_n(x)$. Solving this system by Cramer's rule yields formula (2.28). Formula (2.29) follows from the last row expansion of the determinant in the numerator. ∎

Proposition 2.5.2 *The differential operator D can be written in the following form:*

$$D = \left(\frac{d}{dx} - \frac{g_1'}{g_1}\right)\left(\frac{d}{dx} - \frac{g_2'}{g_2}\right) \cdots \left(\frac{d}{dx} - \frac{g_n'}{g_n}\right), \tag{2.30}$$

where $g_n = f_n$, and

$$g_i = \frac{\mathrm{Wr}(f_n, f_{n-1}, \ldots, f_i)}{\mathrm{Wr}(f_n, f_{n-1}, \ldots, f_{i+1})} \,, \qquad i = 1, \ldots, n-1 \,. \tag{2.31}$$

Proof Denote by D_1 the differential operator in the right hand side of (2.5.2). By Lemma 2.5.1, it is sufficient to prove that $D_1 f_i = 0$ for all $i = 1, \ldots, n$. We will prove it by induction on n.

If $n = 1$, then $g_1 = f_1$ and $D_1 f_1 = (d/dx - f_1'/f_1)\, f_1 = 0$.

Let D_2 be the monic differential operator of order $n-1$ whose kernel is spanned by $f_2, \ldots, f_n$. By induction assumption,

$$D_2 = \left(\frac{d}{dx} - \frac{g_2'}{g_2}\right)\left(\frac{d}{dx} - \frac{g_3'}{g_3}\right) \cdots \left(\frac{d}{dx} - \frac{g_n'}{g_n}\right).$$

Since $D_1 = (d/dx - g_1'/g_1)\, D_2$, we have $D_1 f_i = 0$ for $i = 2, \ldots, n$. Formula (2.29) yields $D_2 f_1 = g_1$, thus $D_1 f_1 = 0$ as well. ∎

2.5.2 Formal conjugate differential operator

Given a differential operator $D = \sum_{i=0}^{n} a_i(x) \left(\frac{d}{dx}\right)^{n-i}$, define its formal conjugate by the formula:

$$D^\dagger h(x) = \sum_{i=0}^{n} \left(-\frac{d}{dx}\right)^{n-i} \big(a_i(x) h(x)\big) \,. \tag{2.32}$$

Clearly, the formal conjugation is an antihomomorphism of the algebra of linear differential operators. In particular, if D is given by formula (2.5.2), then

$$D^{\dagger} = (-1)^n \left(\frac{d}{dx} + \frac{g_n'}{g_n}\right)\left(\frac{d}{dx} + \frac{g_{n-1}'}{g_{n-1}}\right) \dots \left(\frac{d}{dx} + \frac{g_1'}{g_1}\right). \tag{2.33}$$

Proposition 2.5.3 *Let*

$$h_i = \frac{\mathrm{Wr}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)}{\mathrm{Wr}(f_1, \dots, f_n)},$$

Then the functions $h_1, \dots, h_n$ are linearly independent, and $D^{\dagger}h_i = 0$ for all $i = 1$, $\dots, n$.

Proof Since $h_1 = (-1)^{n-1}/g_1$, we have $D^{\dagger}h_1 = 0$ by formula (2.33).

Let σ be a permutation of $\{1, \dots, n\}$. Take a new sequence $f_{\sigma(1)}, \dots, f_{\sigma(n)}$ of n linearly independent solutions of the equation $Df = 0$. Then similarly to the consideration above, we get

$$D^{\dagger} = (-1)^n \left(\frac{d}{dx} + \frac{g_{n,\sigma}'}{g_{n,\sigma}}\right)\left(\frac{d}{dx} + \frac{g_{n-1,\sigma}'}{g_{n-1,\sigma}}\right) \dots \left(\frac{d}{dx} + \frac{g_{1,\sigma}'}{g_{1,\sigma}}\right),$$

cf (2.33), where $g_{n,\sigma} = f_{\sigma(n)}$ and

$$g_{i,\sigma} = \frac{\mathrm{Wr}(f_{\sigma(n)}, f_{\sigma(n-1)}, \dots, f_{\sigma(i)})}{\mathrm{Wr}(f_{\sigma(n)}, f_{\sigma(n-1)}, \dots, f_{\sigma(i+1)})}, \qquad i = 1, \dots, n-1.$$

Taking σ such that $\sigma(1) = i$, we get $D^{\dagger}h_i = 0$.

The linear independence of the functions $h_1, \dots, h_n$ follows from the relation

$$\mathrm{Wr}(h_1, \dots, h_n) = \frac{(-1)^{n(n-1)/2}}{\mathrm{Wr}(f_1, \dots, f_n)}. \tag{2.34}$$

The proof of relation (2.34) is given in Appendix A. ■

2.5.3 Quotient differential operator

Let D and $\widehat{D}$ be monic differential operators such that $\ker D \subset \ker \widehat{D}$. Then there is a differential operator $\check{D}$, such that $\widehat{D} = \check{D}D$. For instance, it can be seen from the factorization formula (2.30). We will call $\check{D}$ the *quotient differential operator*.

Let $f_1, f_2, \ldots, f_n$ be a basis of $\ker D$ and $f_1, f_2, \ldots, f_n, h_1, \ldots, h_k$ be a basis of $\ker \hat{D}$. Define functions $\varphi_1, \ldots, \varphi_k$ by the formula

$$\varphi_a = \frac{\mathrm{Wr}(f_1, \ldots, f_n, h_1, \ldots, h_{a-1}, h_{a+1}, \ldots, h_k)}{\mathrm{Wr}(f_1, \ldots, f_n, h_1, \ldots, h_k)} \, .$$

Proposition 2.5.4 *The functions $\varphi_1, \ldots, \varphi_k$ are linearly independent, and $\check{D}^\dagger \varphi_a = 0$ for all $a = 1, \ldots, k$.*

Proof Set $\tilde{h}_a = Dh_a$, $a = 1, \ldots, k$. The functions $\tilde{h}_1, \ldots, \tilde{h}_k$ are linearly independent. Indeed, if there are numbers $c_1, \ldots, c_k$, not all equal to zero, such that $c_1\tilde{h}_1 + \ldots + c_k\tilde{h}_k = 0$, then $D(c_1h_1 + \ldots + c_kh_k) = 0$. This means that $c_1h_1 + \ldots + c_kh_k$ belongs to the span of $f_1, \ldots, f_n$ contrary to the linear independence of the functions $f_1, \ldots, f_n, h_1, \ldots, h_k$.

Formula (2.29) yields $\tilde{h}_i = \mathrm{Wr}(f_1, \ldots, f_n, h_i)/\,\mathrm{Wr}(f_1, \ldots, f_n)$. Using identities (A.1) and (A.4), one can check that

$$\frac{\mathrm{Wr}(\tilde{h}_1, \ldots, \tilde{h}_{a-1}, \tilde{h}_{a+1}, \ldots, \tilde{h}_k)}{\mathrm{Wr}(\tilde{h}_1, \ldots, \tilde{h}_k)} = \frac{\mathrm{Wr}(f_1, \ldots, f_n, h_1, \ldots, h_{a-1}, h_{a+1}, \ldots, h_k)}{\mathrm{Wr}(f_1, \ldots, f_n, h_1, \ldots, h_k)} = \varphi_a \, .$$

Since $\check{D}\tilde{h}_a = \hat{D}h_a = 0$ for all $a = 1, \ldots, k$, the functions $\tilde{h}_1, \ldots, \tilde{h}_k$ form a basis of $\ker \check{D}$. Since

$$\varphi_a = \frac{\mathrm{Wr}(\tilde{h}_1, \ldots, \tilde{h}_{a-1}, \tilde{h}_{a+1}, \ldots, \tilde{h}_k)}{\mathrm{Wr}(\tilde{h}_1, \ldots, \tilde{h}_k)} \, ,$$

Proposition 2.5.4 follows from Proposition 2.5.3 applied to $\check{D}$. ■

2.5.4 Quotient differential operator and spaces of quasi-exponentials

Let V be a space of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. For Section 2.5.4, we will assume that the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ are reduced, that is, the sequences $\bar{\mu}$ and $\bar{\lambda}$ do not contain zero partitions. For each $i = 1, \ldots, n$, denote $n_i = (\mu^{(i)})'_1$ and $p_i = \mu^{(i)}_1 + n_i$.

Introduce also a larger space $\widehat{V}$ spanned by the functions $x^p e^{\alpha_i x}$ for all $i = 1, \dots,$ n, and $p = 0, \dots, p_i - 1$. Denote

$$\mathrm{Wr}(\widehat{V}) = \mathrm{Wr}(e^{\alpha_1 x}, xe^{\alpha_1 x}, \dots, x^{p_1-1}e^{\alpha_1 x}, \dots, e^{\alpha_n x}, xe^{\alpha_n x}, \dots, x^{p_n-1}e^{\alpha_n x}),$$

$$\mathrm{Wr}_{ij}(\widehat{V}) = \mathrm{Wr}(\dots, \widehat{x^j e^{\alpha_i x}}, \dots).$$

The functions in the second line are the same except the function $x^j e^{\alpha_i x}$ is omitted.

Lemma 2.5.5 *The following holds:*

$$\mathrm{Wr}(\widehat{V}) = e^{\sum_{i=1}^n p_i \alpha_i x} \prod_{i=1}^n \prod_{s=1}^{p_i-1} s! \prod_{1 \leqslant i < j \leqslant n} (\alpha_j - \alpha_i)^{p_i p_j}, \tag{2.35}$$

$$\mathrm{Wr}_{ij}(\widehat{V}) = e^{\sum_{l=1}^n (p_l - \delta_{il})\alpha_l x} r_{ij}(x) \prod_{l=1}^n \prod_{\substack{s=1 \\ (l,s)\neq(i,j)}}^{p_l-1} s! \prod_{1 \leqslant l < l' \leqslant n} (\alpha_{l'} - \alpha_l)^{(p_l - \delta_{li})(p_{l'} - \delta_{l'i})}, \tag{2.36}$$

where $r_{ij}(x)$ is a monic polynomial in x and $\deg r_{ij} = p_i - j - 1$.

Proof We will prove (2.35) by induction on $\sum_{i=1}^n (p_i - 1) = P$. For $P = 0$, equality (2.35) becomes

$$\mathrm{Wr}(e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_n x}) = e^{\sum_{i=1}^n \alpha_i x} \prod_{1 \leqslant i < j \leqslant n} (\alpha_j - \alpha_i),$$

which is equivalent to the Vandermonde determinant formula.

Fix $P_0 \in \mathbb{Z}_{\geqslant 0}$. Suppose that (2.35) is true for all n and all $p_1, \dots, p_n$ such that $\sum_{i=1}^n (p_i - 1) = P_0$. We will indicate the dependence of the space $\widehat{V}$ on $p_1, \dots, p_n$ as follows: $\widehat{V}^{p_1,\dots,p_n}$.

Fix $p_1, \dots, p_n$ such that $\sum_{i=1}^n (p_i - 1) = P_0$. For each $l = 1, \dots, n$, let $\mathrm{Wr}_{(\beta,l)}$ be the Wronski determinant obtained from $\mathrm{Wr}(\widehat{V}^{p_1,\dots,p_n})$ by inserting the exponential $e^{\beta x}$ after the function $x^{p_l-1}e^{\alpha_l x}$. Notice that $(\partial/\partial\beta)^{p_l}|_{\beta=\alpha_l} \mathrm{Wr}_{(\beta,l)} = \mathrm{Wr}(\widehat{V}^{p'_1,\dots,p'_n})$, where $p'_i = p_i$ if $i \neq l$ and $p'_l = p_l + 1$.

By the induction assumption, we have

$$\mathrm{Wr}_{(\beta,l)} = e^{\sum_{i=1}^n (p_i\alpha_i + \beta)x} \prod_{i=1}^n \prod_{s=1}^{p_i-1} s! \prod_{1 \leqslant i < j \leqslant n} (\alpha_j - \alpha_i)^{p_i p_j} \prod_{i=1}^{l} (\beta - \alpha_i)^{p_i} \prod_{i=l+1}^{n} (\alpha_i - \beta)^{p_i},$$

which gives

$$\left(\frac{\partial}{\partial\beta}\right)^{p_l}\Big|_{\beta=\alpha_l}\mathrm{Wr}_{(\beta,l)} \;=\; e^{\sum_{i=1}^n p_i'\alpha_i x}\prod_{i=1}^{n}\prod_{s=1}^{p_i'-1} s!\prod_{1\leqslant i<j\leqslant n}(\alpha_j-\alpha_i)^{p_i'p_j'}\,.$$

This proves the induction step for formula (2.35).

To prove formula (2.36), we fix i and use induction on $s=p_i-j-1$. The base of induction at $s=0$ is given by formula (2.35).

Fix $s_0\in\mathbb{Z}_{\geqslant 0}$. Suppose that (2.36) is true for all n, all $p_1,\dots,p_n$, and j such that $s=s_0$. Fix $p_1,\dots,p_n$, and j such that $p_i-j-1=s_0$. Let $\mathrm{Wr}_{(\beta,i,j)}$ be the Wronski determinant obtained from $\mathrm{Wr}_{ij}(\widehat{V}^{p_1,\dots,p_n})$ by inserting the exponential $e^{\beta x}$ after the function $x^{p_i-1}e^{\alpha_i x}$ if $j\leqslant p_i-1$ or after the function $x^{p_i-2}e^{\alpha_i x}$ if $j=p_i-1$. Notice that

$$(\partial/\partial\beta)^{p_i}|_{\beta=\alpha_i}\,\mathrm{Wr}_{(\beta,i,j)}=\mathrm{Wr}_{ij}(\widehat{V}^{p_1',\dots,p_n'})\,,$$

where $p_l'=p_l$ for $l\neq i$, $p_i'=p_i+1$, and $s'=p_i'-1-j=s_0+1$.

By the induction assumption, we have

$$\begin{aligned}\mathrm{Wr}_{(\beta,i,j)} \;=\; e^{\sum_{l=1}^n(p_l-\delta_{il})\alpha_l x+\beta x}\,r_{ij}(x)\prod_{l=1}^{n}\prod_{\substack{s=1\\(l,s)\neq(i,j)}}^{p_l-1} s!\prod_{1\leqslant l<l'\leqslant n}(\alpha_{l'}-\alpha_l)^{(p_l-\delta_{li})(p_{l'}-\delta_{l'i})}\\ \times\prod_{l=1}^{i}(\beta-\alpha_l)^{p_l-\delta_{il}}\prod_{l=i+1}^{n}(\alpha_l-\beta)^{p_l-\delta_{il}},\end{aligned}$$

where $r_{ij}(x)$ is a monic poynomial and $\deg r_{ij}(x)=p_i-j-1$. The last formula gives

$$\left(\frac{\partial}{\partial\beta}\right)^{p_i}\Big|_{\beta=\alpha_i}\mathrm{Wr}_{(\beta,i,j)} \;=\; e^{\sum_{l=1}^n(p_l'-\delta_{il})\alpha_l x}\,A(x)\prod_{l=1}^{n}\prod_{\substack{s=1\\(l,s)\neq(i,j)}}^{p_l'-1} s!\prod_{l<l'}(\alpha_{l'}-\alpha_l)^{(p_l'-\delta_{li})(p_{l'}'-\delta_{l'i})},$$

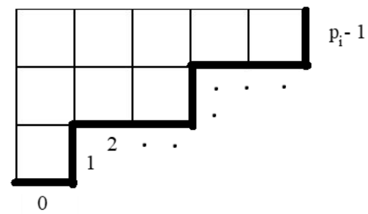
where $A(x)$ is a monic polynomial and $\deg A(x)=\deg r_{ij}(x)+1$. This completes the induction step for formula (2.36). ■

For each $i=1,\dots,n$, set

$$\boldsymbol{d}_i=\{n_i+\mu_j^{(i)}-j\mid j=1,\dots,n_i\}\,,\qquad \boldsymbol{d}_i^c=\{0,1,2,\dots,p_i-1\}\setminus\boldsymbol{d}_i\,. \tag{2.37}$$

Lemma 2.5.6 *We have* $\boldsymbol{d}_i^c=\{n_i-(\mu^{(i)})_j'+j-1\mid j=1,\dots,\mu_1^{(i)}\}\,.$

Proof Consider the Young diagram corresponding to the partition $\mu^{(i)}$. Enumerate, starting from 0, the sides of boxes in this diagram that form the bottom-right boundary, see the picture.



Then by (2.37), the set $\boldsymbol{d}_i$ corresponds to the right-most sides of the rows, which are the vertical sides of the boundary. Thus the complementary set $\boldsymbol{d}_i^c$ corresponds to the horizontal sides of the boundary, which are the bottom sides of the columns. The last observation proves the lemma. ■

Let D_V be the fundamental differential operator of V. Define $\widehat{D} = \prod_{i=1}^{n} (d/dx - \alpha_i)^{p_i}$. Then $\ker \widehat{D} = \widehat{V}$. Therefore, $\ker D_V \subset \ker \widehat{D}$, and there exists a differential operator $\check{D}_V$, such that $\widehat{D} = \check{D}_V D_V$, see Section 2.5.3. Let $\check{V}^\dagger = \ker \check{D}^\dagger$.

Theorem 2.5.7 *The space $\check{V}^\dagger$ is a space of quasi-exponentials with the data $(\bar{\mu}', \bar{\lambda}'; -\bar{\alpha}, \bar{z})$.*

Proof The space V has a basis of the form $\{q_{ij}(x)e^{\alpha_i x} \mid i = 1, \dots, n,\ j = 1, \dots, n_i\}$, where $q_{ij}(x)$ are polynomials and $\deg q_{ij} = n_i + \mu_j^{(i)} - j$. Then the functions $x^l e^{\alpha_i x}$, $i = 1, \dots, n,\ l \in \boldsymbol{d}_i^c$, complement this basis of V to a basis of $\hat{V}$.

By Proposition 2.5.4, the space $\check{V}^\dagger$ has the following basis

$$\frac{\mathrm{Wr}_{ij}(\widehat{V})}{\mathrm{Wr}(\widehat{V})} + \sum_{l=j+1}^{p_i-1} C_{ijl} \frac{\mathrm{Wr}_{il}(\widehat{V})}{\mathrm{Wr}(\widehat{V})}\,, \qquad i = 1, \dots, n\,, \quad j \in \boldsymbol{d}_i^c\,, \tag{2.38}$$

where C_{ijl} are complex numbers. Then by Lemma 2.5.5, for each i, j, the corresponding element of this basis has the form $\tilde{r}_{ij}(x)e^{-\alpha_i x}$, where $\tilde{r}_{ij}(x)$ is a polynomial of degree $p_i - j - 1$.

By Lemma 2.5.6, $j \in \boldsymbol{d}_i^c$ if and only if $j = n_i - (\mu^{(i)})'_l + l - 1$ for some $l \in \{1, \dots, \mu_1^{(i)}\}$. Set $\check{q}_{il}(x) = \tilde{r}_{ij}(x)$. Then $\check{V}^\dagger$ has a basis of the form $\{\check{q}_{il}(x) e^{-\alpha_i x} \mid i = 1, \dots, n,\ l = 1, \dots, \mu_1^{(i)}\}$ and

$$\deg \check{q}_{il} \;=\; \deg \tilde{r}_{ij} \;=\; \mu_1^{(i)} + n_i - (n_i - (\mu^{(i)})'_l + l - 1) - 1 = \mu_1^{(i)} + (\mu^{(i)})'_l - l\,. \quad (2.39)$$

Recall $M' = \dim V = \sum_{i=1}^n (\mu^{(i)})'_1$. Set $M = \dim \check{V}^\dagger = \sum_{i=1}^n \mu_1^{(i)}$. We also have $\dim \hat{V} = M' + M$.

Fix a point $z \in \mathbb{C}$, and let $\boldsymbol{e} = \{e_1 > \ldots > e_{M'}\}$ be the set of exponents of V at z. Then there is a basis $\{\psi_1, \dots, \psi_{M'}\}$ of V such that

$$\psi_i \;=\; (x - z)^{e_i}\bigl(1 + o(1)\bigr)\,, \qquad x \to z\,, \quad (2.40)$$

for any $i = 1, \dots, M'$.

Set $\hat{\boldsymbol{e}} \,=\, \{\,\hat{e}_1 < \hat{e}_2 < \ldots < \hat{e}_M\} = \{0, 1, 2, \dots, M' + M - 1\} \setminus \boldsymbol{e}$. By formula (2.35), the Wronskian $\mathrm{Wr}(\widehat{V})$ has no zeros, thus z is not a singular point of $\widehat{V}$. Therefore, there is a basis $\{\psi_1, \dots, \psi_{M'}, \chi_1, \dots, \chi_M\}$ of $\widehat{V}$ such that

$$\chi_i(x) \;=\; (x - z)^{\hat{e}_i}\bigl(1 + o(1)\bigr)\,, \qquad x \to z\,, \quad (2.41)$$

for any $i = 1, \dots, M$.

By Proposition 2.5.4, the set

$$\left\{ \frac{\mathrm{Wr}(\psi_1, \dots, \psi_{M'}, \chi_1, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_M)}{\mathrm{Wr}(\psi_1, \dots, \psi_{M'}, \chi_1, \dots, \chi_M)} \;\middle|\; i = 1, \dots, M \right\} \quad (2.42)$$

is a basis of $\check{V}^\dagger$. Formulas (2.35), (2.40), (2.41) show that for any $i = 1, \dots, M$,

$$\frac{\mathrm{Wr}(\psi_1, \dots, \psi_{M'}, \chi_1, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_M)}{\mathrm{Wr}(\psi_1, \dots, \psi_{M'}, \chi_1, \dots, \chi_M)} \;=\; C_i\,(x - z)^{M' + M - \hat{e}_i - 1}\bigl(1 + o(1)\bigr)$$

as $x \to z$, where C_i is a nonzero complex number. Therefore, the set of exponents of $\check{V}^\dagger$ at the point z is $\check{\boldsymbol{e}}^\dagger = \{M' + M - \hat{e}_1 - 1 > \ldots > M' + M - \hat{e}_M - 1\}$. In particular, z is a singular point of $\check{V}^\dagger$ if and only if z is a singular point of V.

If a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ corresponds to the set $\boldsymbol{e}$, that is, $\lambda_i = e_i + i - M'$ for $i = 1, \dots, M'$, and $\lambda_i = 0$ for $i > M'$, then similarly to Lemma 2.5.6, $\hat{e}_i =$

$M' - \lambda'_i + i - 1$, and $\check{e}^\dagger_i = M' + M - \hat{e}_i - 1 = \lambda'_i + M - i$. Thus the set $\check{\boldsymbol{e}}^\dagger$ of exponents of $\check{V}^\dagger$ at z corresponds to a partition λ'.

Recall that the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ are reduced, in particular, $\bar{z}$ is the set of singular points of V. To summarize, the consideration above shows that $\bar{z}$ is the set of singular points of $\check{V}^\dagger$ as well, and $\check{V}^\dagger$ is the space of quasi-exponentials with the data $(\bar{\mu}', \bar{\lambda}'; -\bar{\alpha}, \bar{z})$. Theorem 2.5.7 is proved. ■

2.5.5 Proof of Theorem 2.2.2

It is sufficient to prove Theorem 2.2.2, parts (1) and (2) for the case of reduced data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. This is immediate for part (1), since M', L, D_V and $\widetilde{D}_V$ depend only on $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})^{\mathrm{red}}$. And for part (2), the following observation does the job: if $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})^{\mathrm{red}} = (\bar{\mu}^{\mathrm{red}}, \bar{\lambda}^{\mathrm{red}}; \bar{\alpha}^{\mathrm{red}}, \bar{z}^{\mathrm{red}})$, then $(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha})^{\mathrm{red}} = \big((\bar{\mu}^{\mathrm{red}})', (\bar{\lambda}^{\mathrm{red}})'; \bar{z}^{\mathrm{red}}, -\bar{\alpha}^{\mathrm{red}}\big)$.

Let V be a space of quasi-exponentials with the reduced data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. Let $\{f_1, \dots, f_{M'}\}$ be a basis of V and $D_V = \sum_{i=0}^{M'} b_i(x)(d/dx)^{M'-i}$ be the fundamental differential operator of V. For each $i = 1, \dots, M'$, the ratio $\mathrm{Wr}_i(f_1, \dots, f_{M'})/\,\mathrm{Wr}(f_1, \dots, f_{M'})$ is a rational function of x regular at infinity. Together with Lemma 2.5.1, this proves Lemma 2.2.1, and we can consider D_V as an invertible element of $\Psi\mathfrak{D}$.

For any $i = 0, \dots, M'$, let $\sum_{j=0}^{\infty} b_{ij} x^{-j}$ be the Laurent series of $b_i(x)$ at infinity. We will refer to the functions $b_i(x)$ as coefficients of the differential operator D_V, and to b_{ij} as *expansion* coefficients of the differential operator D_V. This terminology also applies to any differential operator with rational coefficients.

Notice that the formal conjugation $(\cdot)^\dagger$ of a differential operator, introduced in Section 2.5.2, is consistent with the formal conjugation on $\Psi\mathfrak{D}$, introduced in Section 2.1. Recall the involutive antiautomorphism $(\cdot)^\ddagger : \Psi\mathfrak{D} \to \Psi\mathfrak{D}$ introduced in Section 2.1.

Let $\widehat{D} = \prod_{i=1}^{n} (d/dx - \alpha_i)^{\mu_1^{(i)} + (\mu^{(i)})'_1}$. Denote by $\check{D}_V$ the quotient differential operator such that $\widehat{D} = \check{D}_V D_V$. Set $D_V^{\times} = \big(\prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} \check{D}_V^{\dagger} \big)^{\ddagger}$. Recall the pseudo-differential operator $\tilde{D}_V$ defined by (2.3). It is straightforward to verify that

$$D_V^{\times} \;=\; (-1)^M \prod_{i=1}^{n} (x + \alpha_i)^{\mu_1^{(i)}} \widetilde{D}_V \,, \tag{2.43}$$

where $M = \mu_1^{(1)} + \ldots + \mu_1^{(n)}$.

The next theorem is proved in [11].

Theorem 2.5.8 *Let D be the fundamental differential operator of a space of quasi-exponentials with the data $(\bar{\mu}', \bar{\lambda}'; -\bar{\alpha}, \bar{z})$. Then the following holds.*

1. *The differential operator $\prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} D$ has polynomial coefficients.*
2. *The differential operator $\prod_{i=1}^{n} (x + \alpha_i)^{-\mu_1^{(i)}} \big(\prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} D \big)^{\ddagger}$ is monic and has order $L = \lambda_1^{(1)} + \ldots + \lambda_1^{(k)}$.*
3. *The kernel of $\big(\prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} D \big)^{\ddagger}$ is a space of quasi-exponentials with the data $(\bar{\lambda}', \bar{\mu}'; \bar{z}, -\bar{\alpha})$.*

By Theorem 2.5.7, one can apply Theorem 2.5.8 to the monic differential operator $(-1)^M \check{D}_V^{\dagger}$. Hence, the differential operator $\prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} \check{D}_V^{\dagger}$ has polynomial coefficients and the pseudo-differential operator $D_V^{\times}$ is actually a differential operator. Furthermore, formula (2.43) and parts (2), (3) of Theorem 2.5.8 yield parts (1) and (2) of Theorem 2.2.2.

To prove part (3) of Theorem 2.2.2, consider a chain of transformations:

$$\begin{aligned} D_V^{\mathrm{aug}} \xrightarrow{(1)} D_V \xrightarrow{(2)} \check{D}_V \xrightarrow{(3)} \check{D}_V^{\dagger} \xrightarrow{(4)} \prod_{a=1}^{k} (x - z_a)^{\lambda_1^{(a)}} \check{D}_V^{\dagger} \xrightarrow{(5)} & \\ \xrightarrow{(5)} D_V^{\times} \xrightarrow{(6)} \widetilde{D}_V \xrightarrow{(7)} \widetilde{D}_V^{\mathrm{aug}} \,. & \end{aligned} \tag{2.44}$$

Lemma 2.5.9 *For each of the transformations in chain* (2.44)*, the expansion coefficients of the transformed operator can be expressed as polynomials in the expansion coefficients of the initial operator.*

Proof Fix $\beta \in \mathbb{C}$. Let $b_0(x), \dots, b_{M'}(x)$, $b_0^\beta(x), \dots, b_{M'+1}^\beta(x)$ be the coefficients of the differential operators D_V and $D_V\,(d/dx - \beta)$:

$$D_V = \sum_{i=0}^{M'} b_i(x) \left(\frac{d}{dx}\right)^{M'-i}, \quad D_V \left(\frac{d}{dx} - \beta\right) = \sum_{i=0}^{M'+1} b_i^\beta(x) \left(\frac{d}{dx}\right)^{M'+1-i}.$$

Then Lemma 2.5.9 for transformation (1) follows from the relations:

$$b_i(x) = \sum_{j=0}^{i} \beta^{i-j} b_j^\beta(x), \quad i = 1, \dots, M'. \tag{2.45}$$

Let $c_0(x), \dots, c_M(x)$, and $a_0, \dots, a_{M'+M}$, be the coefficients of the differential operators $\check{D}_V$ and $\widehat{D}$:

$$\check{D}_V = \sum_{j=0}^{M} c_j(x) \left(\frac{d}{dx}\right)^{M-j}, \quad \widehat{D} = \sum_{l=0}^{M'+M} a_l \left(\frac{d}{dx}\right)^{M'+M-l}.$$

The coefficients $a_0, \dots, a_{M'+M}$ are the elementary symmetric polynomials in $\alpha_1, \dots, \alpha_n$.

