CONTRIBUTIONS TO ROUGH PATHS AND STOCHASTIC PDES

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ABSTRACT

Chakraborty, Prakash PhD, Purdue University, August 2020. Contributions to Rough Paths and Stochastic PDEs. Major Professors: Kiseop Lee and Samy Tindel.

Probability theory is the study of random phenomena. Many dynamical systems with random influence, in nature or artificial complex systems, are better modeled by equations incorporating the intrinsic stochasticity involved. In probability theory, stochastic partial differential equations (SPDEs) generalize partial differential equations through random force terms and coefficients, while stochastic differential equations (SDEs) generalize ordinary differential equations. They are both abound in models involving Brownian motion throughout science, engineering and economics. However, Brownian motion is just one example of a random noisy input. The goal of this thesis is to make contributions in the study and applications of stochastic dynamical systems involving a wider variety of stochastic processes and noises. This is achieved by considering different models arising out of applications in thermal engineering, population dynamics and mathematical finance.

1. Power-type non-linearities in SDEs with rough noise: We consider a noisy differential equation driven by a rough noise that could be a fractional Brownian motion, a generalization of Brownian motion, while the equation's coefficient behaves like a power function [1]. These coefficients are interesting because of their relation to classical population dynamics models [2], while their analysis is particularly challenging because of the intrinsic singularities. Two different methods are used to construct solutions: (i) In the one-dimensional case, a well-known transformation is used; (ii) For multidimensional situations, we find and quantify an improved regularity structure of the solution as it approaches the origin. Our research is the first successful analysis of the system described under a truly rough noise context. We find that the system

is well-defined and yields non-unique solutions. In addition, the solutions possess the same roughness as that of the noise.

2. Parabolic Anderson model in rough environment: The parabolic Anderson model [3] is one of the most interesting and challenging SPDEs used to model varied physical phenomena. Its original motivation involved bound states for electrons in crystals with impurities. It also provides a model for the growth of magnetic field in young stars and has an interpretation as a population growth model. The model can be expressed as a stochastic heat equation with additional multiplicative noise. This noise is traditionally a generalized derivative of Brownian motion. Here we consider a one dimensional parabolic Anderson model which is continuous in space and includes a more general rough noise. See [4] and [5] for previous work in this direction. We first show that the equation admits a solution and that it is unique under some regularity assumptions on the initial condition. In addition, we show that it can be represented using the Feynman-Kac formula, thus providing a connection with the SPDE and a stochastic process, in this case a Brownian motion. The bulk of our study is devoted to explore the large time behavior of the solution, and we provide an explicit formula for the asymptotic behavior of the logarithm of the solution.

3. Heat conduction in semiconductors: Standard heat flow, at a macroscopic level, is modeled by the random erratic movements of Brownian motions starting at the source of heat. However, this diffusive nature of heat flow predicted by Brownian motion is not observed in certain materials (semiconductors, dielectric solids) over short length and time scales [6, 7]. The thermal transport in these materials is more akin to a super-diffusive heat flow, and necessitates the need for processes beyond Brownian motion to capture this heavy tailed behavior. In this context, we propose the use of a well-defined Lévy process, the so-called relativistic stable process to better model the observed phenomenon. This process captures the observed heat dynamics at short length-time scales and is also closely related to the relativistic Schrödinger operator. In addition, it serves as a good candidate for explaining the usual diffusive nature of heat flow under large length-time regimes. The goal is to verify our model against experimental data, retrieve the best parameters of the process and discuss their connections to material thermal properties.

4. Bond-pricing under partial information: We study an information asymmetry problem in a bond market. Especially we derive bond price dynamics of traders with different levels of information. We allow all information processes as well as the short rate to have jumps in their sample paths, thus representing more dramatic movements. In addition we allow the short rate to be modulated by all information processes in addition to having instantaneous feedbacks from the current levels of itself. A fully informed trader observes all information which affects the bond price while a partially informed trader observes only a part of it. We first obtain the bond price dynamic under the full information, and also derive the bond price of the partially informed trader using Bayesian filtering method. The key step is to perform a change of measure so that the dynamic under the new measure becomes computationally efficient. See [8] for a previous work in this direction.

1. ROUGH DIFFERENTIAL EQUATIONS WITH POWER TYPE NONLINEARITIES

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1.1 Introduction

This article is concerned with the following $\mathbb{R}^m$-valued integral equation:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_s)dx_s^j, \quad t \in [0,T] \tag{1.1}$$

where $x : [0,T] \to \mathbb{R}^d$ is a noisy function in the Hölder space $\mathcal{C}^\gamma([0,T];\mathbb{R}^d)$ with $\gamma > \frac{1}{3}$, $a \in \mathbb{R}^m$ is the initial value and σ^j are vector fields on $\mathbb{R}^m$. We shall resort to rough path techniques in order to make sense of the noisy integral in equation (1.1), and we refer to [9, 10] for further details on the rough path theory. Our main goal is to understand how to define solutions to (1.1) when the coefficients σ^j behave like power functions.

Indeed, the rough path theory allows to consider very general noisy signals x as drivers of equation (1.1), but it requires heavy regularity assumptions on the coefficients σ^j in order to get existence and uniqueness of solutions. More specifically, given the regularity of the coefficient σ, a minimal sufficient regularity of the driving signal that guarantees existence and uniqueness of the solution is provided in [9]. However, for differential equations driven by Brownian motion (which means in particular that $x \in \mathcal{C}^{\frac{1}{2}-}$) the condition amounts to the coefficient being twice differentiable. This is obviously far from being optimal with respect to the classical stochastic calculus approach for Brownian motion.

One of the current challenges in rough path analysis is thus to improve the regularity conditions on the coefficients of (1.1), and still get solutions to the differential system at stake. Among the irregular coefficients which can be thought of, power type functions of the form $\sigma^j(\xi) = |\xi|^\kappa$ with $\kappa \in (0,1)$ play a special role. On the one hand these coefficients are related to classical population dynamics models (see e.g [2] for a review), which make them interesting in their own right. On the other hand, the fact that these coefficients vanish at the origin grant them some special properties which can be exploited in order to construct Hölder-continuous solutions. Roughly speaking, equation (1.1) behaves like a noiseless equation when y approaches 0, and one expects existence of a γ-Hölder solution whenever $\gamma + \kappa > 1$. This heuristic argument is explained at length in the introduction of [1], and the current contribution can be seen as the first implementation of such an idea in a genuinely rough context.

Let us now recall some of the results obtained for equations driven by a Brownian motion B. For power type coefficients, most of the results concern one dimensional cases of the form:

$$y_t = a + \int_0^t \sigma(y_s) dB_s, \quad t \in [0,T]. \tag{1.2}$$

The classical result [11, Theorem 2] involves stochastic integrals in the Itô sense, and gives existence and uniqueness for $\sigma(\xi) = |\xi|^\kappa$ with $\kappa \geq \frac{1}{2}$. However, the rough path setting is more related to Stratonovich type integrals in the Brownian case. We thus refer the interested reader to the comprehensive study performed in [12], which studies singular stochastic differential equations and classifies them according to the nature of their solution. Comparing equation (1.2) interpreted in the Stratonovich sense with the systems analyzed in [12], their results can be read as follows: if $\sigma(\xi) = |\xi|^\kappa$ with $\kappa \geq \frac{1}{2}$ and the solution of (1.2) starts at a non-negative location, then it reaches zero almost surely. In addition, among solutions with vanishing local time at 0, there is a non-negative solution which is unique in law. However, in general we do not have uniqueness. The results we will obtain for a general rough path are not as sharp, but are at least compatible with the Brownian case. Let us also mention the works [13, 14], where the authors study existence and uniqueness of solutions in the context of stochastic heat equations with space time white noise and power type coefficients.

As far as power type equations driven by general noisy signals x are concerned, we are only aware of the article [1] exploring equation (1.1) in the Young case $\gamma > 1/2$. The current contribution has thus to be seen as a generalization of [1], allowing to cope with γ-Hölder signals x with $\gamma \in (1/3, 1/2]$. Notice that we have restricted our analysis to $\gamma > 1/3$ in order to keep our computations to a reasonable size. However, we believe that our techniques can be adopted to obtain similar results when $\gamma < 1/3$, at the price of higher order rough path type expansions. As we will see, it turns out that when $\kappa + \gamma > 1$ equation (1.1) is well defined and yields a solution. More specifically, we shall obtain the following theorem in the 1-dimensional case (see Theorem 1.3.7 for a more precise and general formulation).

Theorem 1.1.1 *Consider a 1-dimensional signal $x \in \mathcal{C}^\gamma$, with $\gamma \in (1/3, 1/2]$. Let σ be the power function given by $\sigma(\xi) = |\xi|^\kappa$ and ϕ be the function defined by $\phi(\xi) = \int_0^\xi \frac{ds}{\sigma(s)}$. Assume $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $\kappa + \gamma > 1$. Then the function $y = \phi^{-1}(x + \phi(a))$ is a solution of the equation*

$$y_t = a + \int_0^t \sigma(y_s) dx_s, \quad t \geq 0.$$

In the multidimensional case under a slightly increased regularity assumption on x, namely $x \in \mathcal{C}^{\gamma+}([0, T])$ as well as a roughness assumption (see Hypothesis 1.4.6 for precise statement), the following theorem holds under a few power type hypotheses on σ and its derivatives.

Theorem 1.1.2 *Consider a d-dimensional signal $x \in \mathcal{C}^{\gamma+}$ with $\gamma \in (1/3, 1/2]$, giving raise to a rough path. Assume $\kappa+\gamma > 1$, and that $\sigma(\xi)$ behaves like a power coefficient $|\xi|^\kappa$ near the origin. Then there exists a continuous function y defined on $[0, T]$ and an instant $\tau \leq T$, such that one of the following two possibilities holds:*

(A) $\tau = T$: y is non-zero on $[0, T]$, $y \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ and y solves equation (1.22) on $[0, T]$.

(B) $\tau < T$*: the path* y *sits in* $\mathcal{C}^{\gamma}([0,T];\mathbb{R}^m)$ *and* y *solves equation* (1.22) *on* $[0,T]$*. Furthermore,* $y_s \neq 0$ *on* $[0,\tau)$*,* $\lim_{t\to\tau} y_t = 0$ *and* $y_t = 0$ *on the interval* $[\tau,T]$*.*

As mentioned above, Theorems 1.1.1 and 1.1.2 are the first existence results for power type coefficients in a truly rough context. As in [1], their proofs mainly hinge on a quantification of the regularity gain of the solution y when it approaches the origin. We should mention however that this quantification requires a significant amount of effort in the rough case. Indeed we resort to some discrete type expansions, whose analysis is based on precise estimates inspired by the numerical analysis of rough differential equations (see e.g. [15]).

Having stated the key results, we now describe the outline of this article. In Section 1.2, a short account of the necessary notions of rough path theory is provided. Section 1.3.1 deals with a few hypotheses we assume on the coefficient σ, all of which are satisfied by the power type coefficient $|\xi|^{\kappa}$. Section 1.3.2 proves the existence of a solution in the one-dimensional case. In Section 1.4 we proceed by considering a few stopping times and quantify the regularity gain mentioned above of the solution when it hits 0. We achieve this through discretization techniques as employed in Theorem 1.4.1. Finally we show Hölder continuity of our solution.

Notations. The following notations are used in this article:

1. For an arbitrary real $T > 0$, let $\mathcal{S}_k([0,T])$ be the kth order simplex defined by $\mathcal{S}_k([0,T]) = \{(s_1,\ldots,s_k) : 0 \le s_1 \le \cdots \le s_k \le T\}$.

2. For quantities a and b, let $a \lesssim b$ denote the existence of a constant c such that $a \le cb$.

3. For an element z in the functional space $\mathcal{R}$, let $\mathcal{N}[z;\mathcal{R}]$ denote the corresponding norm of z in $\mathcal{R}$.

1.2 Rough Path Notions

The following is a short account of the rough path notions used in this article, mostly taken from [10]. We review the notion of controlled process as well as their integrals with respect to a rough path. We shall also give a version of an Itô-Stratonovich change of variable formula under reduced regularity condition.

1.2.1 Increments

For a vector space V and an integer $k \geq 1$, let $\mathcal{C}_k(V)$ be the set of functions $g : \mathcal{S}_k([0,T]) \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-increment, and we set $\mathcal{C}_*(V) = \cup_{k\geq 1}\mathcal{C}_k(V)$. Then the operator $\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V)$ is defined as follows

$$\delta g_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1 \cdots \hat{t_i} \cdots t_{k+1}} \tag{1.3}$$

where $\hat{t_i}$ means that this particular argument is omitted. It is easily verified that $\delta\delta = 0$ when considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$.

The size of these k-increments are measured by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ and $\mu > 0$ let

$$\|f\|_\mu = \sup_{(s,t)\in\mathcal{S}_2([0,T])} \frac{\|f_{st}\|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V); \|f\|_\mu < \infty\} \tag{1.4}$$

The usual Hölder space $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = \|\delta g\|_\mu$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite.

Remark 1.2.1 *Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will generally work on spaces for which the initial value of the function is fixed.*

We shall also need to measure the regularity of increments in $\mathcal{C}_3(V)$. To this aim, similarly to (1.4), we introduce the following norm for $h \in \mathcal{C}_3(V)$:

$$\|h\|_\mu = \sup_{(s,u,t)\in\mathcal{S}_3([0,T])} \frac{|h_{sut}|}{|t-s|^\mu}. \tag{1.5}$$

Then the μ-Hölder continuous increments in $\mathcal{C}_3(V)$ are defined as:

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

Notice that the ratio in (1.5) could have been written as $\frac{|h_{sut}|}{|t-u|^{\mu_1}|u-s|^{\mu_2}}$ with $\mu_1+\mu_2 = \mu$, in order to stress the dependence on u of our increment h. However, expression (1.5) is simpler and captures the regularities we need, since we are working on the simplex $\mathcal{S}_3$.

The building block of the rough path theory is the so-called sewing map lemma. We recall this fundamental result here for further use.

Proposition 1.2.1 *Let $h \in \mathcal{C}_3^\mu(V)$ for $\mu > 1$ be such that $\delta h = 0$. Then there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^\mu(V)$ such that $\delta g = h$. Furthermore for such an h, the following relations hold true:*

$$\delta\Lambda(h) = h \quad \textit{and} \quad \|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu.$$

1.2.2 Elementary computations in $\mathcal{C}_2$ and $\mathcal{C}_3$

Consider $V = \mathbb{R}$, and let $\mathcal{C}_k^\gamma$ for $\mathcal{C}_k^\gamma(\mathbb{R})$. Then $(\mathcal{C}_*, \delta)$ can be endowed with the following product: for $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$ we let gh be the element of $\mathcal{C}_{m+n-1}$ defined by

$$(gh)_{t_1,\dots,t_{m+n-1}} = g_{t_1,\cdots,t_n} h_{t_n,\cdots t_{m+n-1}}, \quad (t_1,\dots,t_{m+n-1}) \in \mathcal{S}_{m+n-1}([0,T]).$$

We now label a rule for discrete differentiation of products for further use throughout the article. Its proof is an elementary application of the definition (1.3), ans is omitted for sake of conciseness.

Proposition 1.2.2 *The following rule holds true: Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then $gh \in \mathcal{C}_2$ and*

$$\delta(gh) = \delta g\, h - g\, \delta h.$$

The iterated integrals of smooth functions on $[0,T]$ are particular cases of elements of $\mathcal{C}_2$, which will be of interest. Specifically, for smooth real-valued functions f and g, let us denote $\int f dg$ by $\mathcal{I}(f dg)$ and consider it as an element of $\mathcal{C}_2$: for $(s,t) \in \mathcal{S}_2([0,T])$ we set

$$\mathcal{I}_{st}(f dg) = \left(\int f dg \right)_{st} = \int_s^t f_u dg_u.$$

1.2.3 Weakly controlled processes

One of our basic assumptions on the driving process x of equation (1.1) is that it gives raise to a geometric rough path. This assumption can be summarized as follows.

Hypothesis 1.2.2 *The path $x : [0,T] \to \mathbb{R}^d$ belongs to the Hölder space $\mathcal{C}^\gamma([0,T];\mathbb{R}^d)$ with $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $x_0 = 0$. In addition x admits a Lévy area above itself, that is, there exists a two index map $\mathbf{x}^2 : \mathcal{S}_2([0,T]) \to \mathbb{R}^{d,d}$ which belongs to $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$ and such that*

$$\delta \mathbf{x}^{2;ij}_{sut} = \delta x^i_{su} \otimes \delta x^j_{ut}, \qquad \textit{and} \qquad \mathbf{x}^{2;ij}_{st} + \mathbf{x}^{2;ji}_{st} = \delta x^i_{st} \otimes \delta x^j_{st}.$$

The γ-Hölder norm of x is denoted by:

$$\|\mathbf{x}\|_\gamma = \mathcal{N}(x; \mathcal{C}_1^\gamma([0,T], \mathbb{R}^d)) + \mathcal{N}(\mathbf{x^2}; \mathcal{C}_2^{2\gamma}([0,T], \mathbb{R}^{d,d})).$$

Preparing the ground for the upcoming change of variable formula in Proposition 1.2.5, we now define the notion weakly controlled process as a slight variation of the usual one.

Definition 1.2.1 *Let z be a process in $\mathcal{C}_1^\gamma(\mathbb{R}^n)$ with $1/3 < \gamma \le 1/2$ and consider $\eta > \gamma$. We say that z is weakly controlled by x with a remainder of order η if $\delta z \in \mathcal{C}_2^\gamma(\mathbb{R}^n)$ can be decomposed into*

$$\delta z^i = \zeta^{ii_1} \delta x^{i_1} + r^i, \quad \textit{i.e.} \quad \delta z^i_{st} = \zeta^{ii_1}_s \delta x^{i_1}_{st} + r^i_{st}$$

for all $(s,t) \in \mathcal{S}_2([0,T])$*. In the previous formula we assume* $\zeta \in \mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{n,d})$ *and* r *is a more regular remainder such that* $r \in \mathcal{C}_2^{\eta}(\mathbb{R}^n)$*. The space of weakly controlled paths will be denoted by* $\mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ *and a process* $z \in \mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ *can be considered as a couple* (z,ζ)*. The natural semi-norm on* $\mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ *is given by*

$$\mathcal{N}[z;\mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)] = \mathcal{N}[z;\mathcal{C}_1^{\gamma}(\mathbb{R}^n)] + \mathcal{N}[\zeta;\mathcal{C}_1^{\infty}(\mathbb{R}^{n,d})] + \mathcal{N}[\zeta;\mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{n,d})] + \mathcal{N}[r;\mathcal{C}_2^{\eta}(\mathbb{R}^n)].$$

Let $\mathrm{Lip}^{n+\lambda}$ denote the space of n-times differential functions with $\lambda-$Hölder nth derivative, endowed with the norm:

$$\|f\|_{n,\lambda} = \|f\|_\infty + \sum_{k=1}^{n} \|\partial^k f\|_\infty + \|\partial^n f\|_\lambda.$$

The following gives a composition rule which asserts that our rough path x composed with a $\mathrm{Lip}^{1+\lambda}$ function is weakly controlled.

Proposition 1.2.3 *Let* $f : \mathbb{R}^d \to \mathbb{R}^n$ *be a* $\mathrm{Lip}^{1+\lambda}$ *function and set* $z = f(x)$*. Then* $z \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ *with* $\sigma = \gamma(\lambda+1)$*, where* $\mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ *is introduced in Definition 1.2.1, and it can be decomposed into* $\delta z = \zeta\delta x + r$*, with*

$$\zeta^{ii_1} = \partial_{i_1} f_i(x) \quad \textit{and} \quad r^i = \delta f_i(x) - \partial_{i_1} f_i(x) \delta x_{st}^{i_1}.$$

Furthermore, the norm of z *as a controlled process can be bounded as follows:*

$$\mathcal{N}[z;\mathcal{Q}_{\gamma,\sigma}] \le K\|f\|_{1,\lambda}(1 + \mathcal{N}^{1+\lambda}[x;\mathcal{C}_1^{\gamma}(\mathbb{R}^d)]),$$

where K *is a positive constant.*

Proof The algebraic part of the assertion is straightforward. Just write

$$\delta z_{st} = f(x_t) - f(x_s) = \partial_{i_1} f(x_s) \delta x_{st}^{i_1} + r_{st}$$

The estimate of $\mathcal{N}[z;\mathcal{Q}_{\gamma,\sigma}]$ is obtained from the estimates of $\mathcal{N}[z;\mathcal{C}_1^{\gamma}(\mathbb{R}^n)]$, $\mathcal{N}[\zeta;\mathcal{C}_1^{\infty}(\mathbb{R}^{n,d})]$, $\mathcal{N}[\zeta;\mathcal{C}_1^{\sigma-\gamma}(\mathbb{R}^{n,d})]$ and $\mathcal{N}[r;\mathcal{C}_2^{\sigma}(\mathbb{R}^n)]$. The details are similar to [10, Appendix] and left to the patient reader. ■

More generally, we also need to specify the composition of a controlled process with a $\mathrm{Lip}^{1+\lambda}$ function. The proof of this proposition is similar to Proposition 1.2.3 and omitted for sake of conciseness.

Proposition 1.2.4 *Let $z \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ with decomposition $\delta z = \tilde{\zeta}\delta x + \tilde{r}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathrm{Lip}^{1+\lambda}$ function. Set $w = g(x)$. Then $w \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^m)$ with $\sigma = \gamma(\lambda+1)$ and it can be decomposed into $\delta w = \zeta\delta x + r$, with*

$$\zeta^{ii_1} = \partial_{i_2} f_i(x)\tilde{\zeta}^{i_2,i_1}.$$

The class of weakly controlled paths provides a natural and basic set of functions which can be integrated with respect to a rough path. The basic proposition in this direction, whose proof can be found in [10], is summarized below.

Theorem 1.2.3 *For $1/3 < \gamma \leq 1/2$, let x be a process satisfying Hypothesis 1.2.2. Furthermore let $m \in \mathcal{Q}_{\gamma,\eta}(\mathbb{R}^d)$ with $\eta + \gamma > 1$, whose decomposition is given by $m_0 = b \in \mathbb{R}^d$ and*

$$\delta m^i = \mu^{ii_1}\delta x^{i_1} + r^i \quad \textit{where} \quad \mu \in \mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{d,d}), r \in \mathcal{C}_2^{\eta}(\mathbb{R}^n).$$

Define z by $z_0 = a \in \mathbb{R}^d$ and

$$\delta z = m^i \delta x^i + \mu^{ii_1}\mathbf{x}^{2;i_1 i} - \Lambda(r^i\delta x^i + \delta\mu^{ii_1}\mathbf{x}^{2;i_1 i}).$$

Finally, set

$$\mathcal{I}_{st}(m dx) = \int_s^t \langle m_u, dx_u\rangle_{\mathbb{R}^d} := \delta z_{st}.$$

Then this integral extends Young integration and coincides with the Riemann-Stieltjes integral of m with respect to x whenever these two functions are smooth. Furthermore, $\mathcal{I}_{st}(mdx)$ is the limit of modified Riemann sums:

$$\mathcal{I}_{st}(mdx) = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} [m^i_{t_q}\delta x^i_{t_q t_{q+1}} + \mu^{ii_1}_{t_q}\mathbf{x}^{2;i_1 i}_{t_q t_{q+1}}],$$

for any $0 \leq s < t \leq T$, where the limit is taken over all partitions $\Pi_{st} = \{s = t_0, \ldots, t_n = t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

1.2.4 Itô-Stratonovich formula

We now state a change of variable formula for a function $g(x)$ of a rough path, under minimal assumptions on the regularity of g. To the best of our knowledge, this proposition cannot be found in literature, and therefore a short and elementary proof is included. The techniques of this proof will prove to be useful for the study of our system (1.1) in the one-dimensional case.

Proposition 1.2.5 *Let x satisfy Hypothesis 1.2.2. Let g be a* $\mathrm{Lip}^{2+\lambda}$ *function such that $(\lambda+2)\gamma > 1$. Then*

$$[\delta(g(x))]_{st} = \mathcal{I}_{st}(\nabla g(x) dx) = \int_s^t \langle \nabla g(x_u), dx_u \rangle_{\mathbb{R}^d}, \tag{1.6}$$

where the integral above has to be understood in the sense of Theorem 1.2.3.

Proof Consider a partition $\Pi_{st} = \{s = t_0 < \cdots t_n = t\}$ of $[s,t]$. The following identity holds trivially:

$$\begin{aligned} g(x_t) - g(x_s) &= \sum_{q=0}^{n-1} \left[g(x_{t_{q+1}}) - g(x_{t_q}) \right] \\ &= \sum_{q=0}^{n-1} \left[\sum_i \partial_i g(x_{t_q}) \delta x^i_{t_q t_{q+1}} + \frac{1}{2} \sum_{i_1,i_2} \partial^2_{i_1 i_2} g(x_{t_q}) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}} + r_{t_q t_{q+1}} \right] \end{aligned} \tag{1.7}$$

where

$$r_{t_q t_{q+1}} = g(t_{q+1}) - g(t_q) - \sum_i \partial_i g(x_{t_q}) \delta x^i_{t_q t_{q+1}} - \frac{1}{2} \sum_{i_1,i_2} \partial^2_{i_1 i_2} g(x_{t_q}) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}}.$$

Furthermore, an elementary Taylor type argument shows that for all i_1, i_2 there exists an element $\xi^q_{i_1 i_2}$ of $[x_{t_q}, x_{t_{q+1}}]$ such that

$$\begin{aligned} r_{t_q t_{q+1}} &= \frac{1}{2} \sum_{i_1,i_2} \partial^2_{i_1 i_2} g(\xi^q_{i_1 i_2}) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}} - \frac{1}{2} \sum_{i_1,i_2} \partial^2_{i_1 i_2} g(x_{t_q}) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}} \\ &= \frac{1}{2} \sum_{i_1,i_2} \left(\partial^2_{i_1 i_2} g(\xi^q_{i_1 i_2}) - \partial^2_{i_1 i_2} g(x_{t_q}) \right) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}}. \end{aligned}$$

We now invoke the fact that $g \in \mathrm{Lip}^{2+\lambda}$ in order to get

$$\left|r_{t_q t_{q+1}}\right| \leq C|t_q - t_{q+1}|^{(2+\lambda)\gamma},$$

where C is a constant depending on g and x. Thus, since $(\lambda+2)\gamma > 1$, it is easily seen that

$$\lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} r_{t_q t_{q+1}} = 0. \tag{1.8}$$

In addition, using Hypothesis 1.2.2 and continuity of the partial derivatives, we can write

$$\frac{1}{2}\sum_{i_1,i_2} \partial^2_{i_1 i_2} g(x_{t_q}) \delta x^{i_1}_{t_q t_{q+1}} \delta x^{i_2}_{t_q t_{q+1}} = \sum_{i_1,i_2} \partial^2_{i_1 i_2} g(x_{t_q}) \mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}}. \tag{1.9}$$

Plugging (1.8) and (1.9) into (1.7) we get

$$g(x_t) - g(x_s) = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \partial_i g(x_{t_q}) \delta x^i_{t_q t_{q+1}} + \sum_{q=0}^{n-1} \partial^2_{i_1 i_2} g(x_{t_q}) \mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}}, \tag{1.10}$$

for all $(s,t) \in \mathcal{S}_2\,[0,T])$.

On the other hand looking at the decomposition of $\nabla g(x)$ as a weakly controlled process and using Proposition 1.2.3 we obtain:

$$\delta\left[\nabla g(x)\right]^i_{st} = \delta\partial_i g(x)_{st} = \partial^2_{i_1 i} g(x_s)\delta x^{i_1}_{st} + R^i_{st},$$

where R lies in $\mathcal{C}_2^{(1+\lambda)\gamma}$. Then using the Riemann sum representation (1.2.3) of rough integrals, we have

$$\mathcal{I}_{st}(\nabla g(x) dx) = \lim_{|\Pi_{st}|\to 0} \left[\sum_{q=0}^{n-1} \partial_i g(x_{t_q}) \delta x^i_{t_q t_{q+1}} + \sum_{q=0}^{n-1} \partial^2_{i_1 i_2} g(x_{t_q}) \mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}}\right].$$

Comparing the above formula with (1.10) proves the result. ■

1.3 Differential equations: setting and one-dimensional case

In this section we will give the general formulation and assumptions for equation (1.1). Then we state an existence result in dimension 1, which follows quickly from our preliminary considerations in Section 1.2.

1.3.1 Setting

Recall that we are considering the following rough differential equation:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_s) dx_s^j, \tag{1.11}$$

where x satisfies Hypothesis 1.2.2 and $\sigma^1, \dots, \sigma^d$ are vector fields on $\mathbb{R}^m$. In this section we will specify some general assumptions on the coefficient σ, which will prevail for the remainder of the article.

Let us start with a regularity assumption on σ:

Hypothesis 1.3.1 . *Let $\kappa > 0$ be a constant such that $\gamma + \kappa > 1$, where γ is introduced in Hypothesis 1.2.2. We assume that $\sigma(0) = 0$, and that the following two conditions are valid:*

(i) For all $\xi_1, \xi_2 \in \mathbb{R}^m$ we have the following:

$$|\sigma(\xi_1) - \sigma(\xi_2)| \lesssim |\xi_1 - \xi_2|^\kappa, \tag{1.12}$$

(ii) Consider the function $\Psi = D\sigma \cdot \sigma$ defined on $\mathbb{R}^m$. For all $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\frac{1}{r} \leq \frac{|\xi_1|}{|\xi_2|} \leq r$ for a fixed $r > 1$, there exists a constant $\mathcal{N}_\Psi$ (depending on r, m and κ) satisfying:

$$|\Psi(\xi_1) - \Psi(\xi_2)| \leq \mathcal{N}_\Psi \left| \frac{1}{|\xi_1|^{2(1-\kappa)}} + \frac{1}{|\xi_2|^{2(1-\kappa)}} \right| |\xi_1 - \xi_2|, \tag{1.13}$$

In addition to above, we assume that outside of a neighborhood of 0, σ behaves like a Lip^p_{loc} function with $p > \frac{1}{\gamma}$. In other words, σ is bounded with bounded two derivatives and the second derivative is locally Hölder continuous with order larger than $(\frac{1}{\gamma} - 2)$.

We also need a more specific assumption in dimension 1:

Hypothesis 1.3.2 *Whenever $m = d = 1$, assume σ is positive on $\mathbb{R}_+$ and that ϕ defined by $\phi(\xi) = \int_0^\xi \frac{ds}{\sigma(s)}$ exists. Also consider $\kappa > 0$ as in Hypothesis 1.3.1. Then we assume for all $\xi_1, \xi_2 \in \mathbb{R}$ we have*

$$|F(\xi_1) - F(\xi_2)| \lesssim |\xi_1 - \xi_2|^\lambda,$$

where F stands for the function $(D\sigma \cdot \sigma) \circ \phi^{-1}$ and $\lambda = \frac{2\kappa - 1}{1-\kappa} \wedge 1$.

We now give a typical example of a coefficient σ satisfying our standing assumptions.

Proposition 1.3.1 *Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a smooth cutoff function such that $\chi(z) = 1$ if $|z| \le \frac{M}{2}$ and $\chi(z) = 0$ if $|z| \ge M$, for a given $M > 0$. Assume that $\sigma = (\sigma^1, \dots, \sigma^m)$ where each $\sigma^i : \mathbb{R}^m \to \mathbb{R}$ is defined by the κ^{th} power of the Euclidean norm: $\sigma^i(\xi) = \left(\sum_j (\xi^j)^2\right)^{\kappa/2} \chi(|\xi|)$. Then inequality (1.13) holds true for all $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\frac{1}{r} \le \frac{|\xi_1|}{|\xi_2|} \le r$.*

Proof We only handle inequality (1.13) when ξ_1, ξ_2 are close to 0, which is the relevant case in our situation. We can thus assume that each σ^i is of the form $\sigma^i(\xi) = |\xi|^\kappa$ in the sequel. For notational sake we will set $\tilde{\sigma}(\xi) = |\xi|^\kappa$ in the remainder of the proof.

Observe that $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ defined by $\Psi(\xi) = (D\sigma \cdot \sigma)(\xi)$ satisfies $\Psi^i(\xi) = \sum_k \sigma^k(\xi) \partial_k \sigma^i(\xi)$. Consequently,

$$\begin{aligned} \nabla \Psi^{ij}(\xi) = \partial_j \Psi^i(\xi) = \partial_j &\left[\sum_{k=1}^m \partial_k \sigma^i(\xi) \sigma^k(\xi) \right] \\ &= \sum_{k=1}^m \left[\left(\partial_j \partial_k \sigma^i(\xi)\right) \sigma^k(\xi) + \left(\partial_k \sigma^i(\xi)\right) \left(\partial_j \sigma^k(\xi)\right) \right] \end{aligned} \tag{1.14}$$

The partial derivatives above, when evaluated for $\sigma^k(\xi) = \tilde{\sigma}(\xi) = |\xi|^\kappa = (\sum {\xi^i}^2)^{\kappa/2}$, turn out to be as follows:

$$\partial_k \tilde{\sigma}(\xi) = \kappa |\xi|^{\kappa-2} \xi_k \quad \text{and} \quad \partial_j \partial_k \tilde{\sigma}(\xi) = \kappa(\kappa-2)|\xi|^{\kappa-4} \xi_j \xi_k + \kappa |\xi|^{\kappa-2} \mathbf{1}_{(j=k)}.$$

Plugging these partial derivatives in the formula obtained in (1.14), we get

$$\nabla \Psi^{ij}(\xi) = 2\kappa(\kappa-1)|\xi|^{2\kappa-4} \xi^j (\xi \cdot \mathbf{1}) + \kappa |\xi|^{2(\kappa-1)}, \tag{1.15}$$

where $\xi \cdot \mathbf{1}$ denotes the inner product of ξ and the vector $\mathbf{1} \in \mathbb{R}^m$. Now we use the multivariate mean value theorem in integral form given by:

$$\Psi(\xi_1) - \Psi(\xi_2) = \int_0^1 \nabla \Psi(\xi_t) \cdot (\xi_1 - \xi_2)\, dt,$$

where we have set $\xi_t = (1-t)\xi_2 + t\xi_1$ for $t \in [0,1]$. From (1.15) we thus obtain

$$\Psi^i(\xi_1) - \Psi^i(\xi_2) = \sum_{j=1}^m \int_0^1 \left(2\kappa(\kappa-1)|\xi_t|^{2\kappa-4}\xi_t^j\,(\xi_t\cdot\mathbf{1}) + \kappa|\xi_t|^{2(\kappa-1)}\right)\left(\xi_1^j - \xi_2^j\right)dt,$$

Assume wlog that $|\xi_1| \leq |\xi_2|$, which implies by our assumption on ξ_1, ξ_2 that $1 \leq \frac{|\xi_2|}{|\xi_1|} \leq r$. Now observe

$$\begin{aligned}
&|\xi_1|^{2(1-\kappa)}\left|\Psi^i(\xi_1) - \Psi^i(\xi_2)\right| \\
&= |\xi_1|^{2(1-\kappa)}\left|\sum_{j=1}^m \int_0^1 \left(2\kappa(\kappa-1)|\xi_t|^{2\kappa-4}\xi_t^j\,(\xi_t\cdot\mathbf{1}) + \kappa|\xi_t|^{2(\kappa-1)}\right)\left(\xi_1^j - \xi_2^j\right)dt\right| \\
&\leq \sum_{j=1}^m \int_0^1 \left(2\kappa(\kappa-1)\left|\frac{\xi_t}{|\xi_1|}\right|^{2\kappa-4}\left|\frac{\xi_t^j}{|\xi_1|}\right|\left|\frac{(\xi_t\cdot\mathbf{1})}{|\xi_1|}\right| + \kappa\left|\frac{\xi_t}{|\xi_1|}\right|^{2(\kappa-1)}\right)\left|\xi_1^j - \xi_2^j\right|dt. \quad (1.16)
\end{aligned}$$

Since $\frac{\xi_t}{|\xi_1|} = (1-t)\frac{\xi_2}{|\xi_1|} + t\frac{\xi_1}{|\xi_1|}$ and $1 \leq \frac{|\xi_2|}{|\xi_1|} \leq r$ we must have $1 \leq |\frac{\xi_t}{|\xi_1|}| \leq r$. Using this information in (1.16) we get

$$|\xi_1|^{2(1-\kappa)}\left|\Psi^i(\xi_1) - \Psi^i(\xi_2)\right| \lesssim \sum_{j=1}^m |\xi_1 - \xi_2| \lesssim |\xi_1 - \xi_2|.$$

This yields (1.13). ■

Remark 1.3.3 *A sufficient condition for σ to satisfy Hypothesis 1.3.1 is the boundedness of $|\xi_1|^{2(1-\kappa)}|\nabla\Psi(\tilde{\xi})|$ for any $\tilde{\xi}$ such that $1 \leq \frac{|\tilde{\xi}|}{|\xi_1|} \leq r$.*

Remark 1.3.4 *Let χ be defined as in Proposition 1.3.1. It can be easily shown that perturbations of the power function, e.g. $\sigma(\xi) = (\sigma^1(\xi), \ldots, \sigma^m(\xi))$ where each σ^j is of the form $\sigma^j(\xi) = (|\xi|^\kappa + \sin(|\xi|^\kappa))\chi(\xi)$, also fall under the purvue of Hypothesis 1.3.1.*

Finally we add some assumptions on the first and second order derivatives of σ, which will be mainly invoked in the proof of Proposition 1.4.1.

Hypothesis 1.3.5 *The derivatives of σ satisfy the following: there exists a $\ell_0 > 0$ such that for all ξ with $0 < |\xi| \leq \ell_0$ we have*

$$|D\sigma(\xi)| \lesssim |\xi|^{\kappa-1} \quad \textit{and} \quad |D^2\sigma(\xi)| \lesssim |\xi|^{\kappa-2}. \quad (1.17)$$

Remark 1.3.6 *Observe that Hypotheses 1.3.1 and 1.3.5 imply: there exists a $\ell_0 > 0$ such that for all ξ with $0 < |\xi| \le \ell_0$ we have*

$$|D\sigma \cdot \sigma(\xi)| \lesssim |\xi|^{2\kappa-1}. \tag{1.18}$$

In addition, the reader can check that (1.17) and (1.18) is satisfied for σ as in Proposition 1.3.1.

Definition 1.3.1 *Let $\mathcal{N}_{\alpha,F}$ be defined as:*

$$\mathcal{N}_{\alpha,F} := \sup\left\{\frac{|F(\xi)|}{|\xi|^{\alpha}}; |\xi| \neq 0\right\}, \tag{1.19}$$

where $\alpha = \kappa$ if $F = \sigma$ and $\alpha = 2\kappa - 1$ if $F = \Psi = (D\sigma \cdot \sigma)$.

1.3.2 One-dimensional differential equations

In the one-dimensional case, similarly to what is done for more regular coefficients (See [16]), one can prove that a suitable function of x solves equation (1.11). This stems from an application of our extension of Itô's formula (see Proposition 1.2.5) and is obtained in the following theorem.

Theorem 1.3.7 *Consider equation (1.11) with $m = d = 1$, let $\sigma : \mathbb{R} \to \mathbb{R}$ and assume Hypothesis 1.3.2 to hold true. Assume $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $\kappa + \gamma > 1$. Let ϕ be the function defined in Hypothesis 1.3.2. Then the function $y = \phi^{-1}(x + \phi(a))$ is a solution of the equation*

$$y_t = a + \int_0^t \sigma(y_s)dx_s, \quad t \ge 0. \tag{1.20}$$

Proof Let $\psi(\xi) = \phi^{-1}(\xi + \phi(a))$. Due to the definition of ϕ, some elementary computations show that $\psi'(\xi) = \frac{1}{\phi'(\phi^{-1}(\xi+\phi(a)))} = \sigma(\psi(\xi))$ and thus we are reduced to show

$$\delta\psi(x)_{st} = \int_s^t \psi'(x_u)dx_u. \tag{1.21}$$

To this aim, observe that the second derivative of ψ satisfies

$$\psi''(\xi) = D\sigma(\psi(\xi))\psi'(\xi) = (D\sigma \cdot \sigma)(\psi(\xi)).$$

Using Hypothesis 1.3.2, ψ'' is thus $\lambda-$Hölder continuous where $\lambda = \frac{2\kappa-1}{1-\kappa} \wedge 1$, that is, ψ is a $\mathrm{Lip}^{2+\lambda}$ function. Moreover, since $\kappa+\gamma > 1$ and $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ we find $(\lambda+2)\gamma > 1$. Consequently we can invoke Proposition 1.2.5 and hence we obtain directly (1.21). The result is now proved. ■

Remark 1.3.8 *It is readily checked that the power coefficient* $\sigma(\xi) = |\xi|^\kappa$ *satisfies the conditions of Theorem 1.3.7, with a function* F *defined by* $F(\xi) = c_\kappa |\xi|^\lambda sgn(\xi)$ *and where the exponent* λ *is given by* $\lambda = \frac{2\kappa-1}{1-\kappa}$.

Remark 1.3.9 *If* $a = 0$, *we do not have uniqueness of solution since in addition to the solution defined above,* $y \equiv 0$ *solves equation* (1.20). *This is not in contradiction to the results stated in [12] where the authors deal with equations with non-vanishing coefficients. In our case,* $\sigma(0) = 0$.

Remark 1.3.10 *As the reader might see, Theorem 1.3.7 is an easy consequence of the change of variable formula* (1.6). *This is in contrast with the corresponding proof in [1], which relied on a negative moment estimate and non trivial extensions of Young's integral in the fractional calculus framework.*

1.4 Multidimensional Differential Equations

In the multidimensional case, our strategy in order to construct a solution is based (as in [1]) on quantifying an additional smoothness of the solution y as it approaches the origin. However, our computations here are more involved than in [1], due to the fact that we are handling a rough process x.

1.4.1 Prelude

In this section, we will introduce a sequence of stopping times, similarly to [1]. We assume that each component $\sigma^j : \mathbb{R}^m \to \mathbb{R}^m$ satisfies Hypothesis 1.3.1 and we consider the following equation for a fixed $a \in \mathbb{R}^m \setminus \{0\}$:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_u)\, dx_u^j, \quad t \in [0,T], \tag{1.22}$$

where $T > 0$ is a fixed arbitrary horizon and $\mathbf{x} = (x, \mathbf{x^2})$ is a γ-rough path above x, as given in Hypothesis 1.2.2.

Our considerations start from the fact that, as long as we are away from 0, we can solve equation (1.22) as a rough path equation with regular coefficients. In particular the following can be shown under the above set-up. See [9].

Theorem 1.4.1 *Assume Hypothesis 1.3.1 is fulfilled. Then there exists a continuous function y defined on $[0,T]$ and an instant $\tau \le T$, such that one of the following two possibilities holds:*

(A) $\tau = T$, y is non-zero on $[0,T]$, $y \in \mathcal{C}^\gamma([0,T];\mathbb{R}^m)$ and y solves equation (1.22) on $[0,T]$, where the integrals $\int \sigma^j(y_u)\, dx_u^j$ are understood in the rough path sense.

(B) We have $\tau < T$. Then for any $t < \tau$, the path y sits in $\mathcal{C}^\gamma([0,t];\mathbb{R}^m)$ and y solves equation (1.22) on $[0,t]$. Furthermore, $y_s \neq 0$ on $[0,\tau)$, $\lim_{t\to\tau} y_t = 0$ and $y_t = 0$ on the interval $[\tau, T]$.

Option (A) above leads to classical solutions of equation (1.22). In the rest of this section, we will assume (B), that is the function y given by Theorem 1.4.1 vanishes in the interval $[\tau, T]$. The aim of this section is to prove the following:

- The path y is globally γ-Hölder continuous on $[0,T]$.

To achieve this we will require some additional hypotheses on x (See Hypothesis 1.4.4 below).

Quantification of the increased smoothness of the solution as it approaches the origin would require a partition of the interval $(0, \tau]$ as follows. Let $a_j = 2^{-j}$ and consider the following decomposition of $\mathbb{R}_+$:

$$\mathbb{R}_+ = \bigcup_{j=-1}^{\infty} I_j,$$

where

$$I_{-1} = [1, \infty)\,, \quad \text{and} \quad I_q = [a_{q+1}, a_q), \quad q \geq 0.$$

Also consider:

$$J_{-1} = [3/4, \infty)\,, \quad \text{and} \quad J_q = \left[\frac{a_{q+2} + a_{q+1}}{2}, \frac{a_{q+1} + a_q}{2}\right) =: [\hat{a}_{q+1}, \hat{a}_q)\,, \quad q \geq 0.$$

Observe that owing to the definition of a_q, we have $\hat{a}_q = \frac{3}{2^{q+2}}$. Let q_0 be such that $a \in I_{q_0}$. Define $\lambda_0 = 0$ and

$$\tau_0 = \inf\{t \geq 0 : |y_t| \notin I_{q_0}\}$$

By definition, $y_{\tau_0} \in J_{\hat{q}_0}$ with $\hat{q}_0 \in \{q_0, q_0 - 1\}$. Now define

$$\lambda_1 = \inf\{t \geq \tau_0 : |y_t| \notin J_{\hat{q}_0}\}$$

Thus we get a sequence of stopping times $\lambda_0 < \tau_0 < \cdots < \lambda_k < \tau_k$, such that

$$y_t \in \left[\frac{b_1}{2^{q_k}}, \frac{b_2}{2^{q_k}}\right], \quad \text{for} \quad t \in [\lambda_k, \tau_k] \cup [\tau_k, \lambda_{k+1}], \tag{1.23}$$

where $b_1 = \frac{3}{8}$, $b_2 = \frac{3}{4}$ and $q_{k+1} = q_k + \ell$, with $\ell \in \{-1, 0, 1\}$, for $q_k \geq 1$. If $q_k = 0$ or $q_k = 1$, then we can choose the upper bound b_2 as $b_2 = \infty$.

