EPSILON MULTIPLICITY OF MODULES WITH NOETHERIAN SATURATION ALGEBRAS

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To my grandmother Elizabeth.

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ABSTRACT

Ulloa-Esquivel, Roberto PhD, Purdue University, August 2020. Epsilon multiplicity of modules with Noetherian saturation algebras. Major Professor: Dr. Bernd Ulrich.

In the need of computational tools for ε-multiplicity, we provide a criterion for a module with a rank E inside a free module F to have rational ε-multiplicity in terms of the finite generation of the saturation Rees algebra of E. In this case, the multiplicity can be related to a Hilbert multiplicity of certain graded algebra. A particular example of this situation is provided: it is shown that the ε-multiplicity of monomial modules is Noetherian. Numerical evidence is provided that leads to a conjecture formula for the ε-multiplicity of certain monomial curves in $\mathbb{A}^3$.

1. INTRODUCTION

Given $(R, \mathfrak{m}, k)$ a Noetherian local ring, one would like to study the growth of powers of an R-ideal I. If I is $\mathfrak{m}$-primary, one can look at the function $\lambda_R(R/I^n)$. It is a classical result in commutative algebra that this numerical function is of polynomial type, i.e., there is a polynomial with rational coefficients, $P \in \mathbb{Q}[X]$ such that for n large enough $\lambda_R(R/I^n) = P(n)$ [17]. The leading coefficient of this polynomial (after normalization) is called the multiplicity of I. The same theory can not be applied if I is not $\mathfrak{m}$-primary, since in this case the lengths of the modules R/I^n are not guaranteed to be finite.

A way to generalize the notion of multiplicity is to apply the section function, or zero-th local cohomology to the modules before considering their length, i.e, to look at the modules $H^0_{\mathfrak{m}}(R/I^n)$. For a fixed $n \in \mathbb{N}$, this module is the largest submodule of R/I^n which has finite length, hence we can define the function $\Lambda_I(n) = \lambda_R(H^0_{\mathfrak{m}}(R/I^n))$. This numerical function however is more complicated that its analogue in the $\mathfrak{m}$-primary case. For example, it is not always of polynomial type. If I is a monomial ideal, it is known that this function is of quasipolynomial type, i.e., it behaves cyclically as a polynomial function [13]. In particular, there is no leading coefficient to look at to extract a multiplicity. However, if $\operatorname{depth}(R) > 0$, and I contains a nonzero divisor, then $\Gamma_I(n)$ is bounded above by a polynomial of degree d [24]. This allows to define

$$\varepsilon(I) = \limsup_{n\to\infty} \frac{d!\Lambda_I(n)}{n^d}$$

Cutkosky has also shown that if R is analytically unramified and $\dim R > 0$, then one can replace limsup by an actual limit, or if R is regular. [4] The limit in question appears when studying the asymptotic behaviour of graded families of ideals. The graded family associated to the ε-mutliplicity on an R-ideal is its saturated Rees algebra. The R-ideal $I^n :_R \mathfrak{m}^\infty = \bigcup_{j\geq 0}(I^n :_R \mathfrak{m}^j)$ is called the n-th saturated power of I. It is straight forward to see that $\Lambda_I(n) = \lambda_R(I^n :_R \mathfrak{m}^\infty/I^n)$. This function appears naturally as it is also equal to $\lambda(H^1_{\mathfrak{m}}(I^n))$ whenever $\operatorname{depth}(R) \geq 2$. Moreover, by local duality if R is Gorenstein, $\Lambda_I(n) = \lambda_R(\operatorname{Ext}^d_R(R/I^n, R))$.

Even when the ε-multiplicity exists as a limit, unlike Hilbert's multiplicity, it can be irrational [5]. For example, if $R = \mathbb{C}[x_1, ..., x_4]$ and $\mathfrak{m} = (x_1, ..., x_4)$, Cutkosky has proved that there is a nonsingular projective curve $\mathscr{C} \subseteq \mathbb{P}^3_{\mathbb{C}}$ with defining R-ideal I and

$$\lim_{n\to\infty} \frac{\lambda_R(H^0_{\mathfrak{m}}(R/I^n))}{n^4} \notin \mathbb{Q}$$

There are cases where the ε-multiplicity of the ideal is known to be rational. For example if I is a monomial ideal, $\Lambda_I(n)$ has a quasipolynomial behaviour. Herzog, Puthenpurakal and Verma have proved that the polynomials repeating cyclically have the same degree and the same leading coefficient [13]. In particular, the limit exists and it is a rational number. Jeffries and Montaño have improved this result by actually computing this multiplicity as a volume of certain region [16].