Fix $j = 0, \dots, M$. Equalizing the coefficients for $(d/dx)^{M'+M-j}$ in both sides of the relation $\widehat{D} = \check{D}_V D_V$, we get

$$c_j(x) = a_j - \sum_{i=0}^{j-1} \sum_{l=0}^{i} c_{i-l}(x) \left(\frac{d^l}{dx^l}\, b_{j-i}(x)\right). \tag{2.46}$$

Since the function c_r appears in the right-hand side of formula (2.46) only for $r < j$, we can recursively express $c_j(x)$ as polynomials in $b_i(x)$ and their derivatives. This proves the statement for transformation (2).

Let $\check{c}_j(x)$, $j = 0, \dots, M$, be the coefficients of the differential operator $\check{D}_V^\dagger$:

$$\check{D}_V^\dagger = \sum_{j=1}^{M} \check{c}_j(x) \left(\frac{d}{dx}\right)^{M-j}.$$

Then we have $\check{c}_j(x) = \sum_{l=0}^{j} (-1)^{M-l} \left((d^l/dx^l) c_{j-l}(x)\right)$. This proves the statement for transformation (3).

For transformations (4), (6), and (7), the statement is obvious. For transformation (5), the statement follows from the definition of the antiautomorphism $(\cdot)^\ddagger$ that transforms the coefficients of a pseudo-differential operator $\sum_{i=-\infty}^{I} \sum_{j=-\infty}^{J} C_{ij} x^i (d/dx)^j$ by the rule $C_{ij} \mapsto C_{ji}$. ■

Lemma 2.5.9 provides an algorithm for expressing the coefficients $\tilde{b}_{st}$ of the differential operator $\widetilde{D}_V^{\text{aug}}$ in item (3) of Theorem 2.2.2 via the coefficients b_{ij} of the operator D_V^{aug}. It is clear that this algorithm depends only on the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ and generates polynomial expressions in b_{ij}. This proves the existence of the polynomials P_{st} in item (3) of Theorem 2.2.2.

It is easy to see that for each transformation in chain (2.44), expressions for expansion coefficients of the transformed operator in terms of expansion coefficients of the initial operator are polynomials in $\bar{\alpha}, \bar{z}$. For transformations (1) and (2), it follows from relations (2.45) and (2.46), respectively. Transformations (3) and (5) do not involve $\bar{\alpha}$ and $\bar{z}$ at all. For transformations (4) and (6), notice that multiplication of a differential operator by the factor $\prod_{a=1}^{k}(x-z_a)^{\lambda_1^{(a)}}$ or $\prod_{i=1}^{n}(x+\alpha_i)^{-\mu_1^{(i)}}$ results in multiplication of its expansion coefficients by polynomials in $z_1, \dots, z_k$ or $\alpha_1, \dots, \alpha_n$, respectively. Finally, for transformation (7), notice that for any $\beta \in \mathbb{C}$, multiplication of a differential operator by $(d/dx - \beta)$ from the right results in multiplication of its expansion coefficients by polynomials in β.

Theorem 2.2.2 is proved.

2.5.6 Proof of Lemma 2.2.3

We will first prove the lemma for the case of reduced data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. In this case, the rational functions $\tilde{c}_i(u)$, $i \in \mathbb{Z}_{\geqslant 0}$ are defined by the following formula:

$$\prod_{a=1}^{k}(u-z_a)^{\lambda_1^{(a)}} \sum_{i=0}^{\infty} \tilde{c}_i(u) x^{-i} = u^L + \sum_{a=1}^{L} \tilde{b}_a(x) u^{L-a}, \tag{2.47}$$

where $\tilde{b}_a(x)$ are the coefficients of the differential operator $\widetilde{D}_V = \widetilde{D}_V^{\text{aug}}$.

Let $\check{c}_1(x), \dots, \check{c}_M(x)$, be the coefficients of the differential operator $\check{D}_V^{\dagger}$:

$$\check{D}_V^{\dagger} = \sum_{i=1}^{M} \check{c}_i(x) \left(\frac{d}{dx}\right)^{M-i}.$$

Recall that $D_V^\times = \left(\prod_{a=1}^k (x-z_a)^{\lambda_1^{(a)}} \check{D}_V^\dagger\right)^\ddagger$. Then we have

$$D_V^\times = (\check{D}_V^\dagger)^\ddagger \prod_{a=1}^k \left(\frac{d}{dx} - z_a\right)^{\lambda_1^{(a)}} = \left[\sum_{i=1}^M x^{M-i} \check{c}_i \left(\frac{d}{dx}\right)\right] \prod_{a=1}^k \left(\frac{d}{dx} - z_a\right)^{\lambda_1^{(a)}}. \quad (2.48)$$

Since $\widetilde{D}_V = \prod_{i=1}^n (x+\alpha_i)^{-\mu_1^{(i)}} D_V^\times$, formulae (2.47) and (2.48) give

$$\sum_{i=0}^\infty \tilde{c}_i(u) x^{-i} = \prod_{i=1}^n (x+\alpha_i)^{-\mu_1^{(i)}} \left[\sum_{i=1}^M x^{M-i} \check{c}_i(u)\right],$$

which yields

$$\tilde{c}_1(u) = \check{c}_1(u) + A_1, \quad \tilde{c}_2 = \check{c}_2(u) + A_1 \check{c}_1(u) + A_2$$

for some constants A_1 and A_2.

Using the last two formulas, it is easy to check that

$$\mathrm{Res}_{u=z_a} \left(\frac{\tilde{c}_1^2(u)}{2} - \tilde{c}_2(u)\right) = \mathrm{Res}_{u=z_a} \left(\frac{\check{c}_1^2(u)}{2} - \check{c}_2(u)\right). \quad (2.49)$$

Let $a_0, \ldots, a_{M+M'}$ and $b_0(x), \ldots, b_{M'}(x)$ be the coefficients of the differential operators $\widehat{D}$ and D_V:

$$\widehat{D} = \sum_{i=0}^{M+M'} a_i \left(\frac{d}{dx}\right)^{M+M'-i}, \quad D_V = \sum_{i=0}^{M'} b_i(x) \left(\frac{d}{dx}\right)^{M'-i}.$$

Notice that $a_0, \ldots, a_{M+M'}$ do not depend on x. The relation $\widehat{D} = \check{D}_V D_V$ gives

$$\sum_{i=0}^{M+M'} a_i \left(\frac{d}{dx}\right)^{M+M'-i} = \left[\sum_{i=0}^M \left(-\frac{d}{dx}\right)^{M-i} \check{c}_i(x)\right] \left[\sum_{j=0}^{M'} b_j(x) \left(\frac{d}{dx}\right)^{M'-j}\right].$$

Writing the right hand side of the last equation in the form $\sum_{i=0}^{M+M'} \tilde{a}_i(x)(d/dx)^{M+M'-i}$ with some functions $\tilde{a}_0(x), \ldots, \tilde{a}_{M+M'}(x)$, we have $\tilde{a}_i(x) = a_i$, $i = 0, \ldots, M+M'$. In particular, $\tilde{a}_1(x) = a_1$ and $\tilde{a}_2(x) = a_2$ give

$$\check{c}_1 = b_1(x) - a_1, \quad \check{c}_2(x) = a_2 - b_2(x) - b_1(x)(a_1 - b_1(x)),$$

respectively.

Using the last two formulas, it is easy to check that

$$\mathrm{Res}_{x=z_a} \left(\frac{\check{c}_1^2(x)}{2} - \check{c}_2(x)\right) = -\,\mathrm{Res}_{x=z_a} \left(\frac{b_1^2(x)}{2} - b_2(x)\right). \quad (2.50)$$

By definition,

$$h_a = \mathrm{Res}_{x=z_a}\Big(\frac{b_1^2(x)}{2} - b_2(x)\Big), \qquad \text{and} \qquad \tilde{g}_a = \mathrm{Res}_{u=z_a}\Big(\frac{\tilde{c}_1^2(u)}{2} - \tilde{c}_2(u)\Big).$$

Therefore, formulas (2.49) and (2.50) give $\tilde{g}_a(x) = -h_a(x)$ proving the lemma for the case of reduced data.

Fix a complex number β. Let $b_0^\beta(x), \dots, b_{M'+1}^\beta(x)$ be the coefficients of the differential operator $D_V(d/dx - \beta)$:

$$D_V\left(\frac{d}{dx} - \beta\right) = \sum_{i=0}^{M'+1} b_i^\beta(x)\left(\frac{d}{dx}\right)^{M'+1-i}.$$

It is easy to check that

$$b_1^\beta(x) = b_1(x) - \beta, \quad b_2^\beta(x) = b_2 - \beta b_1(x).$$

Therefore,

$$\mathrm{Res}_{x=z_a}\left(\frac{(b_1^\beta(x))^2}{2} - b_2^\beta(x)\right) = \mathrm{Res}_{x=z_a}\left(\frac{b_1^2(x)}{2} - b_2(x)\right). \tag{2.51}$$

Formula (2.51) means that the number h_a defined for the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ coincides with the one defined for the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})^{\mathrm{red}}$. Similarly, the number $\tilde{g}_a$ is the same for the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$ and its reduction. Therefore, the lemma holds for a non-reduced data as well.

2.5.7 Bispectral duality and quotient differential operator

Let V be a span of functions of the form $e^{\alpha x}p(x)$, where $p(x)$ is a non-constant polynomial. Let D_V be the monic differential operator of order $\dim V$ annihilating V. Using Lemma 2.5.1, one can show that the coefficients of V are rational functions regular at infinity. Let $q(x)$ be the least common denominator of the coefficients of D_V. Then the pseudo-differential operator $(q(x)D_V)^\ddagger$ is actually a differential operator. Define a map $\mathfrak{T}_1$ as follows:

$$\mathfrak{T}_1 : V \mapsto \ker(q(x)D_V)^\ddagger.$$

The map $\mathfrak{T}_1$ was introduced in [11] in relation to the duality of the Bethe algebras acting on the space of polynomials in commuting variables. The space $\mathfrak{T}_1(V)$ is called the bispectral dual of V. If V is a space of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$, then one can check that $q(x) = \prod_{a=1}^{k}(x - z_a)^{(\lambda^{(a)})'_1}$, and by part (3) of Theorem 2.5.8, the bispectral dual of V is a space of quasi-exponentials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$.

Define the quotient differential operator $\check{D}_V$ in the same way it was defined in Section 2.5.4. Define a map $\mathfrak{T}_2$ as follows:

$$\mathfrak{T}_2 : V \mapsto \ker \check{D}_V^{\dagger}.$$

Let V be a space of quasi-exponentials with the data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. By Theorem 2.5.7, the space $\mathfrak{T}_2(V)$ is a space of quasi-exponentials with the data $(\bar{\mu}', \bar{\lambda}'; -\bar{\alpha}, \bar{z})$. Therefore, the least common denominator of the coefficients of $\check{D}_V^{\dagger}$ equals $\prod_{a=1}^{k}(x - z_a)^{\lambda_1^{(a)}}$, and we have $\mathfrak{T}_1 \circ \mathfrak{T}_2(V) = \ker\big(\prod_{a=1}^{k}(x - z_a)^{\lambda_1^{(a)}} \check{D}_V^{\dagger}\big)^{\ddagger}$.

Recall the differential operator $\widetilde{D}_V$ defined in Section 2.2:

$$\widetilde{D}_V = (-1)^{M'} \prod_{i=1}^{n}(x + \alpha_i)^{(\mu^{(i)})'_1} \left(D_V^{-1}\right)^{\#} \prod_{a=1}^{k}\left(\frac{d}{dx} - z_a\right)^{\lambda_1^{(a)}}. \tag{2.52}$$

Then by formula (2.43), we have $\mathfrak{T}_1 \circ \mathfrak{T}_2(V) = \ker \widetilde{D}_V$.

Proposition 2.5.10 *A function $f(x)$ belongs to the space $\mathfrak{T}_1 \circ \mathfrak{T}_2(V)$ if and only if the function $f(-x)$ belongs to the space $\mathfrak{T}_2 \circ \mathfrak{T}_1(V)$.*

Proof As was mentioned above, the least common denominator of the differential operator D_V equals $\prod_{a=1}^{k}(x - z_a)^{(\lambda^{(a)})'_1}$. Therefore, the bispectral dual $\mathfrak{T}_1(V)$ of the space V is the kernel of the differential operator $\Big(\prod_{a=1}^{k}(x - z_a)^{(\lambda^{(a)})'_1} D_V\Big)^{\ddagger} = D_V^{\ddagger} \prod_{a=1}^{k}\left(\frac{d}{dx} - z_a\right)^{(\lambda^{(a)})'_1}$.

By Theorem 2.5.8, part (2), the differential operator

$$D_V^{\mathrm{bsp}} = \prod_{i=1}^{n}(x - \alpha_i)^{-(\mu^{(i)})'_1} D_V^{\ddagger} \prod_{a=1}^{k}\left(\frac{d}{dx} - z_a\right)^{(\lambda^{(a)})'_1}$$

is monic. Therefore, it is the fundamental differential operator of the space $\mathfrak{T}_1(V)$. Since $\mathfrak{T}_1(V)$ is a space of quasi-exponentials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$, similarly to $\check{D}_V$, there exists a differential operator $\check{D}_V^{\mathrm{bsp}}$ such that

$$\prod_{a=1}^{k} \left(\frac{d}{dx} - z_a \right)^{\lambda_1^{(a)} + (\lambda^{(a)})'_1} = \check{D}_V^{\mathrm{bsp}} D_V^{\mathrm{bsp}}.$$

We have $\mathfrak{T}_2 \circ \mathfrak{T}_1(V) = \ker(\check{D}_V^{\mathrm{bsp}})^\dagger$. Also, it is easy to check that

$$(\check{D}_V^{\mathrm{bsp}})^\dagger = \prod_{i=1}^{n} (x - \alpha_i)^{(\mu^{(i)})'_1} \left((D_V^\ddagger)^{-1} \right)^\dagger \left(\frac{d}{dx} - z_a \right)^{\lambda_1^{(a)}}. \tag{2.53}$$

Since $(\cdot)^\ddagger$ is an antiautomorphism of the algebra $\Psi\mathfrak{D}$ of pseudo-differential operators, we have $(D_V^\ddagger)^{-1} = (D_V^{-1})^\ddagger$. Also, for any pseudo-differential operator D, $(D^\ddagger)^\dagger$ can be obtained from $D^\# = (D^\dagger)^\ddagger$ by the substitution $x \mapsto -x$. Using this and comparing formulae (2.52) and (2.53), we see that $(\check{D}_V^{\mathrm{bsp}})^\dagger$ is obtained from $\widetilde{D}_V$ by the substitution $x \mapsto -x$. Since $\mathfrak{T}_1 \circ \mathfrak{T}_2(V) = \ker \widetilde{D}_V$ and $\mathfrak{T}_2 \circ \mathfrak{T}_1(V) = \ker(\check{D}_V^{\mathrm{bsp}})^\dagger$, the proposition is proved. ∎

Proposition 2.5.11 *The following holds: $\mathfrak{T}_2^2(V) = V$*

Proof Recall that the quotient differential operator $\check{D}_V$ satisfies the following relation:

$$\prod_{i=1}^{n} \left(\frac{d}{dx} - \alpha_i \right)^{\mu_1^{(i)} + (\mu^{(i)})'_1} = \check{D}_V D_V.$$

Applying $(\cdot)^\dagger$ to the both sides of the last formula, we get

$$\prod_{i=1}^{n} \left(\frac{d}{dx} + \alpha_i \right)^{\mu_1^{(i)} + (\mu^{(i)})'_1} = (-1)^{M'} D_V^\dagger (-1)^M \check{D}_V^\dagger, \tag{2.54}$$

where $M = \sum_{i=1}^n \mu_1^{(i)}$ and $M' = \sum_{i=1}^n (\mu^{(i)})'_1$. Relation (2.54) yields $\mathfrak{T}_2^2(V) = \ker D_V = V$.

The proposition is proved. ∎

Denote $\mathfrak{T}_\bullet = \mathfrak{T}_1 \circ \mathfrak{T}_2$.

Corollary 2.5.12 *A function $f(x)$ belongs to the space $\mathfrak{T}^2_\bullet(V)$ if and only if the function $f(-x)$ belongs to the space V.*

Proof By Proposition 2.5.11, we have $\mathfrak{T}_2^{-1} = \mathfrak{T}_2$. Also, it immediately follows from the definition of the map $\mathfrak{T}_1$ that $\mathfrak{T}_1^{-1} = \mathfrak{T}_1$. Therefore, $\mathfrak{T}_\bullet^{-1} = \mathfrak{T}_2 \circ \mathfrak{T}_1$. Then the corollary follows from Proposition 2.5.10 applied to the space $\mathfrak{T}_\bullet(V)$. ■

Corollary 2.5.12 is consistent with Theorem 2.4.6 and the remark below Theorem 2.4.2. Let us explain this now.

Let $v \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ be an eigenvector of the action $\pi_{\bar z}^{\langle n\rangle}$ of the algebra $B_{\bar\alpha}^{\langle n\rangle}$. Let $B_{ij,\bar\alpha}^{\langle n\rangle}$, $i = 1, \dots, n$, $j \in \mathbb{Z}_{\geqslant 0}$ be the generators of the algebra $\mathcal{B}_{\bar\alpha}^{\langle n\rangle}$, see formula (2.7). Here, we indicated the dependence on $\bar\alpha$ explicitly. For each $i = 1, \dots, n$, $j \in \mathbb{Z}_{\geqslant 0}$, denote by b_{ij} the eigenvalue of $\pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})$ corresponding to v. Consider a differential operator $D = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^n b_i(x) \left(\frac{d}{dx}\right)^{n-i}$, where for each $i = 1, \dots, n$, $b_i(x)$ is the rational function whose Laurent series at infinity equals $\sum_{j=0}^\infty b_{ij} x^{-j}$. By Theorem 2.3.3, the space $V = \ker D$ is a space of quasi-exponentials with the data $(\bar\mu, \bar\lambda; \bar\alpha, \bar z)$, where $\bar\mu$ and $\bar\lambda$ are defined by $\boldsymbol{m}$ and $\boldsymbol{l}$, like in Section 2.4.4.

Let $\tilde{\tilde{D}} = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^n \tilde{\tilde{b}}_i(x) \left(\frac{d}{dx}\right)^{n-i}$ be the fundamental differential operator of the space $\mathfrak{T}^2_\bullet(V)$. Applying Theorem 2.4.6 and Corollary 2.4.7 two times, we get that for each $i = 1, \dots, n$, the Laurent series of $\tilde{\tilde{b}}_i(x)$ at infinity equals $\sum_{j=0}^\infty \tilde{\tilde{b}}_{ij} x^{-j}$, where $\tilde{\tilde{b}}_{ij}$ is the eigenvalue of $\pi_{-\bar z}^{\langle n\rangle}(B_{ij,-\bar\alpha}^{\langle n\rangle})$ corresponding to the eigenvector v. As noted in the remark below Theorem 2.4.2, we have $\pi_{-\bar z}^{\langle n\rangle}(B_{ij,-\bar\alpha}^{\langle n\rangle}) = (-1)^{n-i-j} \pi_{\bar z}^{\langle n\rangle}(B_{ij,\bar\alpha}^{\langle n\rangle})$. Therefore, the differential operator $\tilde{\tilde{D}}$ can be obtained from the differential operator D by the substitution $x \mapsto -x$. This argument establishes Corollary 2.5.12 for those spaces V, which describe eigenvectors of the Bethe algebra.

3. $(\mathfrak{gl}_K, \mathfrak{gl}_N)$-DUALITY AND QUOTIENT DIFFERENCE OPERATOR

3.1 Spaces of quasi-polynomials

Fix complex numbers $z_1, \dots, z_k$ and nonzero partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$. Assume that $z_a - z_b \notin \mathbb{Z}$ for $a \neq b$. Let V be a vector space of functions in one variable with a basis $\{x^{z_a} q_{ab}(x) \mid a = 1, \dots, k, b = 1, \dots, (\lambda^{(a)})'_1\}$, where $q_{ab}(x)$ are polynomials and $\deg q_{ab} = (\lambda^{(a)})'_1 + \lambda^{(a)}_b - b$. Assume that the space V satisfies the following property: for each $a = 1, \dots, k$ and any $b = 1, \dots, (\lambda^{(a)})'_1$,

1. there exists a linear combination of polynomials $q_{a1}, q_{a2}, \dots, q_{a(\lambda^{(a)})'_1}$ which has a root at $x = 0$ of multiplicity $b - 1$,
2. the space V does not contain the function $x^{z_a + \deg q_{ab}}$.

Denote $L' = \sum_{a=1}^{k} (\lambda^{(a)})'_1 = \dim V$. For $\alpha \in \mathbb{C}^*$, define *the sequence of exponents of V at α* as a unique sequence of integers $\boldsymbol{e} = \{e_1 > \dots > e_{L'}\}$, with the property: for each $a = 1, \dots, L'$, there exists $f \in V$ such that $f(x) = (x - \alpha)^{e_a}\big(1 + o(1)\big)$ as $x \to \alpha$.

We say that $\alpha \in \mathbb{C}^*$ is a singular point of V if the set of exponents of V at α differs from the set $\{0, \dots, L' - 1\}$. The space V has finitely many singular points.

Let $\alpha_1, \dots, \alpha_n$ be all singular points of V and let $\boldsymbol{e}^{(i)} = \{e^{(i)}_1 > \dots > e^{(i)}_{L'}\}$ be the set of exponents of V at α_i. For each $i = 1, \dots, n$, define a partition $\mu^{(i)} = (\mu^{(i)}_1, \mu^{(i)}_2, \dots)$ as follows: $e^{(i)}_a = L' + \mu^{(i)}_a - a$ for $a = 1, \dots, L'$, and $\mu^{(i)}_a = 0$ for $a > L'$. Clearly, all partitions $\mu^{(1)}, \dots, \mu^{(n)}$ are nonzero.

Denote the sequences $(\lambda^{(1)}, \dots, \lambda^{(k)})$, $(\mu^{(1)}, \dots, \mu^{(n)})$, $(z_1, \dots, z_k)$, $(\alpha_1, \dots, \alpha_n)$, as $\bar\lambda$, $\bar\mu$, $\bar z$, $\bar\alpha$, respectively. We will say that V is a *space of quasi-polynomials with the data* $(\bar\lambda, \bar\mu; \bar z, \bar\alpha)$.

Lemma 3.1.1 *Let V be a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Then*

$$\sum_{a=1}^{k} |\lambda^{(a)}| = \sum_{i=1}^{n} |\mu^{(i)}|.$$

Here $|\lambda|$ denotes the number of boxes in the Young diagram corresponding to the partition λ.

The lemma is proved by analyzing the order of zeros of the Wronskian of V and its asymptotics at infinity.

The fundamental monic differential operator of V is a unique monic linear differential operator of order L' annihilating V. Denote the fundamental monic differential operator of V by D_V.

Lemma 3.1.2 *Define the functions $\beta_1(x), \ldots, \beta_{L'}(x)$ by*

$$x^{L'} D_V = \left(x\frac{d}{dx}\right)^{L'} + \sum_{a=1}^{L'} \beta_a(x) \left(x\frac{d}{dx}\right)^{L'-a}.$$

Then $\beta_1(x), \ldots, \beta_{L'}(x)$ are rational functions regular at infinity. Denote $\beta_a(\infty) = \lim_{x\to\infty} \beta_a(x)$, $n_{ab} = (\lambda^{(a)})'_1 + \lambda^{(a)}_b - b$, $a = 1, \ldots, k$, $b = 1, \ldots, (\lambda^{(a)})'_1$. Then

$$u^{L'} + \sum_{a=1}^{L'} \beta_a(\infty) u^{L'-a} = \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})'_1} (u - z_a - n_{ab}). \tag{3.1}$$

Proof The fact that $\beta_1(x), \ldots, \beta_{L'}(x)$ are rational functions regular at infinity follows from Proposition 2.5.1. Notice that $\ker \prod_{b=1}^{(\lambda^{(a)})'_1} (x(d/dx) - z_a - n_{ab})$ is the span of $\{x^{z_a + n_{ab}} \,|\, a = 1, \ldots, k, b = 1, \ldots, (\lambda^{(a)})'_1\}$, which implies formula (3.1). ■

Lemma 3.1.3 *Define*

$$p_V(x) = \prod_{i=1}^{n} (x - \alpha_i)^{(\mu^{(i)})'_1}.$$

Then for each $a = 1, \ldots, L'$, $p_V(x)\beta_a(x)$ is a polynomial in x.

Proof The lemma follows from Proposition 2.5.1 and the following three facts that are easy to check:

1. For each $i = 1, \dots, n$, the Wronskian $\mathrm{Wr}(V)$ of V has a zero at $x = \alpha_i$ of multiplicity $(\mu^{(i)})'_1$.

2. $\mathrm{Wr}(V)$ is regular at every point different from $0, \alpha_1, \dots, \alpha_n$.

3. The functions $\beta_1(x), \dots, \beta_{L'}(x)$ are regular at 0.

■

We will call the operator $\bar{D}_V = p_V(x)x^{L'}D_V$ *the fundamental regularized differential operator* of V.

3.2 Spaces of quasi-exponentials with difference data

Fix nonzero complex numbers $\alpha_1, \dots, \alpha_n$ and non-zero partitions $\mu^{(1)}, \dots, \mu^{(n)}$. Assume that $\alpha_i \neq \alpha_j$ for $i \neq j$. Let W be a vector space of functions in one variable with a basis $\{\alpha_i^x r_{ij}(x) \mid i = 1, \dots, n, j = 1, \dots, (\mu^{(i)})'_1\}$, where $r_{ij}(x)$ are polynomials and $\deg r_{ij} = (\mu^{(i)})'_1 + \mu_j^{(i)} - j$.

Denote $M' = \sum_{i=1}^n (\mu^{(i)})'_1 = \dim W$. For $z \in \mathbb{C}$, define *the sequence of discrete exponents of W at z* as a unique sequence of integers $\boldsymbol{e} = \{e_1 > \dots > e_{M'}\}$, with the property: for each $i = 1, \dots, M'$, there exists $f \in W$ such that $f(z+j) = 0$ for $j = 0, \dots, e_i - 1$ and $f(z + e_i) \neq 0$. We say that $z \in \mathbb{C}$ is a *discrete singular point* of W if the set of exponents of W at z differs from the set $\{0, \dots, M' - 1\}$.

Assume that there exists a sequence of complex numbers $\bar{z} = (z_1, \dots, z_k)$ and a sequence of partitions $\bar{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ such that $z_1, \dots, z_k$ are discrete singular points of W, $z_a - z_b \notin \mathbb{Z}$ for $a \neq b$, sequence $\boldsymbol{e}^{(a)} = \{e_1^{(a)} > \dots > e_{M'}^{(a)}\}$ of discrete exponents at z_a is given by $e_i^{(a)} = M' + \lambda_i^{(a)} - i$ for $i = 1, \dots, M'$, $\lambda_i^{(a)} = 0$ for $i > M'$, and $\sum_{a=1}^k |\lambda^{(a)}| = \sum_{i=1}^n |\mu^{(i)}|$. Here $|\lambda|$ denotes the number of boxes in the Young diagram corresponding to the partition λ.

Denote the sequences $(\mu^{(1)}, \dots, \mu^{(n)})$, $(\lambda^{(1)}, \dots, \lambda^{(k)})$, $(\alpha_1, \dots, \alpha_n)$, $(z_1, \dots, z_k)$ as $\bar{\mu}$, $\bar{\lambda}$, $\bar{\alpha}$, $\bar{z}$, respectively. We will say that W is a *space of quasi-exponentials with the difference data* $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$.

Define an operator T acting on functions of x by the following rule:

$$Tf(x) = f(x+1).$$

The fundamental monic difference operator of W is a unique difference operator S_W^m of the form

$$S_W^m = T^{M'} + \sum_{i=1}^{M'} b_i(x) T^{M'-i} \tag{3.2}$$

annihilating W.

The following lemma is proved similarly to Lemma 3.1.2.

Lemma 3.2.1 *The coefficients $b_i(x)$ in (3.2) are rational functions regular at infinity. Denote $b_i(\infty) = \lim_{x\to\infty} b_i(x)$. Then*

$$u^{M'} + \sum_{i=1}^{M'} b_i(\infty) u^{M'-i} = \prod_{i=1}^{n} (u - \alpha_i)^{(\mu^{(i)})'_1}.$$

For the proof of the following lemma, see [19].

Lemma 3.2.2 *Let $n_{ab} = (\lambda^{(a)})'_1 + \lambda^{(a)}_b - b$. Define*

$$p_W(x) = \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})'_1} (x - z_a - n_{ab} + (\lambda^{(a)})'_1).$$

Then for each $i = 1, \dots, M'$, $p_W(x) b_i(x)$ is a polynomial in x.

We will call the operator $\bar{S}_W = p_W(x) S_W^m$ *the fundamental regularized differential operator* of W.

3.3 Algebra of pseudo-difference operators

A pseudo-difference operator is a formal series of the form

$$\sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km} x^k T^m, \tag{3.3}$$

where C_{km} are some complex numbers. Using the operator relations $T^m x^k = (x+m)^k T^m$, $k, m \in \mathbb{Z}$, and identifying $(x+m)^k$ with its Laurent series at infinity, one can multiply series (3.3). This multiplication is associative. Denote the algebra of pseudo-difference operators as $\Psi\mathfrak{D}_q$.