Remark 1.4.2 *Since this problem relies heavily on radial variables in $\mathbb{R}^m$, we alleviate vectorial notations and carry out the computations below for $m = d = 1$. Generalizations to higher dimensions are straight forward.*

1.4.2 Regularity estimates

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}$ be a partition of the interval $[0, T]$ for $n \in \mathbb{N}$. Denote by $\mathcal{C}_2(\pi)$ the collection of functions R on π such that $R_{t_k t_{k+1}} = 0$ for $k = 0, 1, \dots n-1$. We now introduce some operators on discrete time increments, which are similar to those in Section 1.2. First, we define the operator $\delta : \mathcal{C}_2(\pi) \to \mathcal{C}_3(\pi)$ by

$$\delta R_{sut} = R_{st} - R_{su} - R_{ut} \quad \text{for } s, u, t \in \pi \tag{1.24}$$

The Hölder seminorms we will consider are similar to those introduced in (1.4) and (1.5). Namely, for $R \in \mathcal{C}_2(\pi)$ we set

$$\|R\|_\mu = \sup_{u,v\in\pi} \frac{R_{uv}}{|u-v|^\mu} \quad \text{and} \quad \|\delta R\|_\mu = \sup_{s,u,t\in\pi} \frac{|\delta R_{sut}|}{|t-s|^\mu}$$

We now state a sewing lemma for discrete increments which is similar to [15, Lemma 2.5]. Its proof is included here for completeness.

Lemma 1.4.3 *For $\mu > 1$ and $R \in \mathcal{C}_2(\pi)$, we have*

$$\|R\|_\mu \le K_\mu \|\delta R\|_\mu,$$

where $K_\mu = 2^\mu \sum_{l=1}^{\infty} \frac{1}{l^\mu}$

Proof Consider some fixed $t_i, t_j \in \pi$. Since $R \in \mathcal{C}_2(\pi)$ we have $\sum_{k=i}^{j-1} R_{t_k t_{k+1}} = 0$. Hence, for an arbitrary sequence of partitions $\{\pi_l; 1 \le l \le j-i-1\}$, where each π_l is a subset of $\pi \cap [t_i, t_j]$ with $l+1$ elements, we can write (thanks to a trivial telescoping sum argument):

$$R_{t_i t_j} = R_{t_i t_j} - \sum_{k=i}^{j-1} R_{t_k t_{k+1}} = \sum_{l=1}^{j-i-1} (R^{\pi_l} - R^{\pi_{l+1}}), \tag{1.25}$$

where we have set $R^{\pi_l} = \sum_{k=0}^{l-1} R_{t^l_k t^l_{k+1}}$. We now specify the choice of partitions π_l recursively:

Define $\pi_{j-i} = \pi \cap [t_i, t_j]$. Given a partition π_l with $l+1$ elements, $l = 2, \ldots, j-i$, we can find $t^l_{k_l} \in \pi_l \setminus \{t_i, t_j\}$ such that

$$t^l_{k_l+1} - t^l_{k_l-1} \leq \frac{2(t_j - t_i)}{l}. \tag{1.26}$$

Denote by π_{l-1} the partition $\pi_l \setminus \{t^l_{k_l}\}$. Owing to (1.24), we obtain:

$$|R^{\pi_{l-1}} - R^{\pi_l}| = \left|\delta R_{t^l_{k_l-1} t^l_{k_l} t^l_{k_l+1}}\right| \leq \|\delta R\|_\mu (t^l_{k_l+1} - t^l_{k_l-1})^\mu \leq \|\delta R\|_\mu \frac{2^\mu (t_j - t_i)^\mu}{l^\mu},$$

where the second inequality follows from (1.26). Now plugging the above estimate in (1.25) we get

$$\left|R_{t_i t_j}\right| \leq 2^\mu (t_j - t_i)^\mu \|\delta R\|_\mu \sum_{l=1}^{j-i-1} \frac{1}{(l+1)^\mu} \leq K_\mu (t_j - t_i)^\mu \|\delta R\|_\mu.$$

By dividing both sides by $(t_i - t_j)^\mu$ and taking supremum over $t_i, t_j \in \pi$, we obtain the desired estimate. ∎

Next we define an increment R which is obtained as a remainder in rough path type expansions.

Definition 1.4.1 *Let y and τ be defined as in Proposition 1.4.1. For $(s,t) \in \mathcal{S}_2([0,\tau])$, let R_{st} be defined by the following decomposition:*

$$\delta y_{st} = \sigma(y_s)\delta x_{st} + (D\sigma \cdot \sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st} + R_{st}. \tag{1.27}$$

The theorem below quantifies the regularity improvement for the solution y of equation (1.22) as it gets closer to 0.

Proposition 1.4.1 *Consider a rough path x satisfying Hypothesis 1.2.2. Assume σ and $(D\sigma \cdot \sigma)$ follow Hypothesis 1.3.1. Also assume Hypothesis 1.3.5 holds. Then there exist constants $c_{0,x}$, $c_{1,x}$ and $c_{2,x}$ such that for $s,t \in [\lambda_k, \lambda_{k+1})$ satisfying $|t-s| \leq c_{0,x} 2^{-\alpha q_k}$, with $\alpha := \frac{1-\kappa}{\gamma}$, we have the following bounds:*

$$\mathcal{N}\left[y; \mathcal{C}^\gamma_1([s,t])\right] \leq c_{1,x} 2^{-\kappa q_k} \tag{1.28}$$

and

$$\mathcal{N}\left[R; \mathcal{C}^{3\gamma}_2([s,t])\right] \leq c_{2,x} 2^{(2-3\kappa) q_k}. \tag{1.29}$$

Proof We divide this proof in several steps.

Step 1: Setting. Consider the dyadic partition on $[s,t]$. Specifically, we set

$$[\![s,t]\!] = \left\{ t_i : t_i = s + \frac{i(t-s)}{2^n}; i = 0, \cdots, 2^n \right\}$$

for all $n \in \mathbb{N}$. Define y^n on $[\![s,t]\!]$ by setting $y^n_s = y_s$, and

$$\delta y^n_{t_i t_{i+1}} = \sigma(y^n_{t_i}) \delta x_{t_i t_{i+1}} + (D\sigma \cdot \sigma)(y^n_{t_i}) \mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}}$$

We also introduce a discrete type remainder R^n, defined for all $(u,v) \in \mathcal{S}_2([\![s,t]\!])$, as follows:

$$R^n_{uv} = \delta y^n_{uv} - \sigma(y^n_{su}) \delta x_{uv} - (D\sigma \cdot \sigma)(y^n_u) \mathbf{x}^{\mathbf{2}}_{uv}$$

Since $\gamma > 1/3$ and σ is sufficiently smooth away from zero, a second order expansion argument (see [9, Section 10.3]) shows that δy^n_{st} converges to δy_{st}.

Step 2: Induction hypothesis. Recall that we are working in $[\lambda_k, \lambda_{k+1})$. Hence, using (1.23) we can choose n large enough so that

$$y^n_u \in \left[\frac{a_1}{2^{q_k}}, \frac{a_2}{2^{q_k}} \right] \quad \text{for} \quad u \in [\![s,t]\!], \tag{1.30}$$

where $a_1 = \frac{2}{8}$ and $a_2 = \frac{7}{8}$. In addition, using Hypothesis 1.3.1, (1.19) and (1.30) above, we also have

$$|\sigma(y^n_u)| \leq \mathcal{N}_{\kappa,\sigma} |y^n_u|^\kappa \leq \mathcal{N}_{\kappa,\sigma} \left(\frac{a_2}{2^{q_k}} \right)^\kappa \tag{1.31}$$

as well as:

$$|(D\sigma \cdot \sigma)(y^n_u)| \leq \mathcal{N}_{2\kappa-1,\Psi} |y^n_u|^{2\kappa-1} \leq \mathcal{N}_{2\kappa-1,\Psi} \left(\frac{a_2}{2^{q_k}} \right)^{2\kappa-1}. \tag{1.32}$$

We now assume that s and t are close enough, namely for a given constant $c_0 > 0$, we have

$$|t-s| \leq c_0 2^{-\alpha q_k} = T_0. \tag{1.33}$$

We will proceed by induction on the points of the partition t_i. That is, for $q \leq 2^n - 1$ we assume that R^n satisfies the following relation:

$$\mathcal{N}[R^n; \mathcal{C}^{3\gamma}_2 [\![s, t_q]\!]] \leq c_2 2^{(2-3\kappa) q_k} \tag{1.34}$$

where c_2 is a constant to be fixed later. We will try to propagate this induction assumption to $[\![s, t_{q+1}]\!]$.

Step 3: A priori bounds on y^n. For $(u,v) \in \mathcal{S}_2([\![s, t_q]\!])$ we have:

$$\delta y^n_{uv} = \sigma(y^n_u)\delta x_{uv} + (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv} + R^n_{uv}. \tag{1.35}$$

Hence, using (1.31), (1.32) and our induction assumption (1.34) we get:

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma[\![s, t_q]\!]] \leq \mathcal{N}_{\kappa,\sigma}\left(\frac{a_2}{2^{q_k}}\right)^{\kappa} \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi}\left(\frac{a_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_\gamma |t_q - s|^\gamma$$
$$+\mathcal{N}[R^n; \mathcal{C}_2^{3\gamma}[\![s, t]\!]]|t_q - s|^{2\gamma}$$

Since $|t_q - s| \leq T_0 = c_0 2^{-\alpha q_k}$, we thus have

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma[\![s, t_q]\!]] \leq \mathcal{N}_{\kappa,\sigma}\left(\frac{a_2}{2^{q_k}}\right)^{\kappa} \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi}\left(\frac{a_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_\gamma \left(c_0 2^{-\alpha q_k}\right)^\gamma$$
$$+\mathcal{N}[R^n; \mathcal{C}_2^{3\gamma}[\![s, t]\!]]\left(c_0 2^{-\alpha q_k}\right)^{2\gamma}.$$

Therefore taking into account the fact that $\alpha = \dfrac{1-\kappa}{\gamma}$ and our assumption (1.34), we obtain:

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma[\![s, t_q]\!]] \leq \tilde{c}\ 2^{-\kappa q_k} \tag{1.36}$$

where the constant $\tilde{c}$ is given by:

$$\tilde{c} = \mathcal{N}_{\kappa,\sigma} a_2^\kappa \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi} a_2^{2\kappa-1} c_0^\gamma \|\mathbf{x}\|_\gamma + c_2 c_0^{2\gamma}. \tag{1.37}$$

Step 4: Induction propagation. Recall that $R^n_{uv} = \delta y^n_{uv} - \sigma(y^n_{su})\delta x_{uv} - (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}$. Hence invoking Proposition 1.2.2 we have:

$$\delta R^n_{uvw} = \mathcal{A}^{n,1}_{uvw} + \mathcal{A}^{n,2}_{uvw} + \mathcal{A}^{n,3}_{uvw}, \tag{1.38}$$

with

$$\mathcal{A}^{n,1}_{uvw} = -\delta\sigma(y^n)_{uv}\delta x_{vw}, \qquad \mathcal{A}^{n,2}_{uvw} = -\delta((D\sigma \cdot \sigma)(y^n))_{uv}\mathbf{x}^{\mathbf{2}}_{vw}$$

and

$$\mathcal{A}^{n,3}_{uvw} = (D\sigma \cdot \sigma)(y^n_u)\delta\mathbf{x}^{\mathbf{2}}_{uvw}.$$

We now treat those terms separately. The term $\mathcal{A}^{n,1}_{uvw}$ in (1.38) can be expressed using Taylor expansion, which yields

$$\mathcal{A}^{n,1}_{uvw} = -\left(D\sigma(y^n_u)\delta y^n_{uv} + \frac{1}{2} D^2\sigma(\xi^n)\left(\delta y^n_{uv}\right)^2 \right)\delta x_{vw},$$

for some $\xi^n \in [y^n_u, y^n_v]$. Now, using (1.35) the above becomes

$$\begin{aligned}\mathcal{A}^{n,1}_{uvw} =& - D\sigma(y^n_u)\left(\sigma(y^n_u)\delta x_{uv} + (D\sigma\cdot\sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv} + R^n_{uv}\right)\delta x_{vw} - \frac{1}{2}D^2\sigma(\xi^n)\left(\delta y^n_{uv}\right)^2\delta x_{vw} \\ =& - (D\sigma\cdot\sigma)(y^n_u)\delta x_{uv}\delta x_{vw} - D\sigma(y^n_u)(D\sigma\cdot\sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}\delta x_{vw} \\ & - D\sigma(y^n_u)R^n_{uv}\delta x_{vw} - \frac{1}{2}D^2\sigma(\xi^n)(\delta y^n_{uv})^2\delta x_{vw}.\end{aligned} \tag{1.39}$$

Due to Hypothesis 1.2.2, the first term of (1.39) cancels $\mathcal{A}^{n,3}_{uvw}$ in (1.38). Therefore we end up with:

$$\mathcal{A}^{n,1}_{uvw}+\mathcal{A}^{n,3}_{uvw} = -D\sigma(y^n_u)(D\sigma\cdot\sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}\delta x_{vw} - D\sigma(y^n_u)R^n_{uv}\delta x_{vw} - \frac{1}{2}D^2\sigma(\xi^n_w)(\delta y^n_{uv})^2\delta x_{vw}.$$

Taking into account (1.12), (1.17) and (1.18) (similarly to what we did for (1.31)–(1.32)), as well as Hypothesis 1.2.2 and relation (1.33) for $|t-s|$, plus the induction (1.34) on R^n, we easily get:

$$\begin{aligned}\mathcal{A}^{n,1}_{uvw} + \mathcal{A}^{n,3}_{uvw} \le \Bigg\{ & \left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-1}\left(\frac{\tilde{a}_2}{2^{q_k}}\right)^{2\kappa-1}\|\mathbf{x}\|^2_\gamma + \left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-1}\|\mathbf{x}\|_\gamma T_0^\gamma \mathcal{N}[R^n;\mathcal{C}_2^{3\gamma}[\![s,t_q]\!]] \\ & + \frac{1}{2}\left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-2}\|\mathbf{x}\|_\gamma \mathcal{N}[y^n;\mathcal{C}_1^\gamma[\![s,t_q]\!]]^2\Bigg\}|w-u|^{3\gamma},\end{aligned} \tag{1.40}$$

where we have incorporated the constants on the right hand side of inequalities (1.17) inside $\tilde{a}_1$ and that of inequality (1.18) inside $\tilde{a}_2$.

We are now left with the estimation of $\mathcal{A}^{n,2}$. To bound this last term we first use inequality (1.13) with $r = \frac{7}{2}$. Taking into account (1.30), we get

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \mathcal{N}_\Psi(|y^n_u|^{-2(1-\kappa)} + |y^n_v|^{-2(1-\kappa)})|y^n_v - y^n_u|\|\mathbf{x}\|_\gamma|w-v|^{2\gamma}.$$

Invoking (1.30) again and the definition of $\mathcal{N}[y^n;\mathcal{C}_1^\gamma[\![s,t_q]\!]]$, this yields:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \mathcal{N}_\Psi\left(\frac{2^{q_k}}{a_1}\right)^{2(1-\kappa)}\mathcal{N}[y^n;\mathcal{C}_1^\gamma[\![s,t_q]\!]]|v-u|^\gamma\|\mathbf{x}\|_\gamma|w-v|^{2\gamma}.$$

Finally using the a priori bound on y^n stated in (1.36) we obtain:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \leq \mathcal{N}_\Psi \left(\frac{2^{q_k}}{a_1}\right)^{2(1-\kappa)} \tilde{c}2^{-\kappa q_k}\|\mathbf{x}\|_\gamma |w-u|^{3\gamma}, \tag{1.41}$$

which can be recast as:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \leq \frac{\mathcal{N}_\Psi}{a_1^{2(1-\kappa)}}\ \tilde{c}\ \|\mathbf{x}\|_\gamma 2^{(2-3\kappa)q_k}|w-u|^{3\gamma}. \tag{1.42}$$

We can now plug (1.40) and (1.42) back into (1.38) in order to get:

$$\begin{aligned}\mathcal{N}[\delta R^n; \mathcal{C}_3^{3\gamma}[\![s,t_{q+1}]\!]] \leq & \left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-1}\left(\frac{\tilde{a}_2}{2^{q_k}}\right)^{2\kappa-1}\|\mathbf{x}\|_\gamma^2 + \left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-1}\|\mathbf{x}\|_\gamma T_0^\gamma \mathcal{N}[R^n;\mathcal{C}_2^{3\gamma}[\![s,t_q]\!]] \\ & + \frac{1}{2}\left(\frac{\tilde{a}_1}{2^{q_k}}\right)^{\kappa-2}\|\mathbf{x}\|_\gamma \mathcal{N}[y^n;\mathcal{C}_1^\gamma[\![s,t_q]\!]]^2 + \frac{1}{a_1^{2(1-\kappa)}}\mathcal{N}_\Psi\tilde{c}\|\mathbf{x}\|_\gamma 2^{(2-3\kappa)q_k}.\end{aligned}$$

Therefore, thanks to our induction assumption (1.34) and the a priori bound (1.36), the above becomes

$$\mathcal{N}[\delta R^n; \mathcal{C}_3^{3\gamma}[\![s,t_{q+1}]\!]] \leq d2^{(2-3\kappa)q_k}$$

with

$$d = \left(\tilde{a}_1^{\kappa-1}\tilde{a}_2^{2\kappa-1}\|\mathbf{x}\|_\gamma^2 + \tilde{a}_1^{\kappa-1}\|\mathbf{x}\|_\gamma c_0^\gamma c_2 + \frac{1}{2}\tilde{a}_1^{\kappa-2}\tilde{c}^2\|\mathbf{x}\|_\gamma + \frac{1}{\tilde{a}_1^{2(1-\kappa)}}\mathcal{N}_\Psi\tilde{c}\|\mathbf{x}\|_\gamma\right) \tag{1.43}$$

Then using the discrete sewing Lemma 1.4.3, we obtain

$$\mathcal{N}[R^n; \mathcal{C}_2^{3\gamma}[\![s,t_{q+1}]\!]] \leq K_{3\gamma}\mathcal{N}[\delta R^n; \mathcal{C}_3^{3\gamma}[\![s,t_{q+1}]\!]] \leq \hat{c}2^{(2-3\kappa)q_k}, \tag{1.44}$$

where $K_{3\gamma} = \sum_{l=1}^\infty \frac{1}{l^{3\gamma}}$ and $\hat{c} = dK_{3\gamma}$.

Plugging in the value of $\tilde{c}$ from (1.37) in the expression for d in (1.43) we find that $\hat{c}$ can be decomposed as

$$\hat{c} = dK_{3\gamma} = (d_{1,x} + d_{2,x})K_{3\gamma},$$

where

$$d_{1,x} = \left(\tilde{a}_1^{\kappa-1}\tilde{a}_2^{2\kappa-1}\|\mathbf{x}\|_\gamma^2 + \frac{1}{2}\tilde{a}_1^{\kappa-2}\mathcal{N}_{\kappa,\sigma}^2 a_2^{2\kappa}\|\mathbf{x}\|_\gamma^3 + \frac{1}{a_1^{2(1-\kappa)}}\mathcal{N}_\Psi\mathcal{N}_{\kappa,\sigma}a_2^\kappa\|\mathbf{x}\|_\gamma^2\right)$$

and $d_{2,x}$ consist of terms containing positive powers of c_0, where we recall that c_0 is defined by (1.33).

Looking at inequality (1.44), we need $\hat{c}$ to be less than c_2 in order to complete the induction propagation. Let us now fix $c_2 = \frac{3}{2} d_{1,x} K_{3\gamma} = c_{2,x}$ and choose $c_0 = c_{0,x}$ small enough so that $d_{2,x} < \frac{d_{1,x}}{2}$. This implies $\hat{c} = dK_{3\gamma} = (d_{1,x} + d_{2,x}) K_{3\gamma} < \frac{3}{2} d_{1,x} K_{3\gamma} = c_{2,x}$, which is what we required. Our propagation is hence established.

Step 5: Conclusion. Completing the iterations over t_q in $[\![s,t]\!]$ we get that relation (1.34) is valid for $\mathcal{N}[R^n; \mathcal{C}_3^{3\gamma}[\![s,t]\!]]$. Next, put the values of $c_{0,x}$ and $c_{2,x}$ in $\tilde{c}$ as defined in (1.36) and call this new value $c_{1,x}$. We thus get the following uniform bound over n:

$$\mathcal{N}[y^n; \mathcal{C}_1^{\gamma}[\![s,t]\!]] \le c_{1,x} 2^{-\kappa q_k}.$$

Our claims (1.29) and (1.28) are now achieved by taking limits over n. ■

In order to further analyze the increments of y^n, we need to increase slightly the regularity assumptions on x. This is summarized in the following hypothesis:

Hypothesis 1.4.4 *There exists $\varepsilon_1 > 0$ such that for $\gamma_1 = \gamma + \varepsilon_1$, we have $\|\mathbf{x}\|_{\gamma_1} < \infty$.*

The extra regularity imposed on $\mathbf{x}$ allows us to improve our estimates on remainders (in rough path expansions) in the following way.

Proposition 1.4.2 *Let us assume that Hypothesis 1.4.4 holds, as well as Hypothesis 1.3.1 and Hypothesis 1.3.5. For $k \ge 0$, consider $(s,t) \in \mathcal{S}_2([\lambda_k, \lambda_{k+1}])$ such that $|t-s| \le c_{0,x} 2^{-\alpha q_k}$, where $c_{0,x}$ is defined in Theorem 1.4.1. Then the following second order decomposition for δy is satisfied:*

$$\delta y_{st} = \sigma(y_s)\, \delta x_{st} + r_{st}, \quad \text{with} \quad |r_{st}| \le c_{3,x}\, 2^{-\kappa_{\varepsilon_1} q_k} |t-s|^{\gamma}, \tag{1.45}$$

where we have set $\kappa_{\varepsilon_1} = \kappa + 2\varepsilon_1 \alpha$.

Proof From (1.27) we have

$$|r_{st}| = |(D\sigma \cdot \sigma)(y_s) \mathbf{x}^{\mathbf{2}}_{st} + R_{st}| \le |(D\sigma \cdot \sigma)(y_s)||\mathbf{x}^{\mathbf{2}}_{st}| + |R_{st}| \tag{1.46}$$

Under the constraints we have imposed on s,t, namely $s,t \in [\lambda_k, \lambda_{k+1})$ such that $|t-s| \le c_{0,x} 2^{-\alpha q_k}$, and recalling that we have set $\gamma_1 = \gamma + \varepsilon_1$, we have

$$\sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{\gamma}} = \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma+2\varepsilon_1}} |t-s|^{\gamma+2\varepsilon_1} \le \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma_1}} \sup_{s,t} |t-s|^{\gamma+2\varepsilon_1}$$
$$\le \mathcal{N}\left[\mathbf{x}^{\mathbf{2}}; \mathcal{C}_2^{2\gamma_1}\right] (c_{0,x} 2^{-\alpha q_k})^{\gamma+2\varepsilon_1}. \quad (1.47)$$

where we have used $\sup_{s,t}$ to stand for supremum over the set $\{(s,t) : s,t \in [\lambda_k, \lambda_{k+1})$ and $|t-s| \le c_{0,x} 2^{-\alpha q_k}\}$.

Note that under Hypothesis 1.4.4, the quantity $\|\mathbf{x}\|_{\gamma_1}$ is finite and hence (1.47) can be read as:

$$\sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{\gamma}} \le \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-\alpha(\gamma+2\varepsilon_1)q_k}. \quad (1.48)$$

Moreover, owing to (1.29) applied to $\gamma := \gamma + \varepsilon_1$, and κ as in Hypothesis 1.3.1, we get

$$\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} = \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3(\gamma+\varepsilon_1)}} |t-s|^{2\gamma+3\varepsilon_1} \le \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3\gamma_1}} \sup_{s,t} |t-s|^{2\gamma+3\varepsilon_1}$$
$$\le \tilde{c}_{2,x} 2^{(2-3\kappa)q_k} (c_{0,x} 2^{-\alpha q_k})^{2\gamma+3\varepsilon_1}. \quad (1.49)$$

Here we have used the notation $\tilde{c}_{2,x}$ to stand for the coefficient $c_{2,x}$ in (1.29), with $\|\mathbf{x}\|_{\gamma}$ replaced by $\|\mathbf{x}\|_{\gamma_1}$. Thus we have

$$\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} \le \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\alpha(2\gamma+3\varepsilon_1)+3\kappa-2)q_k} \quad (1.50)$$

Now incorporating (1.48) and (1.50) in (1.46), and recalling that $\alpha = \frac{1-\kappa}{\gamma}$, we easily get:

$$\sup_{s,t} \frac{|r_{st}|}{|t-s|^{\gamma}} \le \mathcal{N}_{2\kappa-1,\Psi} \left(\frac{b_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-\alpha(\gamma+2\varepsilon_1)q_k} + \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\alpha(2\gamma+3\varepsilon_1)+3\kappa-2)q_k}$$
$$= \mathcal{N}_{2\kappa-1,\Psi} b_2^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-(\kappa+2\varepsilon_1\alpha)q_k} + \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\kappa+3\varepsilon_1\alpha)q_k}$$

Collecting terms and recalling that we have set $\kappa_{\varepsilon_1} = \kappa + 2\varepsilon_1\alpha$, we end up with:

$$\sup_{s,t} \frac{|r_{st}|}{|t-s|^{\gamma}} \le c_{3,x} 2^{-(\kappa+2\varepsilon_1\alpha)q_k} = c_{3,x} 2^{-\kappa_{\varepsilon_1} q_k},$$

which is our claim (1.45). ■

Thanks to our previous efforts, we can now slightly enlarge the interval on which our improved regularity estimates hold true:

Corollary 1.4.5 *Let the assumptions of Proposition 1.4.2 prevail, and consider* $0 < \varepsilon_1 < 1-\gamma$ *as in Hypothesis 1.4.4. Then with* $\alpha = \gamma^{-1}(1-\kappa)$*, there exists* $0 < \varepsilon_2 < \alpha$ *and a constant* $c_{4,x}$ *such that for all* $(s,t) \in \mathcal{S}_2([\lambda_k, \lambda_{k+1}))$ *satisfying* $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}$ *we have*

$$|\delta y_{st}| \le c_{5,x} 2^{-q_k \kappa_{\varepsilon_2}^-} |t-s|^{\gamma}, \quad \textit{where} \quad \kappa_{\varepsilon_2}^- = \kappa - (1-\gamma)\varepsilon_2. \tag{1.51}$$

Moreover, under the same conditions on (s,t)*, decomposition* (1.45) *still holds true, with*

$$|r_{st}| \le c_{6,x} 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma}, \quad \textit{where} \quad \kappa_{\varepsilon_1,\varepsilon_2} = \kappa + 2\alpha\varepsilon_1 - \gamma\varepsilon_2 - 2\varepsilon_1\varepsilon_2. \tag{1.52}$$

Proof We split our computations in 2 steps.
Step 1: Proof of (1.51). Start from inequality (1.28), which is valid for $|t-s| \le c_{0,x} 2^{-\alpha q_k}$. Now let $m \in \mathbb{N}$ and consider $s,t \in [\lambda_k, \lambda_{k+1})$ such that $c_{0,x}(m-1)2^{-\alpha q_k} < |t-s| \le c_{0,x} m 2^{-\alpha q_k}$. We partition the interval $[s,t]$ by setting $t_j = s + c_{0,x} j 2^{-\alpha q_k}$ for $j = 0, \ldots, m-1$ and $t_m = t$. Then we simply write

$$|\delta y_{st}| \le \sum_{j=0}^{m-1} |\delta y_{t_j t_{j+1}}| \le c_{1,x} 2^{-q_k \kappa} \sum_{j=0}^{m-1} (t_{j+1} - t_j)^{\gamma} \le c_{1,x} 2^{-q_k \kappa} m^{1-\gamma} |t-s|^{\gamma},$$

where the last inequality stems from the fact that $t_{j+1} - t_j \le (t-s)/m$. Now the upper bound (1.51) is easily deduced by applying the above inequality to a generic $m \le [2^{\varepsilon_2 q_k}] + 1$, where $0 < \varepsilon_2 < \frac{\kappa}{1-\gamma}$. This ensures $\kappa_{\varepsilon_2}^- = \kappa - (1-\gamma)\varepsilon_2 > 0$.
Step 2: Proof of (1.52). We proceed as in the proof of Proposition 1.4.2, but now with a relaxed constraint on (s,t), namely $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}$ where $\varepsilon_2 > 0$ satisfies:

$$\varepsilon_2 < \min\left(\frac{\kappa}{1-\gamma}, \frac{\varepsilon_1 \alpha}{\gamma + \varepsilon_1}\right). \tag{1.53}$$

The equivalent of relation (1.49) is thus

$$\begin{aligned} \sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} = \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3(\gamma+\varepsilon_1)}} |t-s|^{2\gamma+3\varepsilon_1} &\le \sup_{s,t} \frac{|R_{s,t}|}{|t-s|^{3\gamma_1}} \sup_{s,t} |t-s|^{2\gamma+3\varepsilon_1} \\ &\le \tilde{c}_{2,x} 2^{(2-3\kappa)q_k} (c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k})^{2\gamma+3\varepsilon_1} \end{aligned} \tag{1.54}$$

As in Proposition 1.4.2 we have used the notation $\tilde{c}_{2,x}$ to stand for the coefficient $c_{2,x}$ with $\|\mathbf{x}\|_\gamma$ replaced by $\|\mathbf{x}\|_{\gamma_1}$ and $\sup_{s,t}$ to stand for supremum over the set $\{(s,t) : s,t \in [\lambda_k, \lambda_{k+1})$ and $|t-s| \leq c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}\}$. Collecting the exponents in (1.54) we thus end up with:

$$\sup_{s,t} \frac{|R_{st}|}{|t-s|^\gamma} \leq \tilde{c}_{2,x} c_{4,x} 2^{-(\kappa+3\varepsilon_1\alpha-2\varepsilon_2\gamma-3\varepsilon_1\varepsilon_2)q_k}. \tag{1.55}$$

Similarly to (1.47), we also get:

$$\begin{aligned}\sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^\gamma} = \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma+2\varepsilon_1}} |t-s|^{\gamma+2\varepsilon_1} &\leq \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma_1}} \sup_{s,t} |t-s|^{\gamma+2\varepsilon_1} \\ &\leq \|\mathbf{x}\|_{\gamma_1} (c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k})^{\gamma+2\varepsilon_1}. \end{aligned} \tag{1.56}$$

Consequently, owing to Hypothesis 1.3.5, we get the following relation:

$$\begin{aligned}|(D\sigma\cdot\sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st}| &\leq \mathcal{N}_{2\kappa-1,\Psi} \left(\frac{b_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{4,x}^{\gamma+2\varepsilon_1} 2^{-(\alpha-\varepsilon_2)(\gamma+2\varepsilon_1)q_k} \\ &= \mathcal{N}_{2\kappa-1,D\sigma\cdot\sigma} b_2^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{4,x}^{\gamma+2\varepsilon_1} 2^{-(\kappa+2\varepsilon_1\alpha-\varepsilon_2\gamma-2\varepsilon_1\varepsilon_2)q_k}. \end{aligned} \tag{1.57}$$

Notice that under the conditions on ε_2 in (1.53), we have $\kappa + 2\varepsilon_1\alpha - \varepsilon_2\gamma - 2\varepsilon_1\varepsilon_2 < \kappa + 3\varepsilon_1\alpha - 2\varepsilon_2\gamma - 3\varepsilon_1\varepsilon_2$. Therefore incorporating (1.55) and (1.57) we have:

$$|r_{st}| \leq |(D\sigma\cdot\sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st}| + |R_{st}| \lesssim 2^{-q_k\kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^\gamma$$

which is our claim (1.52). ∎

1.4.3 Estimates for stopping times

Thanks to the previous estimates on improved regularity for the solution y to equation (1.22), we will now get a sharp control on the difference $\lambda_{k+1} - \lambda_k$. Otherwise stated we shall control the speed at which y might converge to 0, which is the key step in order to control the global Hölder continuity of y. This section is similar to what has been done in [1], and proofs are included for sake of completeness. We start with a lower bound on the difference $\lambda_{k+1} - \lambda_k$.

Proposition 1.4.3 *Assume σ and $(D\sigma \cdot \sigma)$ follow Hypothesis 1.3.1. Also assume Hypothesis 1.3.5 holds. Then the sequence of stopping times $\{\lambda_k,\, k \geq 1\}$ defined by (1.23) satisfies*

$$\lambda_{k+1} - \lambda_k \geq c_{5,x}\, 2^{-\alpha q_k}, \tag{1.58}$$

where we recall that $\alpha = (1-\kappa)/\gamma$.

Proof We show that the difference $\tau_k - \lambda_k$ satisfies a lower bound of the form

$$\tau_k - \lambda_k \geq c_{6,x}\, 2^{-\alpha q_k}. \tag{1.59}$$

There exists a similar bound for $\lambda_{k+1} - \tau_k$, and consequently we get our claim (1.58).

To arrive at inequality (1.59) we observe that in order to leave the interval $[\lambda_k, \tau_k)$, an increment of size at least $2^{-(q_k+1)}$ must occur. This is because at λ_k the solution lies at the mid point of I_{q_k}, an interval of size 2^{-q_k}. Thus, if $|\delta y_{st}| \geq 2^{-(q_k+1)}$ and $|t-s| \leq c_{0,x} 2^{-\alpha q_k}$, relation (1.28) provides us with:

$$c_{1,x} \frac{|t-s|^\gamma}{2^{\kappa q_k}} \geq \frac{1}{2^{q_k+1}}, \tag{1.60}$$

which implies

$$|t-s| \geq (2c_{1,x})^{-\frac{1}{\gamma}}\, 2^{-\frac{(1-\kappa)q_k}{\gamma}} = (2c_{1,x})^{-\frac{1}{\gamma}}\, 2^{-\alpha q_k}.$$

This completes the proof. ■

In order to sharpen Proposition 1.4.3, we introduce a roughness hypothesis on x, again as in [1]. This assumption is satisfied when x is a fractional Brownian motion.

Hypothesis 1.4.6 *We assume that for $\hat{\varepsilon}$ arbitrarily small there exists a constant $c > 0$ such that for every s in $[0,T]$, every ϵ in $(0, T/2]$, and every ϕ in $\mathbb{R}^d$ with $|\phi| = 1$, there exists t in $[0,T]$ such that $\epsilon/2 < |t-s| < \epsilon$ and*

$$|\langle \phi, \delta x_{st} \rangle| > c\, \epsilon^{\gamma + \hat{\varepsilon}}.$$

The largest such constant is called the modulus of $(\gamma + \hat{\varepsilon})$-Hölder roughness of x, and is denoted by $L_{\gamma,\hat{\varepsilon}}(x)$.

Under this hypothesis, we are also able to upper bound the difference $\lambda_{k+1} - \lambda_k$.

Proposition 1.4.4 *Assume σ and $(D\sigma \cdot \sigma)$ follow Hypothesis 1.3.1. Also assume Hypothesis 1.3.5 holds and $\sigma(\xi) \gtrsim |\xi|^{\kappa}$. Then for all $\varepsilon_2 < \frac{\alpha\varepsilon_1}{\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$ and q_k large enough (that is for k large enough, since $\lim_{k\to\infty} q_k = \infty$ under Assumption (B) of Proposition 1.4.1), the sequence of stopping times $\{\lambda_k,\, k \geq 1\}$ defined by (1.23) satisfies*

$$\lambda_{k+1} - \lambda_k \leq c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}, \tag{1.61}$$

where we recall that $\alpha = (1-\kappa)/\gamma$. Furthermore, inequality (1.51) can be extended as follows: there exists a constant c_x such that for $s, t \in [\lambda_k, \lambda_{k+1})$ we have

$$|\delta y_{st}| \leq c_x 2^{-\kappa_{\bar{\varepsilon}_2} q_k} |t-s|^{\gamma}, \tag{1.62}$$

Proof We prove by contradiction. Assume the contrary, that is, (1.61) does not hold. This implies that for some $\varepsilon_2 < \frac{\alpha\varepsilon_1}{\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$

$$\lambda_{k+1} - \lambda_k \geq C 2^{-q_k(\alpha-\varepsilon_2)} \tag{1.63}$$

holds for infinitely many values of k, for any constant C. Consequently

$$\lambda_{k+1} - \lambda_k \geq C\, 2^{-q_k(1-\kappa)/(\gamma+\hat{\varepsilon})}, \tag{1.64}$$

for an $\hat{\varepsilon}$ small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \geq \alpha - \varepsilon_2$. We now show that there exists $s, t \in [\lambda_k, \lambda_{k+1}]$ such that $|\delta y_{st}| > |J_{q_k}|$ providing us with our contradiction. Here $|J_{q_k}|$ denotes the size of the interval J_{q_k}.

To achieve this we now use Hypothesis 1.4.6. Taking into account we are in the one-dimensional case let us choose

$$\varepsilon := \frac{c_1\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}}}{[L_{\gamma,\hat{\varepsilon}}(x)]^{\frac{1}{\gamma+\hat{\varepsilon}}}} \leq C\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}},$$

where the inequality is true for a fixed constant c_1 and a large enough constant C. Due to (1.63) and Hypothesis 1.4.6 there now exist $s, t \in [\lambda_k, \lambda_{k+1}]$ such that

$$\frac{\varepsilon}{2} \leq |t-s| \leq \varepsilon, \quad \text{and} \quad |\delta x_{st}| \geq c_1^{\gamma+\hat{\varepsilon}}\, 2^{-q_k(1-\kappa)}. \tag{1.65}$$

Moreover, due to our assumptions on σ and because $y_s \geq b_1 2^{-q_k} \geq 2^{-q_k-2}$, we have $|\sigma(y_s)| \geq c2^{-q_k\kappa}$ for $s \in [\lambda_k, \lambda_{k+1}]$. Consequently, for s, t as in (1.65)

$$|\sigma(y_s)\delta x_{st}| \geq cc_1^{\gamma+\hat{\varepsilon}}\, 2^{-q_k}.$$

For fixed ε, c_1 can be chosen arbitrarily large (by increasing k or decreasing $\hat{\varepsilon}$) such that $cc_1^{\gamma+\hat{\varepsilon}} \geq 6$. We thus have

$$|\sigma(y_s)\delta x_{st}| \geq 6 \cdot 2^{-q_k} = 2|J_{q_k}|.$$

In particular the size of this increment is larger than twice the size of J_{q_k} (see relation (1.23)).

Recall, $\hat{\varepsilon}$ is small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \geq \alpha - \varepsilon_2$, so that from the bound on $|t-s|$ in (1.65) we have $|t-s| \leq c_{7,x} 2^{-q_k(\alpha-\varepsilon_2)}$. With s, t as in relation (1.65) we use the fact that $\delta y_{st} = \sigma(y_s)\delta x_{st} + r_{st}$ and the bound (1.52) to get

$$|\delta y_{st}| \gtrsim A^1_{st} - A^2_{st}, \quad \text{with} \quad A^1_{st} = 6 \cdot 2^{-q_k}, \quad A^2_{st} \leq c_{6,x} 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma} \leq c_{9,x} 2^{-q_k \mu_{\varepsilon_2}},$$

where we recall that $\kappa_{\varepsilon_1,\varepsilon_2} = \kappa + 2\alpha\varepsilon_1 - \gamma\varepsilon_2 - 2\varepsilon_1\varepsilon_2$ to obtain

$$\mu_{\varepsilon_2} = \kappa_{\varepsilon_1,\varepsilon_2} + (\alpha - \varepsilon_2)\gamma = 1 + 2\alpha\varepsilon_1 - 2(\gamma + \varepsilon_1)\varepsilon_2.$$

Compared to 2^{-q_k}, A^2_{st} can be made negligible for large enough q_k by making sure that $\mu_{\varepsilon_2} > 1$. One can ensure $\mu_{\varepsilon_2} > 1$ by choosing ε_1 large enough and ε_2 small enough. As a consequence $|\delta y_{st}| \gtrsim A^1_{st} - A^2_{st}$, where A^1_{st} is larger than twice $|J_{q_k}| = 3 \cdot 2^{-q_k}$ and A^2_{st} is negligible compared to A^1_{st} as q_k gets large. That is, $|\delta y_{st}| > |J_{q_k}|$ for k large enough. We now have our contradiction and this proves (1.61). ∎

1.4.4 Hölder continuity

Eventually the control of the stopping times λ_k leads to the main result of this section, that is the existence of a $\mathcal{C}^\gamma$ solution to equation (1.22). The crucial step in this direction is detailed in the proposition below. It is achieved under the additional assumption $\gamma + \kappa > 1$, and yields directly the proof of Theorem 1.1.2.

Proposition 1.4.5 *Suppose that our noise x satisfies Hypotheses 1.4.4 and 1.4.6. Assume σ and $(D\sigma\cdot\sigma)$ follow Hypothesis 1.3.1 and Hypothesis 1.3.5 holds as well. Also assume $\sigma(\xi) \gtrsim |\xi|^{\kappa}$ and that $\gamma+\kappa > 1$. Then, the function y given in Proposition 1.4.1 belongs to $\mathcal{C}^{\gamma}([0,T];\mathbb{R}^m)$.*

Proof We start with the assumption that y satisfies condition (B) in Proposition 1.4.1. We first consider $s = \lambda_k$ and $t = \lambda_l$ with $k < l$ and decompose the increments $|\delta y_{st}|$ as:

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j \lambda_{j+1}}\right|.$$

Due to Proposition 1.4.4 we have $\lambda_{k+1} - \lambda_k \leq c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}$ for a large enough k. An application of Corollary 1.4.5 yields

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j \lambda_{j+1}}\right| \leq c_{5,x} \sum_{j=k}^{l-1} 2^{-q_j \kappa_{\varepsilon_2}^{-}} |\lambda_{j+1} - \lambda_j|^{\gamma}. \tag{1.66}$$

Rewriting inequality (1.58),

$$2^{-\frac{q_j(1-\kappa)}{\gamma}} \leq c_{7,x}^{-1} \left(\lambda_{j+1} - \lambda_j\right)$$

which implies

$$2^{-q_j \kappa_{\varepsilon_2}^{-}} \leq \left(c_{7,x}\right)^{-\frac{\gamma \kappa_{\varepsilon_2}^{-}}{1-\kappa}} \left(\lambda_{j+1} - \lambda_j\right)^{\frac{\gamma \kappa_{\varepsilon_2}^{-}}{1-\kappa}}.$$

Using this inequality in (1.66) and defining $c_{8,x} = c_{5,x}(c_{7,x})^{-\frac{\gamma\kappa_{\varepsilon_2}^{-}}{1-\kappa}}$, we get:

$$|\delta y_{st}| \leq c_{8,x} \sum_{j=k}^{l-1} |\lambda_{j+1} - \lambda_j|^{\tilde{\mu}_{\varepsilon_2}}, \quad \text{where} \quad \tilde{\mu}_{\varepsilon_2} = \gamma\left(1 + \frac{\kappa_{\varepsilon_2}^{-}}{1-\kappa}\right).$$

Recall $\kappa_{\varepsilon_2}^{-} = \kappa - (1-\gamma)\varepsilon_2$, which can be made arbitrarily close to κ. Hence under the assumption $\gamma + \kappa > 1$, $\tilde{\mu}_{\varepsilon_2}$ is of the form $\tilde{\mu}_{\varepsilon_2} = 1 + \varepsilon_3$. We thus obtain

$$|\delta y_{st}| \leq c_{8,x} \sum_{j=k}^{l-1} |\lambda_{j+1} - \lambda_j|^{1+\varepsilon_3} \leq c_{8,x} |\lambda_l - \lambda_k|^{1+\varepsilon_3} \leq c_{8,x}\, \tau^{1+\varepsilon_3-\gamma} |t-s|^{\gamma},$$

where we recall $s = \lambda_k$ and $t = \lambda_l$. Having proved our claim for this special case, the general case for $s < \lambda_k \leq \lambda_l < t$ is obtained by the following decomposition

$$\delta y_{st} = \delta y_{s\lambda_k} + \delta y_{\lambda_k \lambda_l} + \delta y_{\lambda_l t}.$$

Finally, we make use of (1.62) in order to bound $\delta y_{s\lambda_k}$ and $\delta y_{\lambda_l t}$. ∎

2. QUENCHED ASYMPTOTICS FOR A 1-D STOCHASTIC HEAT EQUATION DRIVEN BY A ROUGH SPATIAL NOISE

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2.1 Introduction

The non trivial effects of random perturbations on the spectrum of the Laplace operator has been a fascinating object of research in the recent past. While a direct spectral analysis of perturbed Laplacians is possible in simple and regular enough cases [17, 18], the problem is often addressed through the large time behavior of the so-called parabolic Anderson model. More specifically the parabolic Anderson model is a stochastic heat equation of the following form:

$$\frac{\partial u_t(x)}{\partial t} = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}(x), \tag{2.1}$$

where the noise $\dot{W}$ is a stationary spatial random field. Because of the linear form of the noise term, it is possible under certain regularity conditions to express the solution of (2.1) using a Feynman-Kac representation. Related to this representation, the asymptotic behavior of $u_t(x)$ as t goes to ∞ gives some insight on the spectrum of the operator $\frac{1}{2}\Delta + \dot{W}$.