Given a ring R and two R-ideals $J \subseteq I$, one can define the saturation Rees algebra of I with respect to J as

$$\mathscr{R}_{\text{sat}}(I) = \bigoplus_{n\geq 0}(I^n :_R J^\infty)t^n \subseteq R[t]$$

A crutial result in the proof of Herzog, Puthenpurakal and Verma is that the saturation Rees algebra of a monomial ideal with respect to any other monomial ideal is Noetherian [12]. We prove in Chapter 4, that $\varepsilon(I) \in \mathbb{Q}$ whenever the saturation Rees algebra is finitely generated. The saturation Rees algebra may not be Noetherian in general. For example, if R is a local ring of dimension d and I is a prime ideal of height $d-1$, then $I^n :_R \mathfrak{m}^\infty = I^{(n)}$, the n-th saturated power of I. The study of the Noetherianness of the symbolic Rees algebra is a classical problem. In fact, it is related to a geometric problem via the following theorem: let k be an infinite field and $\mathscr{C}$ is a curve in $\mathbb{A}^n(k)$. If the symbolic Rees algebra of the curve is finitely generated, then $\mathscr{C}$ is a set-theoretical local complete intersection, i.e., it is locally the intersection (as a set) of $n-1$ hyperplanes. Rees already knew of examples where the symbolic Rees algebra was not Noetherian.

A particular family where the saturation Rees algebra correspond to the symbolic Rees algebra are monomial curves in $\mathbb{A}^3(k)$. Let $\mathscr{C}$ be a monomial curve parametrized by the map $\lambda \mapsto (\lambda^{n_1}, \lambda^{n_2}, \lambda^{n_3})$ and let $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3) = \ker(\varphi)$ is the defining ideal. It is an open problem to find a characterization in termns of n_1, n_2 and n_3 of when is $\mathscr{R}_{\text{sat}}(\mathfrak{p})$ Noetherian. However, Herzog and Ulrich have characterized monomial curves with symbolic Rees algebras generated by the first and second symbolic powers [14]. We explore numerically what happens with the growth of $\lambda_R(H^0_{\mathfrak{m}}(R/I^n))$ for some of these monomial curves in Chapter 5.

Ulrich and Validashti have generalized the notion of ε-multiplicity to modules [24]. If $E \subseteq F$ are R-modules with $F \simeq R^r$ free and E a module with a rank, then one can make sense of the funtion $\Lambda_E(n) = \lambda_R(H^0_{\mathfrak{m}}(F^n/E^n))$ as follows: the symbolic algebra of F is just a polynomial ring over R, say $S = R[t_1, ..., t_r]$. The n-th power of F is just the n-th graded component of S. The symmetric algebra of E maps into S by the universal mapping property of symmetric algebras. The image of this map is a standard graded R-subalgebra of S, called the Rees algebra of E and denoted $\mathscr{R}(E)$. Its graded components are the powers of E. We prove in Chapter 4 that if the R-algebra:

$$\mathscr{R}_{\text{sat}}(E) = \mathscr{R}(E) :_S \mathfrak{m}^\infty$$

is Noetherian, then $\varepsilon(E|F)$ is rational, whenever one can realize the ε-multiplicity as a limit.

In the case of R-ideals this reduces to $\mathscr{R}_{\text{sat}}(I)$ being Noetherian. As we mention before, this algebra is known to be Noetherian for monomial ideals. We generalize this to monomial modules. In Chapter 4 we prove that the ε-multiplicity of a monomial module is rational, by showing that their saturation Rees algebra is always Noetherian.

Finally in Chapter 5, we explore some numerical evidence about the ε-multiplicity of monomial curves, and some results for the relative multiplicity of two ideals of finite colength, equality of colength and multiplicity and the effect of quotiening by general elements in the multiplicity.