Lemma 3.3.1 *If* $S = \sum_{m=-\infty}^{M} \sum_{k=-\infty}^{K} C_{km}\, x^k T^m$ *with* $C_{KM} \neq 0$*, then* S *is invertible in* $\Psi\mathfrak{D}_q$*.*

Proof Define $\acute{S}$ by the rule $1 + \acute{S} = C_{KM}^{-1}\, x^{-K} S\, T^{-M}$. Then $\sum_{j=0}^{\infty} (-1)^j \acute{S}^j$ is a well-defined element of $\Psi\mathfrak{D}_q$ and the inverse of S is given by the formula:

$$S^{-1} = C_{KM}^{-1} T^{-M} \left(\sum_{j=0}^{\infty} (-1)^j \acute{S}^j \right) x^{-K}. \qquad \blacksquare$$

We consider a formal series $\sum_{m=-\infty}^{M} f_m(x) T^m$, where all $f_m(x)$ are rational functions, as an element of $\Psi\mathfrak{D}_q$ replacing each $f_m(x)$ by its Laurent series at infinity. In particular, we identify the algebra of linear difference operators with rational coefficients (that is operators of the form $\sum_{i=0}^{M} a_i(x) T^{M-i}$ where all $a_0(x), \dots, a_M(x)$ are rational functions) and the corresponding subalgebra of $\Psi\mathfrak{D}_q$.

Denote by $\bar{\mathfrak{D}}$ the algebra of differential operators of the form

$$\sum_{a=0}^{L} \gamma_a(x) (x \frac{d}{dx})^{L-a}$$

with rational coefficients $\gamma_0(x), \dots, \gamma_L(x)$. One can check that the assignment

$$\tau : \quad x\frac{d}{dx} \mapsto -x, \quad x \mapsto T \tag{3.4}$$

defines a homomorphism of algebras $\tau : \bar{\mathfrak{D}} \to \Psi\mathfrak{D}_q$.

Let V be a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Let $\bar{D}_V \in \bar{\mathfrak{D}}$ be the fundamental regularized differential operator of V, $n_{ab} = (\lambda^{(a)})'_1 + \lambda_b^{(a)} - b$, $a = 1, \dots, k$, $b = 1, \dots, (\lambda^{(a)})'_1$. Define *the fundamental pseudo-difference operator* S_V of V by the following formula:

$$S_V = \frac{1}{\prod_{i=1}^{n} (T - \alpha_i)^{(\mu^{(i)})'_1}}\, \tau(\bar{D}_V)\, \frac{1}{\prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})'_1} (-x - z_a - n_{ab})}. \tag{3.5}$$

Let W be a space of quasi-exponentials with the difference data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z})$. Let $\bar{S}_W$ be the fundamental regularized difference operator of W. Define *the fundamental pseudo-difference operator* S_W of W by the following formula:

$$S_W = \frac{1}{\prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})'_1} (x - z_a - n_{ab} + (\lambda^{(a)})'_1)} \, \bar{S}_W \, \frac{1}{\prod_{i=1}^{n} (T - \alpha_i)^{(\mu^{(i)})'_1}}. \tag{3.6}$$

Notice that both S_V and S_W have the form $1 + \sum_{l,m \leqslant 1} C_{lm} x^l T^m$. Therefore, by Lemma 3.3.1, the operators S_V and S_W are invertible in $\Psi\mathfrak{D}_q$.

For the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$, where $\bar{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$, $\bar{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$ are sequences of partitions, $\bar{z} = (z_1, \ldots, z_k)$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are sequences of complex numbers, denote $\bar{\lambda}' = ((\lambda^{(1)})', (\lambda^{(2)})', \ldots, (\lambda^{(k)})')$, $\bar{\mu}' = ((\mu^{(1)})', (\mu^{(2)})', \ldots, (\mu^{(n)})')$, $-\bar{z} - \bar{\lambda}'_1 + 1 = (-z_1 - (\lambda^{(1)})'_1 + 1, -z_2 - (\lambda^{(2)})'_1 + 1, \ldots, -z_k - (\lambda^{(k)})'_1 + 1)$.

Theorem 3.3.2 *Let V be a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Let S_V be the fundamental pseudo-difference operator of V. Then there exists a space of quasi-exponentials W with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$ such that*

$$S_V^{-1} = S_W,$$

where S_W is the fundamental pseudo-difference operator of W.

The theorem will be proved in Section 3.4.6.

3.4 Quotient difference operator

3.4.1 Factorization of a difference operator

For any functions $g_1, \ldots, g_n$, let

$$\mathcal{W}r(g_1, \ldots, g_n) = \det((T^{j-1} g_i)_{i,j=1}^{n})$$

be their discrete Wronskian. It is easy to show that $g_1, \ldots, g_n$ are linearly independent over the field of 1-periodic functions if and only if $\mathcal{W}r(g_1, \ldots, g_n) \neq 0$. Let $\mathcal{W}r_i(g_1,$

$\dots, g_n)$ be the determinant of the $n \times n$ matrix whose j-th row is $g_j, Tg_j, \dots, T^{n-i-1}g_j$, $T^{n-i+1}g_j, \dots, T^n g_j$.

Consider a monic linear difference operator S of order n with coefficients $a_i(x)$, $i = 1, \dots, n$:

$$S = T^n + \sum_{i=1}^{n} a_i(x) T^{n-i}, \tag{3.7}$$

Let $f_1, f_2, \dots, f_n$ be solutions of the difference equation $Sf = 0$. Assume that f_1, f_2, $\dots, f_n$ are linearly independent over the field of 1-periodic functions.

Lemma 3.4.1 *The coefficients $a_1(x), \dots, a_n(x)$ of the difference operator S are given by the formulas*

$$a_i(x) = (-1)^i \, \frac{\mathcal{W}r_i(f_1, \dots, f_n)}{\mathcal{W}r(f_1, \dots, f_n)} \,, \qquad i = 1, \dots, n \,, \tag{3.8}$$

Moreover, for any function g, we have

$$Sg = \frac{\mathcal{W}r(f_1, \dots, f_n, g)}{\mathcal{W}r(f_1, \dots, f_n)} \,. \tag{3.9}$$

Proof The equations $Sf_1 = 0, \dots, Sf_n = 0$ give a linear system of equations for the coefficients $a_1(x), \dots, a_n(x)$. Solving this system by Cramer's rule yields formula (3.8). Formula (3.9) follows from the last row expansion of the determinant in the numerator. ∎

Proposition 3.4.2 *The difference operator S can be written in the following form:*

$$S = \left(T - \frac{g_1(x+1)}{g_1(x)}\right)\left(T - \frac{g_2(x+1)}{g_2(x)}\right) \dots \left(T - \frac{g_n(x+1)}{g_n(x)}\right), \tag{3.10}$$

where $g_n = f_n$, and

$$g_i = \frac{\mathcal{W}r(f_n, f_{n-1}, \dots, f_i)}{\mathcal{W}r(f_n, f_{n-1}, \dots, f_{i+1})} \,, \qquad i = 1, \dots, n-1 \,. \tag{3.11}$$

Proposition 3.4.2 is proved similarly to Proposition 2.5.2

3.4.2 Formal conjugate difference operator

Denote $T_- = T^{-1}$. Then $(T_- f)(x) = f(x-1)$. Given a difference operator (3.7), define its formal conjugate by the formula:

$$S^\dagger h(x) = (T_-)^n h(x) + \sum_{i=1}^{n} (T_-)^{n-i}\big(a_i(x)h(x)\big)\,.$$

If a difference operator S is given by formula (3.10), then

$$S^\dagger = \left(T_- - \frac{g_n(x+1)}{g_n(x)}\right)\left(T_- - \frac{g_{n-1}(x+1)}{g_{n-1}(x)}\right)\dots\left(T_- - \frac{g_1(x+1)}{g_1(x)}\right). \tag{3.12}$$

Proposition 3.4.3 *Let $f_1, f_2, \dots, f_n$ be solutions of the difference equation $Sf = 0$. Assume that $f_1, f_2, \dots, f_n$ are linearly independent over the field of 1-periodic functions. Define*

$$h_i \;=\; T\frac{\mathcal{W}r(f_1,\dots,f_{i-1},f_{i+1},\dots,f_n)}{\mathcal{W}r(f_1,\dots,f_n)}\,,$$

Then the functions $h_1, \dots, h_n$ are linearly independent over the field of 1-periodic functions, and $S^\dagger h_i = 0$ for all $i = 1, \dots, n$.

Proof Since $h_1 = (-1)^{n-1}/g_1(x+1)$, formula (3.12) immediately gives $S^\dagger h_1 = 0$. To prove that $S^\dagger$ annihilates $h_2, \dots, h_n$, one can use an argument similar to one used in Proposition 2.5.3.

Observe that the coefficient a_n of the difference operator $S = T^n + \sum_{i=1}^n a_i T^{n-i}$ is not identically zero. Indeed, due to Lemma 3.4.1, $a_n = T\mathcal{W}r(f_1,\dots,f_n)/\mathcal{W}r(f_1, \dots, f_n)$, and $\mathcal{W}r(f_1,\dots,f_n)$ is not identically zero. The linear independence of the functions $h_1, \dots, h_n$ follows from the relation:

$$\mathcal{W}r(h_1,\dots,h_n) \;=\; \frac{(-1)^{\frac{n(n+1)}{2}}}{(a_n)^n\,\mathcal{W}r(f_1,\dots,f_n)}\,, \tag{3.13}$$

The proof of relation (3.13) is given in Appendix B.

■

3.4.3 Quotient difference operator

For this subsection, all vector spaces are considered over the field of 1-periodic functions. Let S and $\widehat{S}$ be linear difference operators such that $\ker S \subset \ker \widehat{S}$. Then there is a difference operator $\check{S}$ such that $\widehat{S} = \check{S}S$. For instance, it can be seen from the factorization formula (3.10). We will call $\check{S}$ the *quotient difference operator.*

Let $f_1, f_2, \ldots, f_n$ be a basis of $\ker S$ and $f_1, f_2, \ldots, f_n, h_1, \ldots, h_k$ be a basis of $\ker \hat{S}$. Define functions $\varphi_1, \ldots, \varphi_k$ by the formula

$$\varphi_a = T\frac{\mathcal{W}r(f_1, \ldots, f_n, h_1, \ldots, h_{a-1}, h_{a+1}, \ldots, h_k)}{\mathcal{W}r(f_1, \ldots, f_n, h_1, \ldots, h_k)} \ .$$

Proposition 3.4.4 *The functions $\varphi_1, \ldots, \varphi_k$ are linearly independent, and $\check{S}^\dagger\varphi_a = 0$ for all $a = 1, \ldots, k$.*

Proof Set $\tilde{h}_a = Sh_a$, $a = 1, \ldots, k$. The functions $\tilde{h}_1, \ldots, \tilde{h}_k$ are linearly independent. Indeed, if there are 1-periodic functions $c_1, \ldots, c_k$, not all equal to zero, such that $c_1\tilde{h}_1 + \ldots + c_k\tilde{h}_k = 0$, then $S(c_1h_1 + \ldots + c_kh_k) = 0$. This means that $c_1h_1 + \ldots + c_kh_k$ belongs to the span of $f_1, \ldots, f_n$ contrary to the linear independence of the functions $f_1, \ldots, f_n, h_1, \ldots, h_k$.

Since $\check{S}\tilde{h}_a = \hat{S}h_a = 0$ for all $a = 1, \ldots, k$ and the order of $\check{S}$ equals k, the functions $\tilde{h}_1, \ldots, \tilde{h}_k$ form a basis of $\ker \check{S}$. Then, by Proposition 3.4.3, the functions

$$T\frac{\mathcal{W}r(\tilde{h}_1, \ldots, \tilde{h}_{a-1}, \tilde{h}_{a+1}, \ldots, \tilde{h}_k)}{\mathcal{W}r(\tilde{h}_1, \ldots, \tilde{h}_k)} \ , \quad a = 1, \ldots, k$$

form a basis of $\ker \check{S}^\dagger$.

Formula (3.9) yields $\tilde{h}_i = \mathcal{W}r(f_1, \ldots, f_n, h_i)/\mathcal{W}r(f_1, \ldots, f_n)$. Then the proposition follows from the Wronskian identity

$$\frac{\mathcal{W}r(\tilde{h}_1, \ldots, \tilde{h}_{a-1}, \tilde{h}_{a+1}, \ldots, \tilde{h}_k)}{\mathcal{W}r(\tilde{h}_1, \ldots, \tilde{h}_k)} = \frac{\mathcal{W}r(f_1, \ldots, f_n, h_1, \ldots, h_{a-1}, h_{a+1}, \ldots, h_k)}{\mathcal{W}r(f_1, \ldots, f_n, h_1, \ldots, h_k)} \ . \quad (3.14)$$

The identity (3.14) can be checked in the straightforward way using formulae (B.1) and (B.4). ■

3.4.4 Quotient difference operator and spaces of quasi-exponentials

For the rest of Section 3.4, we will assume that all vector spaces are over $\mathbb{C}$. For every complex vector space W we will be dealing with, the following is true: any subset of W is linearly independent over complex numbers if and only if it is linearly independent over the field of 1-periodic functions. Therefore, we can apply the results of Section 3.4.3 to W just replacing the field of 1-periodic functions by the field $\mathbb{C}$.

Fix nonzero complex numbers $\alpha_1, \dots, \alpha_n$ and nonzero partitions $\mu^{(1)}, \dots, \mu^{(n)}$. Assume that $\alpha_i \neq \alpha_j$ for $i \neq j$. For each $i = 1, \dots, n$, denote $n_i = (\mu^{(i)})'_1$ and $p_i = \mu^{(i)}_1 + n_i$. Let $\widehat{W}$ be the vector space spanned by the functions $\alpha_i^x x^p$ for all $i = 1, \dots, n$, and $p = 0, \dots, p_i - 1$. Denote

$$\mathcal{W}r(\widehat{W}) = \mathcal{W}r(\alpha_1^x, \alpha_1^x x, \dots, \alpha_1^x x^{p_1-1}, \ \dots\ , \alpha_n^x, \alpha_n^x x, \dots, \alpha_n^x x^{p_n-1}),$$
$$\mathcal{W}_{ij}(\widehat{W}) = \mathcal{W}r(\dots, \widehat{\alpha_i^x x^j}, \dots).$$

The functions in the second line are the same except the function $\alpha_i^x x^j$ is omitted.

Lemma 3.4.5 *The following holds:*

$$\mathcal{W}r(\widehat{W}) = \prod_{i=1}^{n} \left(\alpha_i^{p_i x} \prod_{s=1}^{p_i-1} \alpha_i^s s! \right) \prod_{1 \leqslant i < j \leqslant n} (\alpha_j - \alpha_i)^{p_i p_j}, \tag{3.15}$$

$$\mathcal{W}r_{ij}(\widehat{W}) = r_{ij}(x) \prod_{l=1}^{n} \left(\alpha_l^{(p_l - \delta_{il})x} \prod_{\substack{s=1 \\ (l,s)\neq(i,j)}}^{p_l-1} \alpha_l^s s! \right) \prod_{1 \leqslant l < l' \leqslant n} (\alpha_{l'} - \alpha_l)^{(p_l - \delta_{li})(p_{l'} - \delta_{l'i})}, \tag{3.16}$$

where $r_{ij}(x)$ is a monic polynomial in x and $\deg r_{ij} = p_i - j - 1$.

The Lemma is proved similarly to Lemma 2.5.5.

Denote by $\mathcal{E}(\bar{\alpha}, \bar{\mu})$ the set of all vector spaces with a basis of the form $\{\alpha_i^x q_{ij}(x) \mid i = 1, \dots, n, j = 1, \dots, (\mu^{(i)})'_1\}$, where $q_{ij}(x)$ are some polynomials such that $\deg q_{ij} = (\mu^{(i)})'_1 + \mu^{(i)}_j - j$.

Consider a space $W \in \mathcal{E}(\bar{\alpha}, \bar{\mu})$. Let S^m_W the fundamental monic difference operator of W. Define $\widehat{S} = \prod_{i=1}^n (T - \alpha_i)^{p_i}$. Then $\ker \widehat{S} = \widehat{W}$. Therefore, $\ker S^m_V \subset \ker \widehat{S}$, and there exists a differential operator $\check{S}_W$, such that $\widehat{S} = \check{S}_W S^m_W$, see Section 3.4.3. Let $\check{W}^\dagger = \ker \check{S}^\dagger_W$.

Proposition 3.4.6 *The space* $\check{W}^\dagger$ *has a basis of the form*

$$\{\alpha_i^{-x}\check{q}_{ij}(x) \mid i=1,\ldots,n,\ j=1,\ldots,\mu_1^{(i)}\},$$

where $\deg \check{q}_{ij} = \mu_1^{(i)} + (\mu^{(i)})'_j - j$, $i=1,\ldots,n,\ j=1,\ldots,\mu_1^{(i)}$.

Proof For each $i=1,\ldots,n$, set

$$\boldsymbol{d}_i = \{n_i + \mu_j^{(i)} - j \mid j=1,\ldots,n_i\}, \qquad \boldsymbol{d}_i^c = \{0,1,2,\ldots,p_i-1\} \setminus \boldsymbol{d}_i\,. \tag{3.17}$$

Since the space W has a basis of the form $\{\alpha_i^x q_{ij}(x) \mid i=1,\ldots,n,\ j=1,\ldots,n_i\}$, where $q_{ij}(x)$ are polynomials and $\deg q_{ij} = n_i + \mu_j^{(i)} - j$, the functions $\alpha_i^x x^l$, $i=1,\ldots,n,\ l\in \boldsymbol{d}_i^c$, complement this basis of W to a basis of $\widehat{W}$.

By Proposition 3.4.4, the space $\check{W}^\dagger$ has the following basis

$$T\frac{\mathcal{W}r_{ij}(\widehat{W})}{\mathcal{W}(\widehat{W})} + T\sum_{l=j+1}^{p_i-1} C_{ijl}\,\frac{\mathcal{W}r_{il}(\widehat{W})}{\mathcal{W}r(\widehat{W})}\,, \qquad i=1,\ldots,n\,, \quad j\in\boldsymbol{d}_i^c\,, \tag{3.18}$$

where C_{ijl} are complex numbers. Then by Lemma 3.4.5, for each i,j, the corresponding element of this basis has the form $\alpha_i^{-x}\tilde{r}_{ij}(x)$, where $\tilde{r}_{ij}(x)$ is a polynomial of degree $p_i - j - 1$.

By Lemma 2.5.6, $j\in\boldsymbol{d}_i^c$ if and only if $j = n_i - (\mu^{(i)})'_l + l - 1$ for some $l\in\{1,\ldots,\mu_1^{(i)}\}$. Set $\check{q}_{il}(x) = \tilde{r}_{ij}(x)$. Then $\check{W}^\dagger$ has a basis of the form $\{\alpha_i^{-x}\check{q}_{il}(x) \mid i=1,\ldots,n,\ l=1,\ldots,\mu_1^{(i)}\}$ and

$$\deg \check{q}_{il} \;=\; \deg\tilde{r}_{ij} \;=\; \mu_1^{(i)} + n_i - (n_i - (\mu^{(i)})'_l + l - 1) - 1 = \mu_1^{(i)} + (\mu^{(i)})'_l - l\,. \tag{3.19}$$

Proposition 3.4.6 is proved. ■

Recall that $M' = \sum_{i=1}^n (\mu^{(i)})'_1 = \dim W$. We also have $M = \sum_{i=1}^n \mu_1^{(i)} = \dim \check{W}^\dagger$. For $z\in\mathbb{C}$, define *the sequence of* T_-*-discrete exponents of* $\check{W}^\dagger$ *at* z as a unique sequence of integers $\check{\boldsymbol{e}} = \{\check{e}_1 > \ldots > \check{e}_M\}$, with the property: for each $i=1,\ldots,M$, there exists $f\in\check{W}^\dagger$ such that $f(z-j)=0$ for $j=0,\ldots,\check{e}_i - 1$ and $f(z-\check{e}_i)\neq 0$.

Proposition 3.4.7 *Let $\boldsymbol{e} = \{e_1 > \ldots > e_{M'}\}$ be the sequence of discrete exponents of W at some point $z \in \mathbb{C}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition such that $e_i = M' + \lambda_i - i$, $i = 1, \ldots, M'$ and $\lambda_i = 0$ for $i > M'$. Then the sequence $\check{\boldsymbol{e}} = \{\check{e}_1 > \ldots > \check{e}_M\}$ of T_--discrete exponents of $\check{W}^\dagger$ at $z - 1$ is given by $\check{e}_a = M + \eta_a - a$, $a = 1, \ldots, M$, where $\eta = (\eta_1, \eta_2, \ldots)$ is a partition such that $\eta_a \geqslant \lambda'_a$ for all $a = 1, 2, \ldots$.*

Proof Since $\boldsymbol{e} = \{e_1 > \ldots > e_{M'}\}$ is the sequence of discrete exponents of W at $z \in \mathbb{C}$, there is a basis $\{\psi_1, \ldots, \psi_{M'}\}$ of W such that for each $i = 1, \ldots, M'$, $j = 0, \ldots, e_i - 1$, we have $\psi_i(z + j) = 0$ and $\psi_i(z + e_i) \neq 0$.

By formula (3.15), the Wronskian $\mathcal{W}r(\widehat{W})$ has no zeros, thus z is not a singular point of $\widehat{W}$. Therefore, there is a basis $\{f_1, f_2, \ldots, f_{M+M'}\}$ of $\widehat{W}$ such that it contains the set $\{\psi_1, \ldots, \psi_{M'}\}$ and for each $i = 0, \ldots, M + M' - 1$, $j = 0, \ldots, i$, we have $f_{i+1}(z + j) = 0$ and $f_{i+1}(z + i) \neq 0$.

Consider a matrix-valued function

$$F_a(x) = (T^j f_i)_{\substack{i=1,\ldots,M+M', \quad i \neq a \\ j=1,\ldots,M+M'-1}}.$$

Then $F_a(z)$ is an upper-triangular matrix with the diagonal of the form $\{d_1, d_2, \ldots, d_{a-1}, 0, 0 \ldots\}$, where $d_b \neq 0$, $b = 1, \ldots, a - 1$. An example with $M + M' = 6$, $a = 4$ is shown below.

$$F_4(z) = \begin{pmatrix} d_1 & \star & \star & \star & \star \\ 0 & d_2 & \star & \star & \star \\ 0 & 0 & d_3 & \star & \star \\ 0 & 0 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For every $b = 0, \ldots, M + M' - a$, let F_{ab} be an $(M + M' - b - 1) \times (M + M' - b - 1)$ submatrix of $F_a(z)$ located in the upper-left corner. Then F_{ab} is also an upper-triangular matrix with the diagonal of the form $\{d_1, d_2, \ldots, d_{a-1}, 0, 0 \ldots\}$. We have:

$$\det\left[((T_-)^b F_a)(z)\right] = \text{const} \cdot \det(F_{ab}) = 0, \quad b = 0, \ldots, M + M' - a - 1. \qquad (3.20)$$

The relations (3.20) are illustrated by the example with $M+M'=6$, $a=4$ below.

$$((T_-)F_4)(z) = \begin{pmatrix} \star & \boxed{\begin{matrix} d_1 & \star & \star & \star \\ 0 & d_2 & \star & \star \\ 0 & 0 & d_3 & \star \\ 0 & 0 & 0 & 0 \end{matrix}} \\ \boxed{\star} & \begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \end{pmatrix}, \quad ((T_-)^2F_4)(z) = \begin{pmatrix} \star\ \star & \boxed{\begin{matrix} d_1 & \star & \star \\ 0 & d_2 & \star \\ 0 & 0 & d_3 \end{matrix}} \\ \boxed{\begin{matrix} \star & \star \\ \star & \star \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}. \tag{3.21}$$

In each matrix above, we boxed two minors, whose product gives the determinant of the corresponding matrix. The lower-left minor in each case corresponds to the constant in formula (3.20). The upper-right minor in $\det[((T_-)F_4)(z)]$ is the determinant of F_{41} and the upper-right minor in $\det[((T_-)^2F_4)(z)]$ is the determinant of F_{42}. We see that $\det[((T_-)F_4)(z)] = 0$ and $\det[((T_-)^2F_4)(z)]$ might not be zero.

Recall that $\boldsymbol{e} = \{e_1 > \ldots > e_{M'}\}$ is the sequence of discrete exponents of W at $z \in \mathbb{C}$. Set $\boldsymbol{e}^c = \{0,1,2,\ldots,M+M'-1\} \setminus \boldsymbol{e}$. By Lemma 2.5.6, we have

$$\boldsymbol{e}^c = \{M' - \lambda'_a + a - 1 \,|\, a = 1,\ldots,M\}.$$

Denote $e^c_a = M' - \lambda'_a + a - 1$, $a = 1,\ldots,M$.

For each $a = 1,\ldots,M+M'$, denote

$$\mathcal{W}r_a(\widehat{W}) = \det F_a(x) = \mathcal{W}r(f_1,\ldots,f_{a-1},f_{a+1},\ldots,f_{M+M'}).$$

By Proposition 3.4.4, the set

$$\left\{ T\frac{\mathcal{W}r_{e^c_a}(\widehat{W})}{\mathcal{W}r(\widehat{W})} \,|\, a = 1,\ldots,M \right\}$$

form a basis of $\check{W}^\dagger$.

Since $\mathcal{W}r(\widehat{W})$ has no zeros, relations (3.20) give

$$(T_-)^b \left(\frac{\mathcal{W}r_a(\widehat{W})}{\mathcal{W}r(\widehat{W})} \right)(z) = 0 \quad b = 0,\ldots,M+M'-a-1. \tag{3.22}$$

Notice that $M+M'-1-e^c_a = M+M'-1-M'+\lambda'_a-a+1 = M+\lambda'_a-a$. Therefore, formula (3.22) yields

$$(T_-)^b \left(T\frac{\mathcal{W}r_{e^c_a}(\widehat{W})}{\mathcal{W}r(\widehat{W})} \right)(z-1) = 0 \quad b = 0,\ldots,M+\lambda'_a-a. \tag{3.23}$$

This means that the sequence $\check{\boldsymbol{e}} = \{\check{e}_1 > \ldots > \check{e}_M\}$ of T_--discrete exponents of $\check{W}^\dagger$ at $z-1$ is given by $\check{e}_a = M + \eta_a - a$, $a = 1, \ldots, M$, where $\eta = (\eta_1, \eta_2, \ldots)$ is a partition such that $\eta_a \geqslant \lambda'_a$ for all $a = 1, 2, \ldots$.

Proposition 3.4.7 is proved. ■

3.4.5 Quotient for a difference operator with left shifts

Denote $\bar{\alpha}^{-1} = (\alpha_1^{-1}, \ldots, \alpha_n^{-1})$. By Proposition 3.4.6, we have a map $Q_+ : \mathcal{E}(\bar{\alpha}, \bar{\mu}) \to \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}')$ such that

$$Q_+ : W \mapsto \check{W}^\dagger .$$

We will also write $\check{S}^\dagger_W = Q_+(S^m_W)$.

We are going to introduce a map $Q_- : \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}') \to \mathcal{E}(\bar{\alpha}, \bar{\mu})$ and show that Q_- is the inverse of Q_+.

Consider a space $W_- \in \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}')$. Then $\dim W_- = \sum_{i=1}^n \mu_1^{(i)} = M$. Let $S^m_{W_-}$ be a difference operator of the form

$$S^m_{W_-} = (T_-)^M + \sum_{i=1}^{M} b_i(x)(T_-)^{M-i}$$

annihilating W_-.

Introduce a difference operator $\widehat{S}_- = \prod_{i=1}^n (T_- - \alpha_i)^{p_i}$. Then the space $\widehat{W}_- = \ker \widehat{S}_-$ is spanned by the functions $\alpha_i^{-x} x^p$, $p = 0, \ldots, p_i - 1$. We have that $W_- \subseteq \widehat{W}_-$, and there exists a difference operator $\check{S}_{W_-}$ such that $\widehat{S}_- = \check{S}_{W_-} S_{W_-}$. For instance, it can be seen from an analog of factorization formula (3.4.2) for the operator T_-.

For a difference operator $S = \sum_{i=1}^l a_i(x)(T_-)^{l-i}$, define its formal conjugate $S^\dagger$ by the formula

$$S^\dagger h(x) = \sum_{i=1}^{l} T^{l-i}\big(a_i(x)h(x)\big) .$$

Denote $Q_-(S_{W_-}) = (\check{S}_{W_-})^\dagger$.

Proposition 3.4.8 *The space* $\ker(Q_-(S_{W_-}))$ *belongs to the set* $\mathcal{E}(\bar{\alpha}, \bar{\mu})$.

Proposition 3.4.8 is proved similarly to Proposition 3.4.6.

Due to Proposition 3.4.8, we have a map $Q_- : \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}') \to \mathcal{E}(\bar{\alpha}, \bar{\mu})$ such that $Q_- : W_- \mapsto \ker(Q_-(S_{W_-}))$.

Proposition 3.4.9 *For any $W \in \mathcal{E}(\bar{\alpha}, \bar{\mu})$ and $W_- \in \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}')$, the following holds:*

$$Q_-(Q_+S_W) = S_W, \quad Q_+(Q_-S_{W_-}) = S_{W_-}.$$

Proof Recall that $\widehat{S} = \prod_{i=1}^n (T - \alpha_i)^{p_i} = (\widehat{S}_-)^\dagger$ and $\widehat{S} = (Q_+(S_W))^\dagger S_W$. We have

$$\widehat{S}_- = (\widehat{S})^\dagger = (S_W)^\dagger Q_+(S_W). \tag{3.24}$$

In the relation $\widehat{S}_- = (Q_-(S_{W_-}))^\dagger S_{W_-}$, take $W_- = Q_+(W)$. This yields

$$\widehat{S}_- = (Q_-(Q_+S_W))^\dagger Q_+(S_W). \tag{3.25}$$

Comparing formulae (3.24) and (3.25), we have $Q_-(Q_+S_W) = S_W$.