In the spatially discrete setting with a discrete Laplacian, asymptotic equivalents for the solution of equation (2.1) have been studied at length in [19] and [3]. In particular, if $u_t(x)$ is the solution under the discrete setup in $\mathbb{Z}^d$ and $U(t) = \sum_{z\in\mathbb{Z}^d} u_t(x)$ is the total mass, then it has been proven that both $\frac{1}{t}\log(u_t(x))$ and $\frac{1}{t}\log(U(t))$ converge almost surely under certain regularity assumptions. Any information about those limits can then be translated into an information about the principal eigenvalue of $\frac{1}{2}\Delta + \dot{W}$.

In the spatially continuous setting, the picture is not as clear. Indeed, the large time behavior of the solution u to equation (2.1) has been analyzed in [20] and [21]. In particular, when the noise is Gaussian with a smooth covariance structure given by $\gamma(x) = \mathrm{Cov}(\dot{W}(0)\dot{W}(x))$ satisfying $\lim_{|x|\to\infty}\gamma(x) = 0$, then we have for $x \in \mathbb{R}^d$

$$\lim_{t\to\infty} \frac{1}{t\sqrt{\log t}} \log u_t(x) = \sqrt{2d\gamma(0)} \quad \text{a.s.} \tag{2.2}$$

The fact that the renormalization in (2.2) is of the form $t\sqrt{\log t}$ suggests that the principal eigenvalue of $\frac{1}{2}\Delta + \dot{W}$ is divergent, which is confirmed in [22, 23] by asymptotics on large boxes performed for the white noise.

Motivated by the examples above, non-smooth cases of equation (2.1) under the setting of generalized Gaussian fields have been analyzed in [4]. Namely, the reference [4] handles the case of a centered Gaussian noise W whose covariance function Λ is defined informally (see Section 2.2.2 for more precise definition) by

$$\mathbf{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}^d} \phi(x)\psi(y)\Lambda(x-y)dxdy, \tag{2.3}$$

for all infinite differentiable functions ϕ with compact support. The class of functions Λ considered in [4] are continuous on $\mathbb{R}^d\backslash\{0\}$, bounded away from 0 with a singularity at 0 measured by $\Lambda(x) \sim c(\Lambda)|x|^{-\alpha}$ with $\alpha \in (0, 2\wedge d)$ as $x \downarrow 0$. In this framework, the following almost sure renormalization result is proved in [4] for any $x \in \mathbb{R}^d$:

$$\lim_{t\to\infty} \frac{1}{t(\log t)^{\frac{2}{4-\alpha}}} \log u_t(x) = c_\alpha \quad \text{a.s. }, \tag{2.4}$$

with an explicit constant c_α. Notice that this result is also applicable under a fractional white noise with Hurst parameter $H > \frac{1}{2}$. Namely, considering $d = 1$ for simplicity, relation (2.4) holds for a fractional Brownian noise W with $\alpha = 2 - 2H$ (that is a renormalization of the form $t(\log t)^{\frac{1}{1+H}}$).

In this note we aim to carry forward the asymptotic result (2.4) to very singular environments. Specifically, we consider a fractional noise W as in [4], but we allow the

Hurst parameter to be less than $\frac{1}{2}$ (so that our noise is rougher than white noise). Going back to expression (2.3), we assume that Λ is a positive definite distribution whose Fourier transform $\mathcal{F}\Lambda = \mu$ is a tempered measure given by $\mu(d\xi) = C_H|\xi|^{1-2H}d\xi$. That is for test functions ϕ and ψ we have

$$\mathbf{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.5}$$

Let us first notice that equation (2.1) driven by a fBm with $H < \frac{1}{2}$ is not explicitly solved in the literature. As we will see, one can give a pathwise meaning, in a Young type sense, to the solution of equation (2.1). Namely we show that $t \mapsto u_t$ can be seen as a continuous function with values in a weighted Besov space (we refer to [24] for a complete definition of weighted Besov spaces). We will set up a fixed point argument in those weighted spaces and obtain the following result (see Theorem 2.3.10 for a more precise formulation).

Theorem 2.1.1 *Let W be the Gaussian noise considered in (2.5) with $H \in \left(0, \frac{1}{2}\right)$. Let u_0 be an initial condition lying in a weighted Besov Hölder space (see Definition 2.2.4 or a more detailed description). Then there exists a unique solution to (2.1) in a space of continuous functions with values in Besov spaces, and where the integral with respect to W is understood in the Young sense.*

Once we have solved (2.1) , we will give a property of the (formal) operator $\frac{1}{2}\Delta + \dot{W}$ which is reminiscent of the density of states results contained e.g. in [17, 18]. The result we obtain can be summarized informally in the following theorem:

Theorem 2.1.2 *Let $\lambda_{\dot{W}}(Q_t)$ be the principal eigenvalue of the random operator $\frac{1}{2}\Delta + \dot{W}$ over a restricted space of functions having compact support on $Q_t := (-t, t)$. Then the following limit holds:*

$$\lim_{t\to\infty} \frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{\frac{1}{1+H}}} = (2c_H\mathcal{E})^{\frac{1}{1+H}} \quad \text{a.s,} \tag{2.6}$$

with a strictly positive constant $\mathcal{E}$ defined by

$$\mathcal{E} = \sup_{g\in\mathcal{G}(\mathbb{R})} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{\imath\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda \tag{2.7}$$

where $\mathcal{G}(\mathbb{R})$ *is the space of all Schwartz functions satisfying* $\|g\|_2^2 + \frac{1}{2}\|g'\|_2^2 = 1$.

Using a Feynman-Kac representation for the solution u of (2.1), our next step will be to relate the logarithmic behavior of u_t to the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$. This is the content of the following theorem:

Theorem 2.1.3 *Let* W *be the Gaussian noise defined by* (2.5) *for* $H < \frac{1}{2}$*, and consider the unique solution* u *to* (2.1)*. Then for all* $x \in \mathbb{R}$ *we have*

$$\lim_{t\to\infty} \frac{\frac{1}{t}\log(u_t(x))}{\lambda_{\dot{W}}(Q_t)} = 1, \quad \text{a.s.}$$

As the reader might conceive, our main asymptotic result will be a simple consequence of Theorems 2.1.2 and 2.1.3. It gives a generalization of (2.4) to the case $H < \frac{1}{2}$.

Theorem 2.1.4 *Under the same conditions as in Theorem 2.1.3 and for* $H < \frac{1}{2}$ *we have*

$$\lim_{t\to\infty} \frac{\log(u_t(x))}{t(\log t)^{\frac{1}{1+H}}} = (2c_H\mathcal{E})^{\frac{1}{1+H}}, \quad \text{a.s.} \tag{2.8}$$

Remark 2.1.5 *Let us draw the reader's attention to the fact that formula* (2.8) *has already been proved in [4] for* $H \in [1/2, 1)$ *and hence our contributions in this paper imply that formula* (2.8) *holds for all* $H \in (0, 1)$.

Let us say a few words about the methodology we have resorted to in order to get our main results.

(i) Theorem 2.1.2 is obtained by splitting the eigenvalue problem into small intervals, similarly to what is performed in other parabolic Anderson model studies (see e.g [3] and [4]). Then on each subdomain we combine some variational arguments with supremum computations for Gaussian processes. An extra care is required in our case, due to the singularity of our noise.

(ii) Theorem 2.1.3 relies on a Feynman-Kac representation of $u_t(x)$, whose main ingredient is an integrability property established thanks to a subtle sub-additive argument (see Proposition 2.4.1 below). Once this Feynman-Kac representation (involving

a Brownian motion B) is given, a probabilistic cutoff procedure on the underlying Brownian motion B allows to reduce the logarithmic behavior of $u_t(0)$ to the quantity $\lambda_{\dot{W}}(Q_t)$.

(iii) As mentioned above, Theorem 2.1.4 is an easy consequence of Theorems 2.1.2 and 2.1.3.

Eventually let us highlight the fact that Theorem 2.1.4 provides a rather complete description of the asymptotic behavior of $\log(u_t(x))$ in dimension 1. A very challenging situation would be to handle the case of a rough noise in dimension 2 or higher. In this case it is a well known fact that a renormalization procedure is needed to define the solution u of (2.1), as shown e.g. in [25]. The effect of this kind of renormalization procedure on the principal eigenvalue of $\frac{1}{2}\Delta + \dot{W}$ has been partially investigated for the space white noise when $d = 2$ in [22].

This paper is organized as follows. Section 2.2 contains some preliminaries on Besov spaces and the structure of our noise. In Section 2.3 we prove the existence and uniqueness of our solution as outlined in Theorem 2.1.1. The Feynman-Kac representation of the solution is obtained in Section 2.4. The upper and lower bounds to the long-time asymptotics of the principal eigenvalue of the operator $\frac{1}{2}\Delta + \dot{W}$ are obtained in Subsections 2.5.2 and 2.5.3 respectively. The asymptotic relation between the solution and the principal eigenvalue of the previous section is completed in Section 2.6.

Notations. We denote by $p_t(x)$ the one-dimensional heat kernel $p_t(x) = (2\pi t)^{-1/2}\, e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}$. The space of real valued infinitely differentiable functions with compact support is denoted by $\mathcal{D}(\mathbb{R})$. The space of Schwartz functions is denoted by $\mathscr{S}(\mathbb{R})$. Its dual, the space of tempered distributions, is $\mathscr{S}'(\mathbb{R})$. The Fourier transform is defined as

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}} e^{-\iota\langle x,\xi\rangle} u(x)dx.$$

The inverse Fourier transform is $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-1}\mathcal{F}u(-\xi)$. Denote by l the following probability density function in $\mathcal{S}(\mathbb{R})$:

$$l(x) = c\exp\left(-\frac{1}{1-x^2}\right)\mathbf{1}_{(|x|<1)},$$

where c is a normalizing constant such that $\int_{\mathbb{R}} l(x)dx = 1$. For every $\varepsilon > 0$, let the set of mollifiers generated by l be given by $l_\varepsilon(x) = \varepsilon^{-1}l(\varepsilon^{-1}x)$. Observe that, owing to the fact that l is a probability measure, we have $\lim_{\xi\to 0}\mathcal{F}l(\xi) = 1$ and $\mathcal{F}l(\xi) \leq 1$ for all $\xi \in \mathbb{R}$.

2.2 Preliminaries

This section is devoted to introduce the basic Besov spaces notions which will be used in the remainder of the paper. Observe that since we are dealing with a variable x in the whole space $\mathbb{R}$, we will need to deal with weighted Besov spaces. The definitions and main properties of those spaces are borrowed from [24].

2.2.1 Besov spaces

In this subsection we define some classes of weights which are compatible with our heat equation (2.1). Two scales of weights will be used: stretched exponential weights and polynomial weights.

Definition 2.2.1 *Let* $|x|_* = \sqrt{1+|x|^2}$ *and fix* $\delta \in (0,1)$. *Denote by* $\mathcal{W}$ *the class of weights consisting of:*

(i) the weights w_γ *of the form* $w_\gamma(x) = e^{-\gamma|x|_*^\delta}$, *with* $\gamma > 0$.

(ii) the weights $\hat{w}_\sigma$ *of the form* $\hat{w}_\sigma(x) = |x|_*^{-\sigma}$, *with* $\sigma > 0$.

The definition of our Besov spaces depends heavily on a dyadic partition of unity. In order to handle weights as in Definition 2.2.1 we have to work (as done in [24]) with functions in the so-called Gevrey class, that we now proceed to define.

Definition 2.2.2 *Let $\theta \geq 1$. We call $\mathcal{G}^\theta$, the set of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\text{for every compact } K, \text{ there exists } C < \infty \text{ such that for every } n \in \mathbb{N},$$

$$\sup_K |\partial^n f| \leq C^{n+1}(n!)^\theta.$$

We let $\mathcal{G}_c^\theta$ be the set of compactly supported functions in $\mathcal{G}^\theta$.

We are now ready to state the existence of a partition of unity in the Gevrey class $\mathcal{G}_c^\theta$.

Proposition 2.2.1 *One can construct two functions $\tilde{\chi}, \chi \in \mathcal{G}_c^\theta$, taking values in $[0,1]$ and such that*

(i) $\operatorname{Supp} \tilde{\chi} \subseteq \left[0, \frac{4}{3}\right]$ *and* $\operatorname{Supp} \chi \subseteq \left[\frac{3}{4}, \frac{8}{3}\right]$.

(ii) For all $\xi \in \mathbb{R}$, we have $\tilde{\chi}(\xi) + \sum_{k=0}^\infty \chi(2^{-k}\xi) = 1$.

In the sequel we also set $\chi_k(\xi) = \chi(2^{-k}\xi)$ for $k \geq 0$.

With the partition of unity in hand, the blocks $\Delta_k u$ of the Besov type analysis can be defined as follows.

Definition 2.2.3 *Set $\chi_{-1} = \tilde{\chi}$, and define for $k \geq -1$ and $u \in \mathcal{S}(\mathbb{R})$,*

$$\Delta_k u = \mathcal{F}^{-1}(\chi_k \hat{u}).$$

Our analysis will rely on Besov spaces defined through the weighted blocks introduced in Definition 2.2.3.

Definition 2.2.4 *Let χ and $\tilde{\chi}$ be the functions introduced in Proposition 2.2.1. For any $\kappa \in \mathbb{R}$, $w \in \mathcal{W}$, $p, q \in [1, \infty]$ and $f \in \mathcal{S}(\mathbb{R})$, we define weighted norms of f in the following way:*

$$\|f\|_{\mathcal{B}_{p,q}^{\kappa,w}} := \left[\sum_{j=-1}^\infty \left(2^{\kappa j}\|\Delta_j f\|_{L_w^p}\right)^q\right]^{\frac{1}{q}}, \tag{2.9}$$

where L_w^p is the weighted space $L^p(\mathbb{R}, w(x)dx)$. Denote the weighted Besov space $\mathcal{B}_{p,q}^{\kappa,w}$ as

$$\mathcal{B}_{p,q}^{\kappa,w} = \left\{f \in \mathcal{S}(\mathbb{R}); \|f\|_{\mathcal{B}_{p,q}^{\kappa,w}} < \infty\right\}.$$

Remark 2.2.1 *Notice that as in [26], we define* $\|f\|_{L^p(\mathbb{R}^d;w(x)dx)}$ *as* $\|fw\|_{L^p(\mathbb{R}^d)}$*. This is slightly different from [24], but yields similar results.*

In the next section we will solve the heat equation in a weighted Besov space whose weight is varying with time. We now define this kind of space.

Notation 2.2.2 *Let* λ *and* σ *be two strictly positive constants. For* $t \geq 0$ *we define* v *as the function* $v_t = w_{\lambda+\sigma t}$*, where we recall that* w_γ *is introduced in Definition 2.2.1. We consider an additional parameter* $\kappa_u > 0$ *and* $q \in [1, \infty)$*. Then the space* $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ *is defined by*

$$\mathcal{C}_q^{\kappa_u,\lambda,\sigma} = \left\{ f \in \mathcal{C}([0,T] \times \mathbb{R}); \|f_t\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}} \leq c_f \right\}.$$

2.2.2 Description of the noise

The noise driving equation (2.1) is considered as a centered Gaussian family $\{W(\phi), \phi \in \mathcal{D}(\mathbb{R})\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the following covariance structure:

$$\mathbf{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}^2} \phi(x)\psi(y)\Lambda(x-y)dxdy, \tag{2.10}$$

where $\Lambda : \mathbb{R} \mapsto \mathbb{R}_+$ is a non-negative definite distribution. In fact the covariance structure of W is better described in Fourier modes. Indeed, the distribution Λ can be seen as the inverse Fourier transform of a measure μ on $\mathbb{R}$ defined by

$$\mu(d\xi) = c_H |\xi|^{1-2H} d\xi.$$

Then for $\phi, \psi \in \mathcal{D}(\mathbb{R})$ we have

$$\mathbf{E}\left[W(\phi)W(\psi)\right] = \int_{\mathbb{R}} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi). \tag{2.11}$$

It can be shown that (2.11) defines an inner product on $\mathcal{D}(\mathbb{R})$. We call $\mathcal{H}$ the completion of $\mathcal{D}(\mathbb{R})$ with this inner product. It also holds that the variance of our noise W has an alternate direct-coordinate representation (see e.g. [27, relation (2.8)]) in addition to the one suggested by (2.11). Namely for $\phi \in \mathcal{H}$, we have

$$\mathbf{E}[W(\phi)]^2 = c_H \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\phi(x+y) - \phi(x)|^2}{|y|^{2-2H}} dxdy. \tag{2.12}$$

The mapping $\phi \mapsto W(\phi)$ defined in $\mathcal{D}(\mathbb{R})$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by W. This isometry will be denoted by

$$W(\phi) = \int_{\mathbb{R}} \phi(x) W(dx). \tag{2.13}$$

Remark 2.2.3 *Notice that the measure $\mu(d\xi) = c_H |\xi|^{1-2H} d\xi$ satisfies the following condition*

$$\int_{\mathbb{R}} \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}} < \infty, \qquad \textit{for } \alpha < H. \tag{2.14}$$

This relation will be crucial in order to see that W belongs to a weighted Besov space in the proposition below.

Before a complete description of our noise regularity, we state below a Besov embedding result for weighted Besov spaces. Notice that this embedding result is part of the folklore in the analysis literature. However, we include a complete proof here since we haven't been able to spot a precise reference. In particular our result (2.20) doesn't hold true in the setting of [24], for which the weights behave differently from ours. We start off with a version of Bernstein's lemma for our weighted spaces.

Lemma 2.2.4 *Let B be a ball. For every $p \geq q \in [1, \infty]$ and non-negative integer k, there exists $C < \infty$ such that for every $\lambda \geq 1$ and $f \in \mathcal{S}(\mathbb{R})$ we have*

$$\mathrm{Supp} \hat{f} \subset \lambda B \Rightarrow \|\partial^k f\|_{L^p_{\hat{w}_\sigma}} \leq C \lambda^{k+(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q_{\hat{w}_\sigma}}, \tag{2.15}$$

where the weight $\hat{w}_\sigma$ is given in Definition 2.2.1

Proof The proof is similar to that in [24]. Due to the differences in definition of weighted Besov space it is still provided here. Moreover we will only consider the situation where p, q are finite, the proof when at least one of them is infinite being similar.

Let $\phi \in \mathcal{G}_\theta^c$ be such that $\phi = 1$ on B. Define $\phi_\lambda = \phi\left(\frac{\cdot}{\lambda}\right)$. Observe that

$$f = \mathcal{F}^{-1}\left(\hat{f}\phi_\lambda\right) = g_\lambda \star f, \tag{2.16}$$

where the function g_λ is defined by $g_\lambda = \mathcal{F}^{-1}\phi_\lambda = \lambda g_1(\lambda\cdot)$. Writing $g_\lambda^{(k)} := (\partial^k g_1)_\lambda = \lambda(\partial^k g_1)(\lambda\cdot)$, we can differentiate (2.16) in order to get

$$\partial^k f = \lambda^k g_\lambda^{(k)} \star f.$$

Notice that our weight $\hat{w}_\sigma$ satisfies $\hat{w}_\sigma(x+y) \lesssim \hat{w}_{-\sigma}(x)\hat{w}_\sigma(y)$. Using this and the weighted Young inequality [24, Theorem 2.4] we have:

$$\lambda^{-k}\|\partial^k f\|_{L^p_{\hat{w}_\sigma}} = \|(g_\lambda^{(k)} \star f)\hat{w}_\sigma\|_{L^p} \lesssim \|g_\lambda^{(k)}\hat{w}_{-\sigma}\|_{L^r}\|f\hat{w}_\sigma\|_{L^q}, \tag{2.17}$$

where r is such that $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. Since $g_\lambda^{(k)} = \lambda(\partial^k g_1)(\lambda\cdot)$ and $\partial^k g_1$ is the inverse Fourier transform of a function in $\mathcal{G}_c^\theta$ we have due to [24, Proposition 2.2]:

$$\left|g_\lambda^{(k)}(x)\right| \lesssim \lambda e^{-c|\lambda x|^{1/\theta}} \leq \lambda e^{-c|\lambda x|^\delta}.$$

Consequently, recalling the norm $|\cdot|_*$ introduced in Definition 2.2.1, we obtain

$$\begin{aligned}\|g_\lambda^{(k)}\hat{w}_{-\sigma}\|_{L^r} &\lesssim \lambda\Big(\int e^{-cr|\lambda x|^\delta}|x|_*^{\sigma r}dx\Big)^{\frac{1}{r}} \\ &= \lambda^{1-1/r}\left(\int e^{-cr|x|^\delta}\left|\frac{x}{\lambda}\right|_*^{\sigma r}dx\right)^{\frac{1}{r}}.\end{aligned} \tag{2.18}$$

Observe that $\lambda \geq 1$ implies $|x/\lambda|_* \leq |x|_*$. Thus the integral in (2.18) can be bounded above by a constant independent of λ. In addition, it holds that $1 - \frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. We thus end up with the following relation:

$$\|g_\lambda^{(k)}\hat{w}_{-\sigma}\|_{L^r} \leq C\lambda^{(\frac{1}{q}-\frac{1}{p})}. \tag{2.19}$$

Using (2.19) in (2.17) yields our desired result (2.15). ■

The Besov embedding result we need in order to quantify our noise regularity is now a direct consequence of Lemma 2.2.4 and is provided below.

Proposition 2.2.2 *Let $\kappa > \kappa' > 0$. There exists $K < \infty$ and q large enough such that*

$$\|f\|_{\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}} \leq K\|f\|_{\mathcal{B}^{-\kappa',\hat{w}_\sigma}_{2q,2q}}. \tag{2.20}$$

Proof From the definition of $\Delta_k f$, observe that the support of its Fourier transform is contained in $2^k B$ for some ball B in $\mathbb{R}$. Thus we may apply Lemma 2.2.4 with $k = 0$ and $p = \infty$ to obtain:

$$\|\Delta_k f\|_{L^\infty_{\hat{w}_\sigma}} \leq C 2^{\frac{k}{q}} \|\Delta_k f\|_{L^q_{\hat{w}_\sigma}}.$$

Replacing q by $2q$ and premultiplying by $2^{-\kappa k}$ we obtain:

$$\begin{aligned} 2^{-\kappa k}\|\Delta_k f\|_{L^\infty_{\hat{w}_\sigma}} &\leq C 2^{(-\kappa+\frac{1}{2q})k} \|\Delta_k f\|_{L_{\hat{w}_\sigma^{2q}}} \\ &= C 2^{(\kappa'-\kappa+\frac{1}{2q})k} 2^{-\kappa' k} \|\Delta_k f\|_{L_{\hat{w}_\sigma^{2q}}}. \end{aligned} \tag{2.21}$$

Fix a q large enough such that $\omega := \kappa - \kappa' - \frac{1}{2q}$ is positive. Denoting $2^{-\kappa k}\|\Delta_k f\|_{L^\infty_{\hat{w}_\sigma}}$ by x_k and $2^{-\kappa' k}\|\Delta_k f\|_{L^{2q}_{\hat{w}_\sigma}}$ by y_k, Eq (2.21) can be restated as:

$$2^{\omega k} x_k \leq C y_k.$$

This implies that

$$\|y\|_{\ell^{2q}} = \left(\sum_{k=-1}^{\infty} y_k^{2q}\right)^{1/2q} \geq \frac{1}{C}\left(\sum_{k=-1}^{\infty} 2^{2q\omega k} x_k^{2q}\right)^{1/2q}.$$

Since $2^{2q\omega k} \geq \frac{1}{2^{2q\omega}}$ for $k \geq -1$ we obtain

$$\begin{aligned} \|y\|_{\ell^{2q}} &\geq \frac{1}{C}\left(\sum_{k=-1}^{\infty} \frac{1}{2^{2q\omega}} x_k^{2q}\right)^{1/2q} \\ &= \frac{1}{C2^\omega}\left(\sum_{k=-1}^{\infty} x_k^{2q}\right)^{1/2q} \geq \frac{1}{C2^\omega}\|x\|_{\ell^\infty}, \end{aligned} \tag{2.22}$$

where in the last inequality we have used the fact that $\|x\|_{\ell^{2q}} \geq \|x\|_{\ell^\infty}$ for any sequence $x \in \mathbb{R}^{\mathbb{N}}$. Applying inequality (2.22) to the sequences x and y given in (2.21), we now arrive at (2.20). ■

Proposition 2.2.3 *For all $\kappa \in (1 - H, 1)$ and every arbitrary $\sigma > 0$, W has a version in $\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}$, where $\hat{w}_\sigma$ is given in Definition 2.2.1 and $\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}$ is introduced in Definition 2.2.4. In addition the random variable $\|W\|_{\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}}$ has moments of all orders.*

Proof Consider κ, κ' such that $\kappa > \kappa' > 1-\alpha$, where α is defined by (2.14). Due to (2.14) observe that this implies that we can consider any $\kappa > 1-H$. For $q \geq 1$, denote the Besov space $\mathcal{B}^{-\kappa',\hat{w}_\sigma}_{2q,2q}$ by $\mathcal{A}_q$. Invoking Proposition 2.2.2 observe that for large enough q, $\mathcal{A}_q$ is continuously embedded in $\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}$, i.e.,

$$\|\dot{W}\|_{\mathcal{B}^{-\kappa,\hat{w}_\sigma}_{\infty,\infty}} \lesssim \|\dot{W}\|_{\mathcal{A}_q}. \tag{2.23}$$

Hence it is enough to work with $\|\dot{W}\|_{\mathcal{A}_q}$.

Let us now evaluate the quantity $\|\dot{W}\|_{\mathcal{A}_q}$. To this aim, notice that $\Delta_j f(x) = [K_j * f](x)$ where $K_j(z) = 2^j \mathcal{F}^{-1}\chi(2^j z)$. Therefore, using the notation $K_{j,x}(y) = K_j(x-y)$ we obtain:

$$\mathbf{E}\left[\|\dot{W}\|^{2q}_{\mathcal{A}_q}\right] = \sum_{j\geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}} \mathbf{E}\left[|W(K_{j,x})|^{2q}\right] \hat{w}^{2q}_\sigma(x)dx. \tag{2.24}$$

Using the fact that $W(K_{j,x})$ is Gaussian we thus have

$$\mathbf{E}\left[|W(K_{j,x})|^{2q}\right] \leq c_q \mathbf{E}^q\left[|W(K_{j,x})|^2\right].$$

Consequently, (2.24) can be recast as:

$$\mathbf{E}\left[\|\dot{W}\|^{2q}_{\mathcal{A}_q}\right] \leq c_q \sum_{j\geq -1} 2^{-2qj\kappa'} \int_{\mathbb{R}} \mathbf{E}^q\left[|W(K_{j,x})|^2\right] \hat{w}^{2q}_\sigma(x)dx. \tag{2.25}$$

Now let us work with $\mathbf{E}\left[|W(K_{j,x})|^2\right]$. According to (2.10) we have

$$\mathbf{E}\left[|W(K_{j,x})|^2\right] = \int_{\mathbb{R}} |\mathcal{F}K_{j,x}(\xi)|^2 \mu(d\xi).$$

Let us introduce a new measure ν on $\mathbb{R}$ defined by $\nu(d\xi) = \frac{\mu(d\xi)}{1+|\xi|^{2(1-\alpha)}}$. Notice that due to (2.14), ν is a finite measure. Since $K_j = \mathcal{F}^{-1}\chi_j$ and the support of χ is in a closed interval, say $[a,b]$, we obtain:

$$\begin{aligned}\mathbf{E}\left[|W(K_{j,x})|^2\right] &= \int_{\mathbb{R}} \left|\chi(2^{-j}\xi)\right|^2 \mu(d\xi) \leq \int_{\mathbb{R}} \mathbf{1}_{[0,2^j b]}(|\xi|)\left(1+|\xi|^{2(1-\alpha)}\right)\nu(d\xi) \\ &\leq \nu\left(\left[0,2^j b\right]\right)\left(1+(2^j b)^{2(1-\alpha)}\right) \leq c_\mu 2^{2(1-\alpha)j}.\end{aligned} \tag{2.26}$$

Therefore plugging (2.26) into (2.25) and recalling that $1-\alpha<\kappa'<\kappa$, we get:

$$\begin{aligned}\mathbf{E}\left[\|\dot{W}\|_{\mathcal{A}_q}^{2q}\right] &\leq c_q \sum_{j\geq -1} 2^{-2qj\kappa'}\int_{\mathbb{R}} c_\mu^q 2^{(1-\alpha)jq}\hat{w}_\sigma^{2q}(x)dx \\ &= C_{q,\mu}\left(\int_{\mathbb{R}}\hat{w}_\sigma^{2q}(x)dx\right)\sum_{j\geq -1}2^{2qj(1-\alpha-\kappa')}. \end{aligned} \tag{2.27}$$

Owing to the Definition 2.2.1 of $\hat{w}_\sigma$, it is now readily checked that the right hand side of (2.27) is convergent whenever q is large enough.

Similar calculations as the ones leading to (2.27) also show that the random variable $\|W\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_\sigma}}$ has moments of all orders. ■

2.3 Pathwise solution

Now that we have proved that our noise $\dot{W}$ is almost surely an element of $\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_\sigma}$, we will transform our stochastic eq. (2.1) into a deterministic one, which will be solved in the Riemann-Stieltjes sense. We first label an assumption on a general distribution driving the heat equation.

Hypothesis 2.3.1 *Let $\delta\in(0,1)$ be a fixed constant and $\sigma>0$ be an arbitrarily small constant. We consider a distribution $\mathscr{W}$ on $\mathbb{R}$ such that $\mathscr{W}\in\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\sigma\delta}}$ with $\kappa\in(0,1)$.*

Remark 2.3.2 *The constant $\delta\in(0,1)$ in Hypothesis 2.3.1 is related to the exponential weights in Definition 2.2.1.*

We now introduce the notion of solution for equation (2.1) which will be considered in the sequel.

Definition 2.3.1 *Let $\mathscr{W}$ be a distribution satisfying Hypothesis 2.3.1. Let $u\in\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ for $\lambda,\sigma>0$ and $\kappa_u\in(\kappa,1)$, where $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ is introduced in Notation 2.2.2. Consider an initial condition $u_0\in\mathcal{B}_{q,p}^{\kappa_u,v_0}$ where we recall $v_t=w_{\lambda+\sigma t}$. We say that u is a mild solution to equation*

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+u\mathscr{W} \tag{2.28}$$

with initial condition u_0, if it satisfies the following integral equation

$$u_t = p_t u_0 + \int_0^t p_{t-s}(u_s \mathscr{W}) ds. \tag{2.29}$$

Remark 2.3.3 *Observe that the Dirac delta initial condition δ_0 falls beyond the scope of our considerations, as κ_u needs to be negative in order to have $\delta_0 \in \mathcal{B}^{\kappa_u, w_\lambda}_{q,p}$.*

Remark 2.3.4 *In (2.29), we implicitly assume that the product of distributions $u \cdot \mathscr{W}$ is well defined. This will be treated in the forthcoming Lemma 2.3.8.*

Before we can solve equation (2.29), we list a few results which would prove useful later. The first one recalls the action of the heat semigroup on weighted Besov spaces.

Lemma 2.3.5 *The following smoothing effect of the heat flow is valid in Besov spaces: Let $\hat{\kappa} \geq \kappa$ be real numbers, $\gamma_0 > 0$ and $q \in [1, \infty]$. Then there exists $C < \infty$ such that uniformly over $\gamma \leq \gamma_0$ and $t > 0$,*

$$\|p_t f\|_{\mathcal{B}^{\hat{\kappa}, w_\gamma}_{q,\infty}} \leq C t^{-\frac{\hat{\kappa}-\kappa}{2}} \|f\|_{\mathcal{B}^{\kappa, w_\gamma}_{q,\infty}}$$

Proof See [24, Proposition 3.11]. ■

We now give a result on comparison of Besov norms for different weights w.

Lemma 2.3.6 *Let $w_1, w_2 \in \mathcal{W}$ be such that $w_1 \leq w_2$. Then for every $f \in \mathcal{B}^{\kappa, w_2}_{p,q}$ we have*

$$\|f\|_{\mathcal{B}^{\kappa, w_1}_{p,q}} \leq \|f\|_{\mathcal{B}^{\kappa, w_2}_{p,q}}$$

Proof Follows easily from Definition 2.2.4. ■

Our next preliminary lemma is an elementary comparison between the weights corresponding to Definition 2.2.1.

Lemma 2.3.7 *Recall that the weight $v_t = w_{\lambda+\sigma t}$ has been defined for $t \geq 0$ in Notation 2.2.2. Then for $0 \leq s < t$ and for all $\sigma > 0$, there exists a constant c_σ such that*

$$v_t \leq c_\sigma |t-s|^{-\sigma} v_s \hat{w}_{\delta\sigma}.$$

Proof For $0 \leq s < t$, observe that $v_t = v_s e^{-\sigma(t-s)|x|_*^\delta}$. Then we use the fact that there exists a constant c_α such that

$$0 \leq x^\alpha e^{-sx} \leq \frac{c_\alpha}{s^\alpha} \quad \text{for} \quad x, \alpha, s \in \mathbb{R}_+$$

Consequently $e^{-\sigma(t-s)|x|_*^\delta} \leq c_\sigma |t-s|^{-\sigma} |x|_*^{-\sigma\delta}$ which implies $v_t \leq c_\sigma |t-s|^{-\sigma} v_s \hat{w}_{\sigma\delta}$. ∎

Let us recall the definition of products of distributions within the weighted Besov spaces framework.

Lemma 2.3.8 *Let $\alpha < 0 < \beta$ be such that $\alpha + \beta > 0$. In addition, consider $p, q \in [1, \infty]$ and $\nu \in [0,1]$. Let $p_1, p_2 \in [1, \infty]$ be such that*

$$\frac{1}{p_1} = \frac{\nu}{p} \quad , \quad \frac{1}{p_2} = \frac{1-\nu}{p} \qquad \textit{and} \qquad w = w_\gamma \hat{w}_\sigma.$$

Then the mapping $(f, g) \mapsto fg$ can be extended to a continuous linear map from $\mathcal{B}_{p_1,q}^{\alpha,w_\gamma} \times \mathcal{B}_{p_2,q}^{\beta,\hat{w}_\sigma}$ to $\mathcal{B}_{p,q}^{\alpha,w}$. Moreover there exists a constant C such that

$$\|fg\|_{\mathcal{B}_{p,q}^{\alpha,w}} \leq C \|f\|_{\mathcal{B}_{p_1,q}^{\alpha,w_\gamma}} \|g\|_{\mathcal{B}_{p_2,q}^{\beta,\hat{w}_\sigma}}.$$

Proof The proof is similar to that of [24, Corollary 3.21]. ∎

We also include the following extension of Gronwall's Lemma taken from [28, Lemma 15] which will be required in order to show existence of our solution.

Lemma 2.3.9 *Let $g : [0, T] \mapsto \mathbb{R}_+$ be a non-negative function such that $\int_0^T g(s)ds < \infty$. Let $(f_n, n \in \mathbb{N})$ be a sequence of non-negative functions on $[0, T]$ and k_1, k_2 be non-negative numbers such that for $0 \leq t \leq T$,*

$$f_n(t) \leq k_1 + \int_0^t (k_2 + f_{n-1}(s)) g(t-s) ds. \tag{2.30}$$

If $\sup_{0 \leq s \leq T} f_0(s) < \infty$, then $\sup_{n \geq 0} \sup_{0 \leq t \leq T} f_n(t) < \infty$, and if $k_1 = k_2 = 0$, then $\sum_{n \geq 0} f_n(t)$ converges uniformly on $[0, T]$.

We are ready to state our main result about existence and uniqueness of solution for our abstract heat equation (2.28).

Proposition 2.3.1 *Let $\mathscr{W}$ be a distribution as in Hypothesis 2.3.1. Consider $\lambda > 0$ and $q \geq 1$. Then there exists a unique solution to equation (2.29) lying in $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ where $\kappa_u \in (\kappa, 1)$ and where $\mathcal{C}_p^{\kappa_u,\lambda,\sigma}$ is defined in Notation 2.2.2.*

Proof We will follow a standard Picard iteration scheme to prove our result. Consider a small time interval $[0, \tau]$ where τ is to be fixed later. We restrict all spaces and corresponding norms on this time interval. Define $u^{(0)} \equiv u_0$ and for $n \geq 0$ set

$$u_t^{(n+1)} = \int_0^t p_{t-s}(u_s^{(n)}\mathscr{W})ds. \tag{2.31}$$

Fix $\kappa_u \in (\kappa, 1)$ and consider $\delta u_t^{(n)} = u_t^{(n+1)} - u_t^{(n)}$. Observe that from (2.31) and then applying Lemma 2.3.5 we obtain

$$\begin{aligned}\|\delta u_t^{(n+1)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}} &\leq \int_0^t \|p_{t-s}(\delta u_s^{(n)}\mathscr{W})\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}} \\ &\leq C\int_0^t |t-s|^{-\frac{\kappa_u+\kappa}{2}} \|\delta u_s^{(n)}\mathscr{W}\|_{\mathcal{B}_{q,\infty}^{-\kappa,v_t}} ds.\end{aligned}$$

where here and in the following C is a generic constant which may change in subsequent steps. Now applying Lemmas 2.3.6 and 2.3.7 we get

$$\|\delta u_t^{(n+1)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}} \leq C\int_0^t (t-s)^{-\frac{\kappa_u+\kappa}{2}-\sigma} \|\delta u_s^{(n)}\mathscr{W}\|_{\mathcal{B}_{q,\infty}^{-\kappa,v_s\hat{w}_{\delta\sigma}}} ds.$$

Using $\nu = 1$ in Lemma 2.3.8 and observing $\kappa_u > \kappa$, we find

$$\|\delta u_s^{(n)}\mathscr{W}\|_{\mathcal{B}_{q,\infty}^{-\kappa,v_s\hat{w}_{\delta\sigma}}} \leq \|\delta u_s^{(n)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}} \|\mathscr{W}\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\delta\sigma}}}.$$

Consequently,

$$\|\delta u_t^{(n+1)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}} \leq C\|\mathscr{W}\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\delta\sigma}}} \int_0^t \frac{\|\delta u_s^{(n)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}}}{|t-s|^{(\kappa_u+\kappa)/2+\sigma}} ds. \tag{2.32}$$

Observe that

$$\sup_{0\leq s\leq\tau} \|\delta u_s^{(0)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}} = \sup_{0\leq s\leq\tau} \|u_s^{(1)} - u_s^{(0)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}} = \sup_{0\leq s\leq\tau} \|p_s u_0 - u_0\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}}.$$

Also recall that $v_s = w_{\lambda+\sigma s}$, where the weight $w_{\lambda+\sigma s}$ has been defined in Definition 2.2.1 above. Consequently $\|p_s u_0\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}} \leq \|p_s u_0\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_0}}$. Thus, owing to Lemma 2.3.5 and 2.3.6, we have

$$\sup_{0\leq s\leq\tau} \|\delta u_s^{(0)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}} \leq \sup_{0\leq s\leq\tau} \|p_s u_0 - u_0\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_0}} \leq C\|u_0\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_0}}$$

which is finite by our assumption on the initial condition. We can thus apply Gronwall's Lemma as stated in Lemma 2.3.9 to equation (2.32). As a consequence we find $\sum_{n\geq 0}\|\delta u_s^{(n)}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}}$ converges uniformly on $[0,\tau]$ and thus $u^{(n)}$ converges uniformly in $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$. This proves existence of a solution on $[0,\tau]$ (observe that we don't need τ to be small for this step).

In order to prove uniqueness, we can resort to the same techniques. Consider two solutions u^1 and u^2 in $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ and set $u^{12}=u_1-u_2$. We have to show $u^{12}\equiv 0$. Since we have

$$u_t^{12}=\int_0^t p_{t-s}(u_s^{12}\mathscr{W})ds,$$

we obtain similarly to (2.32)

$$\|u_t^{12}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}}\leq C\|\mathscr{W}\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\delta\sigma}}}\int_0^t\frac{\|u_s^{12}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}}}{|t-s|^{(\kappa_u+\kappa)/2+\sigma}}ds. \tag{2.33}$$

Therefore, choosing σ small enough we get:

$$\|u_t^{12}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}}\leq\left(C\|\mathscr{W}\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\delta\sigma}}}\tau^\eta\right)\sup_{0\leq s\leq\tau}\|u_s^{12}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_s}}.$$

where $\eta=1-\left(\frac{\kappa_u+\kappa}{2}+\sigma\right)$. Then choosing τ small enough so that $(\|\mathscr{W}\|_{\mathcal{B}_{\infty,\infty}^{-\kappa,\hat{w}_{\delta\sigma}}}\tau^\eta)<1$, we find $\|u_t^{12}\|_{\mathcal{B}_{q,\infty}^{\kappa_u,v_t}}=0$ for all $t\in[0,\tau]$. This achieves uniqueness on the small interval $[0,\tau)$.

In order to get global existence and uniqueness we observe that our considerations above do not depend on the initial condition of the solution. Hence one can repeat the proof on subsequent intervals of size τ to get the result. ∎

The proof for uniqueness of solution in Proposition 2.3.1 can also be achieved through Picard iterations applied to (2.33) in order to get $\|u_t^{12}\|=0$. This alternative proof would thus avoid the need to consider a small τ. We thank one of the reviewers for drawing our attention to this fact.

We can now apply our general Proposition 2.3.1 in order to solve our original equation (2.1).

Theorem 2.3.10 *Let W be the centered Gaussian noise defined by (2.11), with $H \in \left(0, \frac{1}{2}\right)$ and consider $\kappa \in (1-H, 1)$. Let $u_0 \in \mathcal{B}_{q,\infty}^{\kappa_u, w_\lambda}$ for a given $\lambda > 0$ and $\kappa_u \in (\kappa, 1)$, where $w_\lambda = e^{-\lambda |x|_*^\delta}$ is defined in Definition 2.2.1. Consider the space $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$ introduced in Notation 2.2.2. Then equation (2.1) admits a solution which is unique in $\mathcal{C}_q^{\kappa_u,\lambda,\sigma}$.*

2.4 Feynman-Kac representation

In this section we shall establish a Feynman-Kac representation for the solution of (2.1), which will be at the heart of our Lyapounov computations. We first introduce some additional notations about random environments.

Notation 2.4.1 *Let B be a Brownian motion defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{P})$, independent of the space $(\Omega, \mathcal{F}, \mathbf{P})$ on which W is defined. In the sequel we denote by $\mathbf{E}$ (resp. $\mathbb{E}$) the expectation on $(\Omega, \mathcal{F}, \mathbf{P})$ (resp. $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{P})$). We will also write $\mathbb{E}_x$ when we want to highlight the initial value x of the Brownian motion B.*

We now introduce the Feynman-Kac functional we shall use in order to represent the solution of (2.1).

Notation 2.4.2 *Let W be the Gaussian noise defined by (2.11). For $\varepsilon > 0$ we set*

$$V_t^\varepsilon(x) = \int_0^t \int_{\mathbb{R}} l_\varepsilon(B_r^x - y) W(dy) dr, \tag{2.34}$$

where l_ε stands for the ε-mollifier generated from the standard bump function l as given in the general notation of the Introduction. We will also write, somehow informally,

$$V_t(x) = \int_0^t W(\delta_{B_s^x}) ds = \int_0^t \int_{\mathbb{R}} \delta_0(B_r^x - y) W(dy) dr, \tag{2.35}$$

which will be seen as a L^2-limit of the random variables V_t^ε.

We state the following lemma taken from [29, Theorem 1.3.5] which will be used in the proof for Proposition 2.4.1

Lemma 2.4.3 *For any non-decreasing sub-additive process Z_t defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \mathbb{P})$ with continuous path and with $Z_0 = 0$, the following inequality holds true for all $\theta \geq 0$ and $t > 0$:*

$$\mathbb{E}\left[\exp\left(\theta Z_t\right)\right] < \infty \quad \forall \theta, t > 0.$$

In addition,

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\left[\exp\left(\theta Z_t\right)\right]\right) = \Psi(\theta),$$

where Ψ is a function from $[0,\infty)$ to $[0,\infty)$.

We now give a rigorous meaning to the quantity $V_t(x)$ by showing that it can be seen as a L^2-limit of $V_t^\varepsilon(x)$. We also include some exponential bounds which are crucial for the Feynman-Kac representation of (2.1).

Proposition 2.4.1 *For $\varepsilon > 0$, $t \geq 0$ and $x \in \mathbb{R}$, let $V_t^\varepsilon(x)$ be defined by (2.34). Then*

(i) $\{V_t^\varepsilon(x); \varepsilon > 0\}$ is a convergent sequence in $L^2(\Omega \times \hat{\Omega})$. We call its limit $V_t(x)$, where $V_t(x)$ is defined by (2.35).