2. PRELIMINARIES

2.1 Blowup algebras

Definition 2.1.1 *Let H be an additive monoid and R a ring. We say that R is H-graded, if there is a direct sum decomposition of the additive group of R*

$$R = \bigoplus_{h \in H} R_h$$

such that $1 \in R_0$ and $R_h R_k \subseteq R_{h+k}$ for all h, k in H. We call R_h the h-th graded component of R. An element $x \in R$ is called homogeneous if there is $h \in H$ such that $x \in R_h$ and an R-ideal is called homogeneous if it can be generated by homogeneous elements. If $H = \mathbb{N}_0$ we will just say that R is graded and if $R = R_0[R_1]$ we say that R is ***homogeneous*** *or* ***standard graded****.*

Definition 2.1.2 (Rees algebra, extended Rees algebra) *Let R be a ring and I an R-ideal:*

(a) The ***Rees algebra of*** *R* ***with respect to*** *I is defined to be*

$$\mathscr{R}(I) := R[It] = \left\{ \sum_{i=0}^{n} a_i t^i : n \in \mathbb{N}_0, a_i \in I^i \right\} \subseteq R[t].$$

Note that $\mathscr{R}[It]$ is a standard graded subalgebra of $R[t]$, and $[\mathscr{R}(I)]_i = I^i t^i$,

$$\mathscr{R}(I) = \bigoplus_{i=0}^{\infty} (It)^i = R \oplus It \oplus I^2t^2 \oplus \cdots$$

(b) The ***extended Rees algebra of*** *I is defined as*

$$R[It, t^{-1}] = \left\{ \sum_{i=-n}^{n} a_i t^i : n \in \mathbb{N}_0, a_i \in I^i \right\} \subseteq R[t, t^{-1}].$$

where by convention $I^k = R$ for $k \leq 0$. This is a $\mathbb{Z}$-graded R-subalgebra of $R[t,t^{-1}]$, and $R[It,t^{-1}]_i = I^i t^i$ where $I^i = R$ for $i \leq 0$. With this we can write

$$R[It,t^{-1}] = \bigoplus_{i=-\infty}^{\infty} (It)^i = \cdots \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \cdots$$

Sometimes $R[It,t^{-1}]$ is also denotes by $\mathscr{R}_I^+(R)$.

Definition 2.1.3 (Associated graded ring, fiber cone, analytic spread) *Let R be a ring and I an R-ideal.*

*(a) The **associated graded ring of** R **with respect to** I is defined as $\mathrm{gr}_I(R) := \mathscr{R}(I)/I\mathscr{R}(I)$. Note that $I\mathscr{R}(I)$ is a homogeneous ideal, hence $\mathrm{gr}_I(R)$ is a standard graded R-algebra, and $[\mathrm{gr}_I(R)]_i \simeq I^it^i/I^{i+1}t^i$. With this*

$$\mathrm{gr}_I(R) \simeq R/I \oplus I/I^2 \oplus \cdots$$

*(b) If $(R,\mathfrak{m},k)$ is local, the **fiber cone of** I is the ring*

$$\mathscr{F}_I(R) := \mathscr{R}(I)/\mathfrak{m}\mathscr{R}(I)$$

Again, $\mathfrak{m}\mathscr{R}(I)$ is a homogeneous ideal, hence the fiber cone is a standard graded R-algebra, and:

$$\mathscr{F}_I(R) \simeq k \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus$$

*(c) The dimension of the fiber cone is called the **analytic spread of** I and is denoted by $\ell(I)$.*

Remark 2.1.1 *Let R be a ring and I an R-ideal. The associated graded ring is an epimorphic image of the extended Rees algebra*

$$\mathrm{gr}_I(R) \simeq \frac{R[It,t^{-1}]}{(t^{-1})R[It,t^{-1}]}$$

We can summarize the relations between the Rees algebra, the extended Rees algebra and the associated graded ring in the following diagram:

$$\begin{tikzcd}
\mathscr{R}(I) \arrow[r, hook, "i"] \arrow[dr, "\text{mod } I\mathscr{R}(I)"'] & R[It, t^{-1}] \arrow[d, two heads, "\text{mod } (t^{-1})"] \\
& \mathrm{gr}_I(R)
\end{tikzcd}$$

2.2 Powers of modules

Let R be a commutative ring, M and N R-modules and $g : M \to N$ an R-linear map. By the universal mapping property of symmetric algebras, there is an induced homogeneous R-algebra homomorphism

$$\mathrm{Sym}(g) : \mathrm{Sym}(M) \to \mathrm{Sym}(N)$$

$$x \in [\mathrm{Sym}(M)]_1 \mapsto g(x) \in [\mathrm{Sym}(N)]_1$$

Note that $\mathrm{im}(\mathrm{Sym}(g))$ is a standard graded R-subalgebra of $\mathrm{Sym}(N)$ generated by linear forms. We apply this construction to the case where E is a submodule of a free R-module F. The inclusion $E \xrightarrow{i} F$ induces a homogeneous homomorphism of standard graded R-algebras

$$\mathrm{Sym}(E) \xrightarrow{\mathrm{Sym}(i)} \mathrm{Sym}(F)$$

(which may not be necessarily injective). Now, since $F \simeq \bigoplus_{i\in\Delta} R_i$ is free, $\mathrm{Sym}(F) \simeq R[\{t_i\}_{i\in\Delta}]$.