The relation $Q_+(Q_-S_{W_-}) = S_{W_-}$ is proved in a similar way. ■

Proposition 3.4.10 *Fix $z \in \mathbb{C}$. Let $\boldsymbol{e} = \{e_1 > \ldots > e_{M'}\}$ be the sequence of T_--discrete exponents of $W_- \in \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}')$ at $z-1$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition such that $e_i = M + \lambda_i - i$, $i = 1, \ldots, M$ and $\lambda_i = 0$ for $i > M$. Then the sequence $\check{\boldsymbol{e}} = \{\check{e}_1 > \ldots > \check{e}_{M'}\}$ of discrete exponents of $Q_-(W_-)$ at z is given by $\check{e}_a = M' + \eta_a - a$, $a = 1, \ldots, M'$, where $\eta = (\eta_1, \eta_2, \ldots)$ is a partition such that $\eta_a \geqslant \lambda'_a$ for all $a = 1, 2, \ldots$*

Proposition 3.4.10 is proved similarly to Proposition 3.4.7.

Corollary 3.4.11 *In both Proposition 3.4.7 and Proposition 3.4.10, we have $\eta = \lambda'$.*

Proof Consider a space $W \in \mathcal{E}(\bar{\alpha}, \bar{\mu})$, and let partitions λ and η be like in Proposition 3.4.7. By Proposition 3.4.10 applied to the space $Q_+(W)$, the sequence $\tilde{\boldsymbol{e}} = \{\tilde{e}_1 > \ldots > \tilde{e}_{M'}\}$ of discrete exponents of $Q_-(Q_+(W))$ at z is given by $\tilde{e}_i = M' + \nu_i - i$, $i = 1, \ldots, M'$, where $\nu_i \geqslant \eta'_i$ for all $i = 1, 2, \ldots$. By Proposition 3.4.9, we have $Q_-(Q_+(W)) = W$, thus $\nu_i = \lambda_i$, $i = 1, 2, \ldots$. Notice that $\eta_a \geqslant \lambda'_a$ for all $a = 1, 2, \ldots$ is the same as $\eta'_i \geqslant \lambda_i$ for all $i = 1, 2, \ldots$. Therefore, we have $\lambda_i \leqslant \eta'_i \leqslant \nu_i = \lambda_i$, which yields $\eta'_i = \lambda_i$, or $\eta = \lambda'$.

The equality $\eta = \lambda'$ for Proposition 3.4.10 is proved in a similar way. ■

3.4.6 Proof of Theorem 3.3.2

For any pseudo-difference operator $S=\sum_{i=-\infty}^{N}\sum_{j=-\infty}^{K}C_{ij}x^iT^j$, define a pseudo-difference operator $S^\ddagger$ by

$$S^\ddagger=\sum_{i=-\infty}^{N}\sum_{j=-\infty}^{K}C_{ij}T^j(-x)^i.$$

It is easy to check that $(\cdot)^\ddagger$ is an involutive antiautomorphism on $\Psi\mathfrak{D}_q$.

If S is a difference operator of the form $S=\sum_{i=0}^{l}a_i(x)(T_-)^{l-i}$, define a difference operator $S^\rightarrow$ by

$$S^\rightarrow=\sum_{i=0}^{l}a_i(-x)T^{l-i}.$$

If the coefficients $a_0(x),\dots,a_l(x)$ of S are rational functions, then we identify S with the corresponding element in $\Psi\mathfrak{D}_q$, see Section 3.3. In this case, we have $(S^\rightarrow)^\ddagger=S^\dagger$.

Let V be a space of quasi-polynomials with the data $(\bar\lambda,\bar\mu;\bar z,\bar\alpha)$. Let $\bar D_V$ be the fundamental regularized differential operator of V. Denote $\bar S_V=\tau(\bar D_V)$, where τ is given by formula (3.4).

Let $\bar z=(z_1,\dots,z_k)$ and $\lambda=(\lambda^{(1)},\dots,\lambda^{(k)})$ be the sequences in the data $(\bar\lambda,\bar\mu;\bar z,\bar\alpha)$. Define a sequence of complex numbers $\bar z+\bar\lambda'_1=(z_1+(\lambda^{(1)})'_1,z_2+(\lambda^{(2)})'_1,$ $\dots,z_k+(\lambda^{(k)})'_1)$. The following theorem was proved in [19].

Theorem 3.4.12 *There is a space of quasi-exponentials U with the difference data $(\bar\mu,\bar\lambda;\bar\alpha,\bar z+\bar\lambda'_1)$ with the fundamental regularized difference operator $\bar S_U$ such that*

$$(\bar S_V)^\ddagger=\bar S_U.$$

Let S_U^m be the fundamental monic difference operator of U. Since U has the difference data $(\bar\mu,\bar\lambda;\bar\alpha,\bar z+\bar\lambda'_1)$, denoting $n_{ab}=(\lambda^{(a)})'_1+\lambda_b^{(a)}-b$, $a=1,\dots,k$, $b=1$, $\dots,(\lambda^{(a)})'_1$, we have

$$S_U^m=\frac{1}{\prod_{a=1}^{k}\prod_{b=1}^{(\lambda^{(a)})'_1}(x-z_a-n_{ab})}\bar S_U.$$

Recall that the fundamental pseudo-difference operator S_V of V is defined as follows

$$S_V = \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}}\, \tau(\bar{D}_V)\, \frac{1}{\prod_{a=1}^k \prod_{b=1}^{(\lambda^{(a)})'_1} (-x - z_a - n_{ab})}.$$

Therefore,

$$\begin{aligned} S_V^{\ddagger} &= \left(\frac{1}{\prod_{a=1}^k \prod_{b=1}^{(\lambda^{(a)})'_1} (-x - z_a - n_{ab})} \right)^{\ddagger} (\bar{S}_V)^{\ddagger} \left(\frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}} \right)^{\ddagger} = \\ &= \frac{1}{\prod_{a=1}^k \prod_{b=1}^{(\lambda^{(a)})'_1} (x - z_a - n_{ab})}\, \bar{S}_U\, \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}} = \\ &= S_U^m\, \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}}. \end{aligned}$$

Denote $Q_+^{\rightarrow}(S_U) = (Q_+(S_U^m))^{\rightarrow}$. Then

$$\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)} + (\mu^{(i)})'_1} = (Q_+^{\rightarrow}(S_U^m))^{\ddagger} S_U^m, \tag{3.26}$$

which gives

$$\left(Q_+^{\rightarrow}(S_U^m) \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}}} \right)^{\ddagger} \left(S_U^m \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}} \right) = 1.$$

Therefore,

$$(S_V^{\ddagger})^{-1} = \left(S_U^m \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}} \right)^{-1} = \left(Q_+^{\rightarrow}(S_U^m) \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}}} \right)^{\ddagger},$$

and

$$S_V^{-1} = ((S_V^{\ddagger})^{-1})^{\ddagger} = Q_+^{\rightarrow}(S_U^m) \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}}}. \tag{3.27}$$

We will show now that $Q_+^{\rightarrow}(U) = \ker Q_+^{\rightarrow}(S_U^m)$ is a space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$.

Since U is a space of quasi-exponentials with the difference data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z} + \bar{\lambda}'_1)$, $\dim U = \sum_{i=1}^n (\mu^{(i)})'_1 = M'$, and for every $a = 1, \dots, k$, the sequence $\boldsymbol{e}^{(a)} = (e_1^{(a)} > \dots > e_{M'}^{(a)})$ of discrete exponents of U at $z_a + (\lambda^{(a)})'_1$ is given by $e_i^{(a)} = M' + \lambda_i^{(a)} - i$, $i = 1, \dots, M'$. By Proposition 3.4.7 and Corollary 3.4.11, for each $a = 1, \dots, k$, the sequence of T_--discrete exponents $\check{\boldsymbol{e}}^{(a)} = (\check{e}_1^{(a)} > \dots > \check{e}_{M'}^{(a)})$ of $Q_+(U)$ at $z_a + (\lambda^{(a)})'_1 - 1$

is given by $\check{e}_i^{(a)} = M + (\lambda^{(a)})'_i - i$, $i = 1, \ldots, M$. Also, By Proposition 3.4.6, $Q_+(U) \in \mathcal{E}(\bar{\alpha}^{-1}, \bar{\mu}')$.

Notice that every function from the space $Q_+^{\rightarrow}(U)$ is the image of a function from $Q_+(U)$ under the transformation $x \mapsto -x$, and vice versa. Therefore, $Q_+^{\rightarrow}(U) \in \mathcal{E}(\bar{\alpha}, \bar{\mu}')$, and for each $a = 1, \ldots, k$, $\check{\boldsymbol{e}}^{(a)} = (\check{e}_1^{(a)} > \cdots > \check{e}_{M'}^{(a)})$ is the sequence of discrete exponents of $Q_+^{\rightarrow}(U)$ at $-z_a - (\lambda^{(a)})'_1 + 1$. By Lemma 3.1.1, we have $\sum_{a=1}^{k} |(\lambda^{(a)})| = \sum_{i=1}^{n} |(\mu^{(i)})|$, which is the same as $\sum_{a=1}^{k} |(\lambda^{(a)})'| = \sum_{i=1}^{n} |(\mu^{(i)})'|$. Therefore, $Q_+^{\rightarrow}(U) = \ker Q_+^{\rightarrow}(S_U^m)$ is a space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$.

Put $W = Q_+^{\rightarrow}(U)$. Then $S_W^m = Q_+^{\rightarrow}(S_U^m)$ is the fundamental monic difference operator of W. Let S_W be the fundamental pseudo-difference operator of W. Then, by definition (see formula (3.6)), $S_W = S_W^m (\prod_{i=1}^{n} (T - \alpha_i)^{\mu_1^{(i)}})^{-1}$.

On the other hand, formula (3.27) gives $S_V^{-1} = S_W^m (\prod_{i=1}^{n} (T - \alpha_i)^{\mu_1^{(i)}})^{-1}$, and we have $S_V^{-1} = S_W$.

Theorem 3.3.2 is proved.

3.4.7 Quotient difference operator versus quotient differential operator

In the proof of Theorem 3.3.2, we obtained the space W using the quotient difference operator. There is another way to obtain W, which involves the quotient differential operator.

Denote $l_a = \lambda_1^{(a)} + (\lambda^{(a)})'_1 - 1$. Introduce a differential operator

$$\widehat{D} = \prod_{a=1}^{k} \prod_{b=0}^{l_a} (x \frac{d}{dx} - z_a - b). \tag{3.28}$$

Then

$$\widehat{V} = \ker(\widehat{D}) = \{x^{z_a + b} \,|\, a = 1, \ldots, k,\ b = 0, \ldots, l_a\}.$$

Let V be a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Let D_V be the fundamental monic differential operator of V. Recall that V has a basis of the form $\{x^{z_a} q_{ab}(x) \mid a = 1, \ldots, k, b = 1, \ldots, (\lambda^{(a)})'_1\}$, where $q_{ab}(x)$ are polynomials and

$\deg q_{ab} = (\lambda^{(a)})'_1 + \lambda^{(a)}_b - b$. Therefore, $V \subset \widehat{V}$, and there exists a differential operator $\check{D}_V$ such that $\widehat{D} = \check{D}_V x^k D_V$, see Section 2.5.3. Let $\check{D}^\dagger_V$ be the formal conjugate of $\check{D}_V$, see formula (2.32) for the definition. Denote $\tilde{U} = \ker \check{D}^\dagger_V$.

We will say that V is a space of quasi-polynomials *with non-degenerate terms* if it satisfies the following property: for each $a = 1, \dots, k$, $b = 0, \dots, l_a$, there exists $s = 1, \dots, (\lambda^{(a)})'_1$ such that $q^{(b)}_{as}(0) \neq 0$.

Denote $-\bar{z} - \bar{\lambda}'_1 - \bar{\lambda}_1 + 1 = (-z_1 - (\lambda^{(1)})'_1 - \lambda^{(1)}_1 + 1, -z_2 - (\lambda^{(2)})'_1 - \lambda^{(2)}_1 + 1, \dots,$ $-z_k - (\lambda^{(k)})'_1 - \lambda^{(2)}_1 + 1)$. We have the following theorem.

Theorem 3.4.13

1. *Let V be a space of quasi-polynomials with non-degenerate terms. Then $\tilde{U}$ is a space of quasi-polynomials with the data $(\bar{\lambda}', \bar{\mu}', -\bar{z} - \bar{\lambda}'_1 - \bar{\lambda}_1 + 1)$.*

2. *Let $\tilde{W}$ be the space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$ such that $\left(\tau(\bar{D}_{\tilde{U}})\right)^\ddagger = \bar{S}_{\tilde{W}}$, where $\bar{D}_{\tilde{U}}$ and $\bar{S}_{\tilde{W}}$ are the fundamental regularized differential and difference operators of $\tilde{U}$ and $\tilde{W}$, respectively, see Theorem 3.4.12. Let $S_{\tilde{W}}$ be the fundamental pseudo-difference operator of $\tilde{W}$. Then $S_{\tilde{W}} = S_V^{-1}$.*

We will prove Theorem 3.4.13 later in the section.

Two spaces related like in Theorem 3.4.12 are called bispectral dual. Therefore, the space U used in the proof of Theorem 3.3.2 is bispectral dual to V, and the space $\tilde{W}$ in Theorem 3.4.13 is bispectral dual to $\tilde{U}$. Comparing Theorem 3.4.13 and the proof of Theorem 3.3.2, we see that taking bispectral dual U of the space V and then using the quotient difference operator to get W is the same as first using the quotient differential operator and then taking its bispectral dual, as illustrated on diagram (3.29)

$$\begin{tikzcd} & U \arrow[dr, "\text{quotient of difference operator}"] & \\ V \arrow[ur, "\text{bisp. dual}"] \arrow[dr, "\text{quotient of differential operator}"'] & & \tilde{W} = W \\ & \tilde{U} \arrow[ur, "\text{bisp. dual}"'] & \end{tikzcd} \tag{3.29}$$

The space W from Theorem 3.3.2 is a space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z} - \bar{\lambda}'_1 + 1)$. Then the fundamental regularized difference operator $\bar{S}_W$ of W is a difference operator with polynomial coefficients defined by $\bar{S}_W = p_W(x) S_W^m$, where

$$p_W(x) = \prod_{a=1}^{k} \prod_{b=1}^{\lambda_1^{(a)}} (x + z_a + (\lambda^{(a)})'_1 - (\lambda^{(a)})'_b + b - 1), \tag{3.30}$$

and S_W^m is the fundamental monic difference operator, see Section 3.2

Let $\tilde{p}_W(x)$ be the polynomial of minimal degree such that $\tilde{p}_W(x) S_W^m$ has polynomial coefficients. We will call the space W non-degenerate if $\tilde{p}_W(x) = p_W(x)$.

The following theorem is the converse of Theorem 3.4.12, and it is proved in [19].

Theorem 3.4.14 *Let W be non-degenerate. Then the operator $\tau^{-1}\left(\left(\bar{S}_W\right)^{\ddagger}\right)$ is the fundamental regularized differential operator of a space of quasi-polynomials with the data $(\bar{\lambda}', \bar{\mu}', -\bar{z} - \bar{\lambda}'_1 - \bar{\lambda}_1 + 1)$.*

One way to prove Theorem 3.4.13 is to obtain a basis of $\tilde{U}$ using Wronskians like we did in Sections 3.4.1 - 3.4.5 for $\check{W}^{\dagger}$, and in Sections 2.5.1 - 2.5.4 for $\check{V}^{\dagger}$. We are not going to follow this straightforward way, but rather we will show that $\tilde{U} = \ker \tau^{-1}\left(\bar{S}_W^{\ddagger}\right)$ and use Theorem 3.4.14. To do this we should ensure that W is non-degenerate. In particular, we need the following proposition.

Proposition 3.4.15 *Let V be the space of quasi-polynomials with non-degenerate terms. Let W be the space of quasi-exponentials such that $S_W = S_V^{-1}$. Then W is non-degenerate.*

Proof Let $\tilde{p}_W(x)$ be the polynomial of smallest degree such that $\tilde{p}_W(x)S_W^m$ has polynomial coefficients. Recall that

$$S_V^{-1} = S_W = S_W^m \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}}}.$$

Therefore $\tilde{p}_W S_V^{-1}$ is a pseudo-difference operator of the form

$$\sum_{i=-\infty}^{N} a_i(x)T^i, \tag{3.31}$$

where $a_N, a_{N-1}, \ldots$ are polynomials in x. In particular, $\tilde{p}_W S_V^{-1}$ belongs to the image of the injective homomorphism τ, and $\tau^{-1}(\tilde{p}_W S_V^{-1})$ is a differential operator which is a polynomial in $x(d/dx)$.

From the definition of S_V, we get

$$S_V \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})_1'} (-x - z_a - n_{ab}) = \tau(x^k D_V),$$

which gives

$$\tilde{p}_W(x) \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})_1'} (-x - z_a - n_{ab}) = \tilde{p}_W(x) S_V^{-1} \tau(x^k D_V). \tag{3.32}$$

Applying τ^{-1} to both sides of formula (3.32) we get

$$\tau^{-1}(\tilde{p}_W(x)) \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})_1'} (x\frac{d}{dx} - z_a - n_{ab}) = \tau^{-1}(\tilde{p}_W(x) S_V^{-1}) x^k D_V.$$

The last formula implies that

$$V = \ker x^k D_V \subseteq \tilde{V} = \ker \left(\tau^{-1}(\tilde{p}_W(x)) \prod_{a=1}^{k} \prod_{b=1}^{(\lambda^{(a)})_1'} (x\frac{d}{dx} - z_a - n_{ab}) \right). \tag{3.33}$$

Denote $\Delta_a = \{0, \ldots, l_a\} \setminus \{n_{ab} \,|\, b = 1, \ldots, (\lambda^{(a)})_1'\}$. Let V_1 be the span of $\{x^{z_a + n_{ab}} \,|\, a = 1, \ldots, k, b = 1, \ldots, (\lambda^{(a)})_1'\}$, and let V_2 be the span of $\{x^{z_a + b} \,|\, a = 1, \ldots, k, b \in \Delta_a\}$. We have $\widehat{V} = V_1 \bigoplus V_2$.

By Lemma 2.5.6, $\Delta_a = \{(\lambda^{(a)})'_1 - (\lambda^{(a)})'_b + b - 1 \,|\, b = 1, \ldots, \lambda^{(a)}_1\}$. Using this, one can check that for the polynomial $p_W(x)$ defined in formula (3.30), we have

$$p_W(x) = (-1)^L \prod_{a=1}^{k} \prod_{b \in \Delta_a} (-x - z_a - b),$$

where $L = \sum_{a=1}^{k} \lambda^{(a)}_1$. Therefore $V_2 = \ker \tau^{-1}(p_W(x))$.

Consider the space $\tilde{V}_2 = \ker \tau^{-1}(\tilde{p}_W(x))$. The polynomial $\tilde{p}_W(x)$ divides $p_W(x)$. Therefore, $\tilde{V}_2 \subseteq V_2$, and $\tilde{V}_2$ is spanned by $\{x^{z_a+b} \,|\, a = 1, \ldots, k, b \in \tilde{\Delta}_a\}$, where $\tilde{\Delta}_a$ is a subset of Δ_a for each $a = 1, \ldots, k$.

By definition, see formula (3.33), $\tilde{V} = V_1 \bigoplus \tilde{V}_2$. Therefore, $\tilde{V} \subseteq \widehat{V}$, and $\tilde{V}$ is spanned by $\{x^{z_a+b} \,|\, a = 1, \ldots, k, b \in \Delta'_a\}$, where Δ'_a is a subset of $\{0, \ldots, l_a\}$ for each $a = 1, \ldots, k$.

Suppose that $\tilde{p}_W(x) \neq p_W(x)$, then for some $a = 1, \ldots, k$, Δ'_a is a proper subset of $\{0, \ldots, l_a\}$. But since $V \subseteq \tilde{V}$, this contradicts to the fact that V is a space of quasi-polynomials with non-degenerate terms. Therefore, $\tilde{p}_W(x) = p_W(x)$.

Proposition 3.4.15 is proved. ■

Proof [Proof of Theorem 3.4.13.] Let W be the space of quasi-exponentials such that $S_W = S_V^{-1}$. Let $\bar{S}_W$ be the fundamental regularized difference operator of W. We will show that

$$\prod_{i=1}^{n} (x - \alpha_i)^{\mu_1^{(i)}} (-1)^L \check{D}_V^{\dagger} = \tau^{-1}\left(\left(\bar{S}_W\right)^{\ddagger} \right). \tag{3.34}$$

Then part (1) of Theorem 3.4.13 will follow from Theorem 3.4.14.

Recall that $\bar{S}_V = \tau(\bar{D}_V)$, where $\bar{D}_V$ is the fundamental regularized differential operator of V. We have $x^k D_V = \left(\prod_{i=1}^{n} (x - \alpha_i)^{(\mu^{(i)})'_1}\right)^{-1} \bar{D}_V$. Therefore,

$$\tau(x^k D_V) = \tau\left(\frac{1}{\prod_{i=1}^{n} (x - \alpha_i)^{(\mu^{(i)})'_1}} \right) \tau(\bar{D}_V) = \frac{1}{\prod_{i=1}^{n} (T - \alpha_i)^{(\mu^{(i)})'_1}} \bar{S}_V. \tag{3.35}$$

Applying the homomorphism τ to both sides of the relation $\widehat{D} = \check{D}_V x^k D_V$ and using formula (3.35), we get

$$\prod_{a=1}^{k} \prod_{b=0}^{l_a} (-x - z_a - b) = \tau(\check{D}_V) \frac{1}{\prod_{i=1}^{n} (T - \alpha_i)^{(\mu^{(i)})'_1}} \bar{S}_V. \tag{3.36}$$

Recall that $\Delta_a = \{0, \dots, l_a\} \setminus \{n_{ab} \,|\, b = 1, \dots, (\lambda^{(a)})'_1\}$. Then (3.36) gives

$$\frac{1}{\prod_{a=1}^k \prod_{b\in\Delta_a}(-x - z_a - b)}\,\tau(\check{D}_V)\times \frac{1}{\prod_{i=1}^n (T-\alpha_i)^{(\mu^{(i)})'_1}}\,\bar{S}_V\,\frac{1}{\prod_{a=1}^k \prod_{b=1}^{(\lambda^{(a)})'_1}(-x - z_a - n_{ab})} = 1,$$

or

$$S_V^{-1} = \frac{(-1)^L}{p_W(x)}\tau(\check{D}_V), \tag{3.37}$$

where we used $p_W(x) = (-1)^L \prod_{a=1}^k \prod_{b\in\Delta_a}(-x - z_a - b)$.

By the definition of S_W, see formula (3.6), and the relation $S_W = S_V^{-1}$, we have

$$S_V^{-1} = \frac{1}{p_W(x)}\bar{S}_W \frac{1}{\prod_{i=1}^n (x-\alpha_i)^{\mu_1^{(i)}}}, \tag{3.38}$$

Comparing formulae (3.38) and (3.37), we get

$$(-1)^L \tau(\check{D}_V)\prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}} = \bar{S}_W.$$

Therefore,

$$(\bar{S}_W)^{\ddagger} = \prod_{i=1}^n (T-\alpha_i)^{\mu_1^{(i)}} \left((-1)^L \tau(\check{D}_V)\right)^{\ddagger}.$$

Since $\left(\tau(\check{D}_V)\right)^{\ddagger} = \tau(\check{D}_V^{\dagger})$, the last formula implies relation (3.34).

To prove part (2) of Theorem 3.4.13, notice that relation (3.34) and Theorem 3.4.14 imply $\bar{D}_{\tilde{U}} = \tau^{-1}\left(\left(\bar{S}_W\right)^{\ddagger}\right)$. Therefore $\bar{S}_{\tilde{W}} = \left(\tau(\bar{D}_{\tilde{U}})\right)^{\ddagger} = \bar{S}_W$, and $S_{\tilde{W}} = S_W = S_V^{-1}$. ■

3.5 Duality for trigonometric Gaudin and Dynamical Hamiltonians

3.5.1 $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality for trigonometric Gaudin and Dynamical Hamiltonians

Recall from Section 2.3.3 that for any $g \in U(\mathfrak{gl}_k)$, we denote $g_{(i)} = 1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(n-i)} \in U(\mathfrak{gl}_k)^{\otimes n}$, and that we use the same notation for an element of $U(\mathfrak{gl}_k)$ and its image under the diagonal embedding $g \mapsto \sum_{i=1}^n (g)_{(i)} \in U(\mathfrak{gl}_k)^{\otimes n}$. We will use similar notations for $U(\mathfrak{gl}_n)^{\otimes k}$. Let $e^{\langle k\rangle}_{ab}$, $a,b=1,\dots,k$ be the standard generators of the Lie algebra $\mathfrak{gl}_k$. For any $i,j=1,\dots,n$, $i\neq j$, define the following elements of $U(\mathfrak{gl}_k)^{\otimes n}$

$$\Omega^{+}_{(ij)}=\frac{1}{2}\sum_{a=1}^{k}(e^{\langle k\rangle}_{aa})_{(i)}(e^{\langle k\rangle}_{aa})_{(j)}+\sum_{1\leqslant a<b\leqslant k}(e^{\langle k\rangle}_{ab})_{(i)}(e^{\langle k\rangle}_{ba})_{(j)},$$

$$\Omega^{-}_{(ij)}=\frac{1}{2}\sum_{a=1}^{k}(e^{\langle k\rangle}_{aa})_{(i)}(e^{\langle k\rangle}_{aa})_{(j)}+\sum_{1\leqslant a<b\leqslant k}(e^{\langle k\rangle}_{ba})_{(i)}(e^{\langle k\rangle}_{ab})_{(j)}.$$

Fix sequences of pairwise distinct complex numbers $\bar z=(z_1,\dots,z_k)$ and $\bar\alpha=(\alpha_1,\dots,\alpha_n)$. For each $i=1,\dots,n$, define *the trigonometric Gaudin Hamiltonians* $\hat H^{\langle k,n\rangle}_i(\bar\alpha,\bar z)$ by the following formula:

$$\hat H^{\langle k,n\rangle}_i(\bar\alpha,\bar z)=\sum_{a=1}^{k}\Big(z_a-\frac{e^{\langle k\rangle}_{aa}}{2}\Big)(e^{\langle k\rangle}_{aa})_{(i)}+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha_i\Omega^{+}_{(ij)}+\alpha_j\Omega^{-}_{(ij)}}{\alpha_i-\alpha_j}.$$

Let $e^{\langle n\rangle}_{ij}$, $i,j=1,\dots,n$ be the standard generators of the Lie algebra $\mathfrak{gl}_n$. For each $i=1,\dots,n$, define *the trigonometric Dynamical Hamiltonians* $\hat G^{\langle n,k\rangle}_i(\bar z,\bar\alpha)$ by the following formula:

$$\hat G^{\langle n,k\rangle}_i(\bar z,\bar\alpha)=-\frac{(e^{\langle n\rangle}_{ii})^2}{2}+\sum_{a=1}^{k}z_a(e^{\langle n\rangle}_{ii})_{(a)}+$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha_j}{\alpha_i-\alpha_j}\big(e^{\langle n\rangle}_{ij}e^{\langle n\rangle}_{ji}-e^{\langle n\rangle}_{ii}\big)+\sum_{j=1}^{n}\sum_{1\leqslant a<b\leqslant k}(e^{\langle n\rangle}_{ij})_{(a)}(e^{\langle n\rangle}_{ji})_{(b)}.$$

Recall that both $U(\mathfrak{gl}_k)^{\otimes n}$ and $U(\mathfrak{gl}_n)^{\otimes k}$ act on the space $\mathfrak{P}_{kn}$ of polynomials in kn pairwise anticommuting variables ξ_{ai}, $a=1,\dots,k$, $i=1,\dots,n$. The corresponding actions are:

$$\rho^{\langle k,n\rangle}:\ (e^{\langle k\rangle}_{ab})_{(i)}\mapsto\xi_{ai}\partial_{bi},$$

$$\rho^{\langle n,k\rangle}:\ (e^{\langle n\rangle}_{ij})_{(a)}\mapsto\xi_{ai}\partial_{aj},$$

where ∂_{ai}, $a=1,\dots,k$, $i=1,\dots,n$ are the left derivations, see Section 2.4.1.

Denote $-\bar z+1=(-z_1+1,\dots,-z_k+1)$. The following result can be found in [42]:

Proposition 3.5.1 . *For any $i = 1, \dots, n$, we have*

$$\rho^{\langle k,n\rangle}\big(\hat{H}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z})\big) = -\rho^{\langle n,k\rangle}\big(\hat{G}_i^{\langle n,k\rangle}(-\bar{z}+1, \bar{\alpha})\big).$$

Proof The proof is straightforward. ■

3.5.2 Bethe ansatz method for trigonometric Gaudin model

Let $\mathcal{Z}_{kn}$ be the set defined in Section 2.4.3. Fix a pair $(\boldsymbol{l}, \boldsymbol{m}) \in \mathcal{Z}_{kn}$, where $\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{Z}^k_{\geqslant 0}$, $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$. Recall that $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \subset \mathfrak{P}_{kn}$ is the span of all monomials $\xi_{11}^{d_{11}} \dots \xi_{k1}^{d_{k1}} \dots \xi_{1n}^{d_{1n}} \dots \xi_{kn}^{d_{kn}}$ such that $\sum_{a=1}^k d_{ai} = m_i$ and $\sum_{i=1}^n d_{ai} = l_a$. Assume that $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \neq 0$. We also have that

$$\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] = \{p \in \mathfrak{P}_{kn} \,|\, e_{aa}^{\langle k\rangle} p = l_a p,\ e_{ii}^{\langle n\rangle} p = m_i p,\ a = 1, \dots, k,\ i = 1, \dots, n\}.$$

It is easy to check that all trigonometric Gaudin and Dynamical Hamiltonians commute with elements $e_{11}^{\langle k\rangle}, \dots, e_{kk}^{\langle k\rangle}, e_{11}^{\langle n\rangle}, \dots, e_{nn}^{\langle n\rangle}$. Therefore, $\hat{H}_1^{\langle k,n\rangle}(\bar{\alpha}, \bar{z}), \dots, \hat{H}_n^{\langle k,n\rangle}(\bar{\alpha}, \bar{z})$, $\hat{G}_1^{\langle n,k\rangle}(\bar{z}, \bar{\alpha}), \dots, \hat{G}_n^{\langle n,k\rangle}(\bar{z}, \bar{\alpha})$ act on the subspace $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$. We will be interested in the common eigenvectors of the Hamiltonians in the subspace $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$.