(ii) For all $q \geq 1$ we have

$$\lim_{\varepsilon\downarrow 0} \mathbf{E} \otimes \mathbb{E}\left[\left|e^{qV_t^\varepsilon} - e^{qV_t}\right|\right] = 0.$$

Proof We divide this proof in several steps.
Step 1: Proof of (i). Observe that $V_t^\varepsilon(x)$ can be written as $\int_0^t W(l_\varepsilon(B_r^x - \cdot))dr$, where $W(l_\varepsilon(B_r^x - \cdot))$ has to be understood as a Wiener integral conditionally on B (see (2.13)). In the following we try to find $\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbf{E}\otimes\mathbb{E}\left[V_t^{\varepsilon_1}(x)V_t^{\varepsilon_2}(x)\right]$, which is enough to ensure the L^2 convergence of $V_t^\varepsilon(x)$. To this aim, we invoke the isometry (2.11) in order to get

$$\begin{aligned}
\mathbb{E} \otimes \mathbf{E}\left[V_t^{\varepsilon_1}(x)V_t^{\varepsilon_2}(x)\right] &= \mathbb{E} \otimes \mathbf{E}\left[\int_0^t \int_0^t W(l_{\varepsilon_1}(B_u^x - \cdot))W(l_{\varepsilon_2}(B_v^x - \cdot))du\, dv\right] \\
&= \mathbb{E} \int_0^t \int_0^t \int_{\mathbb{R}} \mathcal{F}l_{\varepsilon_1}(B_u^x - \cdot)(\xi)\overline{\mathcal{F}l_{\varepsilon_2}(B_v^x - \cdot)(\xi)}\mu(d\xi)\ du\ dv.
\end{aligned}$$

Taking into account the expression for $\mathcal{F}l_\varepsilon(B_u^x - \cdot)$ we thus get

$$\begin{aligned}\mathbb{E}\otimes\mathbf{E}\left[V_t^{\varepsilon_1}(x)V_t^{\varepsilon_2}(x)\right] &= \mathbb{E}\left[\int_0^t\int_0^t\int_{\mathbb{R}}\overline{\mathcal{F}l(\varepsilon_1\xi)}e^{-\iota\langle\xi,B_u^x\rangle}\mathcal{F}l(\varepsilon_2\xi)e^{\iota\langle\xi,B_v^x\rangle}\mu(d\xi)\;du\;dv\right]\\ &= \mathbb{E}\left[\int_{\mathbb{R}}\left(\int_{[0,t]^2}e^{-\iota\langle\xi,B_u^x-B_v^x\rangle}du\;dv\right)\overline{\mathcal{F}l(\varepsilon_1\xi)}\mathcal{F}l(\varepsilon_2\xi)\mu(d\xi)\right].\end{aligned} \tag{2.36}$$

We can now use the fact that $B_u^x - B_v^x \sim \mathcal{N}(0, v-u)$ to write

$$\mathbb{E}\otimes\mathbf{E}\left[V_t^{\varepsilon_1}(x)V_t^{\varepsilon_2}(x)\right] = \int_{\mathbb{R}}\left(\int_{[0,t]^2}\psi_{\varepsilon_1,\varepsilon_2}(u,v;\xi)dudv\right)\mu(d\xi), \tag{2.37}$$

where $\psi_{\varepsilon_1,\varepsilon_2}(u,v;\xi)$ is defined by

$$\psi_{\varepsilon_1,\varepsilon_2}(u,v;\xi) = e^{-\frac{1}{2}|\xi|^2|v-u|}\overline{\mathcal{F}l(\varepsilon_1\xi)}\mathcal{F}l(\varepsilon_2\xi).$$

Moreover, setting $\psi(u,v;\xi) = e^{-\frac{1}{2}|\xi|^2|v-u|}$, it is readily seen that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\psi_{\varepsilon_1,\varepsilon_2}(u,v;\xi) = \psi(u,v;\xi), \quad \text{and} \quad |\psi_{\varepsilon_1,\varepsilon_2}(u,v;\xi)| \le |\psi(u,v;\xi)|\,.$$

In addition, the reader can check that

$$\int_{\mathbb{R}}\int_{[0,t]^2}\psi(u,v;\xi)\,du\,dv\,\mu(d\xi) \le c\int_{\mathbb{R}}\frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

Therefore, a standard application of the dominated convergence theorem to relation (2.37) proves that for every sequence ε_n converging to zero, $V_t^{\varepsilon_n}(x)$ converges in L^2 to a limit denoted by $V_t(x)$ as mentioned before.

Step 2: Conditional law of $V_t(x)$. We will next show that V_t is conditionally Gaussian for all $t \ge 0$ with conditional variance given by

$$\mathbf{E}\left[V_t^2\right] = \int_{\mathbb{R}}\left|\int_0^t e^{\imath\xi B_s}ds\right|^2\mu(d\xi). \tag{2.38}$$

This will follow from similar calculations as before. First observe that V_t^ε is conditionally Gaussian, with conditional variance given by

$$\mathbf{E}[V_t^\varepsilon]^2 = \mathbf{E}\left[\left(\int_0^t W(l_\varepsilon(B_r^x - \cdot))dr\right)^2\right] \tag{2.39}$$

The right hand side of (2.39) can be simplified by using the covariance structure of our noise as follows, using the same computations as for (2.36):

$$\mathbf{E}\big[(V_t^\varepsilon)^2\big] = \int_0^t \int_0^t \mathbf{E}\left[W(l_\varepsilon(B_r^x - \cdot))W(l_\varepsilon(B_s^x - \cdot))\right] dr ds = \int_{\mathbb{R}} |\mathcal{F}l(\varepsilon\xi)|^2 \left|\int_0^t e^{\imath\xi B_s} ds\right|^2 \mu(d\xi).$$

Since V_t is the L^2 limit of V_t^ε and L^2 limits of Gaussian processes remain Gaussian, we now have that conditioned on the Brownian motion, V_t is Gaussian with zero mean and variance given by (2.38).

Step 3: Exponential moments of V_t. Our next aim is to show that V_t entertains exponential moments. Specifically we will prove that for all $q > 0$ we have

$$\mathbf{E} \otimes \mathbb{E}\left[e^{qV_t}\right] < \infty. \tag{2.40}$$

Since we have already shown that V_t is conditionally Gaussian, we have

$$\mathbf{E}\left[e^{qV_t}\right] = \exp\left(\frac{q^2}{2}\int_{\mathbb{R}} \left|\int_0^t e^{\imath\xi B_s} ds\right|^2 \mu(d\xi)\right). \tag{2.41}$$

Hence, the unconditional expectation of e^{qV_t} is given by

$$\mathbb{E} \otimes \mathbf{E}\left(e^{qV_t}\right) = \mathbb{E}\left[\exp\left(\frac{q^2}{2}\int_{\mathbb{R}} \left|\int_0^t e^{\imath\xi B_s} ds\right|^2 \mu(d\xi)\right)\right].$$

To see that this quantity is finite let us define the following random variable

$$Z_t = \frac{1}{t}\int_{\mathbb{R}} \left|\int_0^t e^{\imath\lambda B_u} du\right|^2 \mu(d\lambda).$$

Observe that we can write:

$$\begin{aligned}
\frac{Z_{s+t}}{s+t} &= \frac{1}{(s+t)^2}\int_{\mathbb{R}} \left|\int_0^{s+t} e^{\imath\lambda B_u} du\right|^2 \mu(d\lambda) \\
&= \int_{\mathbb{R}} \left|\frac{s}{s+t}\left(\frac{1}{s}\int_0^s e^{\imath\lambda B_u} du\right) + \frac{t}{s+t}\left(\frac{1}{t}\int_s^{s+t} e^{\imath\lambda B_u} du\right)\right|^2 \mu(d\lambda).
\end{aligned} \tag{2.42}$$

Using Jensen's inequality in (2.42) we now obtain:

$$\begin{aligned}
\frac{Z_{s+t}}{s+t} &\le \int_{\mathbb{R}} \left(\frac{s}{s+t}\left|\frac{1}{s}\int_0^s e^{\imath\lambda B_u} du\right|^2 + \frac{t}{s+t}\left|\frac{1}{t}\int_s^{s+t} e^{\imath\lambda B_u} du\right|^2\right) \mu(d\lambda) \\
&= \frac{Z_s + Z_t'}{s+t} \quad \text{where} \quad Z_t' = \frac{1}{t}\int_{\mathbb{R}} \left|\int_s^{s+t} e^{\imath\lambda B_u} du\right|^2 \mu(d\lambda).
\end{aligned}$$

We have thus obtained that Z satisfies the following sub-additive property:

$$Z_{s+t} \leq Z_s + Z'_t. \tag{2.43}$$

Moreover, notice that Z'_t above can be written as

$$Z'_t = \frac{1}{t}\int_{\mathbb{R}} \left| \int_s^{s+t} e^{\imath\lambda(B_u - B_s)} du \right|^2 \mu(d\lambda),$$

due to the fact that $|e^{-i\lambda B_s}|^2 = 1$. Hence, it is readily checked that Z'_t is independent of $\{B_u; 0 \leq u \leq s\}$ and thus also independent of $\{Z_u; 0 \leq u \leq s\}$. In addition, $Z'_t \stackrel{d}{=} Z_t$. Let us now slightly generalize those considerations. Namely, consider a new process $\tilde{Z}$ defined as

$$\tilde{Z}_T = \max_{t \leq T} Z_t. \tag{2.44}$$

It is easily seen that the new process $\tilde{Z}_t$ is also sub-additive in nature. In other words, for all $T_1, T_2 \geq 0$, we have

$$\tilde{Z}_{T_1+T_2} \leq \tilde{Z}_{T_1} + \tilde{Z}'_{T_2},$$

where $\tilde{Z}'_{T_2}$ is independent of $\{\tilde{Z}_t; 0 \leq t \leq T_1\}$ with $\tilde{Z}'_{T_2} \stackrel{d}{=} \tilde{Z}_{T_2}$. In addition, since $\tilde{Z}_0 = 0$ and $\tilde{Z}$ has continuous paths, we can apply Lemma 2.4.3 in order to obtain for all $\theta > 0$ and $t > 0$:

$$\mathbb{E}\left[\exp\left\{\theta \tilde{Z}_t\right\}\right] < \infty,$$

and as a direct consequence we also have:

$$\mathbb{E}\left[\exp\left\{\theta Z_t\right\}\right] < \infty.$$

This proves the boundedness of the unconditional expectation of the exponential moments of V_t as expressed in (2.40).

Step 4: Conclusion. Observe that using the mean value theorem in its integral form and then Cauchy-Schwarz inequality one can write:

$$\begin{aligned}\mathbb{E}\otimes\mathbf{E}\left[\left|e^{qV_t^\varepsilon} - e^{qV_t}\right|\right] &= \mathbb{E}\otimes\mathbf{E}\left[\left|q\left(V_t - V_t^\varepsilon\right)\int_0^1 e^{\lambda q V_t^\varepsilon + (1-\lambda)qV_t} d\lambda\right|\right] \\ &\leq q\big(\mathbb{E}\otimes\mathbf{E}\left[|V_t - V_t^\varepsilon|^2\right]\big)^{\frac{1}{2}}\left(\mathbb{E}\otimes\mathbf{E}\left[\left|\int_0^1 e^{\lambda q V_t^\varepsilon + (1-\lambda)qV_t} d\lambda\right|^2\right]\right)^{\frac{1}{2}}.\end{aligned} \tag{2.45}$$

Using Cauchy-Schwarz inequality on two consecutive occasions separated by Fubini, the object on the right hand side in (2.45) can be further decomposed as:

$$\begin{aligned}\mathbb{E}\otimes\mathbf{E}\left[\left|\int_0^1 e^{\lambda qV_t^\varepsilon+(1-\lambda)qV_t}d\lambda\right|^2\right] &\leq \int_0^1 \mathbb{E}\otimes\mathbf{E}\left[e^{2\lambda qV_t^\varepsilon+2(1-\lambda)qV_t}\right]d\lambda \\ &\leq \int_0^1 \left(\mathbb{E}\otimes\mathbf{E}\left[e^{4\lambda qV_t^\varepsilon}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\otimes\mathbf{E}\left[e^{4(1-\lambda)qV_t}\right]\right)^{\frac{1}{2}}d\lambda\end{aligned} \tag{2.46}$$

Observe from the variance of V_t^ε calculated earlier in (2.41) that

$$\mathbf{E}\left[e^{qV_t^\varepsilon}\right] = \exp\left[\frac{q^2}{2}\int_{\mathbb{R}} e^{-\varepsilon\xi^2}\left|\int_0^t e^{\imath\xi B_s}ds\right|^2\mu(d\xi)\right],$$

and consequently

$$\mathbb{E}\otimes\mathbf{E}\left[e^{qV_t^\varepsilon}\right] \leq \mathbb{E}\otimes\mathbf{E}\left[e^{qV_t}\right].$$

Plugging this observation into (2.46) we obtain

$$\mathbb{E}\otimes\mathbf{E}\left[\left|\int_0^1 e^{\lambda qV_t^\varepsilon+(1-\lambda)qV_t}d\lambda\right|^2\right] \leq \mathbb{E}\otimes\mathbf{E}\left[e^{4qV_t}\right], \tag{2.47}$$

which is finite by our considerations in Step 3 (see (2.40)). Using (2.47) in (2.45) we have

$$\mathbb{E}\otimes\mathbf{E}\left[\left|e^{qV_t^\varepsilon}-e^{qV_t}\right|\right] \leq q\big(\mathbb{E}\otimes\mathbf{E}\left[|V_t-V_t^\varepsilon|^2\right]\big)^{\frac{1}{2}}\big(\mathbb{E}\otimes\mathbf{E}\left[e^{4qV_t}\right]\big)^{\frac{1}{2}}.$$

Since $\{V_t^\varepsilon(x); \varepsilon > 0\}$ is a convergent sequence in $L^2(\Omega\times\hat{\Omega})$, our conclusion now follows by taking limits. ■

With the exponential moments of $V_t(x)$ in hand, we can now obtain the announced Feynman-Kac representation of u.

Proposition 2.4.2 *Consider the Gaussian noise $\dot{W}$ defined by* (2.11)*. Let u be the unique solution of equation* (2.1) *with initial condition $u_0(x) = 1$, written in its mild form as:*

$$u_t(x) = 1 + \int_0^t p_{t-s}(u_s\dot{W})ds. \tag{2.48}$$

Then u can be represented as

$$u_t(x) = \mathbb{E}_x\left[\exp\left(V_t(x)\right)\right], \tag{2.49}$$

where $V_t(x)$ is the Feynman-Kac functional defined by (2.35).

Proof For $\varepsilon > 0$, let l_ε be the approximation of the identity given in the Introduction. We define a smoothed noise $\dot{W}^\varepsilon$ by $\dot{W}^\varepsilon = \dot{W} * l_\varepsilon$, as well as the approximation u^ε of u as the solution of

$$u_t^\varepsilon(x) = 1 + \int_0^t p_{t-s}(u_s^\varepsilon \dot{W}^\varepsilon) ds. \tag{2.50}$$

Along the same lines as for Proposition 2.3.1 we can prove that

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon = u \text{ in } \mathcal{C}_q^{\kappa_u, \lambda, \sigma},$$

where κ_u, λ and q are defined in Proposition 2.3.1. In addition, since u^ε solves (2.50) in the strong sense, it also admits a Feynman-Kac representation of the form

$$u_t^\varepsilon(x) = \mathbb{E}\left[e^{V_t^\varepsilon(x)}\right],$$

where $V_t^\varepsilon(x)$ is defined by (2.34). For any $p \geq 1$, we are now claiming that for all $t > 0$ we have

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[|u_t^\varepsilon(x) - u_t(x)|^p\right] = 0. \tag{2.51}$$

In order to get (2.51), notice that

$$\begin{aligned}
\mathbf{E}\left[|u_t^\varepsilon(x) - u_t(x)|^p\right] &= \mathbf{E}\left[\left|\mathbb{E}\left[e^{V_t(x)} - e^{V_t^\varepsilon(x)}\right]\right|^p\right] \\
&\leq \mathbf{E} \otimes \mathbb{E}\left[|V_t(x) - V_t^\varepsilon(x)|^p \left(e^{pV_t(x)} + e^{pV_t^\varepsilon(x)}\right)\right].
\end{aligned}$$

An elementary application of Cauchy-Schwarz inequality and the fact that $V_t(x)$, $V_t^\varepsilon(x)$ are conditionally Gaussian yield

$$\begin{aligned}
\mathbf{E}\left[|u_t^\varepsilon(x) - u_t(x)|^p\right] \leq c_p \Big(&\mathbf{E} \otimes \mathbb{E}\left[|V_t(x) - V_t^\varepsilon(x)|^2\right]\Big)^{\frac{p}{2}} \\
&\times \left[\left(\mathbf{E} \otimes \mathbb{E}\left[e^{2pV_t^\varepsilon(x)}\right]\right)^{\frac{1}{2}} + \left(\mathbf{E} \otimes \mathbb{E}\left[e^{2pV_t(x)}\right]\right)^{\frac{1}{2}}\right].
\end{aligned}$$

We can now apply directly Proposition 2.4.1 in order to get

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[|u_t^\varepsilon(x) - u_t(x)|^p\right] = 0.$$

The proof of (2.49) is now achieved. ■

2.5 Principal eigenvalues

Recall that we have shown in Proposition 2.4.2 that the unique solution u of our stochastic heat equation (2.48) can be written as

$$u_t(x) = \mathbb{E}_x\left[\exp(V_t(x))\right] = \mathbb{E}_x\left[\exp\left(\int_0^t W(\delta_{B_s})ds\right)\right],$$

where the second identity stems from (2.35).

Furthermore, W being a homogeneous noise, the asymptotic behavior of u does not depend on the space parameter $x \in \mathbb{R}$. For sake of simplicity we will thus consider $x = 0$ and investigate the quantity

$$u_t(0) = \mathbb{E}_0\left[\exp\left(\int_0^t W(\delta_{B_s})ds\right)\right].$$

As we will see later on the following equivalence holds true as $t \to \infty$:

$$\mathbb{E}_0\left[\exp\left(\int_0^t W(\delta_{B_s})ds\right)\right] \approx \exp\left(t\lambda_{\dot{W}}\left(Q_{R_t}\right)\right) \tag{2.52}$$

for a given region R_t and a principal eigenvalue type quantity $\lambda_{\dot{W}}$ defined as

$$\lambda_{\dot{W}}(D) = \sup_{g \in \mathcal{K}(D)} \left\{W(g^2) - \frac{1}{2}\int_D |g'(x)|^2 dx\right\}. \tag{2.53}$$

In (2.53), $\mathcal{K}(D)$ is a set of functions defined by

$$\mathcal{K}(D) = \left\{g \in \mathcal{S}(D) : \|g\|_2 = 1 \text{ and } g' \in L^2(\mathbb{R})\right\}, \tag{2.54}$$

where $\mathcal{S}(D)$ is the space of infinitely smooth functions that vanish at the boundary of an open domain D. Notice that $\mathcal{K}(D)$ can be seen as a subset of the classical Sobolev space $W^{1,2}(\mathbb{R})$. In addition, observe that the set $\mathcal{K}(D)$ is not compact, so that the reader might think that the sup defining $\lambda_{\dot{W}}(D)$ in (2.53) is ill-defined. However, as we will see in the proof of Proposition 2.5.1, our optimization can be reduced by scaling to a compact set $\mathcal{G}(D)$ defined by

$$\mathcal{G}(D) := \left\{ g \in \mathcal{S}(D) : \|g\|_2^2 + \frac{1}{2}\|g'\|_2^2 = 1 \right\}. \tag{2.55}$$

Before establishing relation (2.52), we will try to get some information about the limiting behavior of $\lambda_{\dot{W}}(D)$ as the size of the box D becomes large.

2.5.1 Basic results

In this section we establish some Gaussian and analytic results which will be building blocks in the asymptotics (2.52). We start by noting that $W(g)$ is a well-defined Gaussian field on the space $\mathcal{K}(D)$ defined by (2.54).

Lemma 2.5.1 *Let $g \in \mathcal{K}(D)$ for any $D \subset \mathbb{R}$. Then*

$$W(g^2) - \frac{1}{2}\|g'\|_2^2 < \infty \quad \text{a.s.}$$

Proof Note that the variance of $W(g^2)$ is given by

$$\text{Var}\left[W(g^2)\right] = c_H \int_{\mathbb{R}} \left|\mathcal{F}g^2(\xi)\right|^2 |\xi|^{1-2H} d\xi. \tag{2.56}$$

Also observe that for $g \in \mathcal{K}(D)$ we have

$$\left|\mathcal{F}g^2(\xi)\right| = \left|\int_{\mathbb{R}} e^{-\imath\xi x} g^2(x)dx\right| \le \int_{\mathbb{R}} \left|g^2(x)\right|dx = 1.$$

In addition, an elementary integration by parts argument shows that

$$\int_{\mathbb{R}} e^{-\imath\xi x} g^2(x)dx = -i \int_{\mathbb{R}} \left(\frac{1}{\xi}\frac{dg^2}{dx}\right) e^{-i\xi x}dx.$$

Hence for any $\xi \in \mathbb{R}$ and $g \in \mathcal{K}(D)$ we get

$$\left|\mathcal{F}g^2(\xi)\right| \le |\xi|^{-1} \int_{\mathbb{R}} \left|\frac{dg^2}{dx}(x)\right|dx = 2|\xi|^{-1} \int_{\mathbb{R}} |g(x)||g'(x)|dx \le 2|\xi|^{-1}\|g'\|_2,$$

where the last inequality follows from Cauchy-Schwarz inequality and observing that $\|g\|_2 = 1$ for $g \in \mathcal{K}(D)$. Let us now break up the variance in two parts by utilizing the two bounds just established.

$$\int_{\mathbb{R}} \left|\mathcal{F}g^2(\xi)\right|^2 |\xi|^{1-2H} d\xi \leq \int_{-1}^{1} |\xi|^{1-2H} d\xi + 4\|g'\|_2^2 \int_{|\xi|\geq 1} |\xi|^{-(1+2H)} d\xi = \frac{1}{1-H} + \frac{4\|g'\|_2^2}{H} \tag{2.57}$$

We thus get that the variance of $W(g^2)$ is bounded and consequently $W(g^2)$ is finite almost surely. Coupled with the fact that $\|g'\|_2 < \infty$ whenever g is an element of $\mathcal{K}(D)$, we get that

$$W(g^2) - \frac{1}{2}\|g'\|_2^2 < \infty \quad \text{a.s.}$$

■

Remark 2.5.2 *The following variational quantity will play a prominent role in our limiting results (see also* (2.7) *in Theorem* 2.1.2*):*

$$\mathcal{E} \equiv \sup_{g\in\mathcal{G}(\mathbb{R})} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{\imath\lambda x} g^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda, \;\; \text{where } \mathcal{G} \text{ is defined by (2.55)}. \tag{2.58}$$

The computations of Lemma 2.5.1 *imply that* $\mathcal{E}$ *is a finite quantity. Moreover, if we denote* $\mathcal{E} = \mathcal{E}(\dot{W})$*, then it is easily seen from relation* (2.56) *that*

$$\mathcal{E}(p\dot{W}) = p^2\mathcal{E}(\dot{W}) \tag{2.59}$$

The first result we need on Gaussian processes is an entropy type bound.

Lemma 2.5.3 *Let* $\dot{W}$ *be the noise defined by* (2.11)*, and recall that* $\mathcal{G}(-\varepsilon,\varepsilon)$ *is given by* (2.55) *for all* $\varepsilon > 0$*. Then we have:*

$$\lim_{\varepsilon\to 0^+} \mathbf{E}\left[\sup_{g\in\mathcal{G}(-\varepsilon,\varepsilon)} W(g^2)\right] = 0.$$

Proof The beginning of the proof is similar to [4, Lemma 2.2], and we will skip the details for sake of conciseness. Indeed, one can mimic the entropy arguments developed in [4, Proposition 2.1] and show that

$$\lim_{\delta\downarrow 0} \mathbf{E} \sup\left\{W(g^2); g \in \mathcal{G}(Q_1) \text{ and } \mathbf{E}[W(g^2)]^2 \leq \delta\right\} = 0,$$

where we remind the reader of the notation $Q_t = (-t,t)$. Then, still following the steps of [4, Lemma 2.2], it suffices to show that

$$\lim_{\varepsilon\downarrow 0}\sup_{g\in\mathcal{G}(Q_\varepsilon)} \mathbf{E}\big[W(g^2)\big]^2 = 0. \tag{2.60}$$

To establish (2.60) we use the alternate expression for our covariance function as in (2.12), which yields the following expression for all functions $g \in \mathcal{G}(Q_\varepsilon)$:

$$\mathbb{E}\big[W(g^2)\big]^2 = c_H \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left|g^2(x+y)-g^2(x)\right|^2}{|y|^{2-2H}}dxdy. \tag{2.61}$$

Since the domain of any function $g \in \mathcal{G}(Q_\varepsilon)$ is contained in Q_ε, let us break the right hand side of (2.61) into three parts by integrating over three regions $\{R_i\}_{i=1,2,3}$, where

$$\begin{aligned}
R_1 &= \{(x,y) : |x| \le \varepsilon, |x+y| \le \varepsilon\}, \\
R_2 &= \{(x,y) : |x| \le \varepsilon, |x+y| > \varepsilon\}, \\
R_3 &= \{(x,y) : |x| > \varepsilon, |x+y| \le \varepsilon\}.
\end{aligned}$$

Consequently,

$$\mathbb{E}[W(g^2)]^2 = I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon},$$

where

$$I_{i,\varepsilon} = c_H \int_{R_i} \frac{|g^2(x+y)-g^2(x)|^2}{|y|^{2-2H}}dxdy.$$

Let us now work with each integral $I_{i,\varepsilon}$ in succession. In order to upper-bound $I_{1,\varepsilon}$, observe that

$$\left|g^2(x+y)-g^2(x)\right| = |g(x+y)+g(x)|\,|g(x+y)-g(x)|,$$

and that

$$\begin{aligned}
|g(x+y)-g(x)| &= \left|\int_x^{x+y} g'(z)dz\right| \\
&\le \sqrt{\left|\int_x^{x+y} |g'(z)|^2 dz\right| |y|} \le \|g'\|_2\sqrt{|y|},
\end{aligned} \tag{2.62}$$

by an application of the Cauchy-Schwarz inequality. Thus the integrand in $I_{1,\varepsilon}$ can be upper-bounded as follows:

$$\begin{aligned}\frac{|g^2(x+y)-g^2(x)|^2}{|y|^{2-2H}} &= \frac{|g(x+y)+g(x)|^2|g(x+y)-g(x)|^2}{|y|^{2-2H}} \\ &\leq \frac{2\left(g^2(x+y)+g^2(x)\right)\|g'\|_2^2|y|}{|y|^{2-2H}},\end{aligned} \tag{2.63}$$

where we have used (2.62) and the fact that $|a+b|^2 \leq 2(a^2+b^2)$. Plugging (2.63) in $I_{1,\varepsilon}$ and using the fact that $\|g'\|_2^2 \leq 2$ for every $g \in \mathcal{G}(Q_\varepsilon)$, we obtain:

$$\begin{aligned} I_{1,\varepsilon} &\leq 4c_H \int_{R_1} \frac{g^2(x+y)+g^2(x)}{|y|^{1-2H}} dxdy \\ &= 4c_H \left[\int_{-\varepsilon}^{\varepsilon} dx \int_{-\varepsilon-x}^{\varepsilon-x} dy\ \frac{g^2(x+y)}{|y|^{1-2H}} + \int_{-\varepsilon}^{\varepsilon} dx \int_{-\varepsilon-x}^{\varepsilon-x} dy\ \frac{g^2(x)}{|y|^{1-2H}}\right] \\ &\leq 4c_H \left[\int_{-\varepsilon}^{\varepsilon} dx \int_{-\varepsilon}^{\varepsilon} dz\ \frac{g^2(z)}{|z-x|^{1-2H}} + \int_{-\varepsilon}^{\varepsilon} dx\ g^2(x) \int_{-2\varepsilon}^{2\varepsilon} dy\ \frac{1}{|y|^{1-2H}}\right]. \end{aligned} \tag{2.64}$$

Let us now recall some basic analytic facts taken from [30, Chapter 4]: the Sobolev space $W^{1,2}$ is embedded in any $L^k(\mathbb{R})$ for all $k \geq 2$. More specifically, for all $k \geq 2$ we have

$$\|g\|_{L^k(\mathbb{R})} \leq c_k \|g\|_{W^{1,2}(\mathbb{R})}, \tag{2.65}$$

where c_k is a positive constant independent of g.

We shall invoke (2.65) in order to bound the first integral in the right hand side of (2.64). Namely, apply Hölder's inequality with two conjugate numbers p and q, which gives

$$\int_{(-\varepsilon,\varepsilon)^2} \frac{g^2(z)}{|z-x|^{1-2H}} dxdz \leq \left(\int_{(-\varepsilon,\varepsilon)^2} |g(z)|^{2p} dxdz\right)^{\frac{1}{p}} \left(\int_{(-\varepsilon,\varepsilon)^2} \frac{dxdz}{|z-x|^{(1-2H)q}}\right)^{\frac{1}{q}}.$$

We now take a small constant $\delta > 0$ and $q = \frac{1-\delta}{1-2H}$, which means that $p = \frac{1-\delta}{2H-\delta}$. Then inequality (2.65) plus some elementary computations show that for $\varepsilon < 1$

$$\int_{(-\varepsilon,\varepsilon)^2} \frac{g^2(z)}{|z-x|^{1-2H}} \leq c_{H,\delta} \|g\|_{W^{1,2}(\mathbb{R})}^2 \varepsilon \leq 3c_{H,\delta}\varepsilon,$$

where we resort to the fact that $\|g\|^2_{W^{1,2}(\mathbb{R})} \leq 3$ whenever $g \in \mathcal{G}(Q_\varepsilon)$ for the last inequality. Using this information in (2.64) and noting that the second term in (2.64) is bounded thanks to elementary considerations, we obtain:

$$I_{1,\varepsilon} \leq c_H \left(3c_{H,\delta}\varepsilon + \|g\|_2^2 \varepsilon^{2H}\right) \leq c_{H,\delta}\varepsilon^{2H}. \tag{2.66}$$

Let us now work with $I_{2,\varepsilon}$ and $I_{3,\varepsilon}$. Observe that $I_{2,\varepsilon}$ can be expressed as follows:

$$\begin{aligned}
I_{2,\varepsilon} &= c_H \int_{R_2} \frac{\left|g^2(x+y) - g^2(x)\right|^2}{|y|^{2-2H}} dxdy \\
&= c_H \int_{R_2} \frac{g^4(x)}{|y|^{2-2H}} dxdy \\
&= c_H \int_{-\varepsilon}^{\varepsilon} g^4(x) \left[\int_{-\infty}^{-\varepsilon - x} \frac{dy}{|y|^{2-2H}} + \int_{\varepsilon - x}^{\infty} \frac{dy}{|y|^{2-2H}}\right] dx \\
&= \frac{c_H}{1-2H} \int_{-\varepsilon}^{\varepsilon} g^4(x) \left[\frac{1}{(\varepsilon - x)^{1-2H}} + \frac{1}{(\varepsilon + x)^{1-2H}}\right] dx.
\end{aligned}$$

We let the patient reader check that the same kind of identity holds for $I_{3,\varepsilon}$. Thus, we find that

$$I_{2,\varepsilon} + I_{3,\varepsilon} \leq \frac{2c_H}{1-2H} \int_{-\varepsilon}^{\varepsilon} g^4(x) \left[\frac{1}{(\varepsilon - x)^{1-2H}} + \frac{1}{(\varepsilon + x)^{1-2H}}\right] dx. \tag{2.67}$$

In order to bound the right hand side of (2.67), we use the same strategy as for $I_{1,\varepsilon}$. Namely, for $p, q \geq 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality imply

$$I_{1,\varepsilon} + I_{2,\varepsilon} \leq \frac{2c_H}{1-2H} \left(\int_{-\varepsilon}^{\varepsilon} g^{4p}(x) dx\right)^{\frac{1}{p}} \left[\left(\int_{-\varepsilon}^{\varepsilon} \frac{dx}{(\varepsilon - x)^{(1-2H)q}}\right)^{\frac{1}{q}} + \left(\int_{-\varepsilon}^{\varepsilon} \frac{dx}{(\varepsilon + x)^{(1-2H)q}}\right)^{\frac{1}{q}}\right].$$

As before, let us now fix $q = \frac{1-\delta}{1-2H}$. This implies that the integrals $\int_{-\varepsilon}^{\varepsilon} (\varepsilon \pm x)^{-(1-2H)q} dx$ are finite and each is equal to $c_\delta \varepsilon^{1-\delta}$ for a universal constant c_δ. Putting together this information, we find

$$I_{2,\varepsilon} + I_{3,\varepsilon} \leq c_{H,\delta} \|g\|_{4p}^4 \varepsilon^{\frac{\delta}{q}}.$$

Moreover, a second usage of equation (2.65) plus the fact that $\|g\|_{W^{1,2}(\mathbb{R})} \leq \sqrt{3}$ yield:

$$\|g\|_{4p} \leq c_p \|g\|_{W^{1,2}} \leq c_p \sqrt{3}.$$

Thus, we obtain:

$$I_{2,\varepsilon} + I_{3,\varepsilon} \leq c_{H,\delta} \varepsilon^{\frac{\delta}{q}}. \tag{2.68}$$

Combining the inequalities (2.66) and (2.68) we find that

$$I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} \leq c_{H,\delta}\varepsilon^{\nu},$$

for a given $\nu > 0$, uniformly for all $g \in \mathcal{G}(Q_\varepsilon)$. Therefore we get

$$\lim_{\varepsilon \downarrow 0} I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} = 0.$$

We have thus proved (2.60). ■

We now introduce the scalings which will be needed in our future computations.

Notation 2.5.4 *For fixed $u > 0$ and $t > 0$ we introduce a scaling coefficient h_t defined as:*

$$h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}. \tag{2.69}$$

Then for each $g \in \mathcal{S}(\mathbb{R})$, we define a $L^2(\mathbb{R})$-rescaled function g_t as follows:

$$g_t(x) = \sqrt{h_t}\, g(h_t x). \tag{2.70}$$

Also, denote by Q_r the open interval $(-r, r)$.

We now see how to rescale the principal eigenvalues related to $\dot{W}$ in boxes of the form Q_r.

Lemma 2.5.5 *Let $\dot{W}$ be the Gaussian noise defined by (2.11). For a box $Q_t = (-t, t)$, recall that $\lambda_{\dot{W}}(Q_t)$ is given by formula (2.53). Then the following relation holds true:*

$$\lambda_{\dot{W}}(Q_t) = h_t^2 \sup_{g \in \mathcal{K}(Q_{th_t})} \left\{ \frac{1}{h_t^2} W(g_t^2) - \frac{1}{2} \int_{Q_{th_t}} |g'(x)|^2 dx \right\}, \tag{2.71}$$

where the quantities h_t and the function g_t are introduced in Notation 2.5.4.

Proof Notice that the map $g \mapsto g_t$ when defined from $\mathcal{K}(Q_{th_t})$ to $\mathcal{K}(Q_t)$ is a $L^2(\mathbb{R})$-isomorphism between the two spaces. As a consequence, $\sup_{g \in \mathcal{K}(Q_t)} A(g) = \sup_{g \in \mathcal{K}(Q_{th_t})} A(g_t)$ for any general functional A defined on a domain included in $L^2(\mathbb{R})$. Hence,

$$\lambda_{\dot{W}}(Q_t) = \sup_{g \in \mathcal{K}(Q_t)} \left\{ W(g^2) - \frac{1}{2} \int_{Q_t} |g'(x)|^2 dx \right\} = \sup_{g \in \mathcal{K}(Q_{th_t})} \left\{ W(g_t^2) - \frac{1}{2} \int_{Q_t} |g_t'(x)|^2 dx \right\}.$$

Also, since $g_t'(x) = h_t^{3/2} g'(h_t x)$, we get

$$\int_{Q_t} |g_t'(x)|^2 dx = \int_{Q_t} h_t^3 |g'(h_t x)|^2 dx = h_t^2 \int_{Q_{th_t}} |g'(y)|^2 dy,$$

where the second identity is due to an elementary change of variable. Consequently,

$$\begin{aligned}\lambda_{\dot{W}}(Q_t) &= \sup_{g \in \mathcal{K}(Q_t)} \left\{ W(g^2) - \frac{1}{2} \int_{Q_t} |g'(x)|^2 dx \right\} \\ &= h_t^2 \sup_{g \in \mathcal{K}(Q_{th_t})} \left\{ \frac{1}{h_t^2} W(g_t^2) - \frac{1}{2} \int_{Q_{th_t}} |g'(x)|^2 dx \right\}, \end{aligned} \tag{2.72}$$

which is our claim. ∎

Remark 2.5.6 *One can justify the scaling by h_t given by (2.69) in the following way: let us start with the rescaled version (2.71) of $\lambda_{\dot{W}}(Q_t)$, which is valid for any weight h_t. In addition, we will see in Section 2.5.2 that the main quantity we should handle in (2.71) is the family $\{h_t^{-2} W(g_t^2); t \geq 0\}$ and we want this family of Gaussian random variables to remain stochastically bounded in t as $t \to \infty$. Next an elementary computation (see (2.86) below for more details) reveals that for all $g \in \mathcal{K}$ we have*

$$\sigma_{t,g}^2 \equiv \mathrm{Var}\left[h_t^{-2} W(g_t^2)\right] = c_{H,g} h_t^{-2(1+H)}. \tag{2.73}$$

Due to the Gaussian nature of $h_t^{-2} W(g_t^2)$, we thus have (for all $x > 0$)

$$\mathbf{P}\left(h_t^{-2} \left|W(g_t^2)\right| > x\right) \leq c_1 e^{-c_2 x^2 / \sigma_{t,g}^2}. \tag{2.74}$$

A natural way to have the family $\{h_t^{-2} W(g_t^2); t \geq 0\}$ stochastically bounded is thus to pick the minimal h_t such that one can use Borel-Cantelli in the right hand side of (2.74). It is readily checked that this is achieved as long as $\sigma_{t,g}^{-2}$ is of order $\log t$. Recalling the expression (2.73) for $\sigma_{t,g}^2$, this yields h_t of order $(\log t)^{1/2(1+H)}$.

In the following two subsections we explore the long-time asymptotics of $\lambda_{\dot{W}}(Q_t)$. More precisely, we will try to prove the following:

$$\lim_{t \to \infty} \frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{1/(1+H)}} = (2 c_H \mathcal{E})^{1/(1+H)} \quad \text{a.s.} \tag{2.75}$$

2.5.2 Upper Bound

In order to get the upper bound part of (2.75) we rely on the general idea that principal eigenvalues over a large domain can be essentially bounded by the maximum value among the principal eigenvalues on some sub-domains. See [31, Proposition 1] where this result is proved when the potential is defined pointwise. In [4] the same result is stated to be true for generalized functions as well. We start with an elementary lemma whose proof is very similar to the aforementioned references.

Lemma 2.5.7 *Let $r > 0$. There exists a non-negative continuous function $\Phi(x)$ on $\mathbb{R}$ whose support is contained in the $1-$neighborhood of the grid $2r\mathbb{Z}$, such that for any $R > r$ and any generalized function ξ,*

$$\lambda_{\xi-\Phi^y}(Q_R) \leq \max_{z\in 2r\mathbb{Z}\cap Q_R} \lambda_\xi(z + Q_{r+1}), \qquad \textit{for all } y \in Q_r, \tag{2.76}$$

where $\Phi^y(x) = \Phi(x + y)$. In addition $\Phi(x)$ is periodic with period $2r$, namely

$$\Phi(x + 2rz) = \Phi(x), \quad x \in \mathbb{R},\ z \in \mathbb{Z}, \tag{2.77}$$

and there is a constant $K > 0$ independent of r such that

$$\frac{1}{2r}\int_{Q_r} \Phi(x)dx \leq \frac{K}{r}. \tag{2.78}$$

We now show how to split the upper bound for the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$ into small subsets.

Lemma 2.5.8 *Let W be the noise defined by (2.10) and $Q_t = (-t, t)$. We consider the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$ given by (2.53). Recalling that h_t is given by (2.69), the following inequality holds true:*

$$\lambda_{\dot{W}}(Q_t) \leq h_t^2\left(\frac{K}{r} + \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t)\right), \tag{2.79}$$

where the random field $\{X_z(t); z \in 2r\mathbb{Z}, t \geq 0\}$ is defined by:

$$X_z(t) = \sup_{g\in\mathcal{K}(z+Q_{r+1})}\left\{\frac{W(g_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx\right\}. \tag{2.80}$$

In (2.80), the set $\mathcal{K}(z + Q_{r+1})$ is given by (2.54) and the function g_t is defined by (2.70).

Proof Let $\{W_t(\psi), \psi \in \mathcal{D}(\mathbb{R})\}$ be the generalized Gaussian field defined as $W_t(\psi) = W(\hat{\psi}_t)$ where $\hat{\psi}_t(x) = h_t\psi(h_t x)$. Then with the definition (2.70) of g_t in mind, notice that $W(g_t^2) = W_t(g^2)$. Thus invoking Lemma 2.5.5 we have:

$$
\begin{aligned}
&\frac{1}{h_t^2}\lambda_{\dot{W}}(Q_t) = \sup_{g\in\mathcal{K}(Q_{th_t})}\left\{\frac{1}{h_t^2}W(g_t^2) - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\}\\
&= \sup_{g\in\mathcal{K}(Q_{th_t})}\left\{\frac{1}{h_t^2}W_t(g^2) - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\}\\
&= \sup_{g\in\mathcal{K}(Q_{th_t})}\left\{\left\langle\frac{1}{h_t^2}\dot{W}_t - \frac{1}{2r}\int_{Q_r}\Phi^y dy, g^2\right\rangle + \left\langle\frac{1}{2r}\int_{Q_r}\Phi^y(x)dy, g^2\right\rangle - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\},
\end{aligned}
$$

where $\langle\dot{W}_t, g^2\rangle$ is understood in the distribution sense. Hence inequality (2.78) and the fact that $\langle g^2, \mathbf{1}\rangle = 1$ if $g \in \mathcal{K}(Q_{th_t})$ yields

$$
\frac{1}{h_t^2}\lambda_{\dot{W}}(Q_t) \le \frac{K}{r} + \sup_{g\in\mathcal{K}(Q_{th_t})}\left\{\left\langle\frac{1}{h_t^2}\dot{W}_t - \frac{1}{2r}\int_{Q_r}\Phi^y dy, g^2\right\rangle - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\}.
$$

Therefore bounding $\sup\int$ by $\int\sup$ and invoking the definition (2.53) of the principal eigenvalue, we end up with:

$$
\begin{aligned}
&\frac{1}{h_t^2}\lambda_{\dot{W}}(Q_t) \le \frac{K}{r} + \frac{1}{2r}\int_{Q_r}\sup_{g\in\mathcal{K}(Q_{th_t})}\left\{\left\langle\frac{1}{h_t^2}\dot{W}_t - \Phi^y, g^2\right\rangle - \frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\}dy\\
&\le \frac{K}{r} + \frac{1}{2r}\int_{Q_r}\lambda_{\frac{\dot{W}_t}{h_t^2}-\Phi^y}(Q_{th_t})dy.
\end{aligned}
$$

We can now resort to (2.76) in order to get:

$$
\frac{1}{h_t^2}\lambda_{\dot{W}}(Q_t) \le \frac{K}{r} + \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}}\lambda_{\frac{\dot{W}_t}{h_t^2}}(z + Q_{r+1}).
$$

Recall again that $\dot{W}_t$ is defined by $W_t(\psi) = W(\hat{\psi}_t)$ where $\hat{\psi}_t(x) = h_t\psi(h_t x)$. Thus we have

$$
\frac{1}{h_t^2}\lambda_{\dot{W}}(Q_t) \le \frac{K}{r} + \max_{z\in 2r\mathbb{Z}\cap Q_{th_t}}X_z(t),
$$

where the random fields $\{X_z(t); z \in 2r\mathbb{Z}, t \ge 0\}$ are defined by (2.80). Our claim (2.79) is thus easily deduced. ∎

We are ready to state the desired upper bound on our principal eigenvalue.

Proposition 2.5.1 *Let $\lambda_{\dot W}(Q_t)$ be the principal eigenvalue of the random operator $\frac{1}{2}\Delta+\dot W$ over the restricted space $\mathcal{K}(Q_t)$ of functions having compact support on $(-t,t)$, defined by (2.55). Then the following limit holds:*

$$\limsup_{t\to\infty}\frac{\lambda_{\dot W}(Q_t)}{(\log t)^{\frac{1}{1+H}}}\le (2c_H\mathcal{E})^{\frac{1}{1+H}}, \qquad \text{a.s.}$$

where we recall that $\mathcal{E}$ is defined by (2.58).

Proof We shall rely on relation (2.79) and bound $\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t)$ thanks to Gaussian entropy methods. We divide the proof in several steps.