Definition 2.2.1 (Rees algebra on an embedding [6]) *Let R be a commutative ring, E an R-module, and $i : E \hookrightarrow F$ an embedding of E into a free R-module F. The* ***Rees algebra of the inclusion*** *$E \hookrightarrow F$, $\mathscr{R}(i)$, or sometimes just denoted $\mathscr{R}[E]$,*

$$\mathscr{R}(i) = im(Sym(i)) \subseteq Sym(F)$$

Since $\mathscr{R}(i)$ is a standard graded ring, when realized inside $R[\{t_i\}_{i\in\Delta}]$, it is generated by linear forms.

Remark 2.2.1 *If R is a ring and I is an R-ideal, the previous construction for the inclusion $i : I \hookrightarrow R$ gives the classical definition of Rees algebra for an ideal.*

Definition 2.2.2 (Powers of a module) *Let R be a commutative ring, E an R-module, and $i : E \hookrightarrow F$ an embedding of E into a free R-module F. The n-th graded component of the Rees algebra of the embedding is called the n-th power of E, and denoted E^n, i.e.:*

$$E^n := [\mathscr{R}(i)]_n$$

Note that the *powers of modules* do not have all the properties powers of ideals have. In fact, unlike in the ideal case $E^{i+1} \not\hookrightarrow E^i$, so $\{E^i\}_{i\in\mathbb{N}}$ does not form a filtration. As a consequence, we cannot define the associated graded ring of a module, nor its extended Rees algebra. Furthemore, this construction does depend on the embedding, unless we have some assumptions on the module or the ring.

Definition 2.2.3 (Rank) *Let R be a Noetherian ring and M a finite R-module. We say that M has a rank if $M \otimes_R Quot(R) \simeq Quot(R)^r$ for some $r \geq 0$. In such a case, we say that M has rank r and denote it by* $\operatorname{rank}(M) = r$.

Proposition 2.2.1 *Let R be a Noetherian ring and M a finite R-module. The following are equivalent:*

(a) *M has a rank and* $\operatorname{rank}(M) = r$;

(b) *$M_\mathfrak{p} \simeq R_\mathfrak{p}^r$ for every $\mathfrak{p} \in Ass(R)$.*

For a proof of this, see for example [2], Proposition 1.4.3:

Theorem 2.2.2 (Einsenbud, Huneke, Ulrich) *Let R be a Noetherian ring and E a finite R-module. Assume that for each $\mathfrak{p} \in Ass(R)$, one of the following holds*

(i) *$E_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module (e.g. E has a rank);*

(ii) *$R_\mathfrak{p}$ is Gorenstein;*

(iii) $R_{\mathfrak{p}}$ is $\mathbb{Z}$-torsionfree (e.g. if $\mathbb{Q} \subseteq R_{\mathfrak{p}}$ or if R is a domain).

Then $\mathscr{R}(E)$ does not depend on the embedding. In particular, the powers of E do not depend on the embedding.

For a proof of this, see [6], Theorem 1.6.

2.3 Numerical functions

In this section we introduce the concept and some properties of numerical functions that will be used later on.

Definition 2.3.1 (Numerical function) *A* ***numerical function*** *is a function $F : \mathbb{Z} \to \mathbb{Q}$. If $S \subseteq \mathbb{Z}$ is a set and $F : S \to \mathbb{Q}$ we regard F as a numerical function by extending $G : \mathbb{Z} \to \mathbb{Q}$ with $G(n) = 0$ for $n \notin S$ and $G(n) = F(n)$ for $n \in S$.*

The set of numerical functions V is a $\mathbb{Q}$-vector space. One can regard $\mathbb{Q}[t]$, the space of polynomials over $\mathbb{Q}$, as a subspace of V: given a polynomial $P(t) \in \mathbb{Q}[t]$, one defines $P : \mathbb{Z} \to \mathbb{Q}$ with $n \mapsto P(n)$. We will refer to these as **polynomial numerical functions**.