Recall that for each $m \in \mathbb{Z}_{\geqslant 0}$, ω_m is a partition given by $\omega_m = (1, \dots, 1, 0, 0 \dots)$ with m ones. Define the sequence $\boldsymbol{l}_0 = (l_1^0, \dots, l_k^0)$ by $l_a^0 = \sum_{i=1}^n (\omega_{m_i})_a$. Then $(\boldsymbol{l}_0, \boldsymbol{m}) \in \mathcal{Z}_{kn}$.

For any sequence of integers $(c_1, \dots, c_k)$ and for each $a = 1, \dots, k-1$, define a transformation

$$r_a : (c_1, \dots, c_k) \mapsto (c_1, \dots, c_a - 1, c_{a+1} + 1, \dots, c_k).$$

Since $\sum_{a=1}^k l_a = \sum_{a=1}^k l_a^0 = \sum_{i=1}^n m_i$, there exist integers $\bar{l}_1, \dots, \bar{l}_{k-1}$ such that $\boldsymbol{l} = r_1^{\bar{l}_1} \dots r_{k-1}^{\bar{l}_{k-1}} \boldsymbol{l}_0$. It is easy to check that if $\bar{l}_a < 0$ for some $a = 1, \dots, k-1$, then $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] = 0$. Therefore, we can assume that $\bar{l}_a \geqslant 0$ for all $a = 1, \dots, k-1$.

Put $\bar{l}_0 = \bar{l}_k = 0$. Then we have

$$l_a = \sum_{i=1}^n (\omega_{m_i})_a + \bar{l}_{a-1} - \bar{l}_a, \quad a = 1, \dots, k.$$

Therefore

$$\bar{l}_a = \sum_{b=a+1}^{k} (l_b - \sum_{i=1}^{n} (\omega_{m_i})_b), \quad a = 0, \ldots, k-1. \tag{3.39}$$

Let $\boldsymbol{t}$ be a set of $\bar{l}_1 + \cdots + \bar{l}_{k-1}$ variables:

$$\boldsymbol{t} = (t_1^{(1)}, \ldots, t_{\bar{l}_1}^{(1)}, t_1^{(2)}, \ldots, t_{\bar{l}_2}^{(2)}, \ldots, t_1^{(k-1)}, \ldots, t_{\bar{l}_{k-1}}^{(k-1)}).$$

Fix sequences of pairwise distinct complex numbers $\bar{z} = (z_1, \ldots, z_k)$ and $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$. In [26], the authors introduced the hypergeometric solutions of the trigonometric Knizhnik-Zamolodchikov (KZ) equations. In the case that we need, this solution involves a certain $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$-valued function $\varphi(\boldsymbol{t}, \bar{\alpha})$ and *the master function*:

$$\Phi(\boldsymbol{t}, \bar{\alpha}, \bar{z}, \boldsymbol{l}, \boldsymbol{m}) = \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^{\min(m_i, m_j)} \prod_{i=1}^{n} \prod_{a=1}^{\bar{l}_{m_i}} (t_a^{(m_i)} - \alpha_i)^{-1} \prod_{i=1}^{n} \alpha_i^{\sum_{a=1}^{m_i} z_a + \frac{m_i}{2}} C(\boldsymbol{t}, \bar{z}), \tag{3.40}$$

where $C(\boldsymbol{t}, \bar{z})$ is a function of $\boldsymbol{t}$ and $\bar{z}$ that does not depend on $\bar{\alpha}$. We will not need the explicit formula for $C(\boldsymbol{t}, \bar{z})$.

The following equations are called the Gaudin Bethe ansatz equations:

$$\left(\Phi^{-1} \frac{\partial \Phi}{\partial t_b^{(a)}} \right) (\boldsymbol{t}, \bar{\alpha}, \bar{z}, \boldsymbol{l}, \boldsymbol{m}) = 0, \quad a = 1, \ldots, k-1, \; b = 1, \ldots, \bar{l}_a. \tag{3.41}$$

We will call a solution $\boldsymbol{t}$ of the Gaudin Bethe ansatz equation (3.41) Gaudin admissible if

$$t_i^{(a)} \neq t_j^{(a)}, \quad t_{i'}^{(b)} \neq t_{j'}^{(b+1)}, \quad t_i^{(a)} \neq \alpha_l, \quad t_i^{(a)} \neq 0 \tag{3.42}$$

for all $a = 1, \ldots, k-1$, $i, j = 1, \ldots, \bar{l}_a$, $i \neq j$, $b = 1, \ldots, k-2$, $i' = 1, \ldots, \bar{l}_b$, $j' = 1, \ldots, \bar{l}_{b+1}$, $l = 1, \ldots, n$.

In [27], the authors considered a certain limit of the rational KZ equations. Similar limit for the trigonometric KZ equation gives:

Theorem 3.5.2 *Let $\boldsymbol{t}$ be a Gaudin admissible solution of the Gaudin Bethe ansatz equations* (3.41). *Suppose that $\varphi(\boldsymbol{t}, \bar{\alpha}) \neq 0$. Then $\varphi(\boldsymbol{t}, \bar{\alpha})$ is a common eigenvector of the Gaudin Hamiltonians, and for each $i = 1, \ldots, n$, the corresponding eigenvalue $h_i^{\langle k,n \rangle}(\boldsymbol{t}, \bar{\alpha}, \bar{z}, \boldsymbol{l}, \boldsymbol{m})$ of $\hat{H}_i^{\langle k,n \rangle}(\bar{\alpha}, \bar{z})$ is given by*

$$h_i^{\langle k,n \rangle}(\boldsymbol{t}, \bar{\alpha}, \bar{z}, \boldsymbol{l}, \boldsymbol{m}) = \left(\alpha_i \frac{\partial}{\partial \alpha_i} \ln \Phi \right) (\boldsymbol{t}, \bar{\alpha}, \bar{z} - \boldsymbol{l}, \boldsymbol{l}, \boldsymbol{m}), \tag{3.43}$$

where $\bar{z} - \boldsymbol{l} = (z_1 - l_1, z_2 - l_2, \ldots, z_k - l_k)$.

3.5.3 Spaces of quasi-polynomials and eigenvectors of trigonometric Gaudin Hamiltonians

Fix a pair $(\boldsymbol{l}, \boldsymbol{m}) \in \mathcal{Z}_{kn}$, where $\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k_{>0}$, $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$. Assume that $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \neq 0$. Define the sequence of partitions $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ by $\lambda^{(a)} = (l_a, 0, 0, \ldots)$, $a = 1, \ldots, k$. Recall that for each $m \in \mathbb{Z}_{\geqslant 0}$, ω_m is a partition given by $\omega_m = (1, \ldots, 1, 0, 0 \ldots)$ with m ones. Define a sequence of partitions $\bar{\mu} = (\omega_{m_1}, \ldots, \omega_{m_n})$.

Let $\bar{z} = (z_1, \ldots, z_k)$ be a sequence of complex numbers such that $z_a - z_b \notin \mathbb{Z}$ for $a \neq b$. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a sequence of pairwise distinct non-zero complex numbers. Let V be a space of quasi-polynomials with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Then V has a basis of the form

$$\{x^{z_1} q_1(x), x^{z_2} q_2(x), \ldots, x^{z_k} q_k(x)\},$$

where $q_1(x), \ldots, q_k(x)$ are polynomials and $\deg q_a(x) = l_a$.

For each $a = 1, \ldots, k-1$, $b = 1, \ldots, k$, define

$$T_b(x) = \prod_{\substack{i=1 \\ m_i \geqslant b}}^{n} (x - \alpha_i),$$

$$y_a(x) = \frac{\mathrm{Wr}(x^{z_k} q_k(x), x^{z_{k-1}} q_{k-1}(x), \ldots, x^{z_{a+1}} q_{a+1}(x))}{\prod_{b=a+1}^{k} (x^{z_b - k + b} T_b(x))}. \tag{3.44}$$

One can check that for each $a = 1, \ldots, k-1$, $y_a(x)$ is a polynomial of degree $\bar{l}_a$. The polynomials $r_1(x), \ldots, r_n(x)$ can be normalized in such a way that the polynomials $y_0(x), \ldots, y_{n-1}(x)$ are monic. Write

$$y_a(x) = \prod_{b=1}^{\bar{l}_a} (x - \tilde{t}^{(a)}_b).$$

We will call the space V Gaudin admissible if the tuple

$$\tilde{\boldsymbol{t}} = (\tilde{t}^{(1)}_1, \ldots, \tilde{t}^{(1)}_{\bar{l}_1}, \tilde{t}^{(2)}_1, \ldots, \tilde{t}^{(2)}_{\bar{l}_2}, \ldots, \tilde{t}^{(k-1)}_1, \ldots, \tilde{t}^{(k-1)}_{\bar{l}_{k-1}})$$

satisfies conditions (3.42).

The following theorem was proved in [18].

Theorem 3.5.3 *Let V be Gaudin admissible. Then $\tilde{\boldsymbol{t}}$ is a Gaudin admisible solution of the Gaudin Bethe ansatz equations* (3.41)*.*

Define functions $\beta_1(x), \ldots, \beta_k(x)$ by the following formula:

$$x^k D_V = \left(x\frac{d}{dx}\right)^k + \sum_{a=1}^{k} \beta_a(x) \left(x\frac{d}{dx}\right)^{k-a}.$$

By Lemma 3.1.2, the functions $\beta_1(x), \ldots, \beta_k(x)$ are rational.

Let $\tilde{\boldsymbol{t}}$ be the Gaudin admissible solution of the Gaudin Bethe ansatz equation corresponding to V, like in Theorem 3.5.3. Suppose that $\varphi(\tilde{\boldsymbol{t}}, \bar{\alpha}) \neq 0$. According to Theorem 3.5.2, $\varphi(\tilde{\boldsymbol{t}}, \bar{\alpha})$ is a common eigenvector of the trigonometric Gaudin Hamiltonians, and for each $i = 1, \ldots, n$, the corresponding eigenvalue of $\hat{H}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})$ is $h_i^{\langle k,n\rangle}(\tilde{\boldsymbol{t}}, \bar{\alpha}, \bar{z} + \boldsymbol{l}, \boldsymbol{l}, \boldsymbol{m})$. We will also call $\varphi(\tilde{\boldsymbol{t}}, \bar{\alpha})$ the Bethe vector v_V corresponding to V.

Proposition 3.5.4 *The following holds*

$$h_i^{\langle k,n\rangle}(\tilde{\boldsymbol{t}}, \bar{\alpha}, \bar{z} + \boldsymbol{l}, \boldsymbol{l}, \boldsymbol{m}) = \frac{1}{\alpha_i} \mathrm{Res}_{x=\alpha_i} \left(\frac{1}{2}\beta_1^2(x) - \beta_2(x)\right) + \frac{m_i^2}{2} - m_i. \tag{3.45}$$

Proof For each function g of x, write $\ln'(g) = (\ln(g))'$, where $(\cdot)'$ is the differentiation with respect to x. Comparing formulae (3.44) and (2.30), we have:

$$\begin{aligned} D_V = &\left(\frac{d}{dx} - \ln'\left(\frac{x^{z_1-k+1}T_1(x)}{y_1(x)}\right)\right)\left(\frac{d}{dx} - \ln'\left(\frac{x^{z_2-k+2}T_2(x)y_1}{y_2(x)}\right)\right)\cdots \\ &\cdots\left(\frac{d}{dx} - \ln'\left(\frac{x^{z_{k-1}-1}T_{k-1}(x)y_{k-2}(x)}{y_{k-1}(x)}\right)\right)\left(\frac{d}{dx} - \ln'\left(x^{z_k}T_k y_{k-1}(x)\right)\right). \end{aligned} \tag{3.46}$$

Multiplying each side of (3.46) by x^k, we get

$$\begin{aligned} x^k D_V = &\left(x\frac{d}{dx} - x\ln'\left(\frac{T_1(x)}{y_1(x)}\right) - z_1\right)\left(x\frac{d}{dx} - x\ln'\left(\frac{T_2(x)y_1}{y_2(x)}\right) - z_2\right)\cdots \\ &\cdots\left(x\frac{d}{dx} - x\ln'\left(T_k y_{k-1}(x)\right) - z_k\right). \end{aligned} \tag{3.47}$$

Put $y_0(x) = y_k(x) = 1$. For each $a = 1, \dots, k$, denote

$$Y_a = -x \ln' \left(\frac{T_a(x) y_{a-1}(x)}{y_a(x)} \right) - z_a.$$

By formula (3.47), we have

$$\beta_2(x) = \sum_{1 \leqslant a < b \leqslant k} Y_a(x) Y_b(x) + \sum_{a=1}^{k} x Y_a'(x), \quad \beta_1(x) = \sum_{a=1}^{k} Y_a(x). \tag{3.48}$$

Since $\tilde{\boldsymbol{t}}$ is Gaudin admissible, for each $i = 1, \dots, n$, $a = 1, \dots, k-1$, α_i is not a root of the polynomial $y_a(x)$. Also, for each $i = 1, \dots, n$, α_i is a root of the polynomial $T_a(x)$ if and only if $a \leqslant m_i$. Using this, we can compute:

$$\begin{aligned} &\frac{1}{\alpha_i} \operatorname{Res}_{x=\alpha_i} \left(\sum_{1 \leqslant a < b \leqslant k} Y_a(x) Y_b(x) \right) = \\ &\qquad = \sum_{b=1}^{\bar{l}_a} \frac{\alpha_i}{\alpha_i - \tilde{t}_b^{(m_i)}} + \sum_{a=1}^{m_i} \sum_{\substack{b=1 \\ b \neq a}}^{k} \left(z_b + \sum_{\substack{j=1 \\ m_j \geqslant b}}^{n} \frac{\alpha_i}{\alpha_i - \alpha_j} \right) + m_i(m_i - 1), \end{aligned} \tag{3.49}$$

$$\frac{1}{\alpha_i} \operatorname{Res}_{x=\alpha_i} \left(\sum_{a=1}^{k} x Y_a'(x) \right) = \frac{m_i(m_i-1)}{2}, \tag{3.50}$$

$$\frac{1}{\alpha_i} \operatorname{Res}_{x=\alpha_i} \left(\frac{1}{2} \left(\sum_{a=1}^{k} Y_a(x) \right)^2 \right) = \sum_{a=1}^{m_i} \sum_{b=1}^{k} \left(z_b + \sum_{\substack{j=1 \\ m_j \geqslant b}}^{n} \frac{\alpha_i}{\alpha_i - \alpha_j} \right) + m_i^2. \tag{3.51}$$

From formulae (3.48) - (3.51), we get

$$\begin{aligned} &\frac{1}{\alpha_i} \operatorname{Res}_{x=\alpha_i} (\frac{1}{2} \beta_1^2(x) - \beta_2(x)) = \\ &\qquad = \sum_{b=1}^{\bar{l}_a} \frac{\alpha_i}{\tilde{t}_b^{(m_i)} - \alpha_i} + \sum_{a=1}^{m_i} z_a + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_i \min(m_i, m_j)}{\alpha_i - \alpha_j} - \frac{m_i^2}{2} + \frac{3}{2} m_i. \end{aligned} \tag{3.52}$$

On the other hand, using formula (3.40), we can compute

$$\begin{aligned} &\left(\alpha_i \frac{\partial}{\partial \alpha_i} \ln \Phi \right) (\tilde{\boldsymbol{t}}, \bar{\alpha}, \bar{z}, \boldsymbol{l}, \boldsymbol{m}) = \\ &\qquad = \sum_{b=1}^{\bar{l}_a} \frac{\alpha_i}{\tilde{t}_b^{(m_i)} - \alpha_i} + \sum_{a=1}^{m_i} z_a + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\alpha_i \min(m_i, m_j)}{\alpha_i - \alpha_j} + \frac{m_i}{2} \end{aligned} \tag{3.53}$$

Comparing formulae (3.52), (3.53), and (3.43), we get relation (3.45). ∎

3.5.4 Bethe ansatz method for XXX spin chain model

Fix a pair $(\boldsymbol{l},\boldsymbol{m}) \in \mathcal{Z}_{kn}$, where $\boldsymbol{l} = (l_1,\dots,l_k) \in \mathbb{Z}^k_{\geqslant 0}$, $\boldsymbol{m} = (m_1,\dots,m_n) \in \mathbb{Z}^n_{\geqslant 0}$. Assume that $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}] \neq 0$. For each $i = 0,\dots,n-1$, define

$$\bar{m}_i = \sum_{j=i+1}^{n} (m_j - \sum_{a=1}^{k} (\omega_{l_a})_j). \tag{3.54}$$

The numbers $\bar{m}_1,\dots,\bar{m}_{n-1}$ are the $(\mathfrak{gl}_k,\mathfrak{gl}_n)$-dual analogs of the numbers $\bar{l}_1,\dots,\bar{l}_{k-1}$, see formula (3.39). Recall that non-triviality of $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ implies $\bar{l}_a \geqslant 0$, $a = 0,\dots,k-1$. Similarly, non-triviality of $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ implies $\bar{m}_i \geqslant 0$, $i = 0,\dots,n-1$.

Let $\boldsymbol{t}$ be a set of $\bar{m}_1 + \dots + \bar{m}_{n-1}$ variables:

$$\boldsymbol{t} = (t_1^{(1)},\dots,t_{\bar{m}_1}^{(1)},t_1^{(2)},\dots,t_{\bar{m}_2}^{(2)},\dots,t_1^{(n-1)},\dots,t_{\bar{m}_{n-1}}^{(n-1)}).$$

Fix sequences of pairwise distinct complex numbers $\bar{z} = (z_1,\dots,z_k)$ and $\bar{\alpha} = (\alpha_1,\dots,\alpha_n)$. We have $\bar{m}_0 = 0$. Also, put $\bar{m}_n = 0$. The XXX Bethe ansatz equations is the following system of $\bar{m}_1 + \dots + \bar{m}_{n-1}$ equations:

$$\frac{\alpha_{i+1}}{\alpha_i} = \prod_{\substack{a=1\\ l_a=i}}^{k} \frac{t_b^{(l_a)} - z_a + 1}{t_b^{(l_a)} - z_a} \prod_{a=1}^{\bar{m}_{i-1}} \frac{t_b^{(i)} - t_a^{(i-1)} + 1}{t_b^{(i)} - t_a^{(i-1)}} \prod_{a=1}^{\bar{m}_{i+1}} \frac{t_b^{(i)} - t_a^{(i+1)}}{t_b^{(i)} - t_a^{(i-1)} - 1} \prod_{\substack{a=1\\ a\neq b}}^{\bar{m}_i} \frac{t_b^{(i)} - t_a^{(i)} - 1}{t_b^{(i)} - t_a^{(i)} + 1}, \tag{3.55}$$

where $i = 1,\dots,n-1$, $b = 1,\dots,\bar{m}_i$.

A solution $\boldsymbol{t}$ of the XXX Bethe ansatz equations (3.55) is called XXX-admissible if $t_a^{(i)} \neq t_b^{(i)}$, $t_{a'}^{(j)} \neq t_{b'}^{(j+1)}$ for any $i = 1,\dots,n-1$, $a,b = 1,\dots,\bar{m}_i$, $a \neq b$, $j = 1,\dots,n-2$, $a' = 1,\dots,\bar{m}_j$, $b' = 1,\dots,\bar{m}_{j+1}$.

For each $i,j = 1,\dots,n$, define

$$\mathcal{X}_i(x,\boldsymbol{t},\bar{z},\bar{\alpha}) = \alpha_i \prod_{\substack{a=1\\ l_a\geqslant i}}^{k} \frac{x - z_a + 1}{x - z_a} \prod_{a=1}^{\bar{m}_{i-1}} \frac{x - t_a^{(i-1)} + 1}{x - t_a^{(i-1)}} \prod_{a=1}^{\bar{m}_i} \frac{x - t_a^{(i)} - 1}{x - t_a^{(i)}}, \tag{3.56}$$

$$\tilde{E}_j(x,\boldsymbol{t},\bar{z},\bar{\alpha}) = \sum_{1\leqslant i_1<\dots<i_j\leqslant n} \mathcal{X}_{i_1}(x)\mathcal{X}_{i_2}(x-1)\dots\mathcal{X}_{i_j}(x-j+1). \tag{3.57}$$

In the last formula $\mathcal{X}_i(x) = \mathcal{X}_i(x,\boldsymbol{t},\bar{z},\bar{\alpha})$, $i = 1,\dots,n$.

Introduce a new variable u. Consider the following polynomial in u:

$$E(u,x,\boldsymbol{t},\bar{z},\bar{\alpha}) = u^n + \sum_{j=1}^{n} \tilde{E}_j(x,\boldsymbol{t},\bar{z},\bar{\alpha})u^{n-i},$$

which is also a rational function of x regular at infinity. Let $E_a(u,\boldsymbol{t},\bar{z},\bar{\alpha})$, $a \in \mathbb{Z}_{\geqslant 0}$ be the coefficients of the Laurent series at infinity of $E(u,x)$ as a function of x:

$$E(u,x,\boldsymbol{t},\bar{z},\bar{\alpha}) = \sum_{a=0}^{\infty} x^{-a}E_a(u,\boldsymbol{t},\bar{z},\bar{\alpha}). \tag{3.58}$$

In [10], a certain function $\psi_i(\boldsymbol{t},\bar{z})$ of $\boldsymbol{t}$ called the *universal weight function for the XXX-type model* was defined. This function takes values in tensor products of highest weight $\mathfrak{gl}_n$-modules. In the case that we need, $\psi_i(\boldsymbol{t},\bar{z})$ is a $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$-valued function. If $\boldsymbol{t}$ is an XXX-admissible solution of the XXX Bethe ansatz equations (3.55), and $\psi_i(\boldsymbol{t},\bar{z}) \neq 0$, then $\psi_i(\boldsymbol{t},\bar{z})$ is a common eigenvector of the higher transfer matrices for the XXX spin chain model. Higher transfer matrices are series in x^{-1}, whose coefficients generate a large commutative subalgebra called the XXX Bethe subalgebra inside the Yangian $Y(\mathfrak{gl}_n)$. The trigonometric Dynamical Hamiltonians can be considered as elements of the XXX Bethe subalgebra, see [10, Appendix B]. In particular, if $\boldsymbol{t}$ is an XXX-admissible solution of the XXX Bethe ansatz equations (3.55), and $\psi_i(\boldsymbol{t},\bar{z}) \neq 0$, then $\psi_i(\boldsymbol{t},\bar{z})$ is a common eigenvector of the Dynamical Hamiltonians, and the corresponding eigenvalue can be computed using [10, Proposition B.1]. We will formulate the result in the following theorem:

Theorem 3.5.5 *Let $\boldsymbol{t}$ be an XXX-admissible solution of the XXX Bethe ansatz equations* (3.55). *Then for each $i = 1,\dots,n$, we have:*

$$\hat{G}_i^{\langle n,k\rangle}(\bar{z},\bar{\alpha})\psi_i(\boldsymbol{t},\bar{z}) = \hat{g}_i^{\langle n,k\rangle}(\boldsymbol{t},\bar{z},\bar{\alpha})\psi_i(\boldsymbol{t},\bar{z}),$$

where

$$\hat{g}_i^{\langle n,k\rangle}(\boldsymbol{t},\bar{z},\bar{\alpha}) = -\frac{1}{\alpha_i}\operatorname{Res}_{u=\alpha_i}\frac{E_2(u,\boldsymbol{t},\bar{z},\bar{\alpha})}{\prod_{j=1}^{n}(u-\alpha_i)} + \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha_j m_i m_j}{\alpha_i-\alpha_j} - \frac{m_i^2}{2}, \tag{3.59}$$

and $E_2(u,\boldsymbol{t},\bar{z},\bar{\alpha})$ is the coefficient in the expansion (3.58).

3.5.5 Spaces of quasi-exponentials and eigenvectors of trigonometric Dynamical Hamiltonians

Fix a pair $(\boldsymbol{l}, \boldsymbol{m}) \in \mathcal{Z}_{kn}$, where $\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k_{>0}$, $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{>0}$. Assume that $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \neq 0$. Let the data $(\bar\lambda, \bar\mu; \bar z, \bar\alpha)$ be like in Section 3.5.3, and let W be a space of quasi-exponentials with the difference data $(\bar\mu', \bar\lambda'; \bar\alpha, -\bar z)$. Then W has a basis of the form

$$\{\alpha_1^x r_1(x), \alpha_2^x r_2(x), \ldots, \alpha_n^x r_n(x)\},$$

where $r_1(x), \ldots, r_n(x)$ are polynomials and $\deg r_i(x) = m_i$.

For each $i = 1, \ldots, n$, define

$$T_i(x) = \prod_{\substack{a=1 \\ l_a \geqslant i}}^{k} (x + z_a + l_a - i). \tag{3.60}$$

The following lemma is a special case of Lemma 3.7 in [19]:

Lemma 3.5.6 *For each $i = 0, \ldots, n-1$, $j_1, \ldots, j_{n-i} \in \{1, \ldots, n\}$, the functions*

$$\frac{\mathcal{W}r(\alpha^x_{j_1} r_{j_1}(x), \alpha^x_{j_2} r_{j_2}(x), \ldots, \alpha^x_{j_{n-i}} r_{j_{n-i}}(x))}{\prod_{l=i+1}^{n} (\alpha^x_{j_{n-l+1}} T_j(x))}$$

are polynomials.

For each $i = 0, \ldots, n-1$, $j = 1, \ldots, n$, define

$$y_i(x) = \frac{\mathcal{W}r(\alpha_n^x r_n(x), \alpha_{n-1}^x r_{n-1}(x), \ldots, \alpha_{i+1}^x r_{i+1}(x))}{\prod_{j=i+1}^{n} (\alpha_j^x T_j(x))}, \tag{3.61}$$

$$\tilde T_j(x) = \prod_{\substack{a=1 \\ l_a = j}}^{k} (x + z_a).$$

According to Lemma 3.5.6, the functions $y_0(x), \ldots, y_{n-1}(x)$ are polynomials.

Lemma 3.5.7 *For each $i = 1, \ldots, n-1$, there exists a polynomial $\tilde y_i$ such that*

$$\mathcal{W}r\left(y_i(x), \frac{\alpha_i^x}{\alpha_{i+1}^x} \tilde y_i(x)\right) = \frac{\alpha_i^x}{\alpha_{i+1}^x} \tilde T_i(x) y_{i-1}(x) y_{i+1}(x+1). \tag{3.62}$$

Proof Set

$$\tilde{y}_i(x) = \alpha_{i+1}\frac{\mathcal{W}r(\alpha_n^x r_n(x), \ldots, \alpha_{i+2}^x r_{i+2}(x), \alpha_i^x r_i(x))}{\alpha_n^x \ldots \alpha_{i+2}^x \alpha_i^x \prod_{j=i+1}^n (T_j(x))}, \quad i = 1, \ldots, n-1.$$

By Lemma 3.62, $\tilde{y}_1(x), \ldots, \tilde{y}_{n-1}(x)$ are polynomials, and (3.62) follows from discrete Wronskian identities (B.1) and (B.4). ■

Denote $u_i(x) = y_i(x + i/2)$, $i = 0, \ldots, n-1$. Then equations (3.62) become

$$\mathcal{W}r\left(u_i(x), \frac{\alpha_i^x}{\alpha_{i+1}^x}\tilde{y}_i(x+i/2)\right) = \frac{\alpha_i^x}{\alpha_{i+1}^x}\tilde{T}_i(x+i/2)u_{i-1}(x+1/2)u_{i+1}(x+1/2), \quad (3.63)$$

where $i = 1, \ldots, n-1$.

It is easy to see that for each $i = 0, \ldots, n-1$, $\deg u_i = \deg y_i = \bar{m}_i$, where $\bar{m}_0, \ldots, \bar{m}_{n-1}$ are given by formula (3.54). In particular, $\deg u_0 = \deg y_0 = 0$. One can normalize polynomials $r_1(x), \ldots, r_n(x)$ so that the polynomials $y_0(x), \ldots, y_{n-1}(x)$ (and hence $u_0(x), \ldots, u_{n-1}(x)$) are monic. For each $i = 1, \ldots, n-1$, write

$$u_i(x) = \prod_{a=1}^{\bar{m}_i}(x - s_a^{(i)}).$$

We will call the space W XXX-admissible if for each $i = 1, \ldots, n-1$, the polynomial $u_i(x)$ has only simple roots, different from the roots of the polynomials $u_{i-1}(x+1/2)$, $u_{i+1}(x+1/2)$, $\tilde{T}_i(x+i/2)$, and $u_i(x+1)$.

The following theorem is a part of the Theorem 7.4 in [18]:

Theorem 3.5.8 *Let W be XXX-admissible, then relations* (3.63) *imply*

$$\frac{\alpha_{i+1}}{\alpha_i} = \prod_{\substack{a=1\\ l_a=i}}^{k}\frac{s_b^{(l_a)} - \check{z}_a + 1/2}{s_b^{(l_a)} - \check{z}_a - 1/2}\prod_{|j-i|=1}\prod_{a=1}^{\bar{m}_j}\frac{s_b^{(i)} - s_a^{(j)} + 1/2}{s_b^{(i)} - s_a^{(j)} - 1/2}\prod_{\substack{a=1\\ a\neq b}}^{\bar{m}_i}\frac{s_b^{(i)} - s_a^{(i)} - 1}{s_b^{(i)} - s_a^{(i)} + 1}, \quad (3.64)$$

where $i = 1, \ldots, n-1$, $b = 1, \ldots, \bar{m}_i$, and $\check{z}_a = -z_a - l_a/2 + 1/2$ for each $a = 1, \ldots, k$.