Step 1: Reduction to a Gaussian supremum. By homogeneity of the Gaussian field $\{W(\phi);\phi\in\mathcal{D}(\mathbb{R})\}$, the random variables $\{X_z(t)\}_{z\in 2r\mathbb{Z}\cap Q_{th_t}}$ are identically distributed. Consequently we have

$$\begin{aligned}\mathbf{P}\left[\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t)>1\right] &\le \#\left\{2r\mathbb{Z}\cap Q_{th_t}\right\}\mathbf{P}\left[X_0(t)>1\right]\\ &\le \left(\frac{th_t}{r}\right)\mathbf{P}\left[X_0(t)>1\right]. \end{aligned}\tag{2.81}$$

Recalling the definition (2.80) of $X_0(t)$, we thus get

$$\mathbf{P}\left[\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t)>1\right]\le\left(\frac{th_t}{r}\right)\mathbf{P}\left[\sup_{g\in\mathcal{K}(Q_{r+1})}\left\{\frac{1}{h_t^2}W(g_t^2)-\frac{1}{2}\int_{Q_{th_t}}|g'(x)|^2dx\right\}>1\right].\tag{2.82}$$

Notice that in (2.82) the Gaussian supremum for the family $(W(g_t^2))$ is taken over the set $\mathcal{K}$ given by (2.54). However, this set is not compact, which is not suitable for Gaussian computations (see e.g. the discussion after [32, Lemma 1.3.1]). In the following steps we will reduce our computations to an optimization over a compact set of the form $\mathcal{G}$ (see equation (2.55)). To this aim, for any $g\in\mathcal{K}(Q_{r+1})$, set

$$\phi=\frac{g}{\sqrt{1+\frac{1}{2}\|g'\|_2^2}}.$$

Notice that since $\|g\|_2=1$, we have

$$\phi\in\mathcal{G}(Q_{r+1}),\quad\text{and}\quad \phi_t=\frac{g_t}{\sqrt{1+\frac{1}{2}\|g'\|_2^2}},$$

where the notation ϕ_t is given by (2.70). Therefore the following rough estimate holds true for the parameter h_t defined by (2.69):

$$\frac{1}{h_t^2}W(\phi_t^2) \leq \frac{1}{h_t^2} \sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2).$$

Moreover, recalling that $\phi_t^2 = \left(1 + \frac{1}{2}\|g'\|_2^2\right)^{-1} g_t^2$, we find

$$\frac{1}{h_t^2}W(g_t^2) \leq \frac{\left(1 + \frac{1}{2}\|g'\|_2^2\right)}{h_t^2} \sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2).$$

Thus, subtracting $\|g'\|_2^2$ on both sides of the above equation we get the following relation for all $g \in \mathcal{K}(Q_{r+1})$:

$$\left(\frac{1}{h_t^2}W(g_t^2) - \frac{1}{2}\int_{Q_{r+1}} |g'(x)|^2 dx\right) \leq \frac{\left(1 + \frac{1}{2}\|g'\|_2^2\right)}{h_t^2} \sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2) - \frac{1}{2}\|g'\|_2^2$$

Taking supremum over $g \in \mathcal{K}(Q_{r+1})$, this yields

$$X_0(t) \leq \sup_{g \in \mathcal{K}(Q_{r+1})} \left\{ \frac{\left(1 + \frac{1}{2}\|g'\|_2^2\right)}{h_t^2} \sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2) - \frac{1}{2}\|g'\|_2^2 \right\}.$$

Consequently, if $X_0(t) \geq 1$, we also have

$$\sup_{g \in \mathcal{K}(Q_{r+1})} \left(\frac{\sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2)}{h_t^2} - 1\right)\left(1 + \frac{1}{2}\|g'\|_2^2\right) \geq 0,$$

or otherwise stated:

$$\left[\sup_{f \in \mathcal{G}(Q_{r+1})} \frac{W(f_t^2)}{h_t^2} - 1\right] \sup_{g \in \mathcal{K}(Q_{r+1})} \left(1 + \frac{1}{2}\|g'\|_2^2\right) \geq 0.$$

It is readily checked that the above condition is met iff $\sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2) \geq h_t^2$. Summarizing, we have shown that

$$\{X_0(t) \geq 1\} \subset \left\{ \sup_{f \in \mathcal{G}(Q_{r+1})} W(f_t^2) \geq h_t^2 \right\},$$

which implies

$$\mathbf{P}\left(X_0(t) \geq 1\right) \leq \mathbf{P}\left[\sup_{g \in \mathcal{G}(Q_{r+1})} W(g_t^2) \geq h_t^2\right]. \tag{2.83}$$

We are now reduced to the desired sup over a compact set.

Step 2: Gaussian concentration. We now evaluate the right hand side of (2.83) by standard Gaussian supremum estimates. Namely, some elementary scaling arguments show that for each $g \in \mathcal{G}\left(Q_{r+1}\right)$,

$$\left(1+\frac{(h_t^2-1)}{2}\|g'\|_2^2\right)^{-1/2} g_t \in \mathcal{G}(Q_{(r+1)/h_t}).$$

Moreover by the linearity of Gaussian fields and due to the fact that $\|g'\|_2^2 \leq 2$ whenever $g \in \mathcal{G}(Q_{r+1})$, we get

$$\mathbf{E}\left[\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2)\right] \leq h_t^2\, \mathbf{E}\left[\sup_{f\in\mathcal{G}(Q_{(r+1)/h_t})} W(f^2)\right] \equiv h_t^2 \delta_t. \tag{2.84}$$

In addition, Lemma 2.5.3 asserts that $\lim_{t\to\infty}\delta_t = 0$ (notice that the fact of working on a box with finite size $r+1$ is crucial for this step). We are now in a position to invoke Borell-TIS concentration inequality for Gaussian fields (See [32, Theorem 2.1.2]) and our inequality (2.84), which yields

$$\begin{aligned}
&\mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2) \geq h_t^2\right] \\
&= \mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2) - \mathbf{E}\left(\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2)\right) \geq h_t^2(1-\delta_t)\right] \\
&\leq \exp\left[-\frac{h_t^4(1-\delta_t)^2}{2\sigma_t^2}\right],
\end{aligned} \tag{2.85}$$

where σ_t^2 is a parameter defined by $\sigma_t^2 = \sup_{g\in\mathcal{G}(Q_{r+1})} \mathrm{Var}\left[W(g_t^2)\right]$.

We now find an upper bound for the term σ_t^2 in (2.85). This is achieved as follows: Owing to the definition (2.11) of the covariance of W, we have

$$\begin{aligned}
\sigma_t^2 &= c_H \sup_{g\in\mathcal{G}(Q_{r+1})} \int_{\mathbb{R}} \left|\mathcal{F}g_t^2(\xi)\right|^2 |\lambda|^{1-2H} d\lambda \\
&= c_H \sup_{g\in\mathcal{G}(Q_{r+1})} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{\imath\lambda x} g_t^2(x)dx\right|^2 |\lambda|^{1-2H} d\lambda.
\end{aligned}$$

Therefore, recalling the definition (2.70) of g_t and invoking some easy scaling arguments we obtain:

$$\begin{aligned}\sigma_t^2 &= c_H h_t^{2-2H} \sup_{g\in\mathcal{G}(Q_{r+1})} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{\imath\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda \\ &\leq c_H h_t^{2-2H} \sup_{g\in\mathcal{G}(\mathbb{R})} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{\imath\lambda x} g^2(x)dx \right|^2 |\lambda|^{1-2H} d\lambda = c_H h_t^{2-2H}\mathcal{E}, \end{aligned} \tag{2.86}$$

where we recall that $\mathcal{E}$ is a finite quantity according to (2.58). We can plug our upper bound (2.86) for the item σ_t^2 in (2.85) and replace h_t by its value $\sqrt{u}(\log t)^{1/(2(1+H))}$. We end up with:

$$\mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2) \geq h_t^2\right] \leq \exp\left(-\frac{(1-\delta_t)^2 u^{1+H}}{2\mathcal{E}c_H}\log t\right).$$

We wish the series $\sum_k \mathbf{P}\left(\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_{2^k}^2) \geq h_{2^k}^2\right)$ to be convergent. To this aim, owing to the fact that $\lim_{t\to\infty}\delta_t = 0$, for t sufficiently large we get

$$\mathbf{P}\left(\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2) \geq h_t^2\right) \leq \exp\left[-(1+\nu)\log t\right] = \frac{1}{t^{1+\nu}}, \tag{2.87}$$

where $\nu > 0$ is a small enough constant, provided the following condition is met:

$$u > (2c_H\mathcal{E})^{1/(1+H)}. \tag{2.88}$$

Here we highlight the fact that $t^{-(1+\nu)}$ is obtained in the right hand side of (2.87). This exponent lead to our choice of scaling by $h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}$ in our computations (see Remark 2.5.6).

Step 3: Conclusion. Now, we summarize our steps so far. Thanks to (2.81), (2.83) and (2.87) we have

$$\begin{aligned}\mathbf{P}\left[\max_{z\in 2r\mathbb{Z}\cap Q_{th_t}} X_z(t) \geq 1\right] &\leq \left(\frac{th_t}{r}\right)\mathbf{P}\left[X_0(t)\geq 1\right] \\ &\leq \left(\frac{th_t}{r}\right)\mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{r+1})} W(g_t^2) \geq h_t^2\right] \\ &\leq \left(\frac{th_t}{r}\right)\exp\left(-(1+\nu)\log t\right) = \frac{h_t}{r}\frac{1}{t^\nu}.\end{aligned}$$

Take the sequence $t_k = 2^k$. Then we have

$$\mathbf{P}\left[\max_{z\in 2r\mathbb{Z}\cap Q_{t_k h_{t_k}}} X_z(t_k) \geq 1\right] \leq \frac{\sqrt{u}}{r}\frac{(\log t_k)^{\frac{1}{2(1+H)}}}{t_k^{\nu}} = \frac{\sqrt{u}}{r}(k\log 2)^{\frac{1}{2(1+H)}}2^{-k\nu},$$

and the right hand side of the above inequality is the general term of a convergent series. By Borell-Cantelli Lemma, we thus have

$$\limsup_{k\to\infty}\max_{z\in 2r\mathbb{Z}\cap Q_{t_k h_{t_k}}} X_z(t_k) < 1, \quad \text{a.s.} \tag{2.89}$$

We now draw conclusions on the principal eigenvalue itself. Indeed, from (2.79) and (2.89), it is readily checked that

$$\limsup_{k\to\infty}\frac{\lambda_{\dot{W}}(Q_{t_k})}{(\log t_k)^{\frac{1}{(1+H)}}} < \left(\frac{K}{r}+1\right)u, \quad \text{a.s.}$$

Thus some elementary monotonicity arguments show that

$$\limsup_{t\to\infty}\frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{\frac{1}{(1+H)}}} < \left(\frac{K}{r}+1\right)u \quad \text{a.s.} \tag{2.90}$$

Since the constant K in (2.90) is independent of r, and r can be arbitrarily large, we also get

$$\limsup_{t\to\infty}\frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{\frac{1}{(1+H)}}} \leq u \quad \text{a.s.}$$

Eventually recall that we had to impose the condition (2.88) on u. However u can be taken as close as we wish to the value $(2c_H\mathcal{E})^{\frac{1}{1+H}}$. As a consequence we get

$$\limsup_{t\to\infty}\frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{\frac{1}{(1+H)}}} \leq (2c_H\mathcal{E})^{1/1+H} \quad \text{a.s.}$$

■

2.5.3 Lower Bound

This section is devoted to a lower bound counterpart of Proposition 2.5.1. We start by a lemma asserting that $\lambda_{\dot{W}}(Q_t)$ cannot get too small with respect to an order of magnitude of h_t^2.

Lemma 2.5.9 *Let $\lambda_{\dot{W}}(Q_t)$ be the principal eigenvalue of the random operator $\frac{1}{2}\Delta+\dot{W}$ over the restricted space $\mathcal{K}(Q_t)$ of functions having compact support on $(-t,t)$. Then we have the following upper bound:*

$$\mathbf{P}\left[\lambda_{\dot{W}}(Q_t) \le h_t^2\right] \le \mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{th_t})} \frac{W(g_t^2)}{h_t^2} \le 1\right]. \tag{2.91}$$

Proof Observe that from (2.71),

$$\mathbf{P}\left(\lambda_{\dot{W}}(Q_t) \le h_t^2\right) = \mathbf{P}\left[\sup_{g\in\mathcal{K}(Q_{th_t})} \left\{\frac{W(g_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx\right\} \le 1\right]. \tag{2.92}$$

Moreover,

$$\mathbf{P}\left[\sup_{g\in\mathcal{K}(Q_{th_t})} \left\{\frac{W(g_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx\right\} \le 1\right] \le \mathbf{P}\left[\sup_{g\in\mathcal{G}(Q_{th_t})} \frac{W(g_t^2)}{h_t^2} \le 1\right]. \tag{2.93}$$

This is proved similarly to our considerations in Step 1 of the proof of Proposition 2.5.1, details are included here for sake of clarity.

Namely, in order to prove (2.93), notice that for $g \in \mathcal{G}\left(Q_{th_t}\right)$, we have $\phi = \frac{g}{\|g\|_2} \in \mathcal{K}(Q_{th_t})$. Consequently

$$\frac{W(\phi_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |\phi'(x)|^2 dx \le \sup_{f\in\mathcal{K}(Q_{th_t})} \left\{\frac{W(f_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |f'(x)|^2 dx\right\}.$$

Thus the bound

$$\sup_{f\in\mathcal{K}(Q_{th_t})} \left\{\frac{W(f_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |f'(x)|^2 dx\right\} \le 1 \tag{2.94}$$

implies, still with $\phi = g/\|g\|_2$,

$$\frac{W(\phi_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |\phi'(x)|^2 dx \le 1.$$

This in turn gives the following inequality when we write down ϕ in terms of g:

$$\frac{W(g_t^2)}{h_t^2} - \frac{1}{2}\int_{Q_{th_t}} |g'(x)|^2 dx \le \|g\|_2^2.$$

Therefore we have obtained, for every $g \in \mathcal{G}(Q_{th_t})$,

$$\frac{W(g_t^2)}{h_t^2} \le \|g\|_2^2 + \frac{1}{2}\|g'\|_2^2 = 1,$$

where the last equality follows from the fact that $g \in \mathcal{G}(Q_{th_t})$. Taking supremum and recalling that we have assumed (2.94), we get

$$\left\{ \sup_{g \in \mathcal{K}(Q_{th_t})} \left\{ \frac{W(g_t^2)}{h_t^2} - \frac{1}{2} \int_{Q_{th_t}} |g'(x)|^2 dx \right\} \leq 1 \right\} \subset \left\{ \sup_{g \in \mathcal{G}(Q_{th_t})} \frac{W(g_t^2)}{h_t^2} \leq 1 \right\}.$$

Thus, (2.93) is proved and (2.92) can be further reduced to

$$\mathbf{P}\left[\lambda_{\dot{W}}(Q_t) \leq h_t^2\right] \leq \mathbf{P}\left[\sup_{g \in \mathcal{G}(Q_{th_t})} \frac{W(g_t^2)}{h_t^2} \leq 1 \right],$$

which proves our result (2.91). ∎

Our next lemma is a general bound for Gaussian vectors with nontrivial covariance structure. It is borrowed from [4, Lemma 4.2] and will be used in a discretization procedure which is part of our strategy for the lower bound on $\lambda_{\dot{W}}(Q_t)$.

Lemma 2.5.10 *Let $(\xi_1, \ldots, \xi_n)$ be a mean-zero Gaussian vector with identically distributed components. Write $R = \max_{i \neq j} |\mathrm{Cov}(\xi_i, \xi_j)|$ and assume that $\mathrm{Var}(\xi_1) \geq 2R$. Then for any $A, B > 0$, the following inequality holds true:*

$$\mathbf{P}\left[\max_{k \leq n} \xi_k \leq A\right] \leq \left(\mathbf{P}\left[\xi_1 \leq \sqrt{\frac{2R + \mathrm{Var}(\xi_1)}{\mathrm{Var}(\xi_1)}}(A+B) \right] \right)^n + \mathbf{P}\left[U \geq \frac{B}{\sqrt{2R}} \right]$$

where U is a standard normal random variable.

We can now state our lower bound on the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$.

Proposition 2.5.2 *Under the same conditions as for Lemma 2.5.9, the following lower bound is fulfilled:*

$$\liminf_{t \to \infty} \frac{\lambda_{\dot{W}}(Q_t)}{(\log t)^{\frac{1}{(1+H)}}} \geq (2 c_H \mathcal{E})^{\frac{1}{1+H}} \quad \text{a.s.}$$

Proof We divide the proof in several steps.

Step 1: Reduction to a discrete Gaussian supremum. Let the constant $r > 0$ be fixed but arbitrary and set $\mathcal{N}_t = 2r\mathbb{Z} \cap Q_{t-r}$. When t is large enough (namely $t > r$ and $h_t > 1$), it is readily checked that $h_t z + Q_r \subset Q_{th_t}$ for each $z \in \mathcal{N}_t$. Hence,

$$\sup_{g \in \mathcal{G}(Q_{th_t})} W(g_t^2) \geq \max_{z \in \mathcal{N}_t} \sup_{g \in \mathcal{G}(h_t z + Q_r)} W(g_t^2).$$

and thus owing to (2.91),

$$\mathbf{P}\left[\lambda_{\dot{W}}(Q_t) \le h_t^2\right] \le \mathbf{P}\left(\max_{z\in\mathcal{N}_t}\sup_{g\in\mathcal{G}(h_t z+Q_r)} W(g_t^2) \le h_t^2\right).$$

For any $g \in \mathcal{G}(Q_r)$ and $z \in \mathcal{N}_t$, notice that $g^z(\cdot) \equiv g(\cdot - h_t z) \in \mathcal{G}(h_t z + Q_r)$. Hence $\max_{z\in\mathcal{N}_t}\sup_{g\in\mathcal{G}(h_t z+Q_r)} W(g_t^2) \ge \max_{z\in\mathcal{N}_t} W((g_t^z)^2)$, for any $g \in \mathcal{G}(Q_r)$. The consequent inequality is therefore:

$$\mathbf{P}\left[\lambda_{\dot{W}}(Q_t) \le h_t^2\right] \le \mathbf{P}\left(\max_{z\in\mathcal{N}_t} W((g_t^z)^2) \le h_t^2\right). \tag{2.95}$$

for any given (but arbitrary) $g \in \mathcal{G}(Q_r)$.

Step 2: Control of covariance. For ease of presentation let us denote $W((g_t^z)^2)$ by $\xi_z(t)$. We will try to control the covariance $\mathrm{Cov}(\xi_z(t), \xi_{z'}(t))$ for $z, z' \in \mathcal{N}_t$ in order to show that the assumptions of Lemma 2.5.10 are met. First notice that $\mathcal{F}((g^z)_t^2)$ can be also expressed as:

$$\begin{aligned}\mathcal{F}\left((g^z)_t^2\right)(\xi) = \int_{\mathbb{R}} e^{-\imath\xi x}\{g_t^z(x)\}^2 dx = \int_{\mathbb{R}} e^{-\imath\xi x}\left\{\sqrt{h_t}g^z(h_t x)\right\}^2 dx \\ = \int_{\mathbb{R}} e^{-\imath\xi x} h_t g^2(h_t(x-z))dx.\end{aligned}$$

Therefore, with change of variable $s = h_t(x-z)$ we get:

$$\mathcal{F}\left((g^z)_t^2\right)(\xi) = e^{-\imath\xi z}\int_{\mathbb{R}} e^{-\imath\xi\frac{s}{h_t}} g^2(s)ds = e^{-\imath\xi z}\mathcal{F}g^2\left(\frac{\xi}{h_t}\right) \tag{2.96}$$

Hence, the covariance of the random field $\xi_z(t)$ is given by

$$\begin{aligned}\mathrm{Cov}\left(\xi_z(t), \xi_{z'}(t)\right) &= \int_{\mathbb{R}} \mathcal{F}(g^z)_t^2(\xi)\overline{\mathcal{F}(g^{z'})_t^2(\xi)}\mu(d\xi) \\ &= c_H \int_{\mathbb{R}} e^{\imath\xi(z-z')}\left|\mathcal{F}g^2\left(\frac{\xi}{h_t}\right)\right|^2 |\xi|^{1-2H} d\xi,\end{aligned}$$

where the last equality follows by using (2.96) and by plugging in the value of μ. Changing variable, we can rewrite the covariance as

$$\mathrm{Cov}\left(\xi_z(t), \xi_{z'}(t)\right) = c_H h_t^{2(1-H)} \int_{\mathbb{R}} e^{\imath h_t u(z-z')}\left|\mathcal{F}g^2(u)\right|^2 |u|^{1-2H} du. \tag{2.97}$$

In particular, we have

$$\mathrm{Var}\left(\xi_0(t)\right) = c_H h_t^{2(1-H)} \sigma^2(g) \tag{2.98}$$

where $\sigma^2(g) = \int_{\mathbb{R}} |\mathcal{F}g^2(u)|^2 |u|^{1-2H} du$, which is a finite quantity when $g \in \mathcal{G}(Q_r)$ according to (2.57).

Recall that $h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}$ and thus $h_t \to \infty$ as $t \to \infty$. In addition, $h_t|z - z'| \geq 2h_t r$ uniformly for $z \neq z'$ in $\mathcal{N}_t$. Also observe again that $g \in \mathcal{G}(Q_r)$ and hence $G(u) = |\mathcal{F}g^2(u)|^2 |u|^{1-2H}$ is in L^1 thanks to (2.57). By Riemann-Lebesgue lemma, we get the following assertion uniformly for $z \neq z'$ in $\mathcal{N}_t$:

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^{\imath h_t u(z-z')} \left|\mathcal{F}g^2(u)\right|^2 |u|^{1-2H} du = 0.$$

Therefore, plugging this information into (2.97), we end up with

$$R_t = \max_{\substack{z,z' \in \mathcal{N}_t \\ z \neq z'}} |\mathrm{Cov}\left(\xi_z(t), \xi_{z'}(t)\right)| = o(h_t^{2(1-H)}). \tag{2.99}$$

Furthermore observe that (2.98) implies $\lim_{t\to\infty}[\mathrm{Var}(\xi_0(t))/h_t^{2(1-H)}] = c_H \sigma^2(g) > 0$. Thus we also get $\mathrm{Var}(\xi_0(t)) \geq 2R_t$ for t sufficiently large. Summarizing our considerations so far, we have proved that the family $\{\xi_z(t); z \in \mathcal{N}_t\}$ satisfies the conditions of Lemma 2.5.10 if t is large enough. We now introduce an additional parameter $v > 0$ (to be chosen small enough later on) and we resort to Lemma 2.5.10 with $A = h_t^2$ and $B = vh_t^2$ in order to write:

$$\mathbf{P}\left[\max_{z\in\mathcal{N}_t} \xi_z(t) \leq h_t^2\right] \leq V_t^1 + V_t^2 \tag{2.100}$$

where R_t is defined by (2.99), and

$$V_t^1 = \left(\mathbf{P}\left[\xi_0(t) \leq (1+v)h_t^2 \sqrt{\frac{2R_t + \mathrm{Var}(\xi_0(t))}{\mathrm{Var}(\xi_0(t))}}\right]\right)^{|\mathcal{N}_t|}, \quad V_t^2 = \mathbf{P}\left[U \geq \frac{vh_t^2}{\sqrt{2R_t}}\right].$$

We now bound these two terms separately.

Step 3: Bound on V_t^2. First, we bound the term V_t^2 on the right hand side in (2.100). By a classical bound on the normal tail probabilities:

$$\mathbf{P}\left[U \geq \frac{vh_t^2}{\sqrt{2R_t}}\right] \leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2R_t}}{vh_t^2} \exp\left(-\frac{v^2 h_t^4}{4R_t}\right). \tag{2.101}$$

Since by (2.99), $\frac{R_t}{h_t^{2(1-H)}} \to 0$ as $t \to \infty$ we have for t sufficiently large

$$\frac{1}{\sqrt{2\pi}} \frac{\sqrt{2R_t}}{vh_t^2} < 1. \tag{2.102}$$

As for the term inside the exponential in (2.101), observe that (recall $h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}$ again)

$$\frac{v^2 h_t^4}{4R_t} = \frac{v^2 h_t^{2(1+H)}}{4}\frac{h_t^{2(1-H)}}{R_t} = \frac{v^2 u^{1+H}}{4}\frac{h_t^{2(1-H)}}{R_t}\log t,$$

and that for t sufficiently large

$$\frac{R_t}{h_t^{2(1-H)}}\frac{4}{v^2 u^{1+H}} < \frac{1}{2}. \tag{2.103}$$

Plugging (2.102) and (2.103) into (2.101), for t sufficiently large we have

$$V_t^2 = \mathbf{P}\left[U \geq \frac{v h_t^2}{\sqrt{2R_t}}\right] \leq \frac{1}{\sqrt{2\pi}}\frac{\sqrt{2R_t}}{v h_t^2}\exp\left(-\frac{v^2 h_t^4}{4R_t}\right) \leq \exp\left(-2\log t\right) = \frac{1}{t^2}. \tag{2.104}$$

Step 4: Bound on V_t^1. Let us now bound V_t^1 in (2.100), which can be written as

$$V_t^1 = \left(\mathbf{P}\left[\frac{\xi_0(t)}{\sqrt{\mathrm{Var}(\xi_0(t))}} \leq \frac{(1+v)h_t^2\sqrt{2R_t + \mathrm{Var}(\xi_0(t))}}{\mathrm{Var}(\xi_0(t))}\right]\right)^{|\mathcal{N}_t|}. \tag{2.105}$$

Therefore according to relation (2.98), we have

$$V_t^1 = \left(\mathbf{P}\left[U \leq l_t\right]\right)^{|\mathcal{N}_t|}$$

where we have set

$$l_t = \frac{(1+v)h_t^{1+H}}{c_H\sigma^2(g)}\sqrt{2\frac{R_t}{h_t^{2(1-H)}} + c_H\sigma^2(g)}.$$

In order to work with this last term we can rewrite it as:

$$\begin{aligned} V_t^1 = \left(\mathbf{P}\left[U \leq l_t\right]\right)^{|\mathcal{N}_t|} &= \left(1 - \mathbf{P}\left[U > l_t\right]\right)^{|\mathcal{N}_t|} \\ &= \left[\exp\left(\frac{1}{\mathbf{P}[U > l_t]}\log\left(1 - \mathbf{P}[U > l_t]\right)\right)\right]^{|\mathcal{N}_t|\mathbf{P}[U>l_t]}. \end{aligned} \tag{2.106}$$

Owing to (2.99) and the fact that $\lim_{t\to\infty} h_t = \infty$, it is easily checked that $\lim_{t\to\infty} l_t = \infty$. Therefore,

$$\lim_{t\to\infty}\exp\left(\frac{1}{\mathbf{P}[U > l_t]}\log\left(1 - \mathbf{P}[U > l_t]\right)\right) = e^{-1} \tag{2.107}$$

Let us now concentrate on $|\mathcal{N}_t|\,\mathbf{P}[U > l_t]$ in (2.106). We use the following elementary facts about l_t:

(i) $\lim_{t\to\infty} \frac{R_t}{h_t^{2(1-H)}} = 0$, and thus

$$\lim_{t\to\infty} \left(l_t - c_{v,g} h_t^{1+H}\right) = 0, \text{ where } c_{v,g} = \frac{(1+v)}{c_H^{1/2}\sigma(g)}. \tag{2.108}$$

(ii) Since $\lim_{t\to\infty} l_t = \infty$, we have $\lim_{t\to\infty} l_t e^{l_t^2/2} \mathbf{P}[U > l_t] = \frac{1}{\sqrt{2\pi}}$.

Using this information it is easy to see that

$$\mathbf{P}[U > l_t] \sim \frac{e^{-\frac{l_t^2}{2}}}{l_t} \sim \frac{\exp\left(-\frac{c_{v,g}^2}{2} h_t^{2(1+H)}\right)}{\sqrt{2\pi} c_{v,g} h_t^{1+H}}$$

With the expression $h_t = \sqrt{u}(\log(t))^{\frac{1}{2(1+H)}}$ in mind, this yields

$$\mathbf{P}[U > l_t] \sim \frac{\exp\left(-\frac{c_{v,g}^2 u^{1+H}}{2} \log(t)\right)}{\sqrt{2\pi} c_{v,g} \left[u^{1+H} \log(t)\right]^{1/2}}$$

and thus

$$\mathbf{P}[U > l_t] \sim \frac{1}{\sqrt{2\pi} c_{v,g} [u^{1+H} \log(t)]^{1/2} t^{\frac{c_{v,g}^2 u^{1+H}}{2}}}$$

In addition, we also have $|\mathcal{N}_t| \sim \frac{t}{r}$, which yields

$$|\mathcal{N}_t| \mathbf{P}[U > l_t] \sim \frac{t^{1-\frac{c_{v,g}^2 u^{1+H}}{2}}}{\sqrt{2\pi} c_{v,g} [u^{1+H} \log(t)]^{1/2} r}$$

Recall now that $c_{v,g}$ is given by expression (2.108). Hence, choosing v small enough and provided

$$u < \left(2 c_H \sigma^2(g)\right)^{1/1+H}, \tag{2.109}$$

we can check that $\frac{c_{v,g}^2 u^{1+H}}{2} < 1$. Consequently, for t large enough we obtain:

$$|\mathcal{N}_t| \mathbf{P}[U > l_t] \geq t^{\beta}, \text{ for some } \beta > 0. \tag{2.110}$$

Plugging (2.107) and (2.110) into (2.106), we get the following relation for t sufficiently large:

$$V_t^1 = \left(\mathbf{P}\left[\xi_0(t) \leq (1+v) h_t^2 \sqrt{\frac{2R_t + \mathrm{Var}(\xi_0(t))}{\mathrm{Var}(\xi_0(t))}}\right]\right)^{|\mathcal{N}_t|} \leq e^{-t^{\beta}}. \tag{2.111}$$

Step 5: Conclusion. Reporting inequalities (2.104) and (2.111) into (2.100), we find:

$$\mathbf{P}\left[\max_{z\in\mathcal{N}_t} W((g^z)_t^2) \le h_t^2\right] \le \frac{1}{t^2} + \frac{1}{e^{t^\beta}},$$

for some $\beta > 0$ whenever

$$u < \left(2c_H\sigma^2(g)\right)^{\frac{1}{1+H}}.$$

Now appealing to (2.95) and using Borel-Cantelli Lemma:

$$\liminf_{k\to\infty} h_{t_k}^{-2}\lambda_{\dot{W}}(Q_{t_k}) > 1 \quad \text{a.s.}$$

for some increasing sequence t_k of integers. Thus the expression $h_t = \sqrt{u}(\log t)^{\frac{1}{2(1+H)}}$ and some elementary monotonicity arguments show that

$$\liminf_{t\to\infty} \lambda_{\dot{W}}(Q_t)(\log t)^{-1/(1+H)} > u \quad \text{a.s.}$$

Now thanks to (2.109) and taking $u \uparrow (2c_H\sigma^2(g))^{\frac{1}{1+H}}$, we have for every $g \in \mathcal{G}(Q_r)$,

$$\liminf_{t\to\infty} \lambda_{\dot{W}}(Q_t)(\log t)^{-1/(1+H)} \ge \left(2c_H\sigma^2(g)\right)^{\frac{1}{1+H}} \quad \text{a.s.}$$

Recall that $\mathcal{E} = \sup_{g\in\mathcal{G}(\mathbb{R})} \sigma^2(g)$. Hence taking supremum over $g \in \mathcal{G}(Q_r)$ and letting $r \to \infty$ gives the needed lower bound

$$\liminf_{t\to\infty} \lambda_{\dot{W}}(Q_t)(\log t)^{-1/(1+H)} \ge (2c_H\mathcal{E})^{\frac{1}{1+H}} \quad \text{a.s.}$$

■

2.6 Lyapounov exponent

In this section we will combine the Feynman-Kac representation of u and our preliminary study of the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$ in order to get the logarithmic behavior of $u_t(x)$, achieving the proof of our main Theorem 2.1.4.

2.6.1 Preliminary Results.

Recall that V^ε is defined by (2.34), and observe that one can also write

$$V_t^\varepsilon(x) = \int_0^t \dot{W}^\varepsilon(B_s)ds,$$

where $\dot{W}^\varepsilon$ is the regularized noise given by

$$\dot{W}^\varepsilon(x) = \int_{\mathbb{R}} l_\varepsilon(x-y)W(dy). \tag{2.112}$$

The following lemma will allow us to extend the domains on which principal eigenvalues are computed. The interested reader is referred to [4, Lemma 2.2] for a proof.

Lemma 2.6.1 *Let W be the Gaussian noise whose covariance is defined by (2.10), and let $\dot{W}^\varepsilon$ be defined by relation (2.112). For a bounded measurable set $D \subset \mathbb{R}$ we write $D_\varepsilon = (-\varepsilon, \varepsilon) + D$ and define for positive θ the eigenvalue type quantity $\lambda^+_{\theta\dot{W}}(D)$ by:*

$$\lambda^+_{\theta\dot{W}}(D) := \lim_{\varepsilon \downarrow 0} \lambda_{\theta\dot{W}}(D_\varepsilon)$$

Then $\lambda_{\theta\dot{W}^\varepsilon}(D)$ is bounded as follows:

$$\lambda_{\theta\dot{W}}(D) \le \liminf_{\varepsilon\downarrow 0} \lambda_{\theta\dot{W}^\varepsilon}(D) \le \limsup_{\varepsilon\downarrow 0} \lambda_{\theta\dot{W}^\varepsilon}(D) \le \lambda^+_{\theta\dot{W}}(D) \quad \text{a.s.}$$

The second lemma below is a first relation between Feynman-Kac representations of equation (2.1) and principal eigenvalues. It is stated for a general potential ξ which is pointwise defined but not necessarily bounded.

Lemma 2.6.2 *Let $\xi : \mathbb{R} \mapsto \mathbb{R}$ be a potential, not necessarily bounded. Let τ_D be the stopping time defined by $\tau_D = \inf\{t \ge 0 : B_t \notin D\}$ for a measurable bounded set $D \subset \mathbb{R}$. Then the following inequalities hold where $\lambda_\xi(D)$ is defined similarly to (2.53):*

(i) We have:

$$\int_D \mathbb{E}_x \left[\exp\left\{ \int_0^t \xi(B_s)ds \right\} \mathbf{1}_{\{\tau_D \ge t\}} \right] dx \le |D| \exp\left\{ t\lambda_\xi(D) \right\}. \tag{2.113}$$

(ii) For any α, $\beta > 1$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $\lambda_{(\beta/\alpha)\xi}(D) < \infty$ we have for $0 < \delta < t$:

$$\int_D \mathbb{E}_x \left[\exp\left\{ \int_0^t \xi(B_s)ds \right\} \mathbf{1}_{\{\tau_D \ge t\}} \right] dx \ge (2\pi)^{\alpha/2}\delta^{1/2}t^{\alpha/(2\beta)}|D|^{-2\alpha/\beta} \times$$
$$\exp\left\{ -\delta(\alpha/\beta)\lambda_{(\beta/\alpha)\xi}(D) \right\} \exp\left\{ \alpha(t+\delta)\lambda_{\alpha^{-1}\xi}(D) \right\}. \tag{2.114}$$

Proof The proof of (2.113) relies on classical Feynman-Kac representations of semi-groups. Namely if $T_t g$ is the semigroup on $L^2(D)$ defined by

$$T_t g(x) = \mathbb{E}_x\left[\exp\left\{\int_0^t \xi(B_s)ds\right\} g(B_t)\mathbf{1}_{\{\tau_D \geq t\}}\right], \ t \geq 0, x \in D, \tag{2.115}$$

it can be shown that the generator A of T_t admits a Dirichlet form defined by

$$\langle g, Ag\rangle = \int_D \xi(x)g^2(x)dx - \frac{1}{2}\int_D |\nabla g(x)|^2 dx.$$

One can prove that

$$\lambda_0 \equiv \sup_{\substack{g\in\mathcal{D}(A)\\ \|g\|=1}} \langle g, Ag\rangle = \sup_{g\in\mathcal{K}(D)} \langle g, Ag\rangle = \lambda_\xi(D).$$

Then (2.113) is obtained thanks to some spectral representation techniques. The reader is referred to [29] for further details and to [33] for the lower bound (2.114). ■

The following lemma holds as a consequence of the Markov property for the Brownian motion B, and will yield a second relation between Lyapounov exponent and our principal eigenvalue. It is borrowed from [33, Section 4].

Lemma 2.6.3 *Let $\xi\colon \mathbb{R} \mapsto \mathbb{R}$ be a not necessarily bounded potential and D be a measurable bounded set. Let $0 < \delta < t$ and assume $0 \in D$. Let $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.*

(i) The following upper bound holds true:

$$\mathbb{E}_0\left[\exp\left\{\int_0^t \xi(B_s)ds\right\}\mathbf{1}_{\{\tau_D\geq t\}}\right] \leq \left(\mathbb{E}_0 \exp\left\{\beta\int_0^\delta \xi(B_s)ds\right\}\right)^{1/\beta} \times \left(\frac{1}{(2\pi\delta)^{d/2}}\int_D \mathbb{E}_x\left[\exp\left\{\alpha\int_0^{t-\delta}\xi(B_s)ds\right\}\mathbf{1}_{\{\tau_D\geq t-\delta\}}\right]dx\right)^{1/\alpha}.$$

(ii) We also have the corresponding lower bound:

$$\mathbb{E}_0\left[\exp\left\{\int_0^t \xi(B_s)ds\right\}\right] \geq \left(\mathbb{E}_0 \exp\left\{-\frac{\beta}{\alpha}\int_0^\delta \xi(B_s)ds\right\}\right)^{-\alpha/\beta} \times \left(\int_D p_\delta(x)\mathbb{E}_x\left[\exp\left\{\frac{1}{\alpha}\int_0^{t-\delta}\xi(B_s)ds\right\}\mathbf{1}_{\{\tau_D\geq t-\delta\}}\right]dx\right)^{\alpha},$$

where we recall that p_δ designates the heat kernel in $\mathbb{R}$ (see Notations in the Introduction).

2.6.2 Upper Bound.

We can now apply the preliminary results on exponential functionals of B recalled in the last section, in order to get a first comparison between $\log(u_t(0))$ and the principal eigenvalue $\lambda_{\dot{W}}(Q_t)$. The logarithmic asymptotic behavior of $u_t(x)$ can be upper bounded thanks to the following result.

Proposition 2.6.1 *Let $\{u_t(x); t \geq 0, x \in \mathbb{R}\}$ be the field defined by (2.49). Then the following holds:*

$$\limsup_{t\to\infty} \frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\dot{W}}(Q_t)} \leq 1, \tag{2.116}$$

where $Q_t = (-t, t)$ and $\lambda_{\dot{W}}(D)$ is defined by (2.53) for a domain D.

Proof *Step 1: Decomposition of $u_t(0)$.* To implement the upper bound (2.116), let us introduce a constant M to be specified later on and for $k \geq 1$ let R_k be defined by

$$R_k = \left\{ Mt(\log t)^{\frac{1}{2(1+H)}} \right\}^k. \tag{2.117}$$

Also recall that, according to (2.49), we have $u_t(x) = \mathbb{E}_x\left[\exp\left(V_t(x)\right)\right]$, where $V_t(x)$ and $V_t^\varepsilon(x)$ are defined by (2.35). We now set $V_t(0) = V_t$ and $V_t^\varepsilon(0) = V_t^\varepsilon$. With these notations in hand, we decompose $u_t(0)$ as:

$$\begin{aligned} u_t(0) &= \mathbb{E}_0\left[\exp\left(V_t\right)\right] \\ &= \mathbb{E}_0\left[\exp\left(V_t\right)\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right] + \sum_{k=1}^{\infty} \mathbb{E}_0\left[\exp\left(V_t\right)\mathbf{1}_{\left\{\tau_{Q_{R_k}} < t \leq \tau_{Q_{R_{k+1}}}\right\}}\right] \end{aligned} \tag{2.118}$$

In order to upper bound the terms in our decomposition (2.118), we apply Hölder's inequality to each term in the sum. We get:

$$u_t(0) \leq U_{t,0} + \sum_{k=1}^{\infty} \mathbf{P}_0(\tau_{Q_{R_k}} < t)^{1/2} U_{t,k} \tag{2.119}$$

where

$$U_{t,0} = \mathbb{E}_0\left[e^{V_t}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right], \tag{2.120}$$

and for $k \geq 1$

$$U_{t,k} = \mathbb{E}_0^{1/2}\left[e^{2V_t}\mathbf{1}_{\left\{\tau_{Q_{R_{k+1}}} \geq t\right\}}\right]. \tag{2.121}$$

We will now bound the terms $U_{t,k}$ separately.

Step 2: Regularization. Let us replace the quantities V_t by V_t^ε in the definition of $U_{t,k}$ for $k \geq 0$. The corresponding random variables are denoted by $U_{t,k}^\varepsilon$. We start by getting a uniform bound for $U_{t,0}^\varepsilon$. Namely using Lemma 2.6.3(i) write

$$\begin{aligned}
U_{t,0}^\varepsilon &= \mathbb{E}_0\left[e^{V_t^\varepsilon}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right] = \mathbb{E}_0\left[\exp\left(\int_0^t \dot{W}^\varepsilon(B_r)dr\right)\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right] \\
&\leq \left(\mathbb{E}_0\left[\exp(q\int_0^1 \dot{W}^\varepsilon(B_s)ds)\right]\right)^{1/q} \\
&\times \left(\frac{1}{\sqrt{2\pi}}\int_{Q_{R_1}} \mathbb{E}_x\left[\exp\left(p\int_0^{t-1}\dot{W}^\varepsilon(B_s)ds\right)\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t-1\right\}}\right]dx\right)^{1/p} \\
&= \left(\mathbb{E}_0\left[e^{qV_1^\varepsilon}\right]\right)^{1/q}\left(\frac{1}{\sqrt{2\pi}}\int_{Q_{R_1}} \mathbb{E}_x\left[e^{pV_{t-1}^\varepsilon(x)}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t-1\right\}}\right]dx\right)^{1/p}.
\end{aligned} \tag{2.122}$$

We can now apply Lemma 2.6.2(i) to the right hand side of the above equation. This yields

$$\int_{Q_{R_1}} \mathbb{E}_x\left[e^{pV_{t-1}^\varepsilon(x)}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t-1\right\}}\right]dx \leq |Q_{R_1}|\exp\left[(t-1)\lambda_{p\dot{W}^\varepsilon}(Q_{R_1})\right].$$

Computing the volume $|Q_{R_1}|$ and plugging into (2.122) we obtain:

$$U_{t,0}^\varepsilon \leq \left(\mathbb{E}_0\left[\exp\left(qV_1^\varepsilon\right)\right]\right)^{\frac{1}{q}}\left(\frac{2R_1^2}{\pi}\right)^{\frac{1}{2p}}\exp\left[\frac{(t-1)}{p}\lambda_{p\dot{W}^\varepsilon}(Q_{R_1})\right]. \tag{2.123}$$

We now take limits in equation (2.123). In order to handle the left hand side of (2.123), we observe that the random variable $\mathbb{E}_0\left[e^{V_t^\varepsilon}\right]$ converges in $L^q(\Omega)$ to $\mathbb{E}_0\left[e^{V_t}\right]$ for $q \geq 1$ thanks to Proposition 2.4.1. Therefore for all $q \geq 1$, an easy application of Hölder's inequality shows that

$$L^q(\Omega) - \lim_{\varepsilon\downarrow 0}\mathbb{E}_0\left[e^{V_t^\varepsilon}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right] = \mathbb{E}_0\left[e^{V_t}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right],$$

where $L^q(\Omega)$ is the space of L^q random variables on $(\Omega, \mathcal{F}, \mathbf{P})$, see Notation 2.4.1. It follows that there exists a subsequence $\{\varepsilon_n; n \geq 0\}$ such that $\mathbf{P}$ − a.s. we have

$$\lim_{n\to\infty}\mathbb{E}_0\left[e^{V_t^{\varepsilon_n}}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right] = \mathbb{E}_0\left[e^{V_t}\mathbf{1}_{\left\{\tau_{Q_{R_1}} \geq t\right\}}\right]. \tag{2.124}$$

Similarly there exists a subsequence $\{\varepsilon_n'; n \geq 0\}$ of $\{\varepsilon_n; n \geq 0\}$ such that $\mathbf{P}-$ a.s. we have

$$\lim_{n\to\infty} \mathbb{E}_0\left[e^{qV_1^{\varepsilon_n'}}\right] = \mathbb{E}_0\left[e^{qV_1}\right] \tag{2.125}$$

Incorporating (2.124), (2.125) and Lemma 2.6.1 into the left and right hand sides of (2.123), we obtain the following relation $\mathbf{P}-$ a.s. (recall that $U_{t,0} = \mathbb{E}_0[e^{V_t}\mathbf{1}_{\{\tau_{Q_{R_1}} \geq t\}}]$ according to (2.120)):

$$U_{t,0} \leq \left(\mathbb{E}_0\left[e^{qV_1}\right]\right)^{\frac{1}{q}} \left(\frac{2R_1^2}{\pi}\right)^{\frac{1}{2p}} \exp\left[\frac{(t-1)}{p}\lambda^+_{p\dot{W}}\left(Q_{R_1}\right)\right]. \tag{2.126}$$

We can proceed similarly in order to bound the terms $U_{t,k}$ in (2.121). Indeed applying Cauchy-Schwarz inequality and following the same steps as for (2.122)-(2.123) we get, for all $k \geq 1$

$$\mathbb{E}_0\left[\exp(2V_t)\mathbf{1}_{\left\{\tau_{Q_{R_{k+1}}} \geq t\right\}}\right] \leq \left(\mathbb{E}_0\left[\exp(4V_1)\right]\right)^{\frac{1}{2}} \left(\frac{2R_{k+1}^2}{\pi}\right)^{\frac{1}{4}} \exp\left[\frac{(t-1)}{2}\lambda^+_{4\dot{W}}\left(Q_{R_{k+1}}\right)\right]. \tag{2.127}$$

Consequently plugging (2.126) and (2.127) into (2.119), we end up with:

$$u_t(0) \leq a_{1,p,q}\mathcal{M}_{p,t} + a_2\mathcal{R}_t, \tag{2.128}$$

where

$$\mathcal{M}_{p,t} = \exp\left(\frac{(t-1)}{p}\lambda_{p\dot{W}}\left(Q_{R_1}\right)\right) \text{ and } \mathcal{R}_t = \sum_{k=1}^{\infty} \alpha_k \exp\left(\frac{(t-1)}{4}\lambda^+_{4\dot{W}}\left(Q_{R_{k+1}}\right)\right), \tag{2.129}$$

and where we also recall that R_k is defined by (2.117), and the constants $a_{1,p,q}$ and a_2 are given by

$$a_{1,p,q} = \left(\frac{2R_1^2}{\pi}\right)^{\frac{1}{2p}} \left(\mathbb{E}\left[\exp(qV_1)\right]\right)^{\frac{1}{q}}, \qquad a_2 = \left(\mathbb{E}_0\left[\exp(4V_1)\right]\right)^{\frac{1}{4}}.$$

In (2.128), the constants α_k for $k \geq 1$ are also defined by:

$$\alpha_k = \left(\mathbb{P}_0\left(\tau_{Q_{R_k}} < t\right)\right)^{\frac{1}{2}} \left(\frac{2R_{k+1}^2}{\pi}\right)^{\frac{1}{8}}. \tag{2.130}$$

We will now treat the terms in (2.128) separately.