Define an equivalence relation on V by declaring $F \sim H$ if and only if $F(n) = H(n)$ for $n \gg 0$. If $F \sim P$ for some polynomial numerical function, we say that F is of **polynomial type**. Note that if F is of polynomial type, there is a unique polynomial P such that $F \sim P$. In this case we say that P is the polynomial associated to F and define $\deg F = \deg P$.

Define a $\mathbb{Q}$-linear transformation $\Delta : V \to V$ by:

$$F \mapsto \begin{pmatrix} \Delta F : \mathbb{Z} \to \mathbb{Q} \\ n \mapsto F(n) - F(n-1) \end{pmatrix}$$

Lemma 2.3.1 *Let $B_{-1}(t) := 0$, $B_0(t) := 1$ and for $r \geq 1$:*

$$B_r(t) = \binom{t+r}{r} := \frac{\prod_{i=1}^{r}(t+i)}{r!} \in \mathbb{Q}[t]$$

(a) $\{B_r(t) : r \geq 0\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[t] \subseteq V$;

(b) For $1 \leq d \leq r$, $\Delta^d B_r(t) = B_{r-d}(t)$;

(c) For $n \in \mathbb{Z}$, $B_r(n) \in \mathbb{Z}$.

Proof For (a) note that if $r \geq 0$, $B_r(t)$ is a monic polynomial of degree r and so $\{B_r(t) : r \geq 0\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[t] \subseteq V$. For (b) proceed by induction on d. Let $d = 1$. Note that

$$\Delta B_r(t) = B_r(t) - B_r(t-1) = \frac{\prod_{i=1}^{r}(t+i) - \prod_{i=1}^{r}(t-1+i)}{r!} =$$

$$\prod_{i=1}^{r-1}(t+i) \cdot \frac{(t+r)-t}{r!} = \frac{\prod_{i=1}^{r-1}(t+i)}{(r-1)!} = B_{r-1}(t)$$

Assume $d \geq 2$. Note that

$$\Delta^d B_r(t) = \Delta(\Delta^{d-1} B_r(t)) = \Delta B_{r-d+1}(t) = B_{r-d}(t).$$

Finally for (d), note that if $n + r \geq 0$, then $B_r(n) = \binom{n+r}{r} \in \mathbb{Z}$. If $n + r < 0$, then $n < -r \leq 0$, so $-n-1 \geq 0$ and:

$$B_r(n) = \binom{n+r}{r} = \frac{\prod_{i=1}^{r}(n+i)}{r!} = (-1)^r \frac{\prod_{i=1}^{r}(-n-i)}{r!}$$

$$= (-1)^r \frac{(-n-1)!}{(-n-r-1)!r!} = (-1)^r \binom{-n-1}{r} \in \mathbb{Z}$$

■

Lemma 2.3.2 *Let F be a numerical function and d an integer with $d \geq 0$. The following are equivalent:*

(a) F is of polynomial type of degree d;

(b) ΔF is of polynomial type of degree $d-1$;

(c) $\Delta^d F$ *is eventually a non-zero constant.*

Proof (a)$\Rightarrow$ (b) Since F is of polynomial type of degree d, and $\{B_r(t) : r \geq 0\}$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[t]$, there are $a_i \in \mathbb{Q}$, $0 \leq i \leq d$ such that $F \sim \sum_{i=0}^{d} a_i B_i(t)$ and $a_d \neq 0$. For $n \gg 0$ note that

$$\Delta F(n) = \Delta \left(\sum_{i=0}^{d} a_i B_i(n) \right) = \sum_{i=0}^{d} a_i \Delta B_{i-1}(n)$$

In particular $\Delta F \sim \sum_{i=0}^{d} a_i \Delta B_{i-1}(t)$ and this is a polynomial of degree $d-1$.

(b)$\Rightarrow$ (a) If $d = 0$, then $\Delta F \sim 0$, so F is eventually constant and hence F polynomial type of degree 0. With this assume $d \geq 1$. There is $Q \in \mathbb{Q}[t]$ such that $\Delta F \sim Q$ and $\deg(Q) = d-1$. Write $Q = \sum_{i=0}^{d-1} b_i B_i(t)$, where $b_{d-1} \neq 0$. Consider the polynomial

$$P(t) = \sum_{i=0}^{d-1} b_i B_{i+1}(t)$$

Then $\Delta P(t) = \sum_{i=0}^{d-1} b_i B_i(t) = Q(t) \sim \Delta F$. In particular $\Delta(F - P) \sim 0$, hence $F - P$ must be associated to a constant polynomial, say C, $C \in \mathbb{Q} \subseteq \mathbb{Q}[t]$. Note that $F \sim P + C$, and hence F is of polynomial type of degree d.