A tuple of polynomials $u_1(x), \ldots, u_{n-1}(x)$ such that relations (3.63) hold for some polynomials $\tilde{y}_1(x), \ldots, \tilde{y}_{n-1}(x)$ is called a *fertile* tuple in [18].

Let us call the equations (3.55) the XXX Bethe ansatz equations associated to $\bar{z} = (z_1, \dots, z_k)$. For each $i = 1, \dots, n-1$, $a = 1, \dots, \bar{m}_i$, set $t_a^{(i)} = s_a^{(i)} - i/2$. Then, using (3.64), it easy to check that $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{\bar{m}_{n-1}}^{(n-1)})$ is an XXX-admissible solution of the XXX Bethe ansatz equations associated to $-\bar{z} - \bar{l} + \bar{1} = (-z_1 - l_1 + 1, -z_2 - l_2 + 1, \dots, -z_k - l_k + 1)$. Therefore, to each XXX-admissible space of quasi-exponentials W with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z})$, corresponds a vector $v_W = \psi(\boldsymbol{t}, -\bar{z} - \bar{l} + \bar{1}) \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$, which, provided that $v_W \neq 0$, is an eigenvector of the trigonometric Dynamical Hamiltonians $\hat{G}_1^{\langle n,k \rangle}(-\bar{z} - \bar{l} + \bar{1}, \bar{\alpha}), \dots, \hat{G}_n^{\langle n,k \rangle}(-\bar{z} - \bar{l} + \bar{1}, \bar{\alpha})$, and the associated eigenvalues are given by the formula (3.59), where we should substitute $z_a \to -z_a - l_a + 1$, $a = 1, \dots, k$. We will call v_W the Bethe vector corresponding to W.

We are now going to relate the eigenvalues of the trigonometric Dynamical Hamiltonians associated with eigenvector v_W and the coefficients of the fundamental monic difference operator S_W^m of the space W.

Let $y_0(x), \dots, y_{n-1}(x)$, $T_1(x), \dots, T_n(x)$ be the polynomials given by (3.61) and (3.60), respectively. Put $y_n(x) = 1$. Define

$$Y_i = \alpha_i \frac{T_i(x+1) y_{i-1}(x+1) y_i(x)}{T_i(x) y_{i-1}(x) y_i(x+1)}, \quad i = 1, \dots, n. \tag{3.65}$$

Comparing formulae (3.10), (3.11), and (3.65), we get:

$$S_W^m = (T - Y_1(x))(T - Y_2(x)) \dots (T - Y_n(x)).$$

For each $i = 1, \dots, n-1$, write

$$y_i(x) = \prod_{a=1}^{\bar{m}_i} (x - \tilde{t}_a^{(i)}).$$

Then we have

$$Y_i(x) = \alpha_i \prod_{\substack{a=1 \\ l_a \geqslant i}}^{k} \frac{x + z_a + l_a - i + 1}{x + z_a + l_a - i} \prod_{a=1}^{\bar{m}_{i-1}} \frac{x - \tilde{t}_a^{(i-1)} + 1}{x - \tilde{t}_a^{(i-1)}} \prod_{a=1}^{\bar{m}_i} \frac{x - \tilde{t}_a^{(i)} - 1}{x - \tilde{t}_a^{(i)}}, \quad i = 1, \dots, n.$$

Since $y_i(x) = u_i(x - i/2)$, we have $s_a^{(i)} = \tilde{t}_a^{(i)} - i/2$, $i = 1, \dots, n-1$, $a = 1, \dots, \bar{m}_i$. Therefore, for the solution $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{\bar{m}_{n-1}}^{(n-1)})$ of the XXX Bethe ansatz equations corresponding to the space W, we get $t_a^{(i)} = s_a^{(i)} - i/2 = \tilde{t}_a^{(i)} - i$. Denote this solution as $\tilde{\boldsymbol{t}} - \boldsymbol{i}$.

Comparing the last formula for $Y_i(x)$ with the formula (3.56) for $\mathcal{X}_i(x, \boldsymbol{t}, \bar{z}, \bar{\alpha})$, we have

$$\mathcal{X}_i(x, \tilde{\boldsymbol{t}} - \boldsymbol{i}, -\bar{z} - \bar{l} + \bar{1}, \bar{\alpha}) = Y_i(x + i - 1). \tag{3.66}$$

Let $\check{E}_1(x), \ldots, \check{E}_n(x)$ be the coefficients of the fundamental monic difference operator S_W^m of the space W:

$$S_W^m = T^n + \sum_{i=1}^n \check{E}_i(x) T^{n-i}.$$

For each $i = 1, \ldots, n$, we have

$$\check{E}_i(x) = \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} Y_{i_1}(x + i_1 - 1) Y_{i_2}(x + i_2 - 2) \ldots Y_{i_j}(x + i_j - j). \tag{3.67}$$

Comparing formulae (3.57), (3.67), and (3.66), we get $\tilde{E}_i(x, \tilde{\boldsymbol{t}} - \boldsymbol{i}, -\bar{z} - \bar{l} + \bar{1}, \bar{\alpha}) = \check{E}_i(x)$. This, together with Theorem 3.5.5, proves the following:

Proposition 3.5.9 *Let W be an XXX-admissible space of quasi-exponentials W with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z})$. Let v_W be the Bethe vector corresponding to W. Write the fundamental monic difference operator S_W^m of the space W in the following form:*

$$S_W^m = \sum_{a=0}^{\infty} x^{-a} E_a(T),$$

where $E_1(T), E_2(T), \ldots$ are some polynomials in T. Then we have

$$\hat{G}_i^{\langle n,k \rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) v_W = \hat{g}_i^{\langle n,k \rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) v_W,$$

where

$$\hat{g}_i^{\langle n,k \rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) = -\frac{1}{\alpha_i} \operatorname{Res}_{u=\alpha_i} \frac{E_2(u)}{\prod_{j=1}^n (u - \alpha_i)} + \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\alpha_j m_i m_j}{\alpha_i - \alpha_j} - \frac{m_i^2}{2}. \tag{3.68}$$

3.5.6 Quotient difference operator and duality for trigonometric Gaudin and Dynamical Hamiltonians

Fix a pair $(\boldsymbol{l}, \boldsymbol{m}) \in \mathcal{Z}_{kn}$, where $\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{Z}^k_{>0}$, $\boldsymbol{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Assume that $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}] \neq 0$. Let the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$ be like in Section 3.5.3. Let V be a Gaudin admissible space of quasi-polynomials with non-degenerate terms with the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$. Let S_V be the fundamental pseudo-difference operator of V, see Section 3.3.

By Theorem 3.3.2, S_V^{-1} is the fundamental pseudo-difference operator S_W of a space of quasi-exponentials W with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z})$. In this section, we will relate a map $V \mapsto W$ with the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality of the trigonometric Gaudin and Dynamical Hamiltonians.

Let $v_V \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ be the Bethe vector corresponding to V, see Section 3.5.3. Assume that $v_V \neq 0$. Then the vector v_V is an eigenvector of the trigonometric Gaudin Hamiltonians $\hat{H}_1^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}), \dots, \hat{H}_n^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})$. Denote the associated eigenvalues as $\hat{h}_1^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}), \dots, \hat{h}_n^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})$, respectively.

Assume that the space W is XXX-admissible. Let $v_W \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ be the Bethe vector corresponding to W, see Section 3.5.5. Assume that $v_W \neq 0$. Then the vector v_W is an eigenvector of the trigonometric Dynamical Hamiltonians $\hat{G}_1^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$, $\dots, \hat{G}_n^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$. Denote the associated eigenvalues as $\hat{g}_1^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$, $\dots, \hat{g}_n^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$, respectively.

Theorem 3.5.10 *The following holds:*

$$\hat{h}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}) = -\hat{g}_i^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}), \quad i = 1, \dots, n. \tag{3.69}$$

Before proving the theorem, let us discuss how it explains the relation between the quotient difference operator and the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality. By Proposition 3.5.1, for each $i = 1, \dots, n$, we have

$$\hat{H}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}) v_W = -\hat{G}_n^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) v_W = -\hat{g}_n^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) v_W. \tag{3.70}$$

Therefore, starting with the space V and the corresponding vector v_V, we have two different ways to obtain a common eigenvector of the trigonometric Dynamical Hamiltonians. First, by the $(\mathfrak{gl}_k, \mathfrak{gl}_n)$-duality, v_V is itself a common eigenvector of the Dynamical Hamiltonians, see formula (3.70). Second, the map $V \mapsto W$ (or more explicitly, the construction of the quotient difference operator) gives the vector v_W. Theorem 3.5.10 assures that for generic $\bar{z}, \bar{\alpha}$, these two eigenvectors are the same up to a constant.

Indeed, comparing formulae (3.69) and (3.70), we have

$$\hat{H}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})v_W = \hat{h}_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})v_W. \tag{3.71}$$

Similarly to Lemma 2.4.5, one can show that for generic $\bar{z}, \bar{\alpha}$, the common eigenspaces of the operators $\hat{H}_1^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}), \dots, \hat{H}_n^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l})$ in $\mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ are one-dimensional. Then relation (3.71) implies that v_W is proportional to v_V.

Proof [Proof of Theorem 3.5.10] Let $\bar{D}_V$ be the fundamental regularized differential operator of V. Denote $\bar{S}_V = \tau(\bar{D}_V)$, where τ is given by formula (3.4). Let U be the space of quasi-exponentials with the difference data $(\bar{\mu}, \bar{\lambda}; \bar{\alpha}, \bar{z} + \bar{1})$ such that $\bar{S}_U = (\bar{S}_V)^{\ddagger}$, where $\bar{S}_U$ is the fundamental regularized difference operator of U, see Theorem 3.4.12.

The space U has dimension $M = \sum_{i=1}^n m_i$. By Lemma 3.2.1, the fundamental monic difference operator $S_U^m = T^M + \sum_{i=1}^M b_i(x)T^{M-i}$ of U has rational coefficients $b_1(x), \dots, b_M(x)$, which are regular at infinity. Therefore, there exist polynomials $B_0(u), B_1(u), B_2(u), \dots$ such that

$$S_U^m = \sum_{a=0}^{\infty} x^{-a} B_a(T). \tag{3.72}$$

Moreover, Lemma 3.2.1 gives an explicit formula for the polynomial $B_0(x)$:

$$B_0(u) = \prod_{i=1}^{n} (u - \alpha_i)^{m_i}. \tag{3.73}$$

We have $\bar{S}_U = p_U(x)S_U^m$, where $p_U(x) = \prod_{a=1}^k (x - z_a - l_a)$, see Lemma 3.2.2. In particular, the coefficients of the operator $\bar{S}_U$ are polynomials in x of degree at most k.

Define numbers A_{ia}, $i = 1, \dots, M$, $a = 1, \dots, k$ by $\bar{S}_U = \sum_{i=1}^{M} \sum_{a=1}^{k} A_{ia} x^a T^i$. Then we have

$$S_U^m = \frac{1}{\prod_{a=1}^{k}(x - z_a - l_a)} \sum_{i=1}^{M} \sum_{a=1}^{k} A_{ia} x^a T^i. \tag{3.74}$$

Denote $\sum_{a=1}^{k}(z_a + l_a) = Z$. Comparing formulae (3.72) and (3.74), we get

$$\begin{gathered} B_0(u) = \sum_{i=1}^{M} A_{i,k} u^i, \quad B_1(u) = \sum_{i=1}^{M} \left(A_{i,k-1} + Z A_{i,k}\right) u^i, \\ B_2(u) = \sum_{i=1}^{M} \left(A_{i,k-2} + Z A_{i,k-1} + Z^2 A_{i,k}\right) u^i. \end{gathered} \tag{3.75}$$

Since $\bar{S}_V = (\bar{S}_U)^{\ddagger}$, it holds that $\bar{S}_V = \sum_{i=1}^{M} \sum_{a=1}^{k} A_{ia} T^i (-x)^a$. Therefore,

$$\bar{D}_V = \sum_{i=1}^{M} \sum_{a=1}^{k} A_{ia} x^i \left(x \frac{d}{dx}\right)^a \tag{3.76}$$

Let D_V be the fundamental monic differential operator of V. We have $\bar{D}_V = p_V(x)(x^k D_V)$, where $p_V(x) = \prod_{i=1}^{n}(x - \alpha_i)^{m_i}$, see Lemma 3.1.3. Write

$$x^k D_V = \left(x \frac{d}{dx}\right)^k + \sum_{a=1}^{k} \beta_a(x) \left(x \frac{d}{dx}\right)^{k-a}. \tag{3.77}$$

Then formula (3.76) gives:

$$\beta_a = \frac{\sum_{i=1}^{M} A_{i,k-a} x^i}{\prod_{i=1}^{n}(x - \alpha_i)^{m_i}}, \quad a = 1, \dots, k. \tag{3.78}$$

By Proposition 3.5.4, we have

$$h_i^{\langle k,n \rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}) = \frac{1}{\alpha_i} \operatorname{Res}_{x=\alpha_i} \left(\frac{1}{2}\beta_1^2(x) - \beta_2(x)\right) + \frac{m_i^2}{2} - m_i. \tag{3.79}$$

Using formulas (3.75), (3.73), and (3.78), one can check

$$\operatorname{Res}_{x=\alpha_i} \left(\frac{1}{2}\beta_1^2(x) - \beta_2(x)\right) = \operatorname{Res}_{u=\alpha_i} \left(\frac{1}{2} \frac{B_1^2(u)}{B_0^2(u)} - \frac{B_2(u)}{B_0(u)}\right).$$

Therefore, formula (3.79) gives

$$h_i^{\langle k,n \rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}) = \frac{1}{\alpha_i} \operatorname{Res}_{u=\alpha_i} \left(\frac{1}{2} \frac{B_1^2(u)}{B_0^2(u)} - \frac{B_2(u)}{B_0(u)}\right) + \frac{m_i^2}{2} - m_i. \tag{3.80}$$

Let W be the space of quasi-exponentials with the difference data $(\bar{\mu}', \bar{\lambda}'; \bar{\alpha}, -\bar{z})$ such that $S_W = S_V^{-1}$, where S_V and S_W are the fundamental pseudo-difference operators of V and W, respectively. Let S_W^m be the fundamental monic difference operator of W. Similarly to S_U^m, see formula (3.72), S_W^m can be written in the form

$$S_W^m = \sum_{a=0}^{\infty} x^{-a} E_a(T).$$

If S is a difference operator of the form $S = \sum_{i=0}^{l} a_i(x) T^{l-i}$, define a difference operator $S^{\leftarrow}$ by

$$S^{\leftarrow} = \sum_{i=0}^{l} a_i(-x)(T_{-})^{l-i}.$$

In the proof of Theorem 3.5.10, see Section 3.4.6, the difference operator S_W^m was given in terms of the quotient difference operator:

$$S_W^m = Q_{+}^{\rightarrow}(S_U^m).$$

Then relation (3.26) gives

$$\prod_{i=1}^{n} (T - \alpha_i)^{m_i+1} = ((S_W^m)^{\leftarrow})^{\dagger}\, S_U^m.$$

Since $((S_W^m)^{\leftarrow})^{\dagger} = \sum_{a=0}^{\infty} E_a(T)(-x)^{-a}$, we have

$$\prod_{i=1}^{n} (T - \alpha_i)^{m_i+1} = \left(\sum_{a=0}^{\infty} E_a(T)(-x)^{-a} \right) \left(\sum_{a=0}^{\infty} x^{-a} B_a(T) \right).$$

Writing the right hand side of the last formula in the form $\sum_{a=0}^{\infty} x^{-a} P_a(T)$ with some polynomials $P_0(x), P_1(x), P_2(x), \dots$ and comparing it to the left hand side, we see that $P_a(u) = 0$ for all $a \geqslant 1$, and

$$E_0(u) B_0(u) = P_0(u) = \prod_{i=1}^{n} (u - \alpha_i)^{m_i+1}. \tag{3.81}$$

From $P_1(u) = 0$, we get

$$E_0(u) B_1(u) - E_1(u) B_0(u) = 0. \tag{3.82}$$

From $P_2(u) = 0$, we get

$$E_2(u)B_0(u) + E_0(u)B_2(u) + uE_1'(u)B_0(u) - uE_0'(u)B_1(u) - E_1(u)B_1(u) = 0. \quad (3.83)$$

In the last formula we used that for every polynomial $P(u)$, we have

$$P(T)x^{-1} = x^{-1}P(T) - x^{-2}TP'(T) + \sum_{a\geqslant 3} x^{-a}\tilde{P}_a(T)$$

for some polynomials $\tilde{P}_3(u), \tilde{P}_4(u), \ldots$.

Using relations (3.82) and (3.83), one can check

$$\frac{1}{2}\frac{B_1^2(u)}{B_0^2(u)} - \frac{B_2(u)}{B_0(u)} = -\left(\frac{1}{2}\frac{E_1^2(u)}{E_0^2(u)} - \frac{E_2(u)}{E_0(u)}\right) + u\left(\frac{E_1(u)}{E_0(u)}\right)'.$$

Therefore, formula (3.80) gives

$$\begin{aligned} h_i^{\langle k,n\rangle}(\bar{\alpha}, \bar{z} + \boldsymbol{l}) = &-\frac{1}{\alpha_i}\operatorname{Res}_{u=\alpha_i}\left(\frac{1}{2}\frac{E_1^2(u)}{E_0^2(u)} - \frac{E_2(u)}{E_0(u)}\right) + \\ &+ \operatorname{Res}_{u=\alpha_i}\left(u\left(\frac{E_1(u)}{E_0(u)}\right)'\right) + \frac{m_i^2}{2} - m_i. \end{aligned} \quad (3.84)$$

Let $\hat{g}_1^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha}), \ldots, \hat{g}_n^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})$ be the eigenvalues of the trigonometric Dynamical Hamiltonians $\hat{G}_1^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha}), \ldots, \hat{G}_n^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})$, respectively, associated with the Bethe vector v_W. By Proposition 3.5.9, we have

$$\hat{g}_i^{\langle n,k\rangle}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) = -\frac{1}{\alpha_i}\operatorname{Res}_{u=\alpha_i}\frac{E_2(u)}{\prod_{j=1}^n (u - \alpha_i)} + \sum_{\substack{j=1\\ j\neq i}}^n \frac{\alpha_j m_i m_j}{\alpha_i - \alpha_j} - \frac{m_i^2}{2}. \quad (3.85)$$

We will use again [10, Proposition B.1], which gives the following explicit formula for the quotient $E_1(u)/\prod_{i=1}^n (u - \alpha_i)$:

$$\frac{E_1(u)}{\prod_{i=1}^n (u - \alpha_i)} = \sum_{j=1}^n \frac{\alpha_j m_j}{\alpha_j - u}. \quad (3.86)$$

From formula (3.82) (or Lemma 3.2.1), we get

$$E_0(u) = \prod_{i=1}^n (u - \alpha_i). \quad (3.87)$$

Using (3.86) and (3.87), we can rewrite (3.85) in the following way:

$$\hat{g}_i^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1},\bar{\alpha}) = \frac{1}{\alpha_i}\operatorname{Res}_{u=\alpha_i}\left(\frac{1}{2}\frac{E_1^2(u)}{E_0^2(u)} - \frac{E_2(u)}{E_0(u)}\right) - \frac{m_i^2}{2}. \tag{3.88}$$

Using (3.86) and (3.87) again, we compute

$$\frac{1}{\alpha_i}\operatorname{Res}_{u=\alpha_i}\left(u\left(\frac{E_1(u)}{E_0(u)}\right)'\right) = m_i. \tag{3.89}$$

Comparing formulae (3.84), (3.88), and (3.89), we get

$$\hat{h}_i^{\langle k,n\rangle}(\bar{\alpha},\bar{z}+\boldsymbol{l}) = -\hat{g}_i^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1},\bar{\alpha}), \quad i=1,\dots,n.$$

Theorem 3.5.10 is proved. ■

3.5.7 Non-reduced data

In the previous section, we related the quotient difference operator and the $(\mathfrak{gl}_k,\mathfrak{gl}_n)$-duality of the trigonometric Gaudin and Dynamical Hamiltonians acting on the space $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$, where $\boldsymbol{l}=(l_1,\dots,l_k)$ and $\boldsymbol{m}=(m_1,\dots,m_n)$ are such that $l_a\neq 0$, $a=1,\dots,k$ and $m_i\neq 0$, $i=1,\dots,n$. In this section, we are going to extend this result to all nontrivial subspaces $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$, $(\boldsymbol{l},\boldsymbol{m})\in\mathcal{Z}_{kn}$.

Fix $\boldsymbol{l}=(l_1,\dots,l_k)\in\mathbb{Z}_{\geqslant 0}^k$. For each $a=1,\dots,k$, let $q_a(x)$ be a polynomial of degree l_a such that $q_a(0)\neq 0$. Fix complex numbers $z_1,\dots,z_k$ such that $z_a-z_b\notin\mathbb{Z}$ if $a\neq b$. Denote by V the space spanned by the functions $x^{z_a}q_a(x)$, $a=1,\dots,k$.

Define

$$V^{\mathrm{red}} = \prod_{\substack{a=1\\ l_a=0}}^{k}\left(x\frac{d}{dx}-z_a\right)V.$$

Denote $k'=\dim V^{\mathrm{red}}$. Fix $\alpha\in\mathbb{C}^*$. Let $(e_1>\dots>e_k)$ be the sequence of exponents of V at α, and let $(e_1^{\mathrm{red}}>\dots>e_{k'}^{\mathrm{red}})$ be the sequence of exponents of V^{red} at α.

Lemma 3.5.11 *Define a partition $\mu=(\mu_1,\mu_2,\dots)$ by $e_a^{\mathrm{red}}=k'+\mu_a-a$, $a=1,\dots,$ k', $\mu_{k'+1}=0$. Then $e_a=k+\mu_a-a$, $a=1,\dots,k$.*

Conversely, if a partition μ is such that $e_a=k+\mu_a-a$, $a=1,\dots,k$, then $\mu_{k'+1}=0$ and $e_a^{\mathrm{red}}=k'+\mu_a-a$, $a=1,\dots,k'$.

Proof We are going to use the Frobenius method of solving linear differential equations. It is enough to prove the lemma for the case when $l_1 = 0$, and $l_2, \dots, l_k$ are not zero. Let D_V and $D_{V^{\mathrm{red}}}$ be the monic linear differential operators of order k and $k-1$, respectively, annihilating V and V^{red}, respectively. Then

$$x^k D_V = x^{k-1} D_{V^{\mathrm{red}}} \left(x\frac{d}{dx} - z_1 \right). \tag{3.90}$$

Define functions $b_1(x), \dots, b_k(x)$, $b_1^{\mathrm{red}}(x), \dots, b_{k-1}^{\mathrm{red}}(x)$ by

$$x^k D_V = \sum_{a=0}^{k} \frac{b_a(x)}{(x-\alpha)^a} \left(x\frac{d}{dx} \right)^{k-a},$$

$$x^{k-1} D_{V^{\mathrm{red}}} = \sum_{a=0}^{k-1} \frac{b_a^{\mathrm{red}}(x)}{(x-\alpha)^a} \left(x\frac{d}{dx} \right)^{k-1-a}.$$

Using Lemma 2.5.1, one can check that $b_1(x), \dots, b_k(x)$, $b_1^{\mathrm{red}}(x), \dots, b_{k-1}^{\mathrm{red}}(x)$ are regular at α. Define polynomials $I(r)$ and $I^{\mathrm{red}}(r)$ by

$$I(r) = \sum_{a=1}^{k} b_a(\alpha) \alpha^{k-a} r(r-1)(r-2)\dots(r-k+a+1),$$

$$I^{\mathrm{red}}(r) = \sum_{a=1}^{k-1} b_a^{\mathrm{red}}(\alpha) \alpha^{k-1-a} r(r-1)(r-2)\dots(r-k+a+2).$$

Notice that $\{e_1, \dots, e_k\}$ is the set of roots of the polynomial $I(r)$. Indeed, substituting a series $\sum_{i=0}^{\infty} A_i (x-\alpha)^{i+r}$ into the differential equation $D_V f = 0$, and looking at the coefficient for the lowest power of $(x-\alpha)$, we get $I(r) = 0$. Similarly, $\{e_1^{\mathrm{red}}, \dots, e_{k'}^{\mathrm{red}}\}$ is the set of roots of the polynomial $I^{\mathrm{red}}(r)$. The polynomials $I(r)$ and $I^{\mathrm{red}}(r)$ are called the indical polynomials of the differential equations $D_V f = 0$ and $D_{V^{\mathrm{red}}} f = 0$, respectively.

Using formula (3.90), we obtain the following relations:

$$b_a(x) = b_a^{\mathrm{red}}(x) - z_1 (x-\alpha) b_{a-1}^{\mathrm{red}}(x), \quad a = 1, \dots, k, \tag{3.91}$$

where we assume that $b_k^{\mathrm{red}}(x) = 0$. Relations (3.91) imply $b_a(\alpha) = b_a^{\mathrm{red}}(\alpha)$, $a = 1, \dots, k$. Since D_V and $D_{V^{\mathrm{red}}}$ are monic, we also have $b_0(x) = b_0^{\mathrm{red}}(x) = 1$. Therefore, $I(r) = r I^{\mathrm{red}}(r-1)$, which implies the lemma. ∎

Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set including all singular points of V. Assume that $\alpha_i \neq \alpha_j$ if $i \neq j$, and $\alpha_i \neq 0$ for all $i = 1, \ldots, n$. Suppose that for each $i = 1, \ldots, n$, the sequence of exponents of V at α_i is given by

$$(k, k-1, \ldots, k-m_i+1, k-m_i-1, k-m_i-2, \ldots, 1, 0)$$

for some $m_i \in \mathbb{Z}$, $0 \leqslant m_i \leqslant k$.

Define a sequence of partitions $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ by $\lambda^{(a)} = (l_a, 0, 0, \ldots)$, $a = 1, \ldots, k$. Define a sequence of partitions $\bar{\mu} = (\mu^{(1)}, \ldots, \mu^{(n)})$ by $\mu^{(i)} = (1, 1, \ldots, 1, 0, 0, \ldots)$ with m_i ones, $i = 1, \ldots, n$. Define sequences $\bar{\lambda}^{\text{red}}$, $\bar{\mu}^{\text{red}}$, $\bar{z}^{\text{red}}$, and $\bar{\alpha}^{\text{red}}$ by removing all zero partitions from the sequences $\bar{\lambda}$, $\bar{\mu}$, and removing corresponding numbers from the sequences $\bar{z} = (z_1, \ldots, z_n)$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, respectively. We will call the data $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha})$ reduced if $(\bar{\lambda}, \bar{\mu}; \bar{z}, \bar{\alpha}) = (\bar{\lambda}^{\text{red}}, \bar{\mu}^{\text{red}}; \bar{z}^{\text{red}}, \bar{\alpha}^{\text{red}})$, and non-reduced otherwise.

Proposition 3.5.12 *V^{red} is a space of quasi-polynomials with the data $(\bar{\lambda}^{\text{red}}, \bar{\mu}^{\text{red}}; \bar{z}^{\text{red}}, \bar{\alpha}^{\text{red}})$.*

Proof Recall that V is spanned by the functions $x^{z_a} q_a(x)$, $a = 1, \ldots, k$, where $q_1(x), \ldots, q_k(x)$ are polynomials such that $\deg q_a = l_a$, and $q_a(0) \neq 0$, $a = 1, \ldots, k$. Then the space V^{red} is spanned by the functions $x^{z_a} \tilde{q}_a(x)$, $a = 1, \ldots, k$, where

$$\tilde{q}_b(x) = \prod_{\substack{a=1 \\ l_a=0}}^{k} \left(x\frac{d}{dx} + z_b - z_a \right) q_b(x) \tag{3.92}$$

If $l_b \neq 0$, then for each a in the product on the left hand side of formula (3.92), we have $z_b - z_a \notin \mathbb{Z}$, which yields $\deg \tilde{q}_a(x) = \deg q_a(x)$, $a = 1, \ldots, k$. If $l_b = 0$, then formula (3.92) implies $\tilde{q}_b(x) = 0$. This shows that the space V^{red} has a basis

$$\{x^{z_a} \tilde{q}_a(x) \,|\, z_a \text{ is present in } \bar{z}^{\text{red}}\},$$

and the degrees of the polynomials $\tilde{q}_a(x)$ appearing in this basis correspond to the sequence $\bar{\lambda}^{\text{red}}$.