Step 3: Bound on $u_t(0)$. Let us first bound the constants α_k in (2.130). To this aim, we can invoke the reflection principle for Brownian motions (see e.g. [34, section 2.6]), which asserts that

$$\mathbb{P}_0\left(\tau_{Q_{R_k}} < t\right) \leq \frac{4}{\sqrt{2\pi t}} \int_{R_k}^{\infty} e^{-\frac{y^2}{2t}} dy \leq \frac{4\sqrt{t}}{\sqrt{2\pi} R_k} e^{-\frac{R_k^2}{2t}}.$$

Furthermore when t is large enough, it is readily checked from the expression (2.117) of R_k that

$$\frac{4\sqrt{t}}{\sqrt{2\pi} R_k} \leq 1,$$

uniformly in $k \geq 1$. Therefore we get

$$\mathbb{P}_0\left(\tau_{Q_{R_k}} < t\right) \leq e^{-\frac{R_k^2}{2t}}.$$

Plugging this inequality in equation (2.130) and designating by c a universal constant which can change from line to line, we get

$$\alpha_k \leq c R_{k+1}^{\frac{1}{4}} e^{-\frac{R_k^2}{4t}}. \tag{2.131}$$

We now prove the convergence of the weighted sum defining $\mathcal{R}_t$ in (2.129). To this aim, recalling the asymptotic relation proved in Proposition 2.5.1, we can say that for t sufficiently large:

$$\exp\left[\frac{(t-1)}{4}\lambda^{+}_{4\dot{W}}\left(Q_{R_{k+1}}\right)\right] \leq \exp\left[\frac{(t-1)}{4}\left(16(2c_H\mathcal{E})^{\frac{1}{1+H}}+1\right)\left(\log(R_{k+1})\right)^{\frac{1}{1+H}}\right]. \tag{2.132}$$

Consequently, using our bound (2.131) on α_k and the expression (2.117) for R_k we have the following:

$$\mathcal{R}_t \leq \sum_{k=1}^{\infty} A_{k,t} B_{k,t} C_{k,t}, \tag{2.133}$$

where

$$\begin{aligned} A_{k,t} &= c R_{k+1}^{\frac{1}{4}} = c M^{\frac{k+1}{4}} t^{\frac{k+1}{4}} (\log t)^{\frac{k+1}{8(1+H)}} \leq c M^{\frac{k}{2}} t^{\frac{k}{2}} (\log t)^{\frac{k}{4(1+H)}}, \\ B_{k,t} &= \exp\left(-\frac{R_k^2}{4t}\right) = \exp\left[-\frac{1}{4} M^{2k} t^{2k-1} (\log t)^{\frac{k}{1+H}}\right], \\ C_{k,t} &= \exp\left[\frac{(t-1)}{4}\lambda^{+}_{4\dot{W}}\left(Q_{R_{k+1}}\right)\right]. \end{aligned} \tag{2.134}$$

Furthermore, thanks to (2.132) for t large enough we have:

$$C_{k,t} \leq \exp\left[c\,t(\log R_{k+1})^{\frac{1}{1+H}}\right].$$

Thus, plugging the value (2.117) of R_k into the above inequality we get

$$C_{k,t} \leq \exp\left[c\,t(k+1)^{\frac{1}{1+H}}\left(\log M + \log t + \frac{1}{2(1+H)}\log\log t\right)^{\frac{1}{1+H}}\right].$$

It is then readily checked for large enough t, that

$$C_{k,t} \leq \exp\left[c(k+1)^{\frac{1}{1+H}}t(\log t)^{\frac{1}{1+H}}\right].$$

In addition, it is easily seen that for any arbitrary constant $c > 0$, there exists M large enough such that $M^{2k} > 8c(k+1)^{\frac{1}{1+H}}$ uniformly in k. Therefore for this value of M, for all $k \geq 1$ and t large enough, we have

$$\begin{aligned} B_{k,t}C_{k,t} &\leq \exp\left[c(k+1)^{\frac{1}{1+H}}t(\log t)^{\frac{1}{1+H}} - \frac{1}{4}M^{2k}t^{2k-1}(\log t)^{\frac{2k}{2(1+H)}}\right] \\ &\leq \exp\left[-\frac{1}{8}M^{2k}t^{2k-1}(\log t)^{\frac{2k}{2(1+H)}}\right] \leq \exp\left[-\frac{1}{8}M^{k}t^{k}(\log t)^{\frac{k}{2(1+H)}}\right]. \end{aligned} \tag{2.135}$$

Combining (2.134) and (2.135) we have thus obtained

$$A_{k,t}B_{k,t}C_{k,t} \leq c_1\eta_t^k e^{-c_2\eta_t^{2k}}, \text{ where } \eta_t = \sqrt{Mt(\log t)^{\frac{1}{2(1+H)}}}.$$

Furthermore, we have that $\eta_t^{2k} > k\eta_t$ for all positive integers k if $\eta_t > \sqrt{2}$. Thus, for sufficiently large t:

$$A_{k,t}B_{k,t}C_{k,t} \leq c_1\eta_t^k e^{-c_2 k\eta_t}.$$

Recalling (2.133), the following bound holds true for the term $\mathcal{R}_t$ defined by (2.128)-(2.129):

$$\mathcal{R}_t \leq c_1\sum_{k=1}^{\infty}\left(\eta_t e^{-c_2\eta_t}\right)^k < 2 \tag{2.136}$$

for all sufficiently large t such that $\eta_t e^{-c_2\eta_t} < \frac{1}{2}$.

Now let us work with the term $\mathcal{M}_{p,t}$ in the expression (2.128). Observe that using Theorem 2.1.2 we have:

$$\begin{aligned}\lim_{t\to\infty}\frac{\lambda_{p\dot{W}}\left(Q_{R_1}\right)}{\lambda_{\dot{W}}(Q_t)} &= \lim_{t\to\infty}\frac{\lambda_{p\dot{W}}\left(Q_{R_1}\right)}{(\log(R_1))^{\frac{1}{1+H}}}\frac{(\log(t))^{\frac{1}{1+H}}}{\lambda_{\dot{W}}(Q_t)}\frac{(\log(R_1))^{\frac{1}{1+H}}}{(\log(t))^{\frac{1}{1+H}}}\\ &= \frac{(2c_H p^2\mathcal{E})^{1/(1+H)}}{(2c_H\mathcal{E})^{1/(1+H)}}\\ &= p^{\frac{2}{1+H}},\end{aligned}\tag{2.137}$$

where we have also used the form of R_k from (2.117) to show that the limit of $\log(R_1)/\log(t)$ as t goes to infinity is 1. Plugging this identity into the definition (2.129), we get that

$$\frac{1}{t}\log(\mathcal{M}_{p,t}) \sim p^{\frac{1-H}{1+H}}\lambda_{\dot{W}}(Q_t),\tag{2.138}$$

as t goes to infinity. In particular, owing to Theorem 2.1.2 we have

$$\lim_{t\to\infty}\frac{1}{t}\log(\mathcal{M}_{p,t}) = \infty \quad \text{a.s.}$$

Finally, going back to (2.128) we write

$$\frac{1}{t}\log\left(u_t(0)\right) \le \frac{1}{t}\log\left(\mathcal{M}_{p,t}\right) + \frac{1}{t}\log\left(a_{1,p,q} + a_2\frac{\mathcal{R}_t}{\mathcal{M}_{p,t}}\right).\tag{2.139}$$

Due to (2.136) and the fact that $\lim_{t\to\infty}\mathcal{M}_{p,t} = \infty$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\left(a_{1,p,q} + a_2\frac{\mathcal{R}_t}{\mathcal{M}_{p,t}}\right) = 0.$$

Therefore, thanks to (2.138), relation (2.139) entails the following upper bound:

$$\limsup_{t\to\infty}\frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\dot{W}}(Q_t)} \le p^{\frac{1-H}{1+H}}.\tag{2.140}$$

At the very end, notice that the parameter $p > 1$ in (2.140) can be chosen arbitrarily close to 1. Therefore, taking limits as $p \downarrow 1$ in (2.140), we end up with

$$\limsup_{t\to\infty}\frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\dot{W}}(Q_t)} \le 1, \quad \text{a.s.},$$

which is our claim (2.116). ■

2.6.3 Lower Bound.

This section is devoted to finding a lower bound for $\log(u_t(0))$ matching the upper bound (2.116). Specifically we will get the following result.

Proposition 2.6.2 *Let u_t be the field defined by (2.49). Then the following holds:*

$$\liminf_{t\to\infty} \frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\dot W}(Q_t)} \geq 1, \tag{2.141}$$

where we recall that $Q_t = (-t,t)$ and $\lambda_{\dot W}(D)$ is introduced in (2.53).

Proof Let $p,q>1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$ with p close to 1 and let $0<b<1$ be close to 1. From Lemma 2.6.3(ii), taking $\alpha = p$, $q=\beta$, $\delta = t^b$ and $\xi = \dot W^\varepsilon$, we get

$$\begin{aligned} u_t^\varepsilon(0) = \mathbb{E}_0\left[\exp\left(\int_0^t \dot W^\varepsilon(B_s)ds\right)\right] &\geq \left(\mathbb{E}_0\left[\exp\left(-\frac{q}{p}\int_0^{t^b} \dot W^\varepsilon(B_s)ds\right)\right]\right)^{-\frac{p}{q}} \\ &\times \left\{\int_{Q_{t^b}} p_{t^b}(x)\mathbb{E}_x\left[\exp\left(\frac{1}{p}\int_0^{t-t^b} \dot W^\varepsilon(B_s)ds\right)\mathbf{1}_{\left\{\tau_{Q_{t^b}}\geq t-t^b\right\}}\right]dx\right\}^p, \end{aligned} \tag{2.142}$$

where we recall that p_δ is the heat kernel in $\mathbb{R}$. Hence some elementary bounds on p_δ over Q_{t^b} yield

$$u_t^\varepsilon(0) \geq D_{\varepsilon,b,p,t}F_{\varepsilon,b,p,t}, \tag{2.143}$$

where

$$\begin{aligned} D_{\varepsilon,b,p,t} &= \left(\mathbb{E}_0\left[\exp\left(-\frac{q}{p}\int_0^{t^b} \dot W^\varepsilon(B_s)ds\right)\right]\right)^{-\frac{p}{q}}, \\ F_{\varepsilon,b,p,t} &= \left\{\frac{e^{-t^b/2}}{\sqrt{2\pi t^b}}\int_{Q_{t^b}} \mathbb{E}_x\left[\exp\left(\frac{1}{p}\int_0^{t-t^b} \dot W^\varepsilon(B_s)ds\right)\right]dx\right\}^p. \end{aligned}$$

We will now bound $D_{\varepsilon,b,p,t}$ and $F_{\varepsilon,b,p,t}$ separately. In order to bound $F_{\varepsilon,b,p,t}$ we apply Lemma 2.6.2(ii), taking $\alpha = p$, $\beta = q$, $t = t-t^b$ and $\delta = t^b$:

$$\begin{aligned} &\int_{Q_{t^b}} \mathbb{E}_x\left[\exp\left(\int_0^{t-t^b} \dot W^\varepsilon(B_s)ds\right)\right]dx \\ &\qquad \geq (2\pi)^{\frac{p}{2}} t^{\frac{b}{2}}(t-t^b)^{\frac{p}{2q}}(2t^b)^{-\frac{2p}{q}} \exp\left(-t^b\frac{p}{q}\lambda_{\frac{p}{q}\dot W^\varepsilon}(Q_{t^b})\right)\exp\left(pt\lambda_{\frac{\dot W^\varepsilon}{p}}(Q_{t^b})\right). \end{aligned}$$

Using (2.143) and replacing $e^{-\frac{t^b}{2}}$ by e^{-Ct^b} for a larger C to absorb all bounded-by-polynomial quantities, we thus get

$$F_{\varepsilon,b,p,t} \geq e^{-Ct^b} \exp\left(-\frac{p^2 t^b}{q} \lambda_{\frac{p}{q}\dot{W}^\varepsilon}(Q_{t^b})\right) \exp\left(t\lambda_{\frac{\dot{W}^\varepsilon}{p}}(Q_{t^b})\right). \tag{2.144}$$

We now take limits as $\varepsilon \downarrow 0$ in relation (2.143). Invoking Proposition 2.4.1, we use our bound (2.144) and Lemma 2.6.1 which gives

$$u_t(0) \geq D_{b,p,t} F_{b,p,t} \tag{2.145}$$

with

$$D_{b,p,t} = e^{-Ct^b} \left(\mathbb{E}_0\left[\exp\left\{-\frac{q}{p}\int_0^{t^b} W(\delta_{B_s})ds\right\}\right]\right)^{-\frac{p}{q}},$$
$$F_{b,p,t} = \exp\left(-\frac{p^2 t^b}{q} \lambda^+_{\frac{p}{q}\dot{W}}(Q_{t^b})\right) \exp\left(t\lambda_{\frac{\dot{W}}{p}}(Q_{t^b})\right).$$

We will now prove that

$$\lim_{t\to\infty} \frac{1}{t}\log(D_{b,p,t}) = 0 \tag{2.146}$$

Indeed, it is easily seen that

$$\frac{1}{t}\log(D_{b,p,t}) = -\frac{C}{t^{1-b}} - \frac{p}{qt^{1-b}} \frac{\log\left(\mathbb{E}_0\left[\exp\left\{-\frac{q}{p}\int_0^{t^b} W(\delta_{B_s})ds\right\}\right]\right)}{t^b}. \tag{2.147}$$

Moreover combining (2.116) and Proposition 2.5.1 we get the following bound for t large enough:

$$\frac{\log\left(\mathbb{E}_0\left[\exp\left\{-\frac{q}{p}\int_0^{t^b} W(\delta_{B_s})ds\right\}\right]\right)}{t^b} \leq c(\log t)^{\frac{1}{1+H}}$$

Plugging this information into (2.147), we obtain (2.146).

Let us now analyze the term $F_{b,p,t}$ in (2.145). We have

$$\frac{1}{t}\log(F_{b,p,t}) = -\frac{p^2}{q}\frac{\lambda^+_{\frac{p}{q}\dot{W}}(Q_{t^b})}{t^{1-b}} + \lambda_{\frac{\dot{W}}{p}}(Q_{t^b}).$$

Taking into account the behavior of $\lambda_{\frac{\dot{W}}{p}}(Q_t)$ given by Proposition 2.5.1, we get

$$\liminf_{t\to\infty} \frac{\frac{1}{t}\log(F_{b,p,t})}{\lambda_{\frac{\dot{W}}{p}}(Q_{t^b})} \geq 1 \quad \text{a.s.} \tag{2.148}$$

In conclusion, plugging (2.148) and (2.146) into (2.145), we end up with

$$\liminf_{t\to\infty} \frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\frac{\dot{W}}{p}}(Q_{t^b})} \geq 1 \quad \text{a.s.} \tag{2.149}$$

Now taking $b \uparrow 1$ and then $p \downarrow 1$ in (2.149), and observing that λ is monotonic under both maneuver, we get our desired lower bound (2.141):

$$\liminf_{t\to\infty} \frac{\frac{1}{t}\log(u_t(0))}{\lambda_{\dot{W}}(Q_t)} \geq 1 \quad \text{a.s.}$$

■

3. RELATIVISTIC STABLE PROCESSES IN QUASI-BALLISTIC HEAT CONDUCTION IN THIN FILM SEMICONDUCTORS

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3.1 Introduction

Standard heat flow, at a macroscopic level, is modeled by the random erratic movements of Brownian motions starting at the source of heat. However, this diffusive nature of heat flow predicted by Brownian motion is not seen in certain materials (semiconductors, dielectric solids) over short length and time scales [35]. Experimental data portraying the non-diffusive behavior of heat flow has been observed for transient thermal grating (TTG) [36,37], time domain thermoreflectance (TDTR) [38] and others [39, 40], by altering the physical size of the heat source. The thermal transport in such materials is more akin to a superdiffusive heat flow, and necessitates the need for processes beyond Brownian motion to capture this heavy tail phenomenon. Recent works [41–47] try to explain the physics behind the quasiballistic heat dynamics. But these methods, driven mostly by the Boltzmann transport equation, are infeasible for processing experimental data. Some more recent studies [48, 49] try to explain the non-diffusive heat flow through hyperbolic diffusion equations, however, closer investigation shows that these methods fail to capture the inherent onset of nondiffusive dynamics at short length scales in periodic heating regimes.

The attempts mentioned above fail to provide a stochastic process that would explain the heat dynamics under short length-time regimes. The most natural stochastic process to explain a superdiffusive behavior is an alpha-stable Lévy process [50].

Alpha-stable Lévy processes differ from Brownian motion in that its movements are governed by stable distributions as compared to Gaussian distributions for the latter. In this context, some of us have tried to explain the heat flow dynamics through a "truncated Lévy distribution" approach [6,7], where it has been possible to extract the value of the Lévy superdiffusive coefficient α that regulates the alloy's quasiballistic heat dynamics.

The current contribution can be seen as a further step in this direction. Specifically, let $T(t,x)$ designate the temperature of a semiconductor or dielectric solid in the experimental settings alluded to above, with initial condition $T_0(x)$. Then we shall describe T through the following Feynman-Kac formula (see Section 3.2.1 for more details about Feynman-Kac representations):

$$T(t,x) = \mathbf{E}\left[T_0(x+X_t)\right], \tag{3.1}$$

where X is a well-defined Lévy process that captures the observed quasiballistic heat dynamics, in addition to being a good candidate for explaining the usual diffusive nature under non-special large length-time regimes. We shall see that such a process X can be chosen as a so-called relativistic stable process (see [51], and [52] for properties related to the relativistic Schrödinger operator). It possesses the remarkable property of behaving like an alpha-stable process under short length-time scales while being closer to Brownian motion otherwise. This is reflected in the estimates of the transition density, provided below in Section 3.2.2. Summarizing, our result lays the mathematical foundations of heat flow modeling on short time scales by means of stochastic processes. In addition, in spite of the fact that our computations are mostly one-dimensional, the model we propose allows natural generalizations to multidimensional and multilayer settings.

3.2 Relativistic stable process: a primer

In this section we give a short introduction on relativistic stable processes. We first recall the definition of this family of processes. Then we will give some kernel bounds indicating how relativistic processes transition, as t increases, from an α-stable behavior to a Brownian type behavior (this property being crucial to model quasiballistic heat dynamics in semiconductors).

3.2.1 Characteristics and Feynman-Kac formula

We will consider here some stochastic processes X, that is a family $\{X_t;\, t \geq 0\}$ of random variables indexed by time. In particular, each X_t has to be considered as a random variable. More specifically, we are concerned here with relativistic α-stable processes. Those objects are parametrized by two quantities $M > 0$ and $\alpha \in (0, 2]$, and will be denoted by X^M. For a relativistic process, each random variable X_t^M is $\mathbb{R}^d$-valued. Its probability distribution is described through the so-called characteristic function. Recall that the characteristic function of a $\mathbb{R}^d$ valued random variable X is given by $\phi(\xi) = \mathbb{E}[\exp(\imath\xi \cdot X)]$ for any $\xi \in \mathbb{R}^d$. For the relativistic stable process this is given, for any $t \geq 0$ and $\xi \in \mathbb{R}^d$, by:

$$\phi_M(\xi) \equiv \mathbb{E}\left[\exp\left(\imath\xi \cdot (X_t^M - X_0^M)\right)\right] = \exp\left(-t\left(\left(|\xi|^2 + M^{2/\alpha}\right)^{\alpha/2} - M\right)\right). \qquad (3.2)$$

Some standard stochastic processes are recovered for certain choices of M and α. Observe that we obtain Brownian motion when $\alpha = 2$, while $M = 0$ returns a rotationally symmetric α-stable process. The infinitesimal generator of X^M is given by $\mathcal{L}^M = M - (-\Delta + M^{2/\alpha})^{\alpha/2}$. Under the special choice of $\alpha = 1$, this reduces to the free relativistic Hamiltonian $M - \sqrt{-\Delta + M^2}$, which explains the name of the process. An explicit expression for the Lévy measure of X^M can be found in [53–55]. We omit this formula for sake of conciseness, since it will not be used in the remainder of the paper. Notice that for $\alpha = 1$, the quantity M can be interpreted as a mass [56]. This is no longer true when $\alpha \neq 1$, and M has to be generally seen as a parameter which prevents large random jumps in the paths $t \mapsto X_t$ (cf. the tail estimate (3.6) below).

Lévy processes like X^M are classically used in order to represent solutions of deterministic PDEs. In our case, consider the following equation governing the temperature T in our material:

$$\partial_t T(t,x) = \mathcal{L}^M T(t,x), \text{ with } T(0,x) = T_0(x). \tag{3.3}$$

Then it is a well known fact (see [50]) that the solution T to (3.3) can be represented by the Feynman-Kac formula (3.1), where the process X^M is our relativistic α-stable process. The Feynman-Kac representation is crucial in order to get equation (3.8) below, and is one of the main point of the current contribution. Indeed, we are giving a link here between the physical heat transfer and a proper stochastic process. This is in contrast with the truncated Lévy distribution approach advocated in [6, 7], which was taking into account the transition from stable to Gaussian type distributions but had no related Feynman-Kac representation.

3.2.2 Transition kernel estimates

In this subsection, we identify the behavior of a relativistic stable process with a stable process on short time scales and a Brownian motion on larger time scales. As mentioned above, this will be achieved by observing the patterns exhibited by the transition kernel of X_t^m. Some results will be stated without formal proof, and interested readers are referred to [53–55] for more details.

Since X^M is a Lévy process, it is also a Markov process. As such it admits a transition kernel p_t^M, defined by:

$$\mathbb{P}\left(X_{s+t}^M \in A \,|\, X_s = x\right) = \int_A p_t^M(x,y)\, dy,$$

for all $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$. Notice that p^M is related to the function ϕ_M (see definition (3.2)) as follows:

$$p_t^M(x,y) = \mathcal{F}^{-1}\phi_M(x-y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\imath(x-y)\cdot\xi} e^{-t\{(M^{2/\alpha}+|\xi|^2)^{\alpha/2}-M\}} d\xi. \tag{3.4}$$

We start by a simple bound on p_t^M, exhibiting the stable behavior for small times and the Brownian behavior for large times. Observe that this bound does not depend on the space variables x, y. We include its proof in Appendix 3.6, which is based on elementary considerations involving Fourier transforms, for sake of completeness.

Theorem 3.2.1 *Consider a relativistic α-stable process X^M, and let p^M be its transition kernel. Then there exists $c_1 = c_1(\alpha) > 0$ such that for all $t > 0$ and all $x, y \in \mathbb{R}^d$:*

$$p_t^M(x, y) \leq c_1(M^{d/\alpha - d/2} t^{-d/2} + t^{-d/\alpha}). \tag{3.5}$$

The upper bound (3.5) already captures a lot of the information we need on relativistic stable processes. Invoking sophisticated arguments based on stopping times and Dirichlet forms, one can get upper and lower bounds on the transition kernel p^M involving some exponential decay in the space variables x, y. We summarize those refinements in the following theorem.

Theorem 3.2.2 *Let p^M be the transition kernel defined by (3.4). Then the following estimates hold true.*

(i) Small time estimates. *Let $T > 0$ be a fixed time horizon. Then there exists $C_1 > 0$ such that for all $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,*

$$C_1^{-1} \left(t^{-d/\alpha} \wedge \frac{t e^{-M^{1/\alpha}|x-y|}}{|x-y|^{(d+\alpha+1)/2}} \right) \leq p_t^M(x, y) \leq C_1 \left(t^{-d/\alpha} \wedge \frac{t e^{-M^{1/\alpha}|x-y|}}{|x-y|^{(d+\alpha+1)/2}} \right). \tag{3.6}$$

(ii) Large time estimates. *There exists $C_2 \geq 1$ such that for every $t \in [1, \infty)$ and $x, y \in \mathbb{R}^d$,*

$$C_2^{-1} t^{-d/2} \leq p_t^M(x, y) \leq C_2 t^{-d/2}.$$

3.3 Application of relativistic stable processes to thermal modelling

In this section we show how to apply the mathematical formalism of Section 3.2 to our concrete physical setting. More specifically, in Section 3.3.1 we shall introduce length scales in our Lévy exponent (3.2). Then Section 3.3.2 compares our model to previous work. Finally Section 3.3.3 is devoted to a description of our experimental setting, and also relates our measurements to the Fourier exponents we have put forward.

3.3.1 Formulation in terms of material thermal properties

The "macroscopic" heat dynamics perceived in a solid crystal is the cumulative effect of *microscopic* motions of a wide distribution of discrete heat carriers linked with the material's fundamental phonon properties. Quantitatively, working backwards from the analytical solution of the Boltzmann transport equation under the relaxation time approximation (RTA-BTE), one can show that 3D phonon transport in an isotropic crystal is governed by a characteristic function given by [6, 57]:

$$\psi(\xi) = \sum_k \frac{C_k\,|\xi|^2\Lambda_{k,x}^2}{\tau_k(1+|\xi|^2\Lambda_{k,x}^2)} \Bigg/ \sum_k \frac{C_k}{1+|\xi|^2\Lambda_{k,x}^2} \tag{3.7}$$

In (3.7), the generalized summation index k runs over discretized wavevector space and all phonon branches, while the mode-specific properties C, τ and $\Lambda_x = |v_x|\tau$ signify the heat capacity per volume unit, relaxation time and mean free path measured along the x-axis respectively.

Concrete evaluations of (3.7) with first-principles DFT phonon data indicate that semiconductor alloys exhibit a transition from characteristic Lévy dynamics with $\psi \sim |\xi|^\alpha$ at short length-time scales to regular diffusive transport with $\psi \sim |\xi|^2$ [6]. Thus the evolution of relativistic processes, from alpha-stable behavior at short length and time scales to regular Brownian motion at longer scales, renders them suitable to describing quasiballistic thermal transport in semiconductor alloys. Let us consider such a material, having nominal thermal diffusivity $D = \kappa/C_v$ with κ being the thermal conductivity and C_v the volumetric heat capacity (in $\mathrm{Jm^{-3}K^{-1}}$). The physical quantity we have access to is a slight variation of the function $T(t,x)$ defined by (3.1). Specifically the single pulse response for the excess thermal energy can be expressed as $P(t,x) = C_v\,\Delta T(t,x)$. Under the Lévy flight paradigm the Fourier transform of P is written as

$$P(|\xi|, t) = \exp\left(-t\,\psi(|\xi|)\right), \tag{3.8}$$

for a given Lévy exponent (also called spatial heat propagator) ψ. For the relativistic case under study here, this spatial heat propagator ψ_{RS} is simply a multiple of the exponent in the function ϕ_M introduced in (3.2). Namely ψ_{RS} reads

$$\psi_{\mathrm{RS}}(|\xi|) = D_\alpha \left[\left(|\xi|^2 + M^{2/\alpha} \right)^{\alpha/2} - M \right]. \tag{3.9}$$

The prefactor D_α with unit $\mathrm{m}^\alpha/\mathrm{s}$ denormalizes the characteristic function for dimensionless space and time variables defined by Eq. (3.2) to its physical counterpart, and denotes the fractional diffusivity of the alpha-stable regime as we shall see shortly.

For thermal modeling purposes it is furthermore convenient to reformulate the process mass M, which has an exponent-dependent unit $1/\mathrm{m}^\alpha$, in terms of an associated characteristic length scale x_{RS} around which the transition from alpha-stable (Lévy) to Brownian dynamics takes place. Our analyses in Appendix 3.6 leading up to Eq. (3.18) and (3.19) show that this length should be given by:

$$x_{\mathrm{RS}} = |\xi_0|^{-1} = M^{-1/\alpha}.$$

This means that expression (3.9) can be recast as

$$\begin{aligned}\psi_{\mathrm{RS}}(|\xi|) &= D_\alpha \left[\left(|\xi|^2 + |\xi_0|^2 \right)^{\alpha/2} - |\xi_0|^\alpha \right] \\ &= D_\alpha \, |\xi_0|^\alpha \left[\left(\tilde{\xi}^2 + 1 \right)^{\alpha/2} - 1 \right],\end{aligned} \tag{3.10}$$

where $\tilde{\xi} \equiv |\xi|/|\xi_0|$. With those values of M and ξ_0 in hand, we can translate (3.19) into an asymptotic transport limit as follows:

$$\begin{aligned}\text{alpha-stable regime } \tilde{\xi} \gg 1: \;\; & \psi(|\xi|) \simeq D_\alpha \, |\xi|^\alpha, \\ \text{Brownian regime } \tilde{\xi} \ll 1: \;\; & \psi(|\xi|) \simeq \frac{\alpha D_\alpha}{2|\xi_0|^{2-\alpha}} \, |\xi|^2.\end{aligned} \tag{3.11}$$

The former corresponds to Lévy superdiffusion with characteristic exponent α and fractional diffusivity D_α; the latter should recover to nominal diffusive transport $\psi(|\xi|) \equiv D|\xi|^2$. In order to make the last relation compatible with (3.11) we must set

$$\frac{\alpha D_\alpha}{2|\xi_0|^{2-\alpha}} = D \quad \Longrightarrow \quad D_\alpha = \frac{2D}{\alpha {x_{\mathrm{RS}}}^{2-\alpha}}. \tag{3.12}$$

Finally, plugging (3.12) into (3.10) the heat propagator reads

$$\psi_{\mathrm{RS}}(|\xi|) = \frac{2D}{\alpha {x_{\mathrm{RS}}}^2} \left[\left(1 + {x_{\mathrm{RS}}}^2 |\xi|^2\right)^{\alpha/2} - 1 \right]. \tag{3.13}$$

This formulation contains 3 material dependent parameters, each with an intuitive physical meaning: the characteristic exponent α of the alpha-stable regime; the nominal diffusivity D of the Brownian regime; and the characteristic length scale x_{RS} around which the transition between those two asymptotic limits occurs (Fig. 3.1). In the sections that follow, we determine these parameter values for $\mathrm{In_{0.53}Ga_{0.47}As}$ by fitting a thermal model built upon the propagator (3.13) to time-domain thermoreflectance (TDTR) measurement signals.

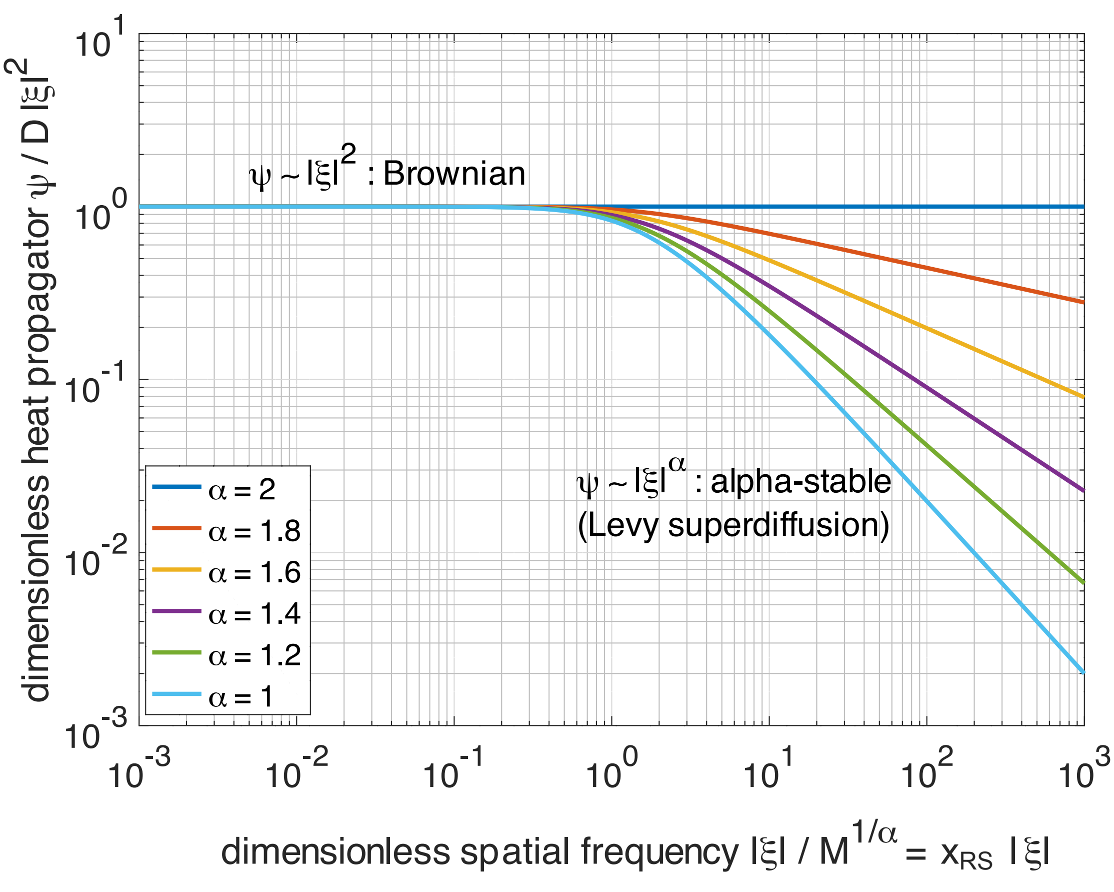


Figure 3.1.: Transition between Brownian and alpha-stable Lévy behavior.

3.3.2 Comparison with previous models

The first publications on this topic [6, 7] used a truncated 1D Lévy process in which the heavy tail of the jump length distribution was attenuated exponentially to enforce the required recovery from Lévy flights to Brownian motion. This approach had been adopted in previous literature in unrelated research disciplines and allowed to carry out most derivations in closed form; however, the resulting expression for ψ is cumbersome and the methodology proved problematic to be extended to higher dimensions. These difficulties were addressed in [57] with an improved "tempered Lévy" (TL) model that rigorously describes isotropic multi-dimensional processes with characteristic function

$$\psi_{\mathrm{TL}}(\xi) = \frac{D|\xi|^2}{(1 + x_{\mathrm{TL}}^2|\xi|^2)^{1-\alpha/2}} \tag{3.14}$$

This function induces a transition from Lévy dynamics with characteristic exponent α and fractional diffusivity $D_\alpha = D/x_{\mathrm{TL}}^{2-\alpha}$ to regular diffusive transport with bulk diffusivity D over characteristic length scale x_{TL} in a compact but merely phenomenological way. However, the main drawback of (3.14) is that it does not correspond to any known Lévy process documented in the literature.

The relativistic stable (RS) processes employed in this work describe a similar transition but through a characteristic function given by (3.2) that can be recast as (3.13), thanks to which we get a fractional diffusivity $D_\alpha = 2D\alpha^{-1}x_{\mathrm{RS}}^{-(2-\alpha)}$, with x_{RS} the characteristic length scale for ballistic-diffusive recovery.

While ψ_{RS} and ψ_{TL} can be shown to be quantitatively quite similar for data-fitting purposes, ψ_{RS} has the advantage to have been extensively studied and characterized by mathematicians and theoretical physicists. It corresponds to a classical RS process, as introduced in [51]. In addition, the Feynman-Kac representation (3.1) links the dynamic thermal profile of the material to averages over sample paths of the relativistic stable process, thus vastly expanding the applicability of this particular model. Our current paper adds to the already existing body of knowledge, but has the additional main objective to forge a collaborative connection between two communities that may otherwise not necessarily interact. On the one hand, our work demonstrates to the mathematical community that RS processes have timely and practical applications in (non-relativistic) physics and engineering contexts, which may help spur further research interest. On the other hand, it introduces solid state physicists and heat conduction specialists to a rich mathematical framework that may help to tackle the open and difficult problem of extending compact semi-analytical models for phonon transport dynamics to thin films and/or multilayer geometries.

Quantitative comparison between relativistic stable and tempered Lévy processes. We can directly compare the TL characteristic function to the RS counterpart by plotting their relative difference $(\psi_{\text{TL}} - \psi_{\text{RS}})/\psi_{\text{RS}}$. In order to make useful comparison, the asymptotes need to be the same. To achieve this, it suffices to impose that both processes have the same D, α and D_α, which for the latter requires that

$$x_{\text{TL}} = (\alpha/2)^{1/(2-\alpha)}\, x_{\text{RS}}. \tag{3.15}$$

The ratio $x_{\text{TL}}/x_{\text{RS}}$ is only weakly α dependent (it monotonically rises from 0.5 for $\alpha = 1$ to $\exp(-1/2) \simeq 0.607$ for $\alpha \to 2$). Having identical asymptotic regimes, the difference between the TL and RS characteristic functions is mainly situated in the transition region (i.e. for length scales slightly above and below x_{TL} and x_{RS}), and remains quite modest ($< 16\%$) for all allowed α values (Fig. 3.2). For typical exponents $\alpha \simeq 1.70$ observed in semiconductor alloys, both functions even remain within 2.2% across their entire domains.

3.3.3 Modelling of TDTR measurement signals

The central principle in TDTR is to heat up the sample with ultrashort *pump* laser pulses, and then monitor the thermal transient decay using a *probe* beam. Pulses from the laser are split into a pump beam and probe beam. The pump pulses pass through an electro-optic modulator (EOM) before being focused onto the sample surface through a microscope objective. A thin (50-100 nm) aluminium film is deposited

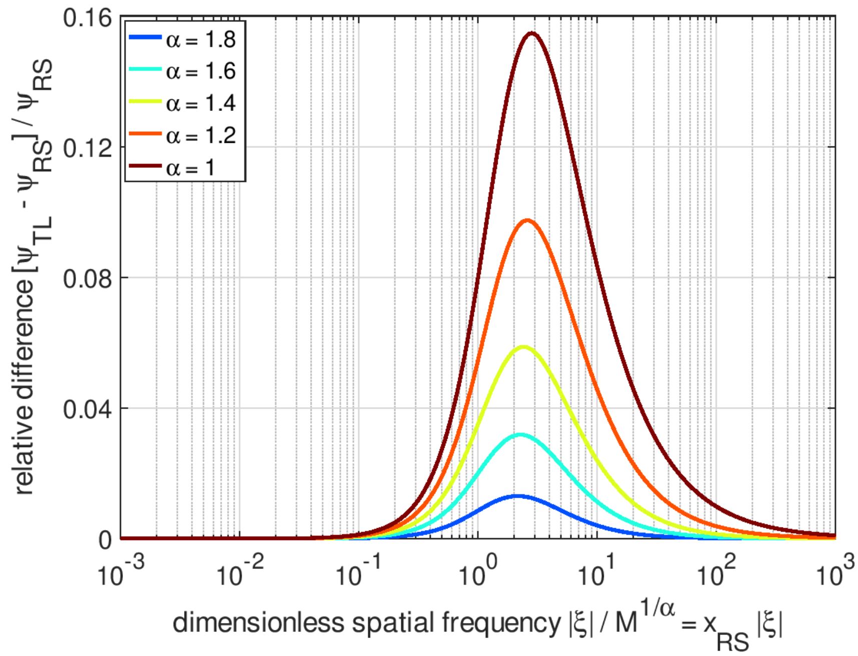


Figure 3.2.: Relative difference between the characteristic functions of a tempered Lévy process (ψ_{TL}) and relativistic stable process (ψ_{RS}).

onto the sample to act as measurement transducer: the metal efficiently absorbs the pump light and converts it to heat, and translates temperature variations to changes in surface reflectivity which can be captured by the probe. Lock-in detection at the pump modulation frequency f_{mod} resolves the thermally induced reflectivity changes captured by the probe beam. A mechanical delay stage allows to vary the relative arrival time of the pump and probe pulses at the sample with picosecond resolution. To minimize the impact of random fluctuations in laser power and the variation of the pump beam induced by the delay stage, thermal characterization is performed not on the raw lock-in signal itself but rather the ratio $-V_{\mathrm{in}}/V_{\mathrm{out}}$ of the in-phase and out-of-phase components as a function of the pump-probe delay.

Theoretical ratio curves $-V_{\mathrm{in}}/V_{\mathrm{out}}$ can be computed semi-analytically through mathematical manipulation of the semiconductor single-pulse response (3.8), as described in detail in Refs. [57–59]. Briefly, we first obtain the surface temperature response of a semi-infinite semiconductor to a cylindrically symmetric energy input via Fourier inversion of (3.8) with respect to the cross-plane coordinate. Next, a matrix formalism that accounts for heat flow in the metal transducer and across the intrinsic thermal resistivity r_{ms} (in K-m^2/W) of the metal-semiconductor interface provides the temperature response, weighted by the Gaussian probe beam, of the transducer top surface induced by a Gaussian pump pulse. Finally, harmonic assembly of this response at frequencies $n \cdot f_{\mathrm{rep}} \pm f_{\mathrm{mod}} \quad (\text{for } n = 0, 1, \ldots)$ accounting for the laser repetition rate f_{rep}, pump modulation frequency f_{mod}, and phase factors induced by the pump-probe delay τ yields the theoretical lock-in ratio signal $-V_{\mathrm{in}}/V_{\mathrm{out}}(\tau)$.

3.4 Experimental analysis

We have applied our model to TDTR measurements taken on a $\simeq$2 μm thick film of $\mathrm{In_{0.53}Ga_{0.47}As}$ ($C_v \simeq 1.55\,\mathrm{MJ/m^3\text{-}K}$) that was MBE-grown on a lattice-matched InP substrate. We note that although the semiconductor alloy under study (the InGaAs layer) is a geometrically thin film, thermally speaking it can still be considered (as is assumed by the thermal model) as a semi-infinite layer with good approximation. This is because the effective thermal penetration length $\ell = \sqrt{D/(\pi f_{\mathrm{mod}})}$ stays firmly within the film over the experimentally probed modulation range $0.8\,\mathrm{MHz} \lesssim f_{\mathrm{mod}} \lesssim 18\,\mathrm{MHz}$. The aluminium transducer deposited onto the sample measured 64 nm in thickness as determined by picoseconds acoustics. We used pump and probe beams with $1/e^2$ radii at the focal plane of 6.5 and 9 μm respectively, with respective powers of 17 and 8 mW at the sample surface.

In the thermal model with relativistic stable heat propagator (3.13), we fixed the heat capacity at the aforementioned $1.55\,\mathrm{MJ/m^3\text{-}K}$. Theoretical ratio curves were then collectively fitted through nonlinear least-square optimization to signals measured at 7 different modulation frequencies to identify the 4 key thermal parameters: the characteristic exponent α of the Lévy superdiffusion regime; the quasiballistic-diffusive transition length scale x_{RS} associated to the mass M; the nominal thermal conductivity $\kappa = C_v D$ of the diffusive regime; and the thermal resistivity r_{ms} of the transducer/semiconductor interface. The resulting best-fitting values $\alpha = 1.695$, $x_{\mathrm{RS}} = 0.86\,\mu\mathrm{m}$, $\kappa = 5.82\,\mathrm{W/m\text{-}K}$, $r_{ms} = 4.28\,\mathrm{nK\text{-}m^2/W}$ yield an excellent agreement with the measured signals (Fig. 3.3). Theoretical curves with parameter values deviating from the best fitting ones (Fig. 3.4) furthermore visually reveal the sensitivity to each of the parameters and illustrate the good quality of the best fit.