(b)$\Rightarrow$ (c) Proceed by induction on d. If $d = 0$, $\Delta F \sim 0$, and $\Delta^0 F = F$ is eventually constant. Assume $d \geq 1$. Since ΔF is a of polynomial type of degree $d-1$, by induction $\Delta^d F = \Delta^{d-1}(\Delta F)$ is eventually constant.

(c)$\Rightarrow$ (b) If $\Delta^0 F$ is eventually constant, then F is eventually constant, and hence $\Delta F \sim 0$. With this we may assume $d \geq 1$. If $\Delta^d F$ is eventually constant, then $\Delta^{d-1}(\Delta F)$ is eventually constant. By induction hypothesis ΔF is of polynomial type of degree $d-1$. ■

Lemma 2.3.3 *Let F be a numerical function of polynomial type with associated polynomial P of degree d. The following are equivalent:*

(a) $F(n) \in \mathbb{Z}$ *for* $n \gg 0$*;*

(b) $P(n) \in \mathbb{Z}$ *for* $n \gg 0$*;*

(c) $P(n) \in \mathbb{Z}$ *for* $n \in \mathbb{Z}$*;*

(d) $P(t) = \sum_{r=0}^{d} a_r B_r(t)$ *with* $a_r \in \mathbb{Z}, a_d \neq 0$ *if* $d \neq -1$.

Proof Note that (a) is equivalent to (b) by definition and (c)$\Rightarrow$ (b) trivially. Also (d)$\Rightarrow$ (c), since Lemma 2.3.1(c) say that for $n \in \mathbb{Z}$, $B_r(n) \in \mathbb{Z}$. Hence it is enough to show that (b)$\Rightarrow$ (d). Write $P(t) = \sum_{r=0}^{d} a_i B_i(t)$, with $a_r \in \mathbb{Q}$ for every r and $a_d \neq 0$. Proceed by induction on d. If $d = 0$, then P is constant and by (b) it takes integer values, so the result holds. Let $d \geq 1$. Note that

$$\Delta P(t) = \sum_{r=0}^{d} a_r \Delta B_r(t) = \sum_{r=0}^{d} a_r B_{r-1}(t) = \sum_{r=-1}^{d-1} a_{r+1} B_r(t) = \sum_{r=0}^{d-1} a_{r+1} B_r(t)$$

Since $P(n) \in \mathbb{Z}$ for $n \gg 0$, $\Delta P(n) \in \mathbb{Z}$ for $n \gg 0$. By induction on d, $a_1, ..., a_d$ are integers. Now, note that $a_0 = P(n) - \sum_{r=1}^{d} a_r B_r(n) \in \mathbb{Z}$ for $n \gg 0$ and so we are done. ∎

Definition 2.3.2 (Integer-valued numerical function) *A numerical function F is said to be* ***integer-valued*** *if $F(n) \in \mathbb{Z}$ for $n \geq 0$. We say that F is* ***non-negative*** *if $F(n) \geq 0$ for $n \gg 0$.*

2.4 Hilbert functions

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a Noetherian graded ring with R_0 Artinian and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated R-module. The R_0-modules M_i are finite for every i. Since R_0 is Artinian, $\lambda(M_i) < \infty$ for $i \geq 0$. With this, the following definition makes sense.

Definition 2.4.1 (Hilbert function) *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a Noetherian graded ring with R_0 Artinian and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely graded R-module.*

(a) The function $H_M : \mathbb{Z} \to \mathbb{N}_0$ given by $H_M(i) = \lambda_{R_0}(M_i)$ is called the ***Hilbert function of*** *M.*

(b) The Laurent series $h_M(t) = \sum_i H_M(i)t^i$ is called the ***Hilbert series of*** *M.*

Theorem 2.4.1 (Hilbert series is rational) *Let R be a Noetherian graded ring with R_0 Artinian, write $R = R_0[x_1, ..., x_n]$ where x_j is homogeneous of degree $d_j > 0$, let $M = \bigoplus_{i=0}^{\infty} M_i$ be a finitely generated graded R-module. Then:*

$$h_M(t) = \frac{q(t)}{\prod_{j=1}^{n}(1 - t^{d_j})},$$

for some $q(t) \in \mathbb{Z}[t, t^{-1}]$. If $M_i = 0$ for $i < 0$, then $q(t) \in \mathbb{Z}[t]$ and $h_M(t)$ is a rational function.