Notice that $\bar{\alpha}^{\mathrm{red}}$ is the set of all singular points of V, and the sequences of exponents of V at these points correspond to the sequence $\bar{\mu}^{\mathrm{red}}$. Therefore, the proposition follows from Lemma 3.5.11. ■

By Proposition 3.5.12 and Theorem 3.3.2, there is a space of quasi-exponentials W^{red} with the difference data $((\bar{\mu}^{\mathrm{red}})', (\bar{\lambda}^{\mathrm{red}})'; \bar{\alpha}^{\mathrm{red}}, -\bar{z}^{\mathrm{red}})$ such that $S_{W^{\mathrm{red}}} = S^{-1}_{V^{\mathrm{red}}}$, where $S_{W^{\mathrm{red}}}$ and $S_{V^{\mathrm{red}}}$ are the fundamental pseudo-difference operators of W^{red} and V^{red}, respectively. We are going to construct a space W such that

$$W^{\mathrm{red}} = \prod_{\substack{i=1\\ m_i=0}}^{n} (T-\alpha_i)W.$$

For this we will need the following lemma:

Lemma 3.5.13 *Fix $\alpha, \beta \in \mathbb{C}^*$, and a polynomial $p(x)$. Assume that $\alpha \neq \beta$. Then there exists a unique polynomial $\tilde{p}(x)$ such that* $\deg \tilde{p}(x) = \deg p(x)$, *and*

$$(T-\beta)\alpha^x \tilde{p}(x) = \alpha^x p(x). \tag{3.93}$$

Proof Relation (3.93) is the same as relation

$$\alpha\tilde{p}(x+1) - \beta\tilde{p}(x) = p(x). \tag{3.94}$$

Let $a_0, \dots, a_m$ be the coefficients of $p(x)$: $p(x) = a_m x^m + a_{m-1}x^{m-1} + \dots + a_1 x + a_0$. Substituting a polynomial $\tilde{p}(x) = \tilde{a}_m x^m + \tilde{a}_{m-1}x^{m-1} + \dots + \tilde{a}_1 x + \tilde{a}_0$ into equation (3.94) and comparing coefficients for powers of x, we get

$$\tilde{a}_{m-i}(\alpha-\beta) = a_{m-i} - \alpha\sum_{j=0}^{i-1} \binom{m-j}{m-i} \tilde{a}_{m-j}, \quad i = 1, \dots, m,$$

which is a recursion that allows to find the numbers $\tilde{a}_1, \dots, \tilde{a}_n$ uniquely. ■

For any $\beta \in \mathbb{C}^*$, define a linear operator $(T-\beta)^{-1}$ on the space spanned by all functions of the form $\alpha^x p(x)$, where $\alpha \in \mathbb{C}^*$, $\alpha \neq \beta$, and $p(x)$ is a polynomial, by the formula

$$(T-\beta)^{-1}\alpha^x p(x) = \alpha^x \tilde{p}(x),$$

where $\tilde{p}(x)$ is the polynomial from Lemma 3.5.13.

Let $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$ be such that $m_i = 0$ if $i = i_s$ for some $s = 1, \ldots, l$, and $m_i \neq 0$ otherwise. Denote by W the space spanned by the functions

$$(T - \alpha_{i_1})^{-1}(T - \alpha_{i_2})^{-1} \ldots (T - \alpha_{i_l})^{-1} f, \quad f \in W^{\mathrm{red}}, \quad \text{and} \quad \alpha_{i_1}^x, \ldots, \alpha_{i_l}^x.$$

Let S_W^m be a unique difference operator of the form $T^n + \sum_{i=1}^n b_i(x) T^{n-i}$ annihilating W. Let $S_{W^{\mathrm{red}}}^m$ be the fundamental monic difference operator of W^{red}. Then we have $W = \ker S_W^m$ and

$$S_W^m = S_{W^{\mathrm{red}}}^m \prod_{\substack{i=1 \\ m_i=0}}^n (T - \alpha_i). \tag{3.95}$$

In particular, this shows that the order of $\alpha_{i_1}, \ldots, \alpha_{i_l}$ in the definition of W does not matter.

Recall that W^{red} is a space of quasi-exponentials with the difference data $((\bar{\mu}^{\mathrm{red}})', (\bar{\lambda}^{\mathrm{red}})'; \bar{\alpha}^{\mathrm{red}}, -\bar{z}^{\mathrm{red}})$. Then the equality $\deg \tilde{p}(x) = \deg p(x)$ in Lemma 3.5.13 implies that the space W has a basis of the form

$$\{\alpha_i^x r_i(x), i = 1, \ldots, n\},$$

where $r_1(x), \ldots, r_n(x)$ are polynomials such that $\deg r_i(x) = m_i$, $i = 1, \ldots, n$.

Fix $z \in \mathbb{C}$. Let $(\tilde{e}_1 > \cdots > \tilde{e}_n)$ be the sequence of discrete exponents of W at z. Denote $n' = n - l = \dim W^{\mathrm{red}}$. Let $(\tilde{e}_1^{\mathrm{red}} > \cdots > \tilde{e}_{n'}^{\mathrm{red}})$ be the sequence of discrete exponents of W^{red} at z.

Lemma 3.5.14 *Define a partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ *by* $\tilde{e}_i^{\mathrm{red}} = n' + \lambda_i - i$, $i = 1, \ldots,$ n', $\lambda_{n'+1} = 0$. *Then* $\tilde{e}_i = n + \lambda_i - i$, $i = 1, \ldots, n$.

Conversely, if a partition λ *is such that* $\tilde{e}_i = n + \lambda_i - i$, $i = 1, \ldots, n$, *then* $\lambda_{n'+1} = 0$ *and* $\tilde{e}_i^{\mathrm{red}} = n' + \lambda_i - i$, $i = 1, \ldots, n'$.

Proof It is enough to prove the Lemma for the case $m_1 = 0$, and $m_2, \ldots, m_n$ are not zero.

Let $f_1(x),\dots,f_{n-1}(x)$ be a basis of W^{red} such that for each $i=1,\dots,n-1$, $T^j f_i(z)=0$, $j=0,\dots,\tilde e_i^{\mathrm{red}}-1$, and $T^{\tilde e_i^{\mathrm{red}}} f_i(z)\neq 0$. Set

$$\tilde f_i(x)=(T-\alpha_1)^{-1}f_i(x)-\alpha_1^{x-z}(T-\alpha_i)^{-1}f_i(z),\quad i=1,\dots,n.$$

Then $\tilde f_i(x)\in W$, $(T-\alpha_1)\tilde f_i(x)=f_i(x)$, and $\tilde f_i(z)=0$, $i=1,\dots,n-1$.

Since $T^j-\alpha_1^j=\left(\sum_{s=0}^{j-1}\alpha_1^{j-1-s}T^s\right)(T-\alpha_1)$, we have

$$T^j\tilde f_i(x)=\alpha_1^j\tilde f_i(x)+\sum_{s=0}^{j-1}\alpha_1^{j-1-s}T^s f_i(x).$$

The last relation implies $T^j\tilde f_i(z)=0$, $j=0,\dots,\tilde e_i^{\mathrm{red}}$, and $T^{\tilde e_i^{\mathrm{red}}+1}\tilde f_i(z)=T^{\tilde e_i^{\mathrm{red}}}f_i(z)\neq 0$.

Since $\{\alpha_1^x,\tilde f_1(x),\dots,\tilde f_{n-1}(x)\}$ is a basis of W, the sequence of discrete exponents of W at z is given by

$$\left(\tilde e_1^{\mathrm{red}}+1>\dots\tilde e_1^{\mathrm{red}}+1>0\right),$$

which implies the lemma. ∎

Notice that for each $a=1,\dots,k$, the sequence of discrete exponents of W^{red} at $-z_a$ is given by

$$(n',n'-1,\dots,n'-l_a+1,n'-l_a-1,\dots,1,0),$$

Therefore, by Lemma 3.5.14, for each $a=1,\dots,k$, the sequence of discrete exponents of W at $-z_a$ is given by

$$(n,n-1,\dots,n-l_a+1,n-l_a-1,\dots,1,0).$$

Consider the space $\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$, where $\boldsymbol{l}=(l_1,\dots,l_k)$ and $\boldsymbol{m}=(m_1,\dots,m_n)$. One can repeat all constructions in Section 3.5.3 for the space V. Assume that V satisfies conditions similar to those for a Gaudin admissible space in Section 3.5.3. Then we obtain a vector $v_V\in\mathfrak{P}_{kn}[\boldsymbol{l},\boldsymbol{m}]$ such that

$$\hat H_i^{\langle k,n\rangle}(\bar\alpha,\bar z+\boldsymbol{l})v_V=\hat h_i(\bar\alpha,\bar z+\boldsymbol{l})v_V,\quad i=1,\dots,n$$

for some numbers $\hat{h}_1(\bar{\alpha}, \bar{z}+\boldsymbol{l}), \dots, \hat{h}_n(\bar{\alpha}, \bar{z}+\boldsymbol{l})$. We will assume that $v_V \neq 0$.

Similarly, one can repeat all constructions in Section 3.5.5 for the space W. Assume that W satisfies conditions similar to those for an XXX-admissible space in Section 3.5.5. Then we obtain a vector $v_W \in \mathfrak{P}_{kn}[\boldsymbol{l}, \boldsymbol{m}]$ such that

$$\hat{G}_i^{\langle n,k\rangle}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})v_W = \hat{g}_i(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})v_W, \quad i=1,\dots,n$$

for some numbers $\hat{g}_1(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha}), \dots, \hat{g}_n(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})$. We will assume that $v_W \neq 0$.

Theorem 3.5.15 *The following holds:*

$$\hat{h}_i(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = -\hat{g}_i(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha}), \quad i=1,\dots,n.$$

Proof Define functions $\beta_0(x), \dots, \beta_k(x)$, $\beta_0^{\mathrm{red}}(x), \dots, \beta_{k'}^{\mathrm{red}}(x)$ by

$$x^k D_V = \sum_{a=0}^{k} \beta_a(x)\left(x\frac{d}{dx}\right)^{k-a}, \quad x^{k'} D_{V^{\mathrm{red}}} = \sum_{a=0}^{k'} \beta_a^{\mathrm{red}}(x)\left(x\frac{d}{dx}\right)^{k'-a}.$$

For each $i=1,\dots,n$, we have

$$\hat{h}_i(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = \frac{1}{\alpha_i}\mathrm{Res}_{x=\alpha_i}\left(\frac{1}{2}\beta_1^2(x) - \beta_2(x)\right) + \frac{m_i^2}{2} - m_i.$$

Define also the following numbers:

$$\hat{h}_i^{\mathrm{red}}(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = \frac{1}{\alpha_i}\mathrm{Res}_{x=\alpha_i}\left(\frac{1}{2}(\beta_1^{\mathrm{red}})^2(x) - \beta_2^{\mathrm{red}}(x)\right) + \frac{m_i^2}{2} - m_i.$$

Suppose that $l_1 = 0$, and $l_2, \dots, l_k$ are not zero. Relation (3.90) implies

$$\beta_1 = \beta_1^{\mathrm{red}} - z_1, \quad \beta_2 = \beta_2^{\mathrm{red}} - z_1\beta_1^{\mathrm{red}}.$$

Using the last two formulas, it is easy to check that

$$\mathrm{Res}_{x=\alpha_i}\left(\frac{1}{2}\beta_1^2(x) - \beta_2(x)\right) = \mathrm{Res}_{x=\alpha_i}\left(\frac{1}{2}(\beta_1^{\mathrm{red}})^2(x) - \beta_2^{\mathrm{red}}(x)\right). \tag{3.96}$$

By induction, formula (3.96) holds for any $l_1, \dots, l_k$. Therefore, we have $\hat{h}_i(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = \hat{h}_i^{\mathrm{red}}(\bar{\alpha}, \bar{z}+\boldsymbol{l})$, $i=1,\dots,n$.

Define polynomials $E_0(u), E_1(u), E_2(u), \dots,\ E_0^{\mathrm{red}}(u), E_1^{\mathrm{red}}(u), E_2^{\mathrm{red}}(u), \dots$ by

$$S_W^m = \sum_{a=0}^{\infty} x^{-a} E_a(T), \quad S_{W^{\mathrm{red}}}^m = \sum_{a=0}^{\infty} x^{-a} E_a^{\mathrm{red}}(T).$$

For each $i = 1, \dots, n$, we have

$$\hat{g}_i(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) = \frac{1}{\alpha_i} \mathrm{Res}_{u=\alpha_i} \left(\frac{1}{2} \frac{E_1^2(u)}{E_0^2(u)} - \frac{E_2(u)}{E_0(u)} \right) - \frac{m_i^2}{2}.$$

Define also the following numbers

$$\hat{g}_i^{\mathrm{red}}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) = \frac{1}{\alpha_i} \mathrm{Res}_{u=\alpha_i} \left(\frac{1}{2} \frac{(E_1^{\mathrm{red}}(u))^2}{(E_0^{\mathrm{red}}(u))^2} - \frac{E_2^{\mathrm{red}}(u)}{E_0^{\mathrm{red}}(u)} \right) - \frac{m_i^2}{2}.$$

Using relation (3.95), we have

$$E_a(u) = E_a^{\mathrm{red}}(u) \prod_{\substack{i=1 \\ m_i=0}}^{n} (u - \alpha_i),$$

which implies $\hat{g}_i(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha}) = \hat{g}_i^{\mathrm{red}}(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$, $i = 1, \dots, n$.

In the proof of Theorem 3.5.10, we already checked that $\hat{h}_i^{\mathrm{red}}(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = -\hat{g}_i^{\mathrm{red}}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha})$ for all i such that $m_i \neq 0$. If $m_i = 0$, then $\hat{h}_i^{\mathrm{red}}(\bar{\alpha}, \bar{z}+\boldsymbol{l}) = \hat{g}_i^{\mathrm{red}}(-\bar{z}-\boldsymbol{l}+\bar{1}, \bar{\alpha}) = 0$. Therefore, we have $\hat{h}_i(\bar{\alpha}, \bar{z} + \boldsymbol{l}) = -\hat{g}_i(-\bar{z} - \boldsymbol{l} + \bar{1}, \bar{\alpha})$, $i = 1, \dots, n$.

Theorem 3.5.15 is proved. ■

4. DUALITY FOR KNIZHNIK-ZAMOLODCHIKOV AND DYNAMICAL EQUATIONS

4.1 KZ, qKZ, DD and qDD equations

Fix a nonzero complex number κ. Consider differential operators $\nabla_1, \dots, \nabla_k$ and $\widehat{\nabla}_1, \dots, \widehat{\nabla}_k$ with coefficients in $U(\mathfrak{gl}_n)^{\otimes k}$ depending on complex variables $\bar{z} = (z_1, \dots, z_k)$ and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$:

$$\nabla_a(\bar{z}, \bar{\alpha}, \kappa) = \kappa \frac{\partial}{\partial z_a} - H_a(\bar{z}, \bar{\alpha}),$$

$$\widehat{\nabla}_a(\bar{z}, \bar{\alpha}, \kappa) = \kappa z_a \frac{\partial}{\partial z_a} - \hat{H}_a(\bar{z}, \bar{\alpha}).$$

Here $H_1(\bar{z}, \bar{\alpha}), \dots, H_k(\bar{z}, \bar{\alpha})$ are the Gaudin Hamiltonians defined in Section 2.3.3, and $\hat{H}_1(\bar{z}, \bar{\alpha}), \dots, \hat{H}_k(\bar{z}, \bar{\alpha})$ are the $\langle n, k\rangle$-analogs of the trigonometric Gaudin Hamiltonians $\hat{H}_1^{\langle k,n\rangle}(\bar{\alpha}, \bar{z}), \dots, \hat{H}_k^{\langle k,n\rangle}(\bar{\alpha}, \bar{z})$ defined in Section 3.5.1

The differential equations $\nabla_a f = 0$ (resp. $\hat{\nabla}_a f = 0$), $a = 1, \dots, k$ are called the *rational* (resp. *trigonometric*) *Knizhnik-Zamolodchikov (KZ) equations.*

Introduce differential operators $D_1, \dots, D_n$ and $\widehat{D}_1, \dots, \widehat{D}_n$ with coefficients in $U(\mathfrak{gl}_n)^{\otimes k}$ depending on complex variables $\bar{z} = (z_1, \dots, z_k)$ and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$:

$$D_i(\bar{z}, \bar{\alpha}, \kappa) = \kappa \frac{\partial}{\partial \alpha_i} - G_i(\bar{z}, \bar{\alpha}),$$

$$\widehat{D}_i(\bar{z}, \bar{\alpha}, \kappa) = \kappa \alpha_i \frac{\partial}{\partial \alpha_i} - \hat{G}_i(\bar{z}, \bar{\alpha}).$$

Here $G_1(\bar{z}, \bar{\alpha}), \dots, G_n(\bar{z}, \bar{\alpha})$ are the Dynamical Hamiltonians defined in Section 2.3.3, and $\hat{G}_1(\bar{z}, \bar{\alpha}), \dots, \hat{G}_n(\bar{z}, \bar{\alpha})$ are the trigonometric Dynamical Hamiltonians $\hat{G}_1^{\langle n,k\rangle}(\bar{z}, \bar{\alpha})$, $\dots, \hat{G}_n^{\langle n,k\rangle}(\bar{z}, \bar{\alpha})$ defined in Section 3.5.1.

The differential equations $D_i f = 0$ (resp. $\widehat{D}_i f = 0$), $i = 1, \dots, n$ are called the *rational* (resp. *trigonometric*) *differential dynamical* (DD) *equations*, see [22, 23].

Let e_{ij}, $i,j=1,\dots,n$ be the standard generators of the Lie algebra $\mathfrak{gl}_n$. For any finite-dimensional irreducible $\mathfrak{gl}_n$-modules L_1 and L_2, there is a distinguished rational function $R_{L_1L_2}(t)$ of t with values in $\mathrm{End}(L_1\otimes L_2)$ called the rational R-matrix. It is uniquely determined by the $\mathfrak{gl}_n$-invariance

$$[R_{L_1L_2}(t), g\otimes 1+1\otimes g]=0 \quad \text{for any} \quad g\in\mathfrak{gl}_n, \tag{4.1}$$

the commutation relations

$$R_{L_1L_2}(t)(te_{ij}\otimes 1+\sum_{l=1}^{n} e_{il}\otimes e_{lj})=(te_{ij}\otimes 1+\sum_{l=1}^{n} e_{lj}\otimes e_{il})R_{L_1L_2}(t), \tag{4.2}$$

and the normalization condition

$$R_{L_1L_2}(t)v\otimes w=v\otimes w, \tag{4.3}$$

where v and w are the highest weight vectors of L_1 and L_2, respectively.

Consider a k-fold tensor product $L_1\otimes\dots\otimes L_k$ of $\mathfrak{gl}_n$-modules. Let $R_{ab}(t)$ be an element of $\mathrm{End}(L_1\otimes\dots\otimes L_k)$ acting as $R_{L_aL_b}(t)$ on factors L_a and L_b, and as identity on all other factors. Denote $z_{ab}=z_a-z_b$. Consider the products $K_1,\dots,K_k$ depending on complex variables $\bar z=(z_1,\dots,z_k)$ and $\bar\alpha=(\alpha_1,\dots,\alpha_n)$:

$$\begin{aligned}K_a(\bar z,\bar\alpha,\kappa)=(R_{ak}(z_{ak})\dots R_{a,a+1}(z_{a,a+1}))^{-1}\times\\ \times\prod_{i=1}^{n}(\alpha_i^{-e_{ii}})_{(a)}R_{1a}(z_{1a}-\kappa)\dots R_{a-1,a}(z_{a-1,a}-\kappa).\end{aligned}$$

Denote by T_u a difference operator acting on a function $f(u)$ by

$$(T_uf)(u)=f(u+\kappa).$$

Introduce difference operators $Z_1,\dots,Z_k$:

$$Z_a(\bar z,\bar\alpha,\kappa)=K_a(\bar z,\bar\alpha,\kappa)T_{z_a}.$$

The difference equations $Z_af=0$ are called (*rational*) *quantized Knizhnik-Zamolodchikov* (qKZ) operators.

For any $i, j = 1, \dots, n$, $i \neq j$, introduce the series $B_{ij}(t)$ depending on a complex variable t:

$$B_{ij}(t) = 1 + \sum_{s=1}^{\infty} \frac{e_{ji}^s e_{ij}^s}{s!} \prod_{b=1}^{s} (t - e_{ii} + e_{jj} - b)^{-1}.$$

The action of this series is well defined on any finite-dimensional $\mathfrak{gl}_n$-module L giving an $\mathrm{End}(L)$-valued rational function of t.

Denote $\alpha_{ij} = \alpha_i - \alpha_j$. Consider the products $X_1, \dots, X_n$ depending on complex variables $\bar{z} = (z_1, \dots, z_k)$, and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$:

$$X_i(\bar{z}, \bar{\alpha}, \kappa) = (B_{in}(\alpha_{in}) \dots B_{i,i+1}(\alpha_{i,i+1}))^{-1} \prod_{a=1}^{k} (z_a^{-e_{ii}})_{(a)} B_{1i}(\alpha_{1i} - \kappa) \dots B_{i-1,i}(\alpha_{i-1,i} - \kappa).$$

The products $X_1, \dots, X_n$ act on any k-fold tensor product $L_1 \otimes \dots \otimes L_k$ of finite-dimensional $\mathfrak{gl}_n$-modules.

Introduce difference operators $Q_1, \dots, Q_n$:

$$Q_i(\bar{z}, \bar{\alpha}, \kappa) = X_i(\bar{z}, \bar{\alpha}, \kappa) T_{\alpha_i}.$$

The difference equations $Q_i f = 0$ are called the (*rational*) *difference dynamical* (qDD) *equations.* [24]

It is known that the introduced operators combine into three commutative families, see [22–24] for more references.

Theorem 4.1.1 *The operators $\nabla_1, \dots, \nabla_k, D_1, \dots, D_n$ pairwise commute.*

Theorem 4.1.2 *The operators $\widehat{\nabla}_1, \dots, \widehat{\nabla}_k, Q_1, \dots, Q_n$ pairwise commute.*

Theorem 4.1.3 *The operators $\widehat{D}_1, \dots, \widehat{D}_n, Z_1, \dots, Z_k$ pairwise commute.*

4.2 ($\mathfrak{gl}_k$,$\mathfrak{gl}_n$)-duality

The operators ∇_a, $\widehat{\nabla}_a$, D_i, $\widehat{D}_i$, Z_a, and Q_i introduced in the previous section are associated with the Lie algebra $\mathfrak{gl}_n$. We will write them now as $\nabla_a^{\langle n \rangle}$, $\widehat{\nabla}_a^{\langle n \rangle}$, $D_i^{\langle n \rangle}$, $\widehat{D}_i^{\langle n \rangle}$, $Z_a^{\langle n \rangle}$, and $Q_i^{\langle n \rangle}$, respectively. Consider also analogous operators $\nabla_i^{\langle k \rangle}$, $\widehat{\nabla}_i^{\langle k \rangle}$, $D_a^{\langle k \rangle}$, $\widehat{D}_a^{\langle k \rangle}$,

$Z_i^{\langle k\rangle}$, and $Q_a^{\langle k\rangle}$ associated with the Lie algebra $\mathfrak{gl}_k$. Using formulas (2.19) and (2.20), consider the action of these operators on $\mathfrak{P}_{kn}$-valued functions of $\bar z = (z_1, \dots, z_k)$ and $\bar\alpha = (\alpha_1, \dots, \alpha_n)$. We will write $F \simeq G$ if the operators F and G act on the $\mathfrak{P}_{kn}$-valued functions in the same way. Let us also write $e_{ij}^{\langle n\rangle}$ for the standard generators of the Lie algebra $\mathfrak{gl}_n$ and $e_{ab}^{\langle k\rangle}$ for the standard generators of the Lie algebra $\mathfrak{gl}_2$.

Denote $-\bar\alpha + 1 = (-\alpha_1 + 1, \dots, -\alpha_n + 1)$, $-\bar z + 1 = (-z_1 + 1, \dots, -z_k + 1)$.

Theorem 4.2.1 *For any $i = 1, ..., n$ and $a = 1, ..., k$, the following relations hold*

$$\nabla_a^{\langle n\rangle}(\bar z, \bar\alpha, \kappa) \simeq D_a^{\langle k\rangle}(\bar\alpha, -\bar z, -\kappa), \tag{4.4}$$

$$\nabla_i^{\langle k\rangle}(\bar\alpha, \bar z, \kappa) \simeq D_i^{\langle n\rangle}(\bar z, -\bar\alpha, -\kappa), \tag{4.5}$$

$$\widehat\nabla_a^{\langle n\rangle}(\bar z, \bar\alpha, \kappa) \simeq -\widehat D_a^{\langle k\rangle}(-\bar\alpha + 1, \bar z, -\kappa), \tag{4.6}$$

$$\widehat\nabla_i^{\langle k\rangle}(\bar\alpha, \bar z, \kappa) \simeq -\widehat D_i^{\langle n\rangle}(-\bar z + 1, \bar\alpha, -\kappa), \tag{4.7}$$

$$Z_a^{\langle n\rangle}(\bar z, \bar\alpha, \kappa) \simeq N_a^{\langle k\rangle}(\bar z) Q_a^{\langle k\rangle}(\bar\alpha, -\bar z, -\kappa), \tag{4.8}$$

$$Z_i^{\langle k\rangle}(\bar\alpha, \bar z, \kappa) \simeq N_i^{\langle n\rangle}(\bar\alpha) Q_i^{\langle n\rangle}(\bar z, -\bar\alpha, -\kappa), \tag{4.9}$$

where

$$N_i^{\langle n\rangle}(\bar\alpha) = \frac{\prod_{1\leqslant j<i} C_{ji}^{\langle n\rangle}(\alpha_{ji} - \kappa)}{\prod_{i<j\leqslant n} C_{ij}^{\langle n\rangle}(\alpha_{ij})}, \quad N_a^{\langle k\rangle}(\bar z) = \frac{\prod_{1\leqslant b<a} C_{ba}^{\langle k\rangle}(z_{ba} - \kappa)}{\prod_{a<b\leqslant k} C_{ab}^{\langle k\rangle}(z_{ab})}, \tag{4.10}$$

and

$$C_{ij}^{\langle n\rangle}(t) = \frac{\Gamma(t + e_{ii}^{\langle n\rangle} + 1)\Gamma(t - e_{jj}^{\langle n\rangle})}{\Gamma(t + e_{ii}^{\langle n\rangle} - e_{jj}^{\langle n\rangle} + 1)\Gamma(t)}, \quad C_{ab}^{\langle k\rangle}(t) = \frac{\Gamma(t + e_{aa}^{\langle k\rangle} + 1)\Gamma(t - e_{bb}^{\langle k\rangle})}{\Gamma(t + e_{aa}^{\langle k\rangle} - e_{bb}^{\langle k\rangle} + 1)\Gamma(t)} \tag{4.11}$$

Proof Relations (4.4) and (4.5) follow from Lemma 2.4.3. Relations (4.6) and (4.7) follow from Proposition 3.5.1. To check (4.8) and (4.9), we have to show that

$$R_{ab}^{\langle n\rangle}(t) \simeq C_{ab}^{\langle k\rangle}(t) B_{ab}^{\langle k\rangle}(-t), \tag{4.12}$$

$$R_{ij}^{\langle k\rangle}(t) \simeq C_{ij}^{\langle n\rangle}(t) B_{ij}^{\langle n\rangle}(-t), \tag{4.13}$$

We will prove relation (4.12). Relation (4.13) can be proved similarly.

Note, that both action of $R^{\langle n\rangle}_{ab}(t)$ on $\mathfrak{P}_{kn}$ and action of $C^{\langle k\rangle}_{ab}(t)B^{\langle k\rangle}_{ab}(-t)$ on $\mathfrak{P}_{kn}$ involve only the variables $\xi_{a1},\dots,\xi_{an}$, $\xi_{b1},\dots,\xi_{bn}$. Therefore, it is sufficient to prove (4.12) for the case of $k=2$, $a=1$, $b=2$.

The $\mathfrak{gl}_n$- module $\mathfrak{P}_{2,n}$ is isomorphic to $\mathfrak{X}_n\otimes\mathfrak{X}_n$. For any $m=0,\dots,n$, let $L^{\langle n\rangle}_m\subset\mathfrak{X}_n$ be the highest-weight $\mathfrak{gl}_n$-module of highest weight $\omega_m=(1,1,\dots,1,0,\dots,0)$, where we have m ones and $n-m$ zeros. Consider the submodule $L^{\langle n\rangle}_{m_1}\otimes L^{\langle n\rangle}_{m_2}\subset\mathfrak{P}_{2,n}$. We have the following decomposition of the $\mathfrak{gl}_n$-module:

$$L^{\langle n\rangle}_{m_1}\otimes L^{\langle n\rangle}_{m_2}=\bigoplus_{m=\max(0,m_1+m_2-n)}^{\min(m_1,m_2)} L^{\langle n\rangle}_{\lambda(m)}. \tag{4.14}$$

Here $\lambda(m)=(2,2,\dots,2,1,\dots,1,0,\dots,0)$, where 2 repeats m times and 1 repeats m_1+m_2-2m times. Denote by v_m the highest weight vector of the summand $L^{\langle n\rangle}_{\lambda(m)}$ given by formula (4.22).

Define the scalar product on $\mathfrak{P}_{2,n}$ by the rule: $\langle f,f\rangle=1$, if $f\in\mathfrak{P}_{2,n}$ is a nonzero monomial, and $\langle f,h\rangle=0$, if $f,h\in\mathfrak{P}_{2,n}$ are two non-proportional monomials.

Lemma 4.2.2 *We have $\langle v_m,v_m\rangle\neq 0$ for every m.*

The proof is straightforward by formula (4.22).

Lemma 4.2.3 $\langle w_1,e^{\langle n\rangle}_{ij}w_2\rangle=\langle e^{\langle n\rangle}_{ji}w_1,w_2\rangle$ *for any $w_1,w_2\in\mathfrak{P}_{2,n}$, and $i,j=1,\dots,n$.*

The proof is straightforward.