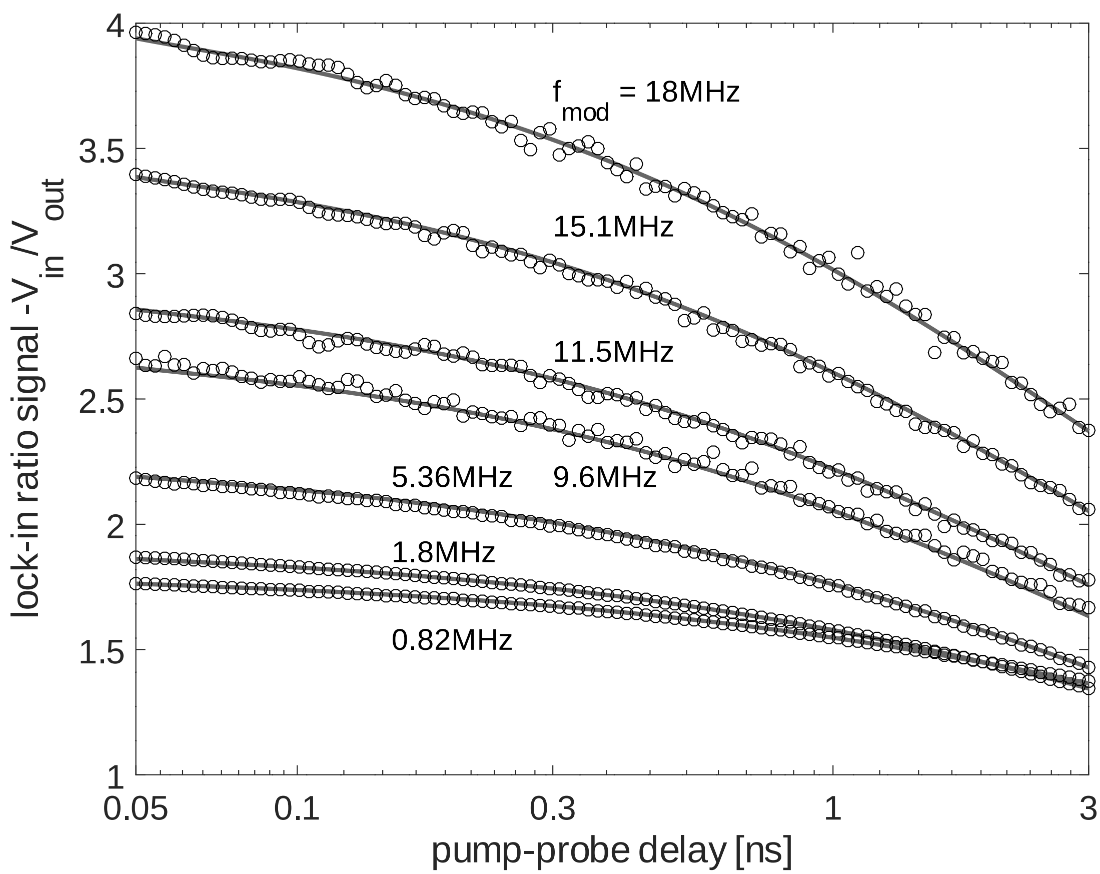
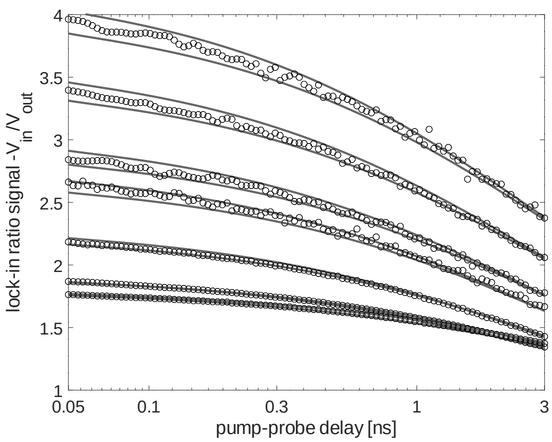
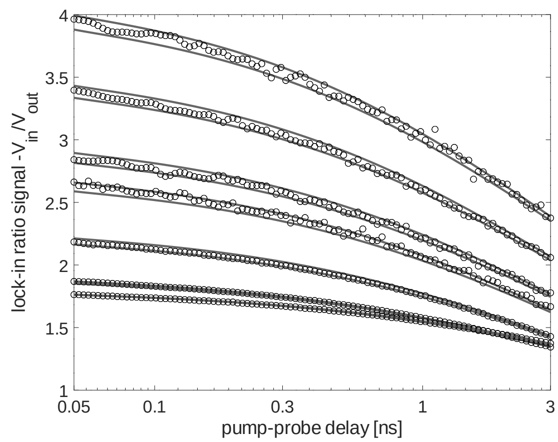


Figure 3.3.: TDTR characterization of Al/InGaAs sample: a thermal model based on a relativistic stable random motion of heat (lines) provides an excellent fit to measured signals (symbols).

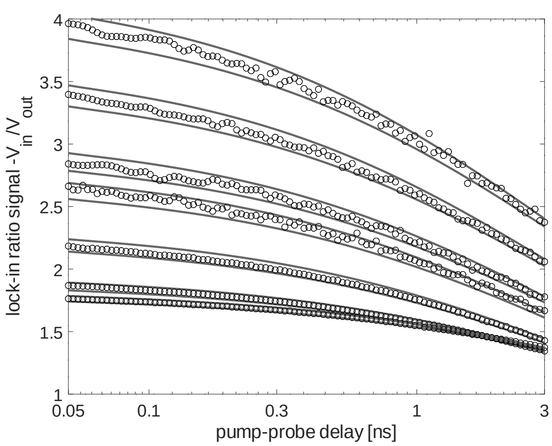
Remark 3.4.1 *The emergence of alpha-stable heat dynamics in alloys originates from the strong dependence of phonon lifetimes on frequency in these materials. An ideal Debye crystal with scattering relation $\tau \sim 1/\omega^n$ where $n \in (3, \infty)$ can be shown*



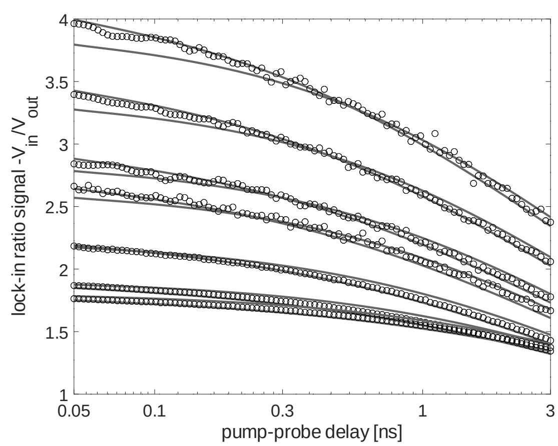
(a) $\alpha \pm 0.02$



(b) $x_{RS} \pm 10\%$



(c) $\kappa \pm 5\%$



(d) $r_{ms} \pm 50\%$

Figure 3.4.: Ratio curve fitting tolerance and sensitivity. The plotted theoretical curves are computed with sub-optimal parameter combinations in which one parameter deviates from its best fitting value as indicated.

to induce Lévy dynamics with characteristic exponent $\alpha = 1 + 3/n$ *(please see Appendix 3.7 for proof). One can therefore expect a generic ideal alloy, being governed by Rayleigh scattering (*$n = 4$*), to yield* $\alpha = 1.75$*. DFT computations for* $In_{0.53}Ga_{0.47}As$ *predict slightly lower exponents* $\alpha \in [1.67, 1.69]$ *(see [6, 60]), in very good agreement with the value of* 1.695 *inferred experimentally in this work.*

Remark 3.4.2 *The quasiballistic-diffusive transition length is related to the characteristic mean free path of the heat carriers. As phonon mean free paths typically span several orders of magnitude (roughly from 1 nm to 10 μm) a single "characteristic" value is not uniquely defined. However, a physically justified average can be obtained by weighing the mean free path of each individual phonon by its relative contribution to the total bulk thermal conductivity κ [60]:*

$$\Lambda_{\text{char}} = \sum_k \frac{\kappa_k}{\kappa} \cdot \Lambda_{k,x} = \frac{\sum_k C_k |v_{k,x}| \Lambda_{k,x}^2}{\sum_k C_k |v_{k,x}| \Lambda_{k,x}}$$

Evaluation of this expression with first-principles phonon data for $In_{0.53}Ga_{0.47}As$ yields $\Lambda_{\text{char}} = 0.73\,\mu m$, once again in good agreement with our experimentally inferred transition length scale $x_{RS} = 0.86\,\mu m$.

Remark 3.4.3 *Note that the truncated [6, 7] or tempered [57] Lévy approaches are in good quantitative agreement with respect to the Lévy exponent α obtained in this work. This complies with the fact that α is directly related to the dominant phonon scattering mechanism (see Remark 3.4.1) and therefore arises as an intrinsic property of the material sample that should be fairly insensitive to the model and fitting details. In addition, note that from (3.15) for $\alpha = 1.695$ we find $x_{\text{TL}} \approx 0.58 x_{\text{RS}}$. For the experimentally inferred value $x_{\text{RS}} = 0.86\mu m$ this implies that $x_{\text{TL}} = 0.50\mu m$, in good agreement with the value of $0.55\mu m$ previously found in [57].*

Remark 3.4.4 *The analysis in the current paper is limited to the TDTR setting, for which the mathematical setting is simple enough. Notice that the tempered Lévy formalism has also been successfully applied to TTG for the $Si_{0.93}Ge_{0.07}$ alloy in [61]. Although FDTR would be worth investigating in order to validate our Lévy type setting, we are not aware of any study in this direction.*

3.5 Conclusions and Outlook

Quasi-ballistic heat propagation in materials can be studied using atomic parameters through a multitude of techniques. First principle calculations and multi spectral phonon Boltzmann transport equations are very powerful in this regard. However, their use in the study of heat propagation in multi-layer/anisotropic materials and materials with complex geometries is limited. The Feynman-Kac representation of solutions to partial differential equations with non-local parameters can potentially provide alternative approaches to explain experimental thermal data. In this article we have replaced the traditional heat equation by a different PDE, whose solution has a Feynman-Kac representation driven by the so-called relativistic stable Lévy process. The transition characteristics of this process is in harmony to the heat propagation behaviour exhibited by TDTR data. In general, numerical approximations of the PDE solution can also be achieved through Monte Carlo simulations of the driving stochastic process in the Feynman-Kac formula. In particular, these numerical computations may provide substitute techniques to optimize materials or source geometry in order to reduce heating from nanoscale and/or ultrafast devices.

Our next challenge in this direction will be to model multidimensional transport in multilayer structures. To this aim, we shall investigate two methods: (i) Monte Carlo simulation according to our Feynman-Kac representation (3.1), taking into account jumps and change of media. (ii) Related PDEs involving the non local operator $\mathcal{L}^M = M - (-\Delta + M^{2/\alpha})^{\alpha/2}$, with boundary terms corresponding to the different layers. Both methods rely crucially on the relativistic Lévy representation advocated in this paper. They will be subject of future publications.

3.6 Proof of Theorem 3.2.1

Proof The strategy of our proof is based on the fact that the characteristic function ϕ_M defined by (3.2) behaves like a Gaussian characteristic function for low frequencies, and like an α-stable characteristic function for high frequencies. We shall quantify this statement below.

Step 1: Elementary inequalities: Let $\beta = \frac{\alpha}{2}$. The following inequality, valid for for $0 \le z \le 1$, is readily checked:

$$z^{\beta} - \beta z \le 1 - \beta. \tag{3.16}$$

Substituting $z = \frac{M^{2/\alpha}}{|\xi|^2 + M^{2/\alpha}}$ in (3.16), we thus have:

$$\frac{M}{(|\xi|^2 + M^{2/\alpha})^{\alpha/2}} - \frac{\alpha\, M^{2/\alpha}}{2(|\xi|^2 + M^{2/\alpha})} \le 1 - \frac{\alpha}{2},$$

which yields:

$$\left(|\xi|^2 + M^{2/\alpha}\right)^{\alpha/2} - M \ge \frac{\alpha |\xi|^2}{2(|\xi|^2 + M^{2/\alpha})^{1-\alpha/2}}. \tag{3.17}$$

Relation (3.17) prompts us to split the frequency domain in two sets:

$$A_1 = \{\xi : |\xi|^2 \le M^{2/\alpha}\}, \quad \text{and} \quad A_2 = \{\xi : |\xi|^2 > M^{2/\alpha}\}. \tag{3.18}$$

Accordingly, we get the following lower bounds:

$$(|\xi|^2 + M^{2/\alpha})^{\alpha/2} - M \ge \begin{cases} \dfrac{\alpha}{2(2M^{2/\alpha})^{1-\alpha/2}}\, |\xi|^2, \\ \qquad \text{when } \xi \in A_1 \\ \dfrac{\alpha}{2^{2-\alpha/2}}\, |\xi|^{\alpha}, \\ \qquad \text{when } \xi \in A_2. \end{cases} \tag{3.19}$$

This relation summarizes the separation between an α-stable and a Gaussian regime alluded to above.

Step 2: Consequence for the transition kernel. Recall relation (3.4) for p^M, that is:

$$p_s^M(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{\imath (x-y)\cdot\xi} e^{-t\{(M^{2/\alpha} + |\xi|^2)^{\alpha/2} - M\}} d\xi.$$

In the integral above, we simply bound $|e^{\imath(x-y)\cdot\xi}|$ by 1 and split the integration domain $\mathbb{R}^d$ into $A_1 \cup A_2$. Taking our relation (3.17) into account, this yields:

$$p_t^M(x,y) \le \frac{1}{(2\pi)^d} \int_{A_1} e^{-\frac{\alpha t}{2(2M^{2/\alpha})^{1-\alpha/2}}\, |\xi|^2} d\xi \\ + \frac{1}{(2\pi)^d} \int_{A_2} e^{-\frac{\alpha t}{2^{2-\alpha/2}}\, |\xi|^{\alpha}} d\xi \le I_t^1 + I_t^2, \tag{3.20}$$

where

$$\begin{aligned} I_t^1 &= \frac{1}{(2\pi)^d}\left[\frac{1}{t^{d/2}}\left(\frac{(2M^{2/\alpha-1})}{\alpha/2}\right)^{d/2}\int_{\mathbb{R}^d} e^{-|\xi|^2}d\xi\right] \\ I_t^2 &= \frac{1}{(2\pi)^d}\left[\frac{1}{t^{d/\alpha}}\left(\frac{(2^{2-\alpha/2})}{\alpha}\right)^{d/\alpha}\int_{\mathbb{R}^d} e^{-|\xi|^\alpha}d\xi\right]. \end{aligned}$$

It is now easily checked that

$$I_t^1 = \frac{c_2}{t^{d/2}}, \quad \text{and} \quad I_t^2 = \frac{c_3}{t^{d/\alpha}}.$$

Plugging this information into (3.20), our claim (3.5) follows. ■

3.7 Proof for emergence of Lévy heat dynamics in semiconductor alloys

Let us consider an isotropic Debye crystal with a single phonon branch. By definition, the branch has a linear dispersion $\omega = v_s k$, where ω is the phonon frequency, v_s the sound velocity, and $k = \|\vec{k}\|$ the phonon wavenumber. Assume furthermore a single dominant phonon scattering mechanism of the form $\tau \sim 1/\omega^n$ in which τ signifies the phonon relaxation time. Due to the linear dispersion, the phonon mean free path $\Lambda = v_s\tau$ then relates to the phonon wavenumber as

$$\Lambda \sim 1/k^n \Leftrightarrow \frac{\mathrm{d}\Lambda}{\mathrm{d}k} \sim 1/k^{n+1}. \tag{3.21}$$

The probability that, at any given moment and location, heat is being carried by a phonon having a wavenumber between k and $k + \mathrm{d}k$ is simply proportional to the phonon density of states:

$$f_K(k)\mathrm{d}k \sim k^2\mathrm{d}k.$$

A change of stochastic variable and invoking (3.21) provides the probability that heat is being carried with a phonon having a mean free path between Λ and $\Lambda + \mathrm{d}\Lambda$:

$$|f_\Lambda(\Lambda)\mathrm{d}\Lambda| = |f_K(k)\mathrm{d}k| \Rightarrow f_\Lambda(\Lambda) = f_K(k)\left|\frac{\mathrm{d}k}{\mathrm{d}\Lambda}\right| \sim k^2 \cdot k^{n+1} \sim k^{n+3} \sim 1/\Lambda^{1+3/n}.$$

We now turn our attention to Poissonian flight processes. These realize random motion by consecutive execution of the following two-step procedure: (i) remain in current position during a time ϑ drawn from an exponential distribution (having average ϑ_0); (ii) perform an instantaneous "jump" in a randomly chosen direction with length U drawn from a jump length distribution $p_U(u)$. It is known [62] that heavy-tailed jump length distributions $p_U(u) \sim 1/u^{1+\alpha}$ where $1 \leq \alpha < 2$ induce Lévy dynamics with characteristic exponent α, i.e. fluid limit of the flight process is governed by a characteristic function $\psi(\xi) \sim |\xi|^\alpha$.

The final step of the proof consists of deriving the jump length distribution $p_U(u)$ that is associated to the mean free path selection probability $f_\Lambda(\Lambda)$ we previously derived for the Debye crystal. While p_u and f_Λ intuitively have a direct qualitative relation, the distributions are quantitatively not the same, due to a subtle but important distinction in how physical heat carriers and random flyers exactly carry out their motion. On the one hand, thermal "jumps" carried out by phonons with long mean free paths take a proportionally longer time to complete than those with short mean free paths, since all phonons propagate with a given finite velocity v_s between scatterings. The Lévy flyer, on the other hand, is characterized by the same average wait time ϑ_0 irrespective of the distance traversed by any given jump. The jump length distribution $p_U(u)$, therefore, acts as a measure for the number of jumps with lengths between u and $u + \mathrm{d}u$ *that are executed per unit of time*. As the number of transitions that a given phonon mode can execute per second is given by its scattering rate $\tau^{-1} \sim 1/\Lambda$, we have

$$p_U(\Lambda) \sim f_\Lambda(\Lambda)/\Lambda \sim 1/\Lambda^{2+3/n} \sim 1/\Lambda^{1+(1+3/n)}.$$

This shows that for n values satisfying $1 \leq 1 + 3/n < 2$ (being $n > 3$ with n not necessarily integer) the jump length distribution is heavy-tailed and gives rise to Lévy dynamics with characteristic exponent $\alpha = 1 + 3/n$. Semiconductor alloys can therefore be expected to produce such behavior, since heat dynamics in these materials are predominantly governed by mass disorder (Rayleigh) scattering $\tau \sim 1/\omega^4$, yielding $n = 4 \leftrightarrow \alpha = 7/4 = 1.75$.

4. BOND PRICES UNDER INFORMATION ASYMMETRY AND A SHORT RATE WITH INSTANTANEOUS FEEDBACK

A version of this chapter has been submitted for review.

4.1 Introduction

In a market, different traders have accesses to different levels of information. This leads to different interpretations of market dynamics under varying degrees of information. Even knowledge of the fact that a trader does not have access to full information greatly alters her strategy. Therefore, understanding difference in dynamics for informed and partially informed traders is a crucial part of modeling dynamics of an information influenced market. Mathematically, additional or exclusive information that only informed traders can access adds extra random sources to a model, and makes the market incomplete as a result. Also, one needs a larger filtration to make additional random variables or stochastic processes adapted.

Information is modeled by a filtration in mathematical finance. A trader with access to more information has a larger filtration than one with access to less. In [63], [64] and [65] the filtration for the fully informed trader is assumed to be of the form $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(L)$ for some fixed random variable L (usually a future price). [66] studied a model that incorporated the availability of additional information to the investor with time. They used a sequence of random variables available only to insiders as additional information at certain points of times. Also see [67], [68] and [69] for enlargement of filtration, utility maximization, and other advancements in this area. This article is a continuation of a series of articles where these studies have been generalized to the case with $\mathcal{F}_t = \mathcal{G}_t \vee \sigma(X_s, 0 \leq s \leq t)$. Here the additional information given to insiders is not a single random variable nor a discrete sequence of random variables, but a continuous time jump-diffusion process.

Let us mention a few cases where some of us have implemented the above notion of information generated by a continuous time process in the context of a hedging problem. [70] considered the case where the information modeled by a diffusion affects the jumps of the price process by influencing the timings of the jumps. The jumps in this study follow a doubly stochastic Poisson process depending on the information. Likewise [71] studied the other scenario where the information affects jump sizes of the price. The two models considered in [70] and [71] may be combined to consider the case where the jump times and sizes of the price process are affected separately by two different information processes. The extension of the model to the scenario when the jump sizes and times of the price process are affected by a single information process is complicated; nevertheless this was implemented in [72].

So far, most studies on information asymmetry are about stock markets. On the other hand, there are only a few studies on information effects on a fixed income market. Information asymmetry is more obvious in a stock market since there are many hidden factors affecting stock prices. On the other hand, there may be less private information in the bond market. However, there are so many factors which affect the interest rate, and a trader can be more informed than others by superior understanding of the market. For instance, a trader with better understanding and interpretation on the effect of COVID19 on the economy in short and long terms is more informed than those who are lack of this insight. In the mathematical modeling, a less informed trader will omit an important information process while an informed one will include. The present article concerns the bond price dynamics for a fully informed versus partially informed trader, where the latter is aware of the deficiency in her knowledge.

There exists a plethora of models for the short rate. The diffusion-type models described to use the short rate can have both a single factor as well as multiple factors. The most common textbook examples from the first category are the time homogeneous models given by Vasicek [73], Rendleman-Bartter [74], Cox-Ingersoll-Ross [75], Ho-Lee [76] and Hull-White [77] among others. They can all be expressed in the general form:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{4.1}$$

where r is the short rate and W is a Brownian motion under a risk neutral measure. The time homogeneous single factor lognormal models, for example the ones by Black-Derman-Toy [78], Black-Karasinski [79], and Kalotay-Williams-Fabozzi [80] can be generally expressed as:

$$d\log r_t = \mu(t, \log r_t) + \sigma(t, \log r_t)dW_t.$$

Single factor models suffer from the drawback of creating all points on the yield curve perfectly correlated. This can be improved by the introduction of multi-factor models for the short rate, where the short rate dynamics is additionally dependent on other factors or prices. They also appear to provide better fitness to empirical data. The most common examples of this model are the Longstaff-Schwarz model [81] and the Chen model [82] with factors or prices assumed to satisfy a diffusion with a general representation:

$$dX_t^i = \mu^i(t, X_t^i)dt + \sigma^i(t, X_t^i)dW_t^i, \; i = 1, \ldots, n,$$

where W^i's are Brownian motions. The short rate is now given by another diffusion akin to (4.1), but where the drift and diffusion coefficients depend on X^i.

In this article we will adopt a multi-factor model for the short rate, that is, n independent factors are assumed to influence the dynamics of the interest rate. In addition we assume only k many of them are known by the partially informed trader. However, we will go beyond the diffusion setting and allow all factors or price processes to have jumps. These jumps are modeled by compound inhomogeneous Poisson processes. The short term interest rate is also given by an Itô-diffusion with jumps, where the latter is modeled the jumps of a compound doubly stochastic Poisson process whose arrival intensity depends on all factors. Thus the short rate receives instantaneous feedback from all factors. Let us mention that a model similar to ours has already been analyzed in [8], where a multi-factor model is assumed for the short rate. The main difference between the model considered in [8] and the one analyzed in this paper is that while only the short rate and no factor is observable in the former, our article allows the factors to be partially observable, in addition to incorporating jumps in the short rate and factor processes.

Considering jump-diffusion type factors gives us a great flexibility and also fits the market better. Some factors or news come continuously like a diffusion, while others arrive at random times. Also, some factors generally move continuously but sometimes have larger shocks that a usual diffusion cannot explain. Therefore, by using jump-diffusion type factors, we can successfully model most types of factors, whether they move continuously, have jumps at random times, or both.

A partially informed trader either observes only a proper subset of these factors or fails to include some of factors in the analysis since she does not know that these are also relevant. The goal of this paper is to find the bond price of this partially informed trader. The partially informed trader knows that she is partially informed and tries to minimize the difference between an informed trader and herself. We consider here the least squares estimate (of the bond price process for the informed trader) amongst all processes adapted to the information filtration of the partially informed trader.

We also provide the bond price dynamic of the fully informed trader. As we will see, this dynamic is given by a partial differential equation. More precisely, we shall obtain the following theorem (also see Theorem 4.4.1).

Theorem 4.1.1 *Let h be the bond price of the fully informed trader. Then we have*

$$\begin{aligned} \partial_t h_t + (\mathcal{A}h)_t - r h_t &= 0, \\ h_T &= 1, \end{aligned} \tag{4.2}$$

where $\mathcal{A}$ is the infinitesimal generator of the multidimensional process formed by combining all factors and the short rate.

The bond price dynamic of the partially informed trader is much more complicated and is given below (see Theorem 4.5.17 for the precise version).

Theorem 4.1.2 *Let π be the bond price of the partially informed trader. Then the dynamic of π is given by a jump diffusion, that is we have:*

$$\pi_t = \pi_0 + \int_{0+}^{t} A_u du + \int_{0+}^{t} D_u d\hat{B}_u + \int_{0+}^{t} J_u du,$$

where $\hat{B}$ is a drifted version of the Brownian motions driving the observed processes, and J is the jump part.

As we will see, the jump part is driven only by the jumps in the observed processes. In addition, the diffusion coefficient D_u depends on the filtered estimates of parameters involved only in the observed process. Theorem 4.1.2 is achieved by applications of techniques in stochastic filtering. This involves working with a new measure to obtain some drifted versions of Brownian motions in the original measure as Brownian motions with no drift in the new measure. In addition, the non-homogeneity of the Poisson processes modeling the location of jumps is removed.

Having stated the key results, we now describe the outline of this article. We explain the model in Section 2. Section 3 introduces necessary mathematical properties essential to our analysis. Our main results are in Section 4 and Section 5, containing the pricing dynamics of a fully informed and a partially informed trader, respectively. Section 6 concludes.

4.2 Model

We restrict ourselves to a finite time horizon $[0, T]$ where the short rate r under the risk neutral measure is given by a jump-diffusion process. However we allow the drift and the jumps of the short rate to depend on n independent information processes or factors X^i, $i = 1, \ldots, n$. Each of these factors is again given by a jump-diffusion process.

4.2.1 Description of factors.

For each $i = 1, \ldots, n$, the information process X^i is given by:

$$dX_t^i = \alpha_X^i dt + \sigma_X^i dW_t^i + dR_t^i, \tag{4.3}$$

where α_X^i and $\sigma_X^i > 0$ are constants for simplicity. W^i's are independent standard Brownian motions and R^i's account for the jumps in the information process X^i. These jumps are modeled by a compound Poisson process given by

$$R_t^i = \sum_{j=1}^{N_t^i} U_j^i.$$

Here N_t^i is a Poisson process with intensity function λ^i, while the U_j^i's are iid random variables drawn from some distribution ν^i, having finite second moment. All Poisson processes and U_j^i's are assumed to be independent within themselves and of each other, and the Brownian motions W^i. In addition, for simplicity in presentation we assume that on $[0, T]$ each intensity function λ^i is bounded and bounded away from 0.

4.2.2 Description of short rate.

The risk neutral dynamics of the short rate is given by

$$dr_t = \mu_t(X_{t-}^{1:n}, r_{t-})dt + \sigma_t(r_{t-})dW_t + dR_t, \tag{4.4}$$

where $R_t = \sum_{j=1}^{N_t} U_j$. However we make the jumps depend on the information processes by now modeling N_t as a doubly stochastic inhomogeneous Poisson process with intensity function $\lambda_t(X_{t-}^{1:n}, r_{t-})$, while as before in (4.3), the U_j's are iid random variables drawn from distribution ν with finite second moment. W appearing in (4.4) is a standard Brownian motion. Both N and W are independent of X^i for all i. We also need some restrictions on the drift and diffusion terms in (4.4). Namely, we assume μ (resp. σ) as functions from $\mathbb{R}^+ \times \mathbb{R}^{n+1}$ (resp. $\mathbb{R}^+ \times \mathbb{R}$) to $\mathbb{R}$ (resp. $\mathbb{R}^+$) such that each is Lipschitz (so that (4.4) makes sense) and $\frac{\mu}{\sigma}$ is bounded. Finally for sake of simplicity in presentation we again assume that the intensity process λ is bounded and bounded away from 0.

Remark 4.2.1 *Note that the short rate r_t and all factors X_t^i have jumps, so that the model can capture any unusual movement. Each factor X_t^i is composed of three parts, a drift part, a Brownian motion part, and a jump part. The first two move continuously, while the last moves at random times. The arrival rate $\lambda_t(X_{t-}^{1:n}, r_{t-})$ gets instantaneous feedback from the current short rate r_t and the current factor levels $X_{t-}^{1:n}$. Therefore factors give instantaneous feedback to the short rate.*

4.2.3 Description of information asymmetry.

Next we consider two traders. The fully informed trader observes all the information processes X^i, $i = 1, \ldots, n$. However out of these n information processes only k of them are known to the partially informed trader. Without loss of generality the partially informed trader observes only X^j, $j = 1, \ldots, k$. Since we are already in the risk neutral world, the bond price at time t for the fully informed trader is calculated by

$$h_t(X_t^{1:n}, r_t) = \mathbb{E}\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right] \tag{4.5}$$

where $\mathcal{F}_t = \sigma(X_s^{1:n}, r_s; 0 \leq s \leq t)$. Note that due to the Markovian nature of all processes in $\mathcal{F}_t$,

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} | \mathcal{F}_t\right] = \mathbb{E}\left[e^{-\int_t^T r_s ds} | X_t^{1:n}, r_t\right].$$

Thus the relation (4.5) is well-defined. Let us define $\mathcal{G}_t = \sigma(X_s^{1:k}, r_s; 0 \leq s \leq t)$ which is the filtration generated by the processes observed by the partially informed trader and corresponds to the information available to her. The bond price of the partially informed trader is given by $\mathbb{E}[h_t(X_t, Y_t, r_t)|\mathcal{G}_t]$. This is the least squares estimate of the bond price chosen among all $\mathcal{G}_t$-adapted processes. The aim of this article is to calculate this conditional expectation.

4.3 Preliminaries

In this section we will gather some essential results from stochastic calculus which would prove to be useful in later sections. We first state the Girsanov-Meyer theorem borrowed from [83] which discusses the change in a semimartingale decomposition under a change of measure.

Theorem 4.3.1 (Girsanov-Meyer) *Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\leq t<\infty}, \mathbb{P})$ be a filtered probability space. In addition let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures (that is, each is absolutely continuous with respect to each other) on $(\Omega, \mathcal{F})$. Denote Z_t to be the right continuous version of $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$. Let X be a classical semimartingale under $\mathbb{P}$ with decomposition $X = M + A$. Then X is also a classical semimartingale under $\mathbb{Q}$ and has a decomposition $X = L + C$, where*

$$L_t = M_t - \int_0^t \frac{1}{Z_s} d[Z, M]_s$$

is a $\mathbb{Q}$ local martingale, and $C = X - L$ is a finite variation process under $\mathbb{Q}$.

The most ubiquitous result in stochastic calculus is perhaps Itô's formula. In this text, we utilize a general version of this important result involving semimartingales. For completeness, the result is stated below and the interested reader is directed to [83] for further information.

Theorem 4.3.2 (Itô's formula) *Let $X = (X^1, \ldots, X^n)$ be an $n-$tuple of semimartingales, and let $f : \mathbb{R}^n \mapsto \mathbb{R}$ have continuous second order partial derivatives. Then $f(X)$ is a semimartingale and the following formula holds:*

$$f(X_t) - f(X_0) = \sum_{i=1}^n \int_{0+}^t \frac{\partial f}{\partial x_i}(X_{s-})dX_s^i + \frac{1}{2}\sum_{1\leq i,j\leq n}\int_{0+}^t \frac{\partial^2}{\partial x_i \partial x_j}(X_{s-})d[X^i, X^j]_s^c$$
$$+ \sum_{0<s\leq t}\left\{f(X_s) - f(X_{s-}) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\right\}.$$

One essential result when dealing with change of numeraire or change of measure is the Bayes formula. In the sequel we will be working with a transformed measure which is not an equivalent martingale measure. Usually in asset pricing, the purpose of changing the measure is to find a measure which makes the price process a martingale. However, our purpose of changing the measure is different. This formula helps in the computation of conditional expectations under the original measure in terms of conditional expectations under the transformed measure which may exhibit more amenable forms.

Lemma 4.3.3 (Bayes formula) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{P} \ll \mathbb{Q}$ for some probability measure $\mathbb{Q}$. Then for any σ-algebra $\mathcal{G} \subset \mathcal{F}$ and for any integrable random variable X (i.e., $\mathbb{E}_{\mathbb{P}}|X| < \infty$), the Bayes formula holds:*

$$\mathbb{E}_{\mathbb{P}}\left(X|\mathcal{G}\right) = \frac{\mathbb{E}_{\mathbb{Q}}\left[X\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{G}\right]}{\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{G}\right]} \quad \mathbb{P}-\text{a.s.} \tag{4.6}$$

The expression is well-defined, as $\mathbb{P}\left(\mathbb{E}_{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{G}\right) = 0\right) = 0$.

4.3.1 Counting measure and compensator.

For ease of presentation in the subsequent sections it would be beneficial to consider the random measures associated with the jump processes $\{R_t^i\}_{1\le i\le n}$ and R_t.

Notation 4.3.4 *Let η^i (resp. η^r) denote the random counting measure on $[0,T]\times\mathbb{R}$ generated by the jumps of R_t^i (resp. R_t).*

Consequently for functions $W : [0,T]\times\mathbb{R} \mapsto \mathbb{R}$ we have

$$\int_0^T \int_{\mathbb{R}} W(\omega; u, x)\eta^i(du, dx) = \sum_{n=1}^{\infty} W(\omega; T_n^i(\omega), U_n^i(\omega))\mathbf{1}_{\{T_n^i(\omega)\le t\}},$$

where T_n^i is the n^{th} jump time of the Poisson process N_t^i. A similar expression holds for η^r. From [84, Ch. 8] η^i and η^r admit the respective compensators $\hat{\eta}^i$ and $\hat{\eta}^r$ given by

$$\hat{\eta}^i(du, dx) = \lambda_u^i \nu^i(dx)du, \quad \text{and} \quad \hat{\eta}^r(du, dw) = \lambda_u(X_{u-}^{1:n}, r_{u-})\nu(dw)du.$$

This means that for predictable mappings $W : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ such that $\int_0^T \int_{\mathbb{R}} |W(\omega; u, w)| \hat{\eta}^r(du, dw) < \infty$ $\mathbb{P}-$a.s. we have

$$\int_0^T \int_{\mathbb{R}} W(\omega; u, w)(\eta^r - \hat{\eta}^r)(du, dw) \quad \text{is a } \mathbb{P} - \text{local martingale.}$$

A similar expression holds for η^i and $\hat{\eta}^i$. We can also consider all jump processes together as an $(n+1)$-dimensional process $J_t = (R_t^1, \ldots, R_t^n, R_t)$ which also admits an associated counting measure.

Notation 4.3.5 *Let η denote the random counting measure on $[0,T] \times \mathbb{R} \mapsto \mathbb{R}$ generated by the jumps of $J_t = (R_t^1, \ldots, R_t^n, R_t)$.*

Because of the inter-independence of $\{N_t^i\}_{1 \le i \le n}$ and N_t, all jumps of J_t are in precisely one coordinate $\mathbb{P}$-almost surely. However since the time component is common to all, superposition of $\{N_t^i\}_{1 \le i \le n}$ and N_t yields the combined intensity $\zeta_u = \sum_{i=1}^n \lambda_u^i + \lambda_u(X_{u-}^{1:n}, r_{u-})$. For every $u \in [0,T]$ an arrival in this superimposed Poisson process is associated to R^i (resp. R) with probability $\frac{\lambda_u^i}{\zeta_u}$ (resp. $\frac{\lambda_u(X_{u-}^{1:n}, r_{u-})}{\zeta_u}$). Thus the compensator of η is given by

$$\hat{\eta}(du, dy^{1:n}, dw) \tag{4.7}$$

$$= \zeta_u \left(\sum_{i=1}^n \frac{\lambda_u^i}{\zeta} \nu^i(y^i) \mathbf{1}_{\{y^j = 0 \ \forall j \neq i, w=0\}} + \frac{\lambda_u(X_{u-}^{1:n}, r_{u-})}{\zeta_u} \nu(dw) \mathbf{1}_{\{y^{1:n} = \mathbf{0}\}} \right) du$$

$$= \left(\sum_{i=1}^n \lambda_u^i \nu^i(y^i) \mathbf{1}_{\{y^j = 0 \ \forall j \neq i, w=0\}} + \lambda_u(X_{u-}^{1:n}, r_{u-}) \nu(dw) \mathbf{1}_{\{y^{1:n} = \mathbf{0}\}} \right) du. \tag{4.8}$$

4.4 Dynamic under full information

It is important to understand the bond price dynamic under full information in order to contrast it with the dynamic under partial information. We first present a preliminary result that provides the dynamic of a generic function of all processes involved in our model. Since the bond price itself is such a function the following result also applies to h given by (4.5).

Lemma 4.4.1 *Let $f:\mathbb{R}^+\times\mathbb{R}^{n+1}\mapsto\mathbb{R}$ have continuous second partial derivatives. Then the dynamic of $f_t(X_t^{1:n},r_t)$ is given by*

$$
\begin{aligned}
f_t(X_t^{1:n},r_t) &= f_0(X_0^{1:n},r_0)+\int_{0+}^{t}(\mathcal{L}f)_u(X_{u-}^{1:n},r_{u-})du+\int_{0+}^{t}(\mathcal{M}f)_u(X_{u-}^{1:n},r_{u-})dB_u\\
&+\int_{0+}^{t}\int_{\mathbb{R}^{n+1}}\left[f_u(X_{u-}^{1:n}+y^{1:n},r_{u-}+w)-f_u(X_{u-}^{1:n},r_{u-})\right](\eta-\hat{\eta})\,(du,dy^{1:n},dw),
\end{aligned}
\tag{4.9}
$$

where $\mathcal{L}$ is given by

$$
\begin{aligned}
&(\mathcal{L}f)_u(x^{1:n},r)\\
&\quad=\left(\left(\partial_u+\sum_{i=1}^{n}\alpha_X^i\partial_{x_i}+\mu\partial_r+\frac{1}{2}\left(\sum_{i=1}^{n}(\sigma_X^i)^2\partial_{x_i}^2+\sigma^2\partial_r^2\right)\right)f\right)_u(x^{1:n},r)\\
&\qquad+\int_0^t\int_{\mathbb{R}^{n+1}}\big(f_u(x^{1:n}+y^{1:n},r+w)-f_u(x^{1:n},r)\big)\,\hat{\eta}(du,dy^{1:n},dw),
\end{aligned}
\tag{4.10}
$$

and

$$
(\mathcal{M}h)_u(x,y,z)=\big((\sigma_X^1\partial_{x_1},\ldots,\sigma_X^n\partial_{x_n},\sigma\partial_r)\,h\big)_u(x,y,z).
$$

In addition, η is the random counting measure given by Notation 4.3.5 and $\hat{\eta}$ is its compensator given by relation (4.7).

Proof The proof is an application of Itô's formula. Observe that from Sections 4.2.1 and 4.2.2 that because of independence of the underlying processes, $[X^i,X^j]=0$ for $i\neq j$ and $[X^i,r]=0$ for all i. Thus we obtain from Theorem 4.3.2 that:

$$
\begin{aligned}
f_t(X_t^{1:n},r_t) &= f_0(X_0^{1:n},r_0)+\int_{0+}^{t}(\partial_u f)_u(X_{u-}^{1:n},r_{u-})du\\
&+\sum_{i=1}^{n}\int_{0+}^{t}(\partial_{x_i}f)_u(X_{u-}^{1:n},r_{u-})dX_u^i+\int_{0+}^{t}(\partial_r f)_u(X_{u-}^{1:n},r_{u-})dr_u\\
&+\frac{1}{2}\sum_{i=1}^{n}\int_{0+}^{t}(\partial_{x_i^2}^2 f)_u(X_{u-}^{1:n},r_{u-})d[X^i,X^i]_u^c+\frac{1}{2}\int_{0+}^{t}(\partial_{r^2}^2 f)_u(X_{u-}^{1:n},r_{u-})d[r,r]_u^c \qquad (4.11)\\
&+\sum_{0<s\leq t}\Big[f_u(X_u^{1:n},r_u)-f_u(X_{u-}^{1:n},r_{u-})\\
&-\sum_{i=1}^{n}(\partial_{x_i}f)_u(X_{u-}^{1:n},r_{u-})\Delta X_u^i-(\partial_r f)_u(X_{u-}^{1:n},r_{u-})\Delta r_u\Big]. \qquad (4.12)
\end{aligned}
$$

Observe that

$$d[X^i, X^i]^c_u = (\sigma^i_X)^2 du \quad \text{and} \quad d[r,r]^c_u = \sigma^2_u(r_{u-})du. \tag{4.13}$$

In addition

$$\begin{aligned}\int_{0+}^{t} (\partial_{x_i} f)_u(X^{1:n}_{u-}, r_{u-})dX^i_u - \sum_{0<s\leq t} (\partial_{x_i} f)_u(X^{1:n}_{u-}, r_{u-})\Delta X^i_u \\ = \int_{0+}^{t} (\partial_{x_i} f)_u(X^{1:n}_{u-}, r_{u-}) \left(\alpha^i_X du + \sigma^i_X dW^i_u\right),\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\int_{0+}^{t} (\partial_r f)_u(X^{1:n}_{u-}, r_{u-})dr_u - \sum_{0<s\leq t} (\partial_r f)_u(X^{1:n}_{u-}, r_{u-})\Delta r_u \\ = \int_{0+}^{t} (\partial_r f)_u(X^{1:n}_{u-}, r_{u-}) \left(\mu_u(X_{u-}, r_{u-})du + \sigma_u(r_{u-})dW_u\right).\end{aligned} \tag{4.15}$$

Finally using the marked point process η we have

$$\begin{aligned}&\sum_{0<s\leq t} \left[f_u(X^{1:n}_u, r_u) - f_u(X^{1:n}_{u-}, r_{u-})\right] \\ &= \int_{0+}^{t}\int_{\mathbb{R}^{n+1}} \left[f_u(X^{1:n}_{u-} + y^{1:n}, r_{u-} + w) - f_u(X^{1:n}_{u-}, r_{u-})\right] \eta(du, dy^{1:n}, dw) \\ &= \int_{0+}^{t}\int_{\mathbb{R}^{n+1}} \left[f_u(X^{1:n}_{u-} + y^{1:n}, r_{u-} + w) - f_u(X^{1:n}_{u-}, r_{u-})\right] \hat{\eta}(du, dy^{1:n}, dw) \\ &+ \int_{0+}^{t}\int_{\mathbb{R}^{n+1}} \left[f_u(X^{1:n}_{u-} + y^{1:n}, r_{u-} + w) - f_u(X^{1:n}_{u-}, r_{u-})\right] (\eta - \hat{\eta})(du, dy^{1:n}, dw).\end{aligned} \tag{4.16}$$

Using (4.13), (4.14), (4.15) and (4.16) inside relation (4.11) we obtain our desired result (4.9). ■

Recall the bond price h in (4.5) which is a conditional expectation given the full information. This is exploited to find its dynamic, well-known in literature to be given by a partial differential equation. The following proposition provides the partial differential equation corresponding to our model as described in Section 4.2.

Proposition 4.4.1 *The bond price process h solves the following partial differential equation:*

$$(\mathcal{L}h)_t(x^{1:n}, r) - rh_t(x^{1:n}, r) = 0,$$
$$h_T(x^{1:n}, r) = 1, \tag{4.17}$$

where $\mathcal{L}$ is given in Lemma 4.4.1.

Proof This is an application of a more general Feynman-Kac formula. Since the proof is a simple application of Itô's formula, it is provided below. Let $0 \le s \le T$ and consider

$$v_s = \exp\left(-\int_0^s r_u du\right).$$

Notice v_s is differentiable and hence of bounded variation. Hence Itô's formula dictates

$$d(v_s h_s) = v_s dh_s + h_s dv_s, \tag{4.18}$$

since the covariation term $[v, h] = 0$. Observe that

$$dv_s = -r_s v_s ds. \tag{4.19}$$

In addition from Lemma 4.4.1 we obtain

$$dh_s = (\mathcal{L}h)_s ds + (\mathcal{M}h)_s dB_s + \int_{\mathbb{R}^{n+1}} (\Delta h)_s d(\eta - \hat{\eta}). \tag{4.20}$$

Plugging (4.19) and (4.20) in (4.18) we have

$$d(v_s h_s) = v_s \left[(\mathcal{L}h)_s - r_s h_s\right] ds + v_s \left[(\mathcal{M}h)_s dB_s + \int (\Delta h)_s d(\eta - \hat{\eta})\right] \tag{4.21}$$

However

$$v_s h_s = \exp\left(-\int_0^s r_u du\right) \mathbb{E}\left[\exp\left(-\int_s^T r_u du\right) |\mathcal{F}_s\right] = \mathbb{E}\left[\exp\left(-\int_0^T r_u du\right) |\mathcal{F}_s\right]$$

is a martingale and hence the drift term in (4.21) must equal zero, that is

$$(\mathcal{L}h)_s - r_s h_s = 0,$$

as asserted in (4.17). ■

4.5 Dynamic under partial information

Our approach, using techniques borrowed from the theory of stochastic filtering, relies on creating a new measure under which the observed and unobserved Wiener processes (or transforms of) become independent of each other. In addition the non-homogeneity of the Poisson processes involved are removed. In order to achieve this we will be using Theorem 4.3.1 stated in the prequel.