Proof Proceed by induction on $n \geq 0$.

(Case $n = 0$**)** If $n = 0$, $R = R_0$ and hence because M is a finite R-module, it has finite length. Since $M = \bigoplus_{i \in \mathbb{Z}} M_i$, it follows that $M_i = 0$ for $|i| >> 0$. In particular, there is N such that

$$h_M(t) = \sum_{i=-N}^{N} \lambda_R(M_i)t^i \in \mathbb{Z}[t, t^{-1}].$$

Note that if $M_i = 0$ for $i < 0$ then $h_M(t) = q(t) \in \mathbb{Z}[t]$.

(Case $n > 0$**)** Let $n > 0$. Then $M(-d_n) \xrightarrow{x_n} M$ is a homogeneous R-linear map since $x_n M_{i-d_n} \subseteq M_i$. Its kernel and cokernel are graded modules giving rise to an exact sequence of homogeneous R-linear maps

$$0 \to K \to M(-d_n) \xrightarrow{x_n} M \to L \to 0$$

where $x_n K = 0 = x_n L$. By the latter, K and L are graded modules over the graded ring $R/(x_n) \simeq R_0[X_1, ..., X_{n-1}]$, where X_i is the class of x_i in $R/(x_n)$. Thus by induction hypothesis

$$h_K(t) = \frac{q'(t)}{\prod_{j=1}^{n-1}(1 - t^{d_j})} \qquad \text{and} \qquad h_L(t) = \frac{q''(t)}{\prod_{j=1}^{n-1}(1 - t^{d_j})}$$

for some $q'(t), q''(t)$ in $\mathbb{Z}[t, t^{-1}]$ (or $\mathbb{Z}[t]$ if $M_i = 0$ for $i < 0$). Now, by additivity:

$$\frac{q''(t) - q'(t)}{\prod_{j=1}^{n-1}(1-t)^{d_j}} = h_L(t) - h_K(t)$$

$$= h_M(t) - h_{M(-d_n)}(t) = h_M(t) - t^{d_n}h_M(t) = (1 - t^{d_n})h_M(t).$$

■

Proposition 2.4.1 *Let R be a standard graded Noetherian ring with R_0 an Artinian ring and let M be a finitely generated graded R-module. If $M \neq 0$, then $h_M(t)$ can be written uniquely as:*

$$h_M(t) = \frac{q_M(t)}{(1-t)^d}$$

where $d = d(M) \geq 0$ and $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $q_M(1) \neq 0$.

Proof By Theorem 2.4.1, $h_M(t) = q(t)/(1-t)^n$ for some $q(t) \in \mathbb{Z}[t, t^{-1}]$. Furthermore

$$h_M(t) = \frac{(1-t)^\ell}{(1-t)^n}q_M(t) = (1-t)^{\ell-n}q_M(t)$$

for some $q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $q_M(1) \neq 0$. Set $d = n - \ell$. If $d < 0$, then

$$0 = h_M(1) = \sum_{i \in \mathbb{Z}} \lambda_{R_0}(M_i) \neq 0$$

since $M \neq 0$. We condude $d \geq 0$. ■

We focus now on the case where $M \neq 0$ is non-negatively graded, i.e., $M_i = 0$ for $i < 0$. Write $\deg(q_M) = s$. Since $d(M)$ is the order of the pole of $h_M(t)$ at $t = 1$, we want to express $q_M(t)$ in the basis $\{(1-t)^n : n \geq 0\}$ which we can do by considering a Taylor expansion around $t = 1$ for $q_M(t)$. Note that

$$q_M(t) = \sum_{i=0}^{s} \frac{q_M^{(i)}(1)}{i!}(t-1)^i = \sum_{i=0}^{s} \frac{(-1)^i q_M^{(i)}(1)}{i!}(1-t)^i$$

Write $q_M(t) = a_0 + \cdots + a_s t^s$. Then

$$q_M^{(i)}(1) = \sum_{n=i}^{s} \prod_{j=0}^{i-1}(n-j)a_n = \sum_{n=i}^{s} \frac{n!}{(n-i)!}a_n = \sum_{n=i}^{s} \binom{n}{i} \cdot i! \cdot a_n = i! \cdot \sum_{n=i}^{s} \binom{n}{i} a_n$$

In particular

$$q_M(t) = \sum_{i=0}^{s} \left[\sum_{n=i}^{s} \binom{n}{i} a_n \right] (-1)^i (1-t)^i$$