Corollary 4.2.4 *If vectors w and $\tilde w$ belong to distinct summands of decomposition* (4.14)*, then $\langle w,\tilde w\rangle=0$.*

Proof The summands of decomposition (4.14) are eigenspaces of the operator $I^{\langle n\rangle}=\sum_{i,j=1}^n e^{\langle n\rangle}_{ij}e^{\langle n\rangle}_{ji}$, and the corresponding eigenvalues are distinct. By Lemma 4.2.3, $I^{\langle n\rangle}$ is symmetric with respect to the scalar product $\langle\cdot,\cdot\rangle$, which implies the statement. ■

Denote

$$S_{ij}(t)=t(e^{\langle n\rangle}_{ij})_{(1)}+\sum_{l=1}^n(e^{\langle n\rangle}_{il})_{(1)}(e^{\langle n\rangle}_{lj})_{(2)},$$

$$P_{ij}(t) = t(e^{\langle n\rangle}_{ij})_{(1)} + \sum_{l=1}^{n} (e^{\langle n\rangle}_{lj})_{(1)} (e^{\langle n\rangle}_{il})_{(2)},$$

$$\alpha_m(t) = \langle S_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1}\rangle, \quad \beta_m(t) = \langle P_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1}\rangle.$$

Lemma 4.2.5 *The functions $\alpha_m(t)$ and $\beta_m(t)$ are nonzero, and*

$$\frac{\alpha_m(t)}{\beta_m(t)} = \frac{t+1+m_1-m}{t-1+m-m_2}. \tag{4.15}$$

The proof is given in Section 4.3.

Due to relation (4.1), for any m, there exists a scalar function $\rho_m(t)$ such that $R^{\langle n\rangle}_{12}(t)w = \rho_m(t)w$ for any $w \in L^{\langle n\rangle}_{\lambda(m)}$.

Lemma 4.2.6 *It holds that*

$$\frac{\rho_m(t)}{\rho_{m-1}(t)} = \frac{\alpha_m(t)}{\beta_m(t)}. \tag{4.16}$$

Proof Let us single out the term $L^{\langle n\rangle}_{\lambda(m-1)}$ in the decomposition (4.14): $L^{\langle n\rangle}_{m_1} \otimes L^{\langle n\rangle}_{m_2} = L^{\langle n\rangle}_{\lambda(m-1)} \bigoplus \tilde{L}$. Then we can write $S_{m_1+m_2-m+1,m}(t) \cdot v_m = w + \tilde{w}$, where $w \in L^{\langle n\rangle}_{\lambda(m-1)}$ and $\tilde{w} \in \tilde{L}$. By the definition of $S_{m_1+m_2-m+1,m}(t)$, the vector w has weight $\lambda(m-1)$. Therefore, $w = av_{m-1}$ for some scalar a. By Corollary 4.2.4, we have

$$\alpha_m(t) = \langle S_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1}\rangle = a\langle v_{m-1}, v_{m-1}\rangle.$$

Notice that $R^{\langle n\rangle}_{12}(t)\tilde{w} \in \tilde{L}$, because R-matrix $R^{\langle n\rangle}_{12}(t)$ acts as a multiplication by a scalar function on each summand of the decomposition (4.14). Then, by Corollary 4.2.4, $\langle R^{\langle n\rangle}_{12}(t)\tilde{w}, v_{m-1}\rangle = 0$, and

$$\begin{aligned}\langle R^{\langle n\rangle}_{12}(t) S_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1}\rangle &= \langle R^{\langle n\rangle}_{12}(t)w, v_{m-1}\rangle = \\ &= \rho_{m-1}(t)a\langle v_{m-1}, v_{m-1}\rangle = \rho_{m-1}(t)\alpha_m(t).\end{aligned}$$

On the other hand, relation (4.2) gives

$$\begin{aligned}\langle R^{\langle n\rangle}_{12}(t) S_{m_1+m_2-m+1,m}(t) \cdot v_m, v_{m-1}\rangle &= \langle P_{m_1+m_2-m+1,m}(t) R^{\langle n\rangle}_{12}(t) \cdot v_m, v_{m-1}\rangle \\ &= \rho_m(t)\beta_m(t).\end{aligned}$$

Thus we get $\alpha_m(t)\rho_{m-1}(t) = \rho_m(t)\beta_m(t)$, which is relation (4.16). ■

By formulae (4.16), (4.15),

$$\rho_m(t) = \prod_{s=1}^{m} \frac{\rho_s(t)}{\rho_{s-1}(t)} = \prod_{s=0}^{m-1} \frac{t+m_1-s}{t-m_2+s}, \tag{4.17}$$

where we used that $\rho_0 = 1$ by the normalization condition (4.3).

Let $L_0^{\langle 2\rangle}, L_1^{\langle 2\rangle}$, and $L_2^{\langle 2\rangle} \subset \mathfrak{X}_2$ be the irreducible highest-weight $\mathfrak{gl}_2$-modules of highest weight $(0,0)$, $(1,0)$, and $(1,1)$, respectively. For each $i = 1,\dots,n$, let s_i be such that $s_i = 0, 1$, or 2, and $\sum_{i=1}^n s_i = m_1 + m_2$. Consider a decomposition of the $\mathfrak{gl}_2$-module:

$$L_{s_1}^{\langle 2\rangle} \otimes \cdots \otimes L_{s_n}^{\langle 2\rangle} = \bigoplus_{0 \leqslant m \leqslant (m_1+m_2)/2} L_{(m_1+m_2-m,m)}^{\langle 2\rangle} \otimes W_m^{\langle 2\rangle},$$

where $W_m^{\langle 2\rangle}$ are multiplicity spaces.

Let $(L_{(m_1+m_2-m,m)}^{\langle 2\rangle} \otimes W_m^{\langle 2\rangle})_{(m_1,m_2)} \subset L_{(m_1+m_2-m,m)}^{\langle 2\rangle} \otimes W_m^{\langle 2\rangle}$ be the subspace of weight (m_1, m_2)

Lemma 4.2.7 *It holds that*

$$B_{12}^{\langle 2\rangle}(t)|_{(L_{(m_1+m_2-m,m)}^{\langle 2\rangle} \otimes W_m^{\langle 2\rangle})_{(m_1,m_2)}} = \prod_{s=m}^{m_2-1} \frac{t+m_2-s}{t-m_1+s}, \tag{4.18}$$

Proof The modules $L_0^{\langle 2\rangle}$ and $L_2^{\langle 2\rangle}$ are one-dimensional, and the elements $e_{12}^{\langle 2\rangle}$, $e_{21}^{\langle 2\rangle}$, and $e_{11}^{\langle 2\rangle} - e_{22}^{\langle 2\rangle}$ act there by zero. Thus, it is enough to consider the case when $s_i = 1$ for all $i = 1,\dots,n$, so formula (4.18) follows from [25, formula (5.13)]. ■

Comparing formulas (4.17), (4.18), and (4.11), we conclude:

$$\rho_m(t) = (B_{12}^{\langle 2\rangle}(-t) C_{12}^{\langle 2\rangle}(t))|_{(L_{(m_1+m_2-m,m)}^{\langle 2\rangle} \otimes W_m^{\langle 2\rangle})_{(m_1,m_2)}}. \tag{4.19}$$

Recall that we write $F \backsimeq G$ if the operators F and G act on $\mathfrak{P}_{kn}$ in the same way.

Lemma 4.2.8 *For the Casimir elements* $I^{\langle 2\rangle} = \sum_{a,b=1}^2 e_{ab}^{\langle 2\rangle} e_{ba}^{\langle 2\rangle}$ *and* $I^{\langle n\rangle} = \sum_{i,j=1}^n e_{ij}^{\langle n\rangle} e_{ji}^{\langle n\rangle}$, *we have*

$$I^{\langle 2\rangle} - 2\sum_{a=1}^{2} e_{aa}^{\langle 2\rangle} \backsimeq -I^{\langle n\rangle} + n\sum_{i=1}^{n} e_{ii}^{\langle n\rangle}. \tag{4.20}$$

The proof is straightforward.

Let $L_\lambda^{\langle n\rangle}$ be the irreducible $\mathfrak{gl}_n$-module with the highest weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. It is easy to check that the element $I^{\langle n\rangle}$ acts on $L_\lambda^{\langle n\rangle}$ as a multiplication by $(\lambda, \lambda+\rho)$, where $(\cdot,\cdot)$ is the dot product, and $\rho = (n-1, n-3, \dots, 1-n)$. The similar statement is true for the Lie algebra $\mathfrak{gl}_2$. Using this, one can verify that

$$(I^{\langle 2\rangle} - 2\sum_{a=1}^{2} e_{aa}^{\langle 2\rangle})|_{L^{\langle 2\rangle}_{(m_1+m_2-m,m)}\otimes W^{\langle 2\rangle}_m} = (-I^{\langle n\rangle} + n\sum_{i=1}^{n} e_{ii}^{\langle n\rangle})|_{L^{\langle n\rangle}_{\lambda(m')}} \tag{4.21}$$

if and only if $m = m'$.

Comparing formulae (4.20) and (4.21), we get that under isomorphisms $\psi_1 : (\mathfrak{X}_n)^{\otimes 2} \to \mathfrak{P}_{2,n}$ and $\psi_2 : (\mathfrak{X}_2)^{\otimes n} \to \mathfrak{P}_{2,n}$ defined in formulas (2.12) and (2.13), the respective images of $L^{\langle n\rangle}_{\lambda(m)}$ and $L^{\langle 2\rangle}_{(m_1+m_2-m,m)} \otimes W^{\langle 2\rangle}_m$ in $\mathfrak{P}_{2,n}$ coincide. To indicate that, we will write $L^{\langle n\rangle}_{\lambda(m)} \backsimeq L^{\langle 2\rangle}_{(m_1+m_2-m,m)} \otimes W^{\langle 2\rangle}_m$.

Recall that $(M)_\lambda$ denotes the weight subspace of a module M with the weight $\lambda \in \mathbb{C}^n$. We have $(L^{\langle n\rangle}_{m_1} \otimes L^{\langle n\rangle}_{m_2})_{(s_1,\dots,s_n)} \backsimeq (L^{\langle 2\rangle}_{s_1} \otimes \cdots \otimes L^{\langle 2\rangle}_{s_n})_{(m_1,m_2)}$. Therefore, $(L^{\langle n\rangle}_{\lambda(m)})_{(s_1,\dots,s_n)} \backsimeq (L^{\langle 2\rangle}_{(m_1+m_2-m,m)} \otimes W^{\langle 2\rangle}_m)_{(m_1,m_2)}$. Now we see that (4.19) gives us a relation between actions of operators $B^{\langle 2\rangle}_{12}(t)$, $C^{\langle 2\rangle}_{12}(t)$, and $R^{\langle n\rangle}_{12}(t)$ on one particular submodule of $\mathfrak{P}_{2,n}$ proving the theorem. ■

4.3 Proof of Lemma 4.2.5

Let

$$v_m = \prod_{i=1}^{m} \xi_{1i}\xi_{2i} \sum_{\{\varepsilon\}} \xi_{\varepsilon_1,m+1}\xi_{\varepsilon_2,m+2}\dots\xi_{\varepsilon_{m_1+m_2-2m},m_1+m_2-m} \tag{4.22}$$

with $\{\varepsilon\} = \left\{(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{m_1+m_2-2m}) : \varepsilon_i = 1 \text{ or } 2, \sum_i \varepsilon_i = m_1 - m + 2(m_2-m)\right\}$. One can easily prove that v_m is a highest weight vector of weight $\lambda(m)$.

It follows from the construction of the scalar product $\langle\cdot,\cdot\rangle$ that $\alpha_m(t)$ (resp. $\beta_m(t)$) equals the sum of the coefficients of those monomials presented in $S_{m_1+m_2-m+1,m}(t)\cdot v_m$ (resp. $P_{m_1+m_2-m+1,m}(t)\cdot v_m$) that also appear in v_{m-1}. In fact, all monomials either in $S_{m_1+m_2-m+1,m}(t)\cdot v_m$ or in $P_{m_1+m_2-m+1,m}(t)\cdot v_m$ appear in v_{m-1} as well.

We will start with α_m. Denote $r = m_1 + m_2 - m + 1$. We will write C_a^b for the binomial coefficient $\binom{b}{a}$. Let us inspect what happens when we apply various terms of $L_{rm}(t)$ to v_m. For the sum $\sum_{l=1}^{n}(e_{rl})_{(1)}(e_{lm})_{(2)}$, we can assume that $m \leqslant l < r$. If $l > m$, then the operator $(e_{rk})_{(1)}(e_{km})_{(2)}$ will send a monomial in v_m to zero if and only if this monomial does not depend on ξ_{1l}. That is, we look at all terms in v_m corresponding to $\varepsilon_{l-m} = 1$. There are $C^{m_2-m}_{m_1+m_2-2m-1}$ such terms with the same contribution $(-1)^{m_1+m_2+m}$. We leave the details of this calculation to a reader. Under the assumption $m < l < r$, there are $m_1 + m_2 - 2m$ different values of l, which yield the overall contribution $(-1)^{m_1+m_2+m}(m_1 + m_2 - 2m)C^{m_2-m}_{m_1+m_2-2m-1}$ to $\alpha_m(t)$.

If $l = m$, then we have $(e_{rl})_{(1)}(e_{lm})_{(2)} \cdot v_m = (e_{rm})_{(1)} \cdot v_m$. Therefore, all $C^{m_1-m}_{m_1+m_2-2m}$ terms in v_m equally contribute $(-1)^{m_1+m_2+m}(m_1 + m_2 - 2m)$.

Finally, the term $t(e_{ij})_{(1)}$ in $L_{rm}(t)$ generates the contribution $t(-1)^{m_1+m_2+m}C^{m_1-m}_{m_1+m_2-2m}$ to $\alpha_m(t)$, which can be seen similarly to the case $l = m$ considered above.

Thus we obtained

$$(-1)^{m_1+m_2+m}\alpha_m = (t+1)\, C^{m_1-m}_{m_1+m_2-2m} + (m_1 + m_2 - 2m)C^{m_2-m}_{m_1+m_2-2m-1}. \tag{4.23}$$

The similar arguments give us

$$(-1)^{m_1+m_2+m}\beta_m = (t-1)\, C^{m_1-m}_{m_1+m_2-2m} - (m_1 + m_2 - 2m)C^{m_1-m}_{m_1+m_2-2m-1}. \tag{4.24}$$

Since

$$(m_1 + m_2 - 2m)C^{m_1-m}_{m_1+m_2-2m-1} = (m_1 - m)C^{m_1-m}_{m_1+m_2-2m}$$

and

$$(m_1 + m_2 - 2m)C^{m_2-m}_{m_1+m_2-2m-1} = (m_2 - m)C^{m_1-m}_{m_1+m_2-2m},$$

Lemma 4.2.5 is proved.

APPENDICES

APPENDIX A. WRONSKIAN IDENTITIES

In this section, we will collect Wronskian identities that were used in the dissertation. Identities (A.1) - (A.4) with proofs can also be found in [44] and in the Appendix of [12].

Recall that for any functions $f_1, \dots, f_n$ with sufficiently many derivatives, the Wronskian $\mathrm{Wr}(f_1, \dots, f_n)$ is the determinant of the matrix $(f_i^{(j-1)})_{i,j=1}^n$. Throughout this section, we will assume that all functions are such that the corresponding Wronskians are well-defined. Using elementary column operations, it is straightforward to check that

$$\mathrm{Wr}(hf_1, \dots, hf_n) = h^n \,\mathrm{Wr}(f_1, \dots, f_n) \quad \text{for any } h \neq 0, \tag{A.1}$$

$$\mathrm{Wr}(1, f_1, \dots, f_n) = \mathrm{Wr}(f_1', \dots, f_n'). \tag{A.2}$$

Combining formulae (A.1) and (A.2), we get

$$\mathrm{Wr}(f_1, \dots, f_n) = f_1^n \,\mathrm{Wr}\left(\left(\frac{f_2}{f_1}\right)', \dots, \left(\frac{f_n}{f_1}\right)'\right). \tag{A.3}$$

Proposition A.1 *For any functions $f_1, \dots, f_n$, $h_1, \dots, h_m$, where $f_1 \neq 0$, the following holds:*

$$\begin{aligned} &\mathrm{Wr}(\mathrm{Wr}(f_1, \dots, f_n, h_1), \dots, \mathrm{Wr}(f_1, \dots, f_n, h_m)) = \\ &\quad = (\mathrm{Wr}(f_1, \dots, f_n))^{m-1} \,\mathrm{Wr}(f_1, \dots, f_n, h_1, \dots, h_m). \end{aligned} \tag{A.4}$$

Proof We will prove the proposition by induction on n.

Let $n = 1$. Denote $f_1 = f$. Using formula (A.3), we compute

$$\mathrm{Wr}(f, h_i) = f^2 \,\mathrm{Wr}\left(\left(\frac{h_i}{f}\right)'\right) = f^2 \left(\frac{h_i}{f}\right)', \quad i = 1, \dots, m.$$

Therefore,

$$\begin{aligned} \mathrm{Wr}(\mathrm{Wr}(f, h_1), \dots, \mathrm{Wr}(f, h_m)) &= f^{2m} \,\mathrm{Wr}\left(\left(\frac{h_1}{f}\right)', \dots, \left(\frac{h_m}{f}\right)'\right) = \\ &= f^{m-1} \,\mathrm{Wr}(h_1, \dots, h_m). \end{aligned}$$

Assume that formula (A.4) is true for some $n \geqslant 1$. For functions $f_1, \dots, f_{n+1}, h_1, \dots, h_m$, where $f_1 \neq 0$, define $\tilde{f}_i = (f_i/f_1)'$, $\tilde{h}_j = (h_j/f_1)'$, $i = 2, \dots, n+1$, $j = 1, \dots, m$. Then we compute

$$\begin{aligned} &\mathrm{Wr}(\mathrm{Wr}(f_1, \dots, f_{n+1}, h_1), \dots, \mathrm{Wr}(f_1, \dots, f_{n+1}, h_m)) = \\ &= f_1^{m+n+2}\,\mathrm{Wr}(\mathrm{Wr}(\tilde{f}_2, \dots, \tilde{f}_{n+1}, \tilde{h}_1), \dots, \mathrm{Wr}(\tilde{f}_2, \dots, \tilde{f}_{n+1}, \tilde{h}_m)) = \\ &= f_1^{m+n+2}(\mathrm{Wr}(\tilde{f}_2, \dots, \tilde{f}_{n+1}))^{m-1}\,\mathrm{Wr}(\tilde{f}_2, \dots, \tilde{f}_{n+1}, \tilde{h}_1, \dots, \tilde{h}_m) = \\ &= (\mathrm{Wr}(f_1, \dots, f_{n+1}))^{m-1}\,\mathrm{Wr}(f_1, \dots, f_{n+1}, h_1, \dots, h_m). \end{aligned} \tag{A.5}$$

Here, on the first step, we used formulas (A.1) and (A.3), on the second step, we used the assumption hypothesis, and on the third step, we used formula (A.3) again.

Computation (A.5) proves the induction step finishing the proof of the proposition.

■

Let $f_1, f_2, \dots, f_n$ be solutions of the differential equation $Df = 0$, where $D = (d/dx)^n + \sum_{i=1}^n a_i (d/dx)^{n-i}$. Assume that $f_1, f_2, \dots, f_n$ are linearly independent. Define

$$h_i \;=\; \frac{\mathrm{Wr}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)}{\mathrm{Wr}(f_1, \dots, f_n)}\,.$$

Proposition A.2 *The following holds:*

$$\mathrm{Wr}(h_1, \dots, h_n) \;=\; \frac{(-1)^{n(n-1)/2}}{\mathrm{Wr}(f_1, \dots, f_n)}\,.$$

Proof Let $p_i = \mathrm{Wr}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$. Denote by b_{ij} the ij-minor of the matrix $A = (f_i^{(j-1)})_{i.j=1}^n$. Then we have $p_i = b_{in}$ and $p_i' = b_{i,n-1}$.

Since $Df_i = 0$ for any $i = 1, \dots, n$, we have $f_i^{(n)} = -\sum_{l=1}^n a_l f_i^{(n-l)}$, where the functions $a_1, \dots, a_n$ do not depend on i. Using this observation, one can check that

$$b'_{i,n-j} = b_{i,n-j-1} + (-1)^j a_{j+1} b_{in} - a_1 b_{i,n-j}.$$

Therefore, by induction on j, we have

$$p_i^{(j)} = b_{i,n-j} + \sum_{k=0}^{j-1} C_{jk} b_{i,n-k}\,, \qquad i = 1, \dots, n\,,$$

for certain functions C_{jk} that do not depend on i. Hence,

$$\mathrm{Wr}(p_1,\ldots,p_n) = \det(p_i^{(j)})_{\substack{i=1,\ldots,n\\ j=0,\ldots,n-1}} = \det(b_{i,n-j})_{\substack{i=1,\ldots,n\\ j=0,\ldots,n-1}}$$

and

$$\begin{aligned}
\mathrm{Wr}(h_1,\ldots,h_n) &= \mathrm{Wr}\Big(\frac{p_1}{\mathrm{Wr}(f_1,\ldots,f_n)},\ldots,\frac{p_n}{\mathrm{Wr}(f_1,\ldots,f_n)}\Big) = \frac{\mathrm{Wr}(p_1,\ldots,p_n)}{(\mathrm{Wr}(f_1,\ldots,f_n))^n}\\
&= \frac{\det(b_{i,n-j})}{(\mathrm{Wr}(f_1,\ldots,f_n))^n} = (-1)^{n(n-1)/2}\,\frac{\det((-1)^{i+j}\,b_{i,j})}{(\mathrm{Wr}(f_1,\ldots,f_n))^n}\\
&= (-1)^{n(n-1)/2}\,\frac{\det(A^{-1}\det A)}{(\det A)^n} = \frac{(-1)^{n(n-1)/2}}{\mathrm{Wr}(f_1,\ldots,f_n)}\ .
\end{aligned}$$

$\blacksquare$

APPENDIX B. DISCRETE WRONSKIAN IDENTITIES

In this section, we will collect discrete Wronskian identities that were used in the dissertation. Identities (B.1) - (B.4) with proofs can also be found in Appendix B of [45].

Recall that T is the shift operator defined by $Tf(x) = f(x+1)$. Recall that for any functions $f_1, \dots, f_n$, the discrete Wronskian $\mathcal{W}r(f_1, \dots, f_n)$ is the determinant of the matrix $(T^{j-1}f_i)_{i,j=1}^n$. Denote $T^{(n)}f = f(Tf)(T^2f)\dots(T^{n-1}f)$. We have the following obvious relations:

$$\mathcal{W}r(hf_1, \dots, hf_n) = (T^{(n)}h)\mathcal{W}r(f_1, \dots, f_n) \quad \text{for any } h, \tag{B.1}$$

$$\mathcal{W}r(1, f_1, \dots, f_n) = \mathcal{W}r((T-1)f_1, \dots, (T-1)f_n). \tag{B.2}$$

Assume that $f_1 \neq 0$. Combining formulae (B.1) and (B.2), we get

$$\mathcal{W}r(f_1, f_2, \dots, f_n) = (T^{(n)}f_1)\mathcal{W}r\left((T-1)\frac{f_2}{f_1}, \dots, (T-1)\frac{f_n}{f_1}\right). \tag{B.3}$$

Proposition B.1 *For any functions $f_1, \dots, f_n$, $h_1, \dots, h_m$, where $f_1 \neq 0$, the following holds:*

$$\begin{aligned} &\mathcal{W}r(\mathcal{W}r(f_1, \dots, f_n, h_1), \dots, \mathcal{W}r(f_1, \dots, f_n, h_m)) = \\ &\qquad = (T^{(m-1)}\mathcal{W}r(Tf_1, \dots, Tf_n))\mathcal{W}r(f_1, \dots, f_n, h_1, \dots, h_m). \end{aligned} \tag{B.4}$$

Proof We will prove the proposition by induction on n.

Let $n = 1$. Denote $f_1 = f$. Using formula (B.3), we compute

$$\mathcal{W}r(f, h_i) = (T^{(2)}f)\mathcal{W}r\left((T-1)\frac{h_i}{f}\right) = (T^{(2)}f)\,(T-1)\frac{h_i}{f}, \quad i = 1, \dots, m.$$

Therefore,

$$\mathcal{W}r(\mathcal{W}r(f,h_1),\ldots,\mathcal{W}r(f,h_m)) = (T^{(m)}T^{(2)}f)\mathcal{W}r\left((T-1)\frac{h_1}{f},\ldots,(T-1)\frac{h_m}{f}\right) =$$
$$= (T^{(m-1)}Tf)(T^{(m+1)}f)\mathcal{W}r\left((T-1)\frac{h_1}{f},\ldots,(T-1)\frac{h_m}{f}\right) =$$
$$= (T^{(m-1)}Tf)\mathcal{W}r(h_1,\ldots,h_m).$$

Assume that formula (B.4) is true for some $n \geqslant 1$. For functions $f_1,\ldots,f_{n+1}$, $h_1,\ldots,$ h_m, define $\tilde{f}_i = (T-1)(f_i/f_1)$, $\tilde{h}_j = (T-1)(h_j/f_1)$, $i = 2,\ldots,n+1$, $j = 1,\ldots,m$. Then we compute

$$\begin{aligned}
&\mathcal{W}r(\mathcal{W}r(f_1,\ldots,f_{n+1},h_1),\ldots,\mathcal{W}r(f_1,\ldots,f_{n+1},h_m)) = \\
&= (T^{(m)}T^{(n+2)}f_1)\mathcal{W}r(\mathcal{W}r(\tilde{f}_2,\ldots,\tilde{f}_{n+1},\tilde{h}_1),\ldots,\mathcal{W}r(\tilde{f}_2,\ldots,\tilde{f}_{n+1},\tilde{h}_m)) = \\
&= (T^{(m)}T^{(n+2)}f_1)(T^{(m-1)}\mathcal{W}r(T\tilde{f}_2,\ldots,T\tilde{f}_{n+1}))\mathcal{W}r(\tilde{f}_2,\ldots,\tilde{f}_{n+1},\tilde{h}_1,\ldots,\tilde{h}_m) = \\
&= \left(T^{(m-1)}\left[(T^{(n+1)}Tf_1)\mathcal{W}r(T\tilde{f}_2,\ldots,T\tilde{f}_{n+1})\right]\right)\times \\
&\qquad\qquad \times (T^{(n+m+1)}f_1)\mathcal{W}r(\tilde{f}_2,\ldots,\tilde{f}_{n+1},\tilde{h}_1,\ldots,\tilde{h}_m) = \\
&= (T^{(m-1)}\mathcal{W}r(Tf_1,\ldots,Tf_{n+1}))\mathcal{W}r(f_1,\ldots,f_{n+1},h_1,\ldots,h_m).
\end{aligned} \tag{B.5}$$

Here, on the first step, we used formulas (B.1) and (B.3), on the second step, we used the assumption hypothesis, on the third step, we used

$$T^{(m)}T^{(n+2)}f_1 = (T^{(m-1)}T^{(n+1)}Tf_1)(T^{(n+m+1)}f_1),$$

and on the forth step, we used formula (B.3) again.

Computation (B.5) proves the induction step finishing the proof of the Proposition.

■

Let $f_1, f_2, \ldots, f_n$ be solutions of the difference equation $Sf = 0$, where $S = T^n + \sum_{i=1}^n a_i T^{n-i}$. Assume that $f_1, f_2, \ldots, f_n$ are linearly independent over the field of 1-periodic functions. Then the function a_n is not identically zero, see the proof of Proposition 3.4.3. Define

$$h_i = T\frac{\mathcal{W}r(f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_n)}{\mathcal{W}(f_1,\ldots,f_n)},$$

Proposition B.2 *The following holds:*

$$\mathcal{W}r(h_1, \ldots, h_n) \;=\; \frac{(-1)^{\frac{n(n+1)}{2}}}{(a_n)^n\, \mathcal{W}r(f_1, \ldots, f_n)}\;,$$

Proof Let $p_i = \mathcal{W}r(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$. Denote by b_{ij} the ij-minor of the matrix $A = (T^{j-1} f_i)_{i.j=1}^n$. Then we have $Tp_i = b_{i1}$.

Since $Sf_i = 0$ for any $i = 1, \ldots, n$, we have $T^n f_i = -\sum_{l=1}^n a_l T^{n-l} f_i$, where the functions $a_1, \ldots, a_n$ do not depend on i. Using this observation, one can check that

$$Tb_{ij} = (-1)^{n-1} a_n b_{i,j+1} + (-1)^{n-j} a_{n-j+1} b_{i,1}.$$

Therefore, by induction on j, we have

$$T^j p_i = (-1)^{(n-1)(j-1)} a_n^{j-1} b_{i,j} + \sum_{j'<j} C_{j'j} b_{i,j'}\,, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n,$$

for certain functions $C_{j'j}$, which do not depend on i. Hence,

$$\mathcal{W}r(Tp_1, \ldots, Tp_n) = \det(T^j p_i)_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} = (-1)^{\frac{n(n-1)^2}{2}} a_n^{\frac{n(n-1)}{2}} \det((-1)^{i+j} b_{i,j})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$$

and

$$\mathcal{W}r(h_1, \ldots, h_n) = \mathcal{W}r\Big(T\frac{p_1}{\det A}, \ldots, T\frac{p_n}{\det A}\Big) = \frac{\mathcal{W}r(Tp_1, \ldots, Tp_n)}{T^{(n)} T \det A}$$

$$= \frac{(-1)^{\frac{n(n-1)^2}{2}} a_n^{\frac{n(n-1)}{2}} \det((-1)^{i+j} b_{i,j})}{T^{(n)} T \det A} = \frac{(-1)^{\frac{n(n-1)^2}{2}} a_n^{\frac{n(n-1)}{2}} \det(A^{-1} \det A\,)}{(-1)^{\frac{n^2(n+1)}{2}} a_n^{\frac{n(n+1)}{2}} (\det A)^n}$$

$$= \frac{(-1)^{\frac{n(n+1)}{2}} (\det A\,)^n}{(a_n)^n (\det A)^{n+1}} = \frac{(-1)^{\frac{n(n+1)}{2}}}{(a_n)^n \mathcal{W}r(f_1, \ldots, f_n)}\;.$$

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