As already mentioned in the Introduction, observe that equation (4.4) can be understood as an extended case of the Vasicek, Ho-Lee and Hull-White models. The Hull-White model which also includes the Vasicek and Ho-Lee models in its generality is usually represented as:

$$dr_t = (\mu_t - \alpha_t r_t)\, dt + \sigma_t dW_t. \tag{4.22}$$

Comparing equation (4.22) with that of (4.4), we find that (4.4) is indeed more general than the three models mentioned above, and in the rest of this section we will provide one way of inferring about the dynamics of the bond price under this more general interest rate dynamics, with the additional constraint of partial information.

Our next goal is to find the bond price under partial information, that is, the conditional expectation $\pi_t = \mathbb{E}[h_t(X_t^{1:n}, r)|\mathcal{G}_t]$, where h is given by Proposition 4.4.1. Conditional expectations of this kind are usually found using well-established techniques in the stochastic filtering literature and here we will try to do the same. More precisely, we will use the change of measure approach in stochastic filtering to find our desired quantity π_t.

4.5.1 Measure change and dynamics under changed measure

We now mention this necessary change of measure under which the Wiener processes driving the unobserved processes become independent of drifted versions of the Wiener processes driving the observed processes. Recall the regularity assumption on μ and σ stated in Section 4.2, namely we assume that in addition to the usual assumptions for the existence of the Itô jump-diffusion (4.4) we also have that $\frac{\mu}{\sigma}$ is bounded. The following hypothesis defines our new measure.

Hypothesis 4.5.1 *Let Z_t be given by*

$$Z_t = 1 - \int_{0+}^{t} Z_{s-} \left[\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1;n}, r_{s-})}{\sigma_s(r_{s-})} dW_s \right. \\ \left. + \sum_{i=1}^{k} \left(1 - \frac{1}{\lambda_s^i}\right)(dN_s^i - \lambda_s ds) + \left(1 - \frac{1}{\lambda_s(X_{s-}^{1:n}, r_{s-})}\right)\left(dN_s - \lambda_s(X_{s-}^{1:n}, r_{s-})ds\right)\right]. \tag{4.23}$$

Define $\mathbb{Q}_T$ by $\frac{d\mathbb{Q}_T}{d\mathbb{P}} = Z_T$ and $\mathbb{Q}_t$ by $\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \mathbb{E}[Z_T|\mathcal{F}_t]$.

Remark 4.5.2 *Observe that Z can be represented as $Z_t = \mathcal{E}(M)_t$ where M is given by*

$$M_t = -\left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n})}{\sigma_s(r_{s-})} dW_s + \sum_{i=1}^{k}\left(1 - \frac{1}{\lambda_s^i}\right) d\tilde{N}_s^i \right. \\ \left. + \left(1 - \frac{1}{\lambda_s(X_{s-}^{1:n}, r_{s-})}\right) d\tilde{N}_s\right), \tag{4.24}$$

and $\mathcal{E}(M)$ is the stochastic exponential of M. In (4.24) we have denoted $\tilde{N}_s^i$ and $\tilde{N}_s$ to stand for the compensated Poisson processes under $\mathbb{P}$. Using the formula for stochastic exponential of semimartingales we have

$$Z_t = \mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M, M]_t^c\right) \prod_{0<s\le t} (1 + \Delta M_s)\exp(-\Delta M_s). \tag{4.25}$$

Using (4.24) in (4.25) it is readily checked that

$$Z_t = \exp\left(- \int_{0+}^{t} \left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n})}{\sigma_s(r_{s-})} dW_s\right) - \frac{1}{2}\left(\sum_{i=1}^{k} \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} t \right.\right. \\ \left. + \int_{0+}^{t} \frac{\mu_s^2(X_{s-}^{1:n}, r_{s-})}{\sigma_s^2(r_{s-})} ds\right) - \sum_{i=1}^{k} \int_0^t \log(\lambda_s^i) dN_s^i - \int_0^t \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) dN_s \\ \left. - \sum_{i=1}^{k} \int_0^t (1 - \lambda_s^i) ds - \int_0^t (1 - \lambda_s(X_{s-}^{1:n}, r_{s-})) ds \right). \tag{4.26}$$

Remark 4.5.3 *$\mathbb{Q}_t$ is a valid probability measure. To see this observe from (4.23) that Z is a $\mathbb{P}$-local martingale. Using [83, Cor. 3, p 73] Z would be a square integrable martingale if and only if*

$$\mathbb{E}[Z,Z]_t < \infty, \qquad \forall t \geq 0.$$

Taking into account the independence of W^i, W, N^i and N we obtain:

$$[Z,Z]_t = \int_{0+}^t Z_{s-}^2 \left(\left(\sum_{i=1}^k \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} + \frac{\mu_s^2(X_{s-}^{1:n}, r_{s-})}{\sigma_s^2(r_{s-})} \right) ds \right.$$
$$\left. + \sum_{i=1}^k \left(1 - \frac{1}{\lambda_s^i}\right)^2 dN_s^i + \left(1 - \frac{1}{\lambda_s(X_{s-}^{1:n}, r_{s-})}\right)^2 dN_s \right).$$

Boundedness of $\frac{\alpha^i}{\sigma^i}$, $\frac{\mu}{\sigma}$, $\frac{1}{\lambda^i}$ and $\frac{1}{\lambda}$ implies there exists a constant c such that:

$$[Z,Z]_t \leq c \left[\int_{0+}^t Z_{s-}^2 ds + \sum_{i=1}^k \int_{0+}^t Z_{s-}^2 dN_s^i + \int_{0+}^t Z_{s-}^2 dN_s \right].$$

Taking expectation on both sides , using the fact that $\mathbb{E}[\int h_s dN_s] = \mathbb{E}[\int h_s \lambda_s ds]$ and a further use of the boundedness of λ^i, λ we obtain

$$\mathbb{E}[Z,Z]_t \leq c' \mathbb{E}\left[\int_{0+}^t Z_{s-}^2 ds\right] = c' \int_{0+}^t \mathbb{E}[Z_{s-}^2] ds.$$

From (4.26) we have

$$\mathbb{E}[Z_t^2] \leq c_1 \mathbb{E}\left[\exp\left(c_2 \sum_{i=1}^k W_t^i + W_t + \sum_{i=1}^k N_t^i + N_t \right) \right] \leq C,$$

for some constant C, by computing the exponential moments explicitly. Having thus obtained that $\mathbb{E}[Z,Z]_t < \infty$ we conclude that Z is a square integrable martingale. Hence

$$\mathbb{Q}_t(\Omega) = \mathbb{E}_{\mathbb{P}}\left[\frac{d\mathbb{Q}_t}{d\mathbb{P}}\right] = \mathbb{E}_{\mathbb{P}}[Z_t] = \mathbb{E}_{\mathbb{P}}[Z_0] = 1.$$

Thus $\mathbb{Q}_t$ is a probability measure for each $t \in [0,T]$.

Remark 4.5.4 *Notice that the measures $\mathbb{P}$ and $\mathbb{Q}_T$ are absolutely continuous with each other. This is evident from the fact that the Radon-Nikodym derivative $\frac{d\mathbb{Q}_T}{d\mathbb{P}}$ is* *strictly positive. In addition* $\mathbb{Q}_t(A) = \mathbb{Q}_T(A)$ *for any* $A \in \mathcal{F}_t$*. To see this observe that for* $A \in \mathcal{F}_t$*:*

$$\mathbb{Q}_T(A) = \mathbb{E}_{\mathbb{P}}\left[Z_T \mathbf{1}_A\right] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[Z_T \mathbf{1}_A | \mathcal{F}_t\right]\right] = \mathbb{E}_{\mathbb{P}}\left[Z_t \mathbf{1}_A\right] = \mathbb{Q}_t(A), \tag{4.27}$$

where we have used the fact that $A \in \mathcal{F}_t$ *to bring* $\mathbf{1}_A$ *out of the inner expectation, and the fact that* Z_t *is a martingale under* $\mathbb{P}$ *for the final conclusion.*

We now provide our result on how some drifted versions of our original Brownian motions behave under the new measure $\mathbb{Q}$ defined in Hypothesis 4.5.1.

Lemma 4.5.5 *Under* $\mathbb{Q}$ *the following are independent standard Brownian motions:*

$$\begin{aligned}
\bar{W}_t^i &= W_t^i + \frac{\alpha_X^i}{\sigma_X^i} t, \quad i = 1, \ldots, k, \\
\bar{W}_t^i &= W_t^i, \quad i = k+1, \ldots, n, \\
\bar{W}_t &= W_t + \int_0^t \left(\frac{\mu}{\sigma}\right)_s (X_{s-}^{1:n}, r_{s-}) ds.
\end{aligned}$$

Proof From Theorem 4.3.1 we know that for $i = 1, \ldots, n$,

$$V_t^i = W_t^i - \int_{0+}^t \frac{1}{Z_{s-}} d[Z, W^i]_s$$

is a $\mathbb{Q}$-local martingale. For $i \leq k$ we have

$$d[Z, W^i]_s = -Z_{s-} \frac{\alpha_X^i}{\sigma_X^i} ds,$$

while $d[Z, W^i] = 0$ for $i > k$. This implies $V_t^i = \bar{W}_t^i$. In addition $[\bar{W}^i, \bar{W}^i]_t = [W^i, W^i]_t = t$. Consequently $\bar{W}^i$ is a standard Brownian motion for $i = 1, \ldots, n$. Further note that

$$V_t = W_t - \int_{0+}^t \frac{1}{Z_{s-}} d[Z, W]_s$$

is a $\mathbb{Q}$-local martingale with

$$d[Z, W]_s = -Z_{s-} \frac{\mu_s(X_{s-}^{1:n}, r_{s-})}{\sigma_s(r_{s-})} ds.$$

This implies $V_t = \bar{W}_t$. In addition $[\bar{W}, \bar{W}]_t = [W, W]_t = t$. Thus $\bar{W}$ is also a standard Brownian motion. Finally observe $[\bar{W}^i, \bar{W}^j] = [W^i, W^j] = 0$ for $i \neq j$, in addition to $[\bar{W}^i, \bar{W}] = [W^i, W] = 0$ for $i = 1, \ldots, n$. Thus $\bar{W}^i$, $i = 1, \ldots, n$ and $\bar{W}$ are independent. ∎

For simplicity in presentation we consider the following notation.

Notation 4.5.6 *Let B and $\bar{B}$ denote the $(n+1)$ dimensional Wiener processes given by*

$$B_t = \left(W_t^1, \ldots, W_t^n, W_t\right)^\intercal, \text{ and } \bar{B}_t = \left(\bar{W}_t^1, \ldots, \bar{W}_t^n, \bar{W}_t\right)^\intercal.$$

In addition, we introduce the notation for the $(n+1)$ length vector

$$\bar{\mu}_t(X_{t-}^{1:n}, r_{t-}) = \left(\frac{\alpha_X^1}{\sigma_X^1}, \ldots, \frac{\alpha_X^k}{\sigma_X^k}, 0, \ldots, 0, \frac{\mu_t(X_{t-}^{1:n}, r_{t-})}{\sigma_t(r_{t-})}\right),$$

so that $d\bar{B}_t = dB_t + \bar{\mu}_t dt$.

We now state a lemma to help find the new compensator of the random measure η under $\mathbb{Q}$. It is adapted from [85, Ch 3 Theorem 3.17], so we omit the proof.

Lemma 4.5.7 *Let $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R})$ be the σ-field of predictable sets in $\tilde{\Omega} = \Omega \times [0,T] \times \mathbb{R}$ and $M_\eta^{\mathbb{P}} = \eta(\omega, du, dx)\mathbb{P}(d\omega)$ be the positive measure on $(\Omega \times [0,T] \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}))$ defined by*

$$M_\eta^{\mathbb{P}}(W) = \mathbb{E}(W * \eta)_T = \mathbb{E} \int_0^T \int_{\mathbb{R}} W(\omega; s, x)\eta(\omega; ds, dx),$$

for measurable non-negative functions $W = W(\omega; t, x)$ given on $\omega \times [0,T] \times \mathbb{R}$. The conditional expectation $M_\eta^{\mathbb{P}}(\frac{Z}{Z_-}|\tilde{\mathcal{P}})$ is defined to be the $M_\eta^{\mathbb{P}}-$a.s. unique $\tilde{\mathcal{P}}-$measurable function Y with the property

$$M_\eta^{\mathbb{P}}\left(\frac{Z}{Z_-}U\right) = M_\eta^{\mathbb{P}}(YU),$$

for all non-negative $\tilde{\mathcal{P}}-$measurable functions $U = U(\omega; t, x)$. Let $\hat{\eta}$ be the compensator of η under $\mathbb{P}$. Then the compensator of η under $\mathbb{Q}$ is given by

$$\tilde{\eta} = \hat{\eta} M_\eta^{\mathbb{P}}\left(\frac{Z}{Z_-}|\tilde{\mathcal{P}}\right)$$

The next lemma gives us the compensator of the random measure under the changed probability $\mathbb{Q}$.

Lemma 4.5.8 *Under $\mathbb{Q}$ the compensator $\hat{\eta}^Q$ of the random measure η is given by*

$$\hat{\eta}^Q(du, dy^{1:n}, dw)$$
$$= \left(\sum_{i=1}^{k} \nu^i(y^i) \mathbf{1}_{\{y^j=0 \ \forall j \neq i\}} + \sum_{i=k+1}^{n} \lambda_u^i \nu^i(y^i) \mathbf{1}_{\{y^j=0 \ \forall j \neq i\}} + \nu(dw) \mathbf{1}_{\{y^{1:n}=\mathbf{0}\}} \right) du.$$

Proof The compensator for η under $\mathbb{P}$ is given by (4.7). In addition according to (4.23)

$$Z_t = 1 + \int_{0+}^{t} Z_{s-} dM_s,$$

where M is given by (4.24). In order to apply Lemma 4.5.7 we need to find $M_\eta^{\mathbb{P}}(\frac{Z}{Z_-}|\tilde{\mathcal{P}})$. To that effect observe that

$$\frac{Z_t}{Z_{t-}} = 1 + \Delta M_t = 1 - \sum_{i=1}^{k} \left(1 - \frac{1}{\lambda_s^i}\right) \Delta N_t^i - \left(1 - \frac{1}{\lambda_t(X_{t-}^{1:n}, r_{t-})}\right) \Delta N_t. \quad (4.28)$$

Because of independence, $\{N^i\}_{1 \le i \le k}$ and N do not share common jumps almost surely. This implies from (4.28)

$$\frac{Z_t}{Z_{t-}} = \sum_{i=1}^{k} \frac{\Delta N_s^i}{\lambda_s^i} + \frac{\Delta N_s}{\lambda_s(X_{s-}^{1:n}, r_{s-})}, \quad \text{a.s.}$$

Taking conditional expectation $M_\eta^{\mathbb{P}}(\cdot|\tilde{\mathcal{P}})$ we obtain that

$$M_\eta^{\mathbb{P}} \left(\frac{Z_t}{Z_{t-}} |\tilde{\mathcal{P}} \right) = \sum_{i=1}^{k} \frac{1}{\lambda_s^i} M_\eta^{\mathbb{P}} \left(\Delta N_s^i |\tilde{\mathcal{P}} \right) + \frac{1}{\lambda_s(X_{s-}^{1:n}, r_{s-})} M_\eta^{\mathbb{P}} \left(\Delta N_s |\tilde{\mathcal{P}} \right). \quad (4.29)$$

For $\tilde{\mathcal{P}}-$measurable U we have

$$M_\eta^{\mathbb{P}}(\Delta N_s U) = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \mathbf{1}_{\{y^{1:n}=0\}} U(u,y) \eta(du, dy, dw) \right] = M_\eta^{\mathbb{P}}(\mathbf{1}_{\{y^{1:n}=0\}} U).$$

Since the indicator functions are $\tilde{\mathcal{P}}-$measurable we get

$$M_\eta^{\mathbb{P}} \left(\Delta N_s |\tilde{\mathcal{P}} \right) = \mathbf{1}_{\{y^{1:n}=0\}}. \quad (4.30)$$

Similarly we have

$$M_\eta^{\mathbb{P}} \left(\Delta N_s^i |\tilde{\mathcal{P}} \right) = \mathbf{1}_{\{y^j=0 \forall j \neq i, w=0\}}. \quad (4.31)$$

Plugging in relations (4.30) and (4.31) in (4.29) we obtain

$$M_\eta^{\mathbb{P}}\left(\frac{Z_t}{Z_{t-}}|\tilde{\mathcal{P}}\right) = \sum_{i=1}^{k} \frac{1}{\lambda_t^i}\mathbf{1}_{\{y^j=0\forall j\neq i, w=0\}} + \frac{1}{\lambda_t(X_{t-}^{1:n}, r_{t-})}\mathbf{1}_{\{y^{1:n}=0\}}.$$

Now appealing to Lemma 4.5.7 we find that the compensator for η under $\mathbb{Q}$ is given by

$$\tilde{\eta} = \hat{\eta} M_\eta^{\mathbb{P}}\left(\frac{Z}{Z_{-}}|\tilde{\mathcal{P}}\right),$$

where $\hat{\eta}$ is given by (4.7). Since the indicators are orthogonal we obtain our desired result. ■

Remark 4.5.9 *It is useful to represent h_t in terms of the martingales under $\mathbb{Q}$.*

$$\begin{aligned}
&h_t(X_t^{1:n}, r_t)\\
&= h_0(X_0^{1:n}, r_0) + \int_{0+}^{t} (\mathcal{L}^Q h)_u(X_{u-}^{1:n}, r_{u-})du + \int_{0+}^{t} (\mathcal{M}^Q h)_u(X_{u-}^{1:n}, r_{u-})d\bar{B}_u\\
&+ \int_{0+}^{t}\int_{\mathbb{R}^{n+1}} \left[h_u(X_{u-}^{1:n} + y^{1:n}, r_{u-} + w) - h_u(X_{u-}^{1:n}, r_{u-})\right]\left(\eta - \hat{\eta}^Q\right)(du, dy^{1:n}, dw),
\end{aligned}$$

where

$$\begin{aligned}
&(\mathcal{L}^Q f)_u(x^{1:n}, r)\\
&\quad = \left(\left(\partial_u + \sum_{i=k+1}^{n} \alpha_X^i \partial_{x_i} + \frac{1}{2}\left(\sum_{i=1}^{n} (\sigma_X^i)^2 \partial_{x_i}^2 + \sigma^2 \partial_r^2\right)\right) f\right)_u (x^{1:n}, r)\\
&\qquad + \int_0^t \int_{\mathbb{R}^{n+}} \left(f_u(x^{1:n} + y^{1:n}, r + w) - f_u(x^{1:n}, r)\right)\hat{\eta}^Q(du, dy^{1:n}, dw),
\end{aligned}$$

and

$$\mathcal{M}^Q = \mathcal{M}.$$

4.5.2 Bridge to the partially informed trader's price

To obtain the partially informed trader's price, we will use Bayesian filtering methods. Recall Z_t defined in Hypothesis 4.5.1. In order to use the Bayes formula in Lemma 4.3.3 and obtain the bond price dynamics for the partially informed trader, it would be useful to obtain the dynamics of Z_t^{-1}. This is the content of the following lemma.

Lemma 4.5.10 *Let $\Lambda_t = \frac{1}{Z_t}$, where Z_t is defined in Hypothesis 4.5.1. Then the dynamics of Λ_t is given by*

$$\Lambda_t = 1 + \int_{0+}^{t} \Lambda_{s-} \left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n}, r_{s-})}{\sigma_s(r_{s-})} dW_s + \sum_{i=1}^{n} \left(\frac{\alpha_X^i}{\sigma_X^i}\right)^2 ds \right.$$
$$\left. + \frac{\mu_s^2(X_{s-}^{1:n}, r_s)}{\sigma_s^2(r_{s-})} ds + \sum_{i=1}^{k} \left(\lambda_s^i - 1\right)\left(dN_s^i - ds\right) + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right)(dN_s - ds) \right)$$
$$= 1 + \int_{0+}^{t} \Lambda_{s-} \left(\bar{\mu}_t(X_{t-}^{1:n}, r_{t-}) d\bar{B}_u + \sum_{i=1}^{k} \left(\lambda_s^i - 1\right) d\bar{N}_s^i + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right) d\bar{N}_s \right),$$

where $\{\bar{N}^i\}_{1\le i\le k}$ and $\bar{N}$ denotes the compensated Poisson processes under $\mathbb{Q}$.

Proof From (4.26) we have

$$\Lambda_t = \exp\left(\left(\int_{0+}^{t} \left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n})}{\sigma_s(r_{s-})} dW_s \right) + \frac{1}{2} \left(\sum_{i=1}^{k} \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} t \right.\right.\right.$$
$$\left. + \int_{0+}^{t} \frac{\mu_s^2(X_{s-}^{1:n}, r_{s-})}{\sigma_s^2(r_{s-})} ds \right) + \sum_{i=1}^{k} \int_0^t \log(\lambda_s^i) dN_s^i + \int_0^t \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) dN_s$$
$$\left. + \sum_{i=1}^{k} \int_0^t (1 - \lambda_s^i) ds + \int_0^t (1 - \lambda_s(X_{s-}^{1:n}, r_{s-})) ds \right). \quad (4.32)$$

Now applying Itô's formula we obtain that

$$\Lambda_t = 1 + \int_{0+}^{t} \Lambda_{s-} \left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n}, r_{s-})}{\sigma_s(r_{s-})} dW_s + \frac{1}{2} \left(\sum_{i=1}^{k} \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} \right.\right.$$
$$\left. + \frac{\mu_s^2(X_{s-}^{1:n}, r_{s-})}{\sigma_s^2(r_{s-})} \right) ds + \sum_{i=1}^{k} \log(\lambda_s^i) dN_s^i + \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) dN_s + \sum_{i=1}^{k} (1 - \lambda_s^i) ds$$
$$\left. + (1 - \lambda_s(X_{s-}^{1:n}, r_{s-})) ds \right) + \frac{1}{2} \int_{0+}^{t} \Lambda_{s-} \left(\sum_{i=1}^{k} \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} + \frac{\mu_s^2(X_{s-}^{1:n}, r_{s-})}{\sigma_s^2(r_{s-})} \right) ds$$
$$+ \sum_{0<s\le t} \left(\Lambda_s - \Lambda_{s-} - \Lambda_{s-} \left(\sum_{i=1}^{k} \log(\lambda_s^i) \Delta N_s^i + \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) \Delta N_s \right) \right) \quad (4.33)$$

Observe that

$$\sum_{0<s\le t} \Lambda_{s-} \left(\sum_{i=1}^{k} \log(\lambda_s^i) \Delta N_s^i + \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) \Delta N_s \right)$$
$$= \int_{0+}^{t} \Lambda_{s-} \left(\sum_{i=1}^{k} \log(\lambda_s^i) dN_s^i + \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) dN_s \right). \quad (4.34)$$

Furthermore from (4.32) itself

$$\sum_{0<s\leq t} (\Lambda_s - \Lambda_{s-}) = \sum_{0<s\leq t} \left(\Lambda_{s-} \exp\left(\sum_{i=1}^{k} \log(\lambda_s^i) \Delta N_s^i + \log(\lambda_s(X_{s-}^{1:n}, r_{s-})) \Delta N_s \right) - \Lambda_{s-} \right).$$

Since the $\{N^i\}_{1\leq i\leq k}$ and N do not share common jumps, for fixed s only one of $\{\Delta N_s^i\}_{1\leq i\leq k}$ and ΔN_s equals 1. This simplifies the above as

$$\sum_{0<s\leq t} (\Lambda_s - \Lambda_{s-}) = \sum_{0<s\leq t} \Lambda_{s-} \left(\sum_{i=1}^{k} (\lambda_s^i - 1) \Delta N_s^i + (\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1) \Delta N_s \right). \quad (4.35)$$

Plugging in (4.34) and (4.35) in (4.33) we obtain

$$\Lambda_t = 1 + \int_{0+}^{t} \Lambda_{s-} \left(\sum_{i=1}^{k} \frac{\alpha_X^i}{\sigma_X^i} dW_s^i + \frac{\mu_s(X_{s-}^{1:n}, r_{s-})}{\sigma_s(r_{s-})} dW_s + \sum_{i=1}^{k} \frac{(\alpha_X^i)^2}{(\sigma_X^i)^2} ds \right.$$
$$\left. + \frac{\mu_s^2(X_{s-}^{1:n}, r_s)}{\sigma_s^2(r_{s-})} ds + \sum_{i=1}^{k} \left(\lambda_s^i - 1\right) \left(dN_s^i - ds\right) + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right) (dN_s - ds) \right),$$

as desired. ■

Remark 4.5.11 *Similar to (4.27) we also have:*

$$\mathbb{E}_{\mathbb{Q}_t}\left[h_t(X_t^{1:n}, r_t) \Lambda_t \middle| \mathcal{G}_t \right] = \mathbb{E}_{\mathbb{Q}_T}\left[h_t(X_t^{1:n}, r_t) \Lambda_t \middle| \mathcal{G}_t \right] = \mathbb{E}_{\mathbb{Q}_T}\left[h_t(X_t^{1:n}, r_t) \Lambda_T \middle| \mathcal{G}_t \right], \quad (4.36)$$

where one might find it convenient to use the fact that Λ is a martingale and the tower property of conditional expectations in order to obtain the last equality.

Notation 4.5.12 *Let $\bar{\eta}^i$ and $\bar{\eta}$ respectively denote the compensated versions under $\mathbb{Q}$ of the random measures η^i and η. More precisely, we denote*

$$\bar{\eta}^i(du, dy) = \left(\eta^i(du, dy) - \nu(dy) du\right), \quad \textit{and} \quad \bar{\eta}(du, dy) = (\eta(du, dy) - \nu(dy) du).$$

It is now necessary to calculate the numerator corresponding to relation (4.6). In order to achieve that we first find the dynamics of the integrand, whose conditional expectation we require.

Lemma 4.5.13 *The dynamics of $h_t(X_t^{1:n}, r_t)\Lambda_t$ is given by*

$$\begin{aligned} h_t(X_t^{1:n}, r_t)\Lambda_t &= h_0(X_0^{1:n}, r_0)\Lambda_0 + \int_{0+}^t \Lambda_{u-}(\mathcal{L}h)_u(X_{u-}^{1:n}, r_{u-})du \\ &+ \int_{0+}^t \Lambda_{u-}((\mathcal{M}+\bar{\mu})\,h)_u(X_{u-}^{1:n}, r_{u-})d\bar{B}_u + \int_{0+}^t \Lambda_{u-}(\Delta h_u)(\eta - \hat{\eta}^Q)(du, dy^{1:n}, dw) \\ &+ \int_{0+}^t \Lambda_{u-}\left(\sum_{i=1}^k (\lambda_u^i - 1)\int_{\mathbb{R}} h_u(X_{u-}^{1:n} + y^i e_{i,n}, r_{u-})\bar{\eta}^i(du, dy^i)\right. \\ &\left. + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right)\int_{\mathbb{R}} h_u(X_{u-}^{1:n}, r_{u-} + w)\bar{\eta}^r(du, dw)\right). \end{aligned} \tag{4.37}$$

Proof We use integration by parts. Recall for two semimartingales U and V we have

$$U_tV_t = U_0V_0 + \int_{0+}^t U_{s-}dV_s + \int_{0+}^t V_{s-}dU_s + [U,V]_t. \tag{4.38}$$

Taking $U_t = h_t(X_t^{1:n}, r_t)$ and $V_t = \Lambda_t$ we have from Remark 4.5.9 and Lemma 4.5.10 that the covariation process is given by

$$\begin{aligned} [U,V]_t &= \int_{0+}^t \Lambda_{u-}((\bar{\mu}^\intercal \mathcal{M})\,h)_u(X_{u-}^{1:n}, r_{u-})du \\ &+ \int_{0+}^t \Lambda_{u-}\left(\sum_{i=1}^k (\lambda_u^i - 1)\int_{\mathbb{R}} \left[h_u(X_{u-}^{1:n} + y^i e_{i,n}, r_{u-}) - h_u(X_{u-}^{1:n}, r_{u-})\right]\eta^i(du, dy^i)\right. \\ &\left. + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right)\int_{\mathbb{R}} \left[h_u(X_{u-}^{1:n}, r_{u-} + w) - h_u(X_{u-}^{1:n}, r_{u-})\right]\eta^r(du, dw)\right) \end{aligned} \tag{4.39}$$

In addition we have

$$\begin{aligned} \int_{0+}^t V_{s-}dU_s &= \int_{0+}^t \Lambda_{s-}(\mathcal{L}^Q h)_u(X_{u-}^{1:n}, r_{u-})du + \int_{0+}^t \Lambda_{s-}(\mathcal{M}^Q h)_u(X_{u-}^{1:n}, r_{u-})d\bar{B}_u \\ &+ \int_{0+}^t \Lambda_{s-}\int_{\mathbb{R}^{n+1}} \Delta h_u(\eta - \hat{\eta}^Q)(du, dy^{1:n}, dw). \end{aligned} \tag{4.40}$$

Finally

$$\begin{aligned} \int_{0+}^t U_{s-}dV_s &= \int_{0+}^t \Lambda_{s-}(\bar{\mu}h)_s(X_{s-}^{1:n}, r_{s-})d\bar{B}_s \\ &+ \int_{0+}^t \Lambda_{s-}\left(\sum_{i=1}^k (\lambda_s^i - 1)\int_{\mathbb{R}} \bar{\eta}^i(ds, dx) + \left(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\right)\int_{\mathbb{R}} \bar{\eta}^r(ds, dx)\right). \end{aligned} \tag{4.41}$$

Plugging relations (4.39), (4.40), (4.41) in (4.38) we have

$$
\begin{aligned}
h_t(X_t^{1:n}, r_t) = h_0(X_0^{1:n}, r_0) + \int_{0+}^{t} \Lambda_{s-} \Big[\big(\big(\bar{\mu}^\intercal \mathcal{M}^Q + \mathcal{L}^Q\big)\, h\big)_u (X_{u-}^{1:n}, r_{u-}) \\
+ \sum_{i=1}^{k} (\lambda_u^i - 1) \int_{\mathbb{R}} \left[h_u(X_{u-}^{1:n} + y^i e_{i,n}, r_{u-}) - h_u(X_{u-}^{1:n}, r_{u-}) \right] \nu^i(dy^i) \\
+ \big(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\big) \int_{\mathbb{R}} \left[h_u(X_{u-}^{1:n}, r_{u-} + w) - h_u(X_{u-}^{1:n}, r_{u-}) \right] \nu^r(dw) \Big]\, du \\
+ \int_{0+}^{t} \Lambda_{u-} \big(\big(\mathcal{M}^Q + \bar{\mu}\big)\, h\big)_u (X_{u-}^{1:n}, r_{u-}) d\bar{B}_u \\
+ \int_{0+}^{t} \Lambda_{u-} \Bigg(\sum_{i=1}^{k} (\lambda_u^i - 1) \int_{\mathbb{R}} h_u(X_{u-}^{1:n} + y^i e_{i,n}, r_{u-}) \bar{\eta}^i(du, dy^i) \\
+ \big(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\big) \int_{\mathbb{R}} h_u(X_{u-}^{1:n}, r_{u-} + w) \bar{\eta}^r(du, dw) \Bigg) \\
+ \int_{0+}^{t} \Lambda_{u-} (\Delta h_u)(\eta - \hat{\eta}^Q)(du, dy^{1:n}, dw)
\end{aligned}
$$

Simplifying the drift and diffusion coefficients above we obtain

$$
\begin{aligned}
h_t(X_t^{1:n}, r_t)\Lambda_t = h_0(X_0^{1:n}, r_0)\Lambda_0 + \int_{0+}^{t} \Lambda_{u-} (\mathcal{L}h)_u (X_{u-}^{1:n}, r_{u-}) du \\
+ \int_{0+}^{t} \Lambda_{u-} ((\mathcal{M} + \bar{\mu})\, h)_u (X_{u-}^{1:n}, r_{u-}) d\bar{B}_u + \int_{0+}^{t} \Lambda_{u-} \big((\Delta h_u)(\eta - \hat{\eta}^Q)(du, dy^{1:n}, dw) \\
+ \sum_{i=1}^{k} (\lambda_u^i - 1) \int_{\mathbb{R}} h_u(X_{u-}^{1:n} + y^i e_{i,n}, r_{u-}) \bar{\eta}^i(du, dy^i) \\
+ \big(\lambda_s(X_{s-}^{1:n}, r_{s-}) - 1\big) \int_{\mathbb{R}} h_u(X_{u-}^{1:n}, r_{u-} + w) \bar{\eta}^r(du, dw) \Big)
\end{aligned}
$$

■

4.5.3 Price of the partially informed trader

We are ready to put all the pieces together to calculate the partially informed trader's price. For simplicity, we define some notations.

Notation 4.5.14 *Let the functional operators $\Delta^i(z)$ and $\Delta(z)$ be defined as follows*

$$\left(\Delta^i(z)f\right)_u(x^{1:n}, r) = f_u(x^{1:n} + ze_{i,n}, r) - f_u(x^{1:n}, r), \text{ for } i = 1, \dots, k,$$
$$\left(\Delta(z)f\right)_u(x^{1:n}, r) = f_u(x^{1:n}, r + z) - f_u(x^{1:n}, r).$$

In addition let

$$\hat{B}_t = (\bar{W}_t^1, \dots, \bar{W}_t^k, \bar{W}_t),$$

and

$$\hat{\mu}_t(X_{t-}^{1:n}, r_{t-}) = \left(\frac{\alpha_X^1}{\sigma_X^1}, \dots, \frac{\alpha_X^k}{\sigma_X^k}, \frac{\mu_t(X_{t-}^{1:n}, r_{t-})}{\sigma_t(r_{t-})} \right).$$

The following lemma will be useful if we are to take expectation in Lemma 4.5.13 in order to obtain γ_t. The proof of this result is standard and can be found in most stochastic calculus textbooks.

Lemma 4.5.15 *Let $\tilde{W}^1$ and $\tilde{W}^2$ be two independent $\mathcal{F}_t$-Brownian motions such that the generated sigma algebras satisfy:*

(i) $\mathcal{F}_t^{\tilde{W}^1} \subseteq \mathcal{G}_t \subset \mathcal{F}_t$,

(ii) $\mathcal{F}_t^{\tilde{W}^2}$ *and* $\mathcal{G}_t$ *are independent.*

Then for an $\mathcal{F}_t$-adapted bounded process F we have:

$$\mathbb{E}\left[\int_0^t F_s \, d\tilde{W}_s^1 \middle| \mathcal{G}_t \right] = \int_0^t \mathbb{E}\left[F_s \middle| \mathcal{G}_t \right] d\tilde{W}_s^1,$$
$$\mathbb{E}\left[\int_0^t F_s \, ds \middle| \mathcal{G}_t \right] = \int_0^t \mathbb{E}\left[F_s \middle| \mathcal{G}_t \right] ds,$$
$$\mathbb{E}\left[\int_0^t F_s \, d\tilde{W}_s^2 \middle| \mathcal{G}_t \right] = 0.$$

Lemma 4.5.16 *The unnormalized filter $\gamma_t(h) = \mathbb{E}_{\mathbb{Q}_t}[h_t(X_t^{1:n}, r_t)\Lambda_t | \mathcal{G}_t]$ is given by*

$$\gamma_t(h) = \gamma_0(h) + \int_{0+}^t \gamma_{u-}(\mathcal{L}h)du + \int_{0+}^t \gamma_{u-}(\hat{\mathcal{M}}h)d\hat{B}_u$$
$$+ \sum_{i=1}^k \int_{\mathbb{R}} \left(\gamma_{u-}(\Delta^i(y^i)h) + (\lambda_u^i - 1)\gamma_{u-}(\Delta^i(y^i)h + h) \right) \bar{\eta}^i(du, dy^i)$$
$$+ \int_{\mathbb{R}} \left(\gamma_{u-}(\Delta(w)h) + \gamma_{u-}((\lambda - 1)(\Delta(w)h + h) \right) \bar{\eta}^r(du, dw),$$

where

$$(\hat{\mathcal{M}}h)_u(x^{1:n},r) = \left(\left(\sigma_X^1\partial_{x_1} + \frac{\alpha_X^1}{\sigma_X^1}, \ldots, \sigma_X^k\partial_{x_k} + \frac{\alpha_X^k}{\sigma_X^k}, \sigma\partial_r + \frac{\mu}{\sigma}\right) h\right)_u (x^{1:n},r).$$

Proof We obtain this result by taking conditional expectation $E_{\mathbb{Q}_t}[\cdot|\mathcal{G}_t]$ in equation (4.37). Using Lemma 4.5.15 we obtain

$$\begin{aligned} E_{\mathbb{Q}_t}\left[\int_{0+}^t \Lambda_{u-}(\mathcal{L}h)_u(X_{u-}^{1:n},r_{u-})du|\mathcal{G}_t\right] &= \int_{0+}^t \mathbb{E}_{\mathbb{Q}_t}\left[\Lambda_{u-}(\mathcal{L}h)_u(X_{u-}^{1:n},r_{u-})|\mathcal{G}_u\right]du \\ &= \int_{0+}^t \mathbb{E}_{\mathbb{Q}_u}\left[\Lambda_{u-}(\mathcal{L}h)_u(X_{u-}^{1:n},r_{u-})|\mathcal{G}_u\right]du \\ &= \int_{0+}^t \gamma_{u-}(\mathcal{L}h)du. \end{aligned} \tag{4.42}$$

Using Lemma 4.5.15 and Lemma 4.5.5 we have

$$E_{\mathbb{Q}_t}\left[\int_{0+}^t \Lambda_{u-}((\mathcal{M}+\bar{\mu})h)_u(X_{u-}^{1:n},r_{u-})d\bar{B}_u|\mathcal{G}_t\right] = \int_{0+}^t \gamma_{u-}(\hat{\mathcal{M}}h)d\hat{B}_u, \tag{4.43}$$

since the Brownian motions $\bar{W}_t^i$, $i>k$ are independent of $\mathcal{G}_t$ under Q_t. Similarly

$$E_{Q_t}\left[\int_{0+}^t \Lambda_{u-}(\Delta h)_u(\eta-\hat{\eta}^Q)(du,dy^{1:n},dw)\right] = \sum_{i=1}^k \int_{\mathbb{R}} \gamma_{u-}(\Delta^i(y^i)h)\bar{\eta}^i(du,dy^i), \tag{4.44}$$

since η^i, $i>k$ are independent of $\mathcal{G}_t$ under Q_t. Combining (4.42), (4.43) and (4.44) together we obtain our result. ■

Finally, the next theorem gives us the bond price of the partially informed trader.

Theorem 4.5.17 *The normalized filter* $\pi_t(h) = \frac{\gamma_t(h)}{\gamma_t(1)}$ *satisfies*

$$\begin{aligned} \pi_t(h) &= \pi_0(h) \\ &+ \int_{0+}^t \Bigg[\pi_{u-}(\mathcal{L}h) - \left\langle \pi_{u-}(\hat{\mathcal{M}}h), \pi_{u-}(\hat{\mu})\right\rangle + \left(\sum_{i=1}^k(\lambda_u^i-1) + \pi_{u-}(\lambda-1)\right)\pi_{u-}(h) \\ &\qquad - \Bigg(\sum_{i=1}^k \int_{\mathbb{R}} \left(\pi_{u-}\left(\Delta^i(y^i)h + (\lambda_u^i-1)\pi_{u-}(\Delta^i(y^i)h+h)\right)\right)\nu^i(dy^i) \\ &\qquad + \int_{\mathbb{R}} \left(\pi_{u-}(\Delta(w)h)\pi_{u-}((\lambda-1)(\Delta(w)h+h))\right)\nu(dw)\Bigg)\Bigg] du \\ &\qquad + \int_{0+}^t \left[\pi_{u-}(\hat{\mathcal{M}}h) - \pi_{u-}(h)\pi_{u-}(\hat{\mu})\right] d\hat{B}_u + \sum_{0<s\le t}\left[\frac{A_s(h)}{B_s(h)} - \pi_{s-}(h)\right], \end{aligned}$$

where

$$A_s(h) = \pi_{s-}(h) + \sum_{i=1}^{k} \int_{\mathbb{R}} \left\{\pi_{s-}(\Delta^i(y^i)h) + (\lambda_s^i - 1)\pi_{s-}(\Delta^i(y^i)h + h)\right\} \eta^i(ds, dy^i)$$
$$+ \int_{\mathbb{R}} \{\pi_{s-}(\Delta(w)h) + \pi_{s-}((\lambda - 1)(\Delta(w)h + h))\} \eta(ds, dw)$$

and

$$B_s(h) = 1 + \sum_{i=1}^{k} (\lambda_s^i - 1) dN_s^i + \pi_{s-}(\lambda - 1) dN_s.$$

Proof From Lemma 4.5.16 the dynamic of $\gamma_t(1)$ is given by

$$\gamma_t(1) = \gamma_0(1) + \int_{0+}^{t} \gamma_{u-}(\hat{\mu}) d\hat{B}_u + \sum_{i=1}^{k} (\lambda_u^i - 1)\gamma_{u-}(1)(dN_u^i - du) + \gamma_{u-}(\lambda - 1)(dN_u - du).$$

We can now apply Itô's formula in Theorem 4.3.2 in order to obtain the dynamics of $\frac{\gamma_t(h)}{\gamma_t(1)}$. To that effect we consider the function f given by

$$f(x, y) = \frac{x}{y}.$$

The partial derivatives of f now satisfy:

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \ \frac{\partial f}{\partial y} = -\frac{x}{y^2}, \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2}. \tag{4.45}$$

Theorem 4.3.2 now imply

$$\begin{aligned}\frac{\gamma_t(h)}{\gamma_t(1)} = \frac{\gamma_0(h)}{\gamma_0(1)} &+ \int_{0+}^{t} \frac{\partial f}{\partial x}(\gamma_u(h), \gamma_u(1)) d\gamma_u(h) + \int_{0}^{t} \frac{\partial f}{\partial y}(\gamma_u(h), \gamma_u(1)) d\gamma_u(1) \\ &+ \int_{0}^{t} \frac{\partial^2 f}{\partial x \partial y}(\gamma_u(h), \gamma_u(1)) d[\gamma(h), \gamma(1)]_u^c \\ + \sum_{0<s\le t} &\left\{ \frac{\gamma_s(h)}{\gamma_s(1)} - \frac{\gamma_{s-}(h)}{\gamma_{s-}(1)} - \frac{\partial f}{\partial x}(\gamma_s(h), \gamma_s(1)) \Delta\gamma_s(h) - \frac{\partial f}{\partial y}(\gamma_s(h), \gamma_s(1)) \Delta\gamma_s(1) \right\}.\end{aligned} \tag{4.46}$$

Notice that

$$\Delta\gamma_s(h) = \sum_{i=1}^{k} \int_{\mathbb{R}} \left\{\gamma_{s-}(\Delta^i(y^i)h) + (\lambda_s^i - 1)\gamma_{s-}(\Delta^i(y^i)h + h)\right\} \eta^i(ds, dy^i)$$
$$+ \int_{\mathbb{R}} \{\gamma_{s-}(\Delta(w)h) + \gamma_{s-}((\lambda - 1)(\Delta(w)h + h))\} \eta(ds, dw),$$

while

$$\Delta\gamma_s(1) = \sum_{i=1}^{k}(\lambda_s^i - 1)\gamma_{s-}(1)dN_s^i + \gamma_{s-}(\lambda - 1)dN_s$$
$$= \gamma_{s-}(1)\left(\sum_{i=1}^{k}(\lambda_s^i - 1)dN_s^i + \pi_{s-}(\lambda - 1)dN_s\right).$$

Using the fact that one can represent

$$\frac{\gamma_s(h)}{\gamma_s(1)} = \frac{\gamma_{s-}(h) + \Delta\gamma_s(h)}{\gamma_{s-}(1) + \Delta\gamma_s(1)},$$

one now has

$$\frac{\gamma_s(h)}{\gamma_s(1)} = \frac{A_s(h)}{B_s(h)}.$$

Combining the above together we have:

$$\begin{aligned}
\pi_t(h) &= \pi_0(h) \\
&+ \int_{0+}^{t}\Bigg[\pi_{u-}(\mathcal{L}h) - \left\langle \pi_{u-}(\hat{M}h), \pi_{u-}(\hat{\mu})\right\rangle + \left(\sum_{i=1}^{k}(\lambda_u^i - 1) + \pi_{u-}(\lambda - 1)\right)\pi_{u-}(h) \\
&\qquad - \left(\sum_{i=1}^{k}\int_{\mathbb{R}}\left(\pi_{u-}\left(\Delta^i(y^i)h + (\lambda_u^i - 1)\pi_{u-}(\Delta^i(y^i)h + h)\right)\right)\nu^i(dy^i)\right. \\
&\qquad\quad \left. + \int_{\mathbb{R}}\left(\pi_{u-}(\Delta(w)h)\pi_{u-}((\lambda - 1)(\Delta(w)h + h))\right)\nu(dw)\right)\Bigg]\,du \\
&\qquad\qquad + \int_{0+}^{t}\left[\pi_{u-}(\hat{\mathcal{M}}h) - \pi_{u-}(h)\pi_{u-}\hat{\mu}\right]d\hat{B}_u + \sum_{0<s\leq t}\left[\frac{A_s(h)}{B_s(h)} - \pi_{s-}(h)\right].
\end{aligned}$$

■

4.6 Conclusion

We have studied bond prices of the fully informed trader and the partially informed trader. The fully informed trader's bond price is given by a partial differential equation, while the partially informed trader's problem is more complicated. We employed a Bayesian filtering method to obtain the partially informed trader's price which included a useful change of measure technique.

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