Let $e_i = \sum_{n=i}^{s} \binom{n}{i} a_n = \frac{q_M^{(i)}(1)}{i!}$. Note that $e_0 = q_M(1) \neq 0$. Also $e_s = a_s \neq 0$ and $e_i \in \mathbb{Z}$ for $0 \leq i \leq s$. With this notation

$$q_M(t) = \sum_{i=0}^{s} e_i (-1)^i (1-t)^i.$$

With the Taylor expansion, we can now write $h_M(t)$ as

$$h_M(t) = \frac{q_M(t)}{(1-t)^d} = \frac{\sum_{i=0}^{s} e_i (-1)^i (1-t)^i}{(1-t)^d}$$

Now we will use this form of $h_M(t)$ to find a polynomial P_M with $H_M \sim P_M$. Recall that if $d \in \mathbb{N}$. Then:

$$\frac{1}{(1-t)^d} = \sum_{i=0}^{\infty} \begin{pmatrix} i+d-1 \\ d-1 \end{pmatrix} t^i = \sum_{i=0}^{\infty} B_{d-1}(i) t^i$$

Theorem 2.4.2 *Let R be a standard graded Noetherian ring with R_0 Artinian and let $0 \neq M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded R-module which is non-negatively graded. For $d \geq 1$ write*

$$P(t) = \sum_{i=0}^{d-1} (-1)^i e_i B_{d-i-1}(t).$$

Then $H_M(n) = P_M(n)$ for $n \geq s - d + 1$. In particular H_M is of polynomial type of degree $d-1$. If $d = 0$, then H_M is of polynomial type of degree -1.

Proof If $d = 0$, then $h_M(t) = q_M(t)$ and hence $H_M(n) = 0$ for $i \geq s+1$. Then $H_M(n)$ is of polynomial type of degree -1. Assume then $d \geq 1$. Consider the case where $s < d$. Then:

$$h_M(t) = \sum_{i=0}^{s} (-1)^i e_i \frac{1}{(1-t)^{d-i}}$$

and $d - i \geq d - s > 0$. Using the expansion for $1/(1-t)^d$ we have:

$$h_M(t) = \sum_{i=0}^{s} (-1)^i e_i \sum_{k=0}^{\infty} B_{d-i-1}(k) t^k = \sum_{k=0}^{\infty} \sum_{i=0}^{s} (-1)^i e_i B_{d-i-1}(k) t^k$$

and hence, comparing coefficients

$$H_M(n) = \lambda(M_n) = \sum_{i=0}^{s}(-1)^i e_i B_{d-i-1}(n) = \sum_{i=0}^{d-1}(-1)^i e_i B_{d-i-1}(n)$$

since $e_i = 0$ for $i > s$. Note that this holds for $n \geq 0 \geq s - d + 1$.

With this we may assume $s \geq d$. Then one can write

$$h_M(t) = \sum_{i=0}^{d-1}(-1)^i e_i \frac{1}{(1-t)^{d-i}} + \sum_{i=d}^{s}(-1)^i e_i (1-t)^{i-d}$$

Using the arguments from the previous case, one can rewrite the first sumand to get

$$h_M(t) = \sum_{k=0}^{\infty}\sum_{i=0}^{d-1}(-1)^i e_i B_{d-i-1}(k) t^k + \sum_{i=d}^{s}(-1)^i e_i (1-t)^{i-d}.$$

Note that $\sum_{i=d}^{s}(-1)^i e_i (1-t)^{i-d}$ is a polynomial of degree at most $s - d$, hence for $n \geq s - d + 1$ we have:

$$H_M(n) = \sum_{i=0}^{d-1}(-1)^i e_i B_{d-i-1}(n) = P_M(n)$$

■

Definition 2.4.2 (Hilbert polynomial, multiplicity) *Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a Noetherian standard graded ring with R_0 Artinian, and $M = \bigoplus_{n=0}^{\infty} M_n$ a finitely generated graded R-module.*

*(a) $P_M(t)$ is called the **Hilbert polynomial of** M;*

*(b) The **Hilbert multiplicity** of M, denoted $e(M)$, is defined as follows:*

$$e(M) = \begin{cases} e_0 & \text{if } d \geq 1; \\ \lambda(M) & \text{if } d = 0. \end{cases}$$

Theorem 2.4.3 (Hilbert) *Let R be a Noetherian standard graded ring with $(R_0, \mathfrak{m}_0, k)$ Artinian and M a finite graded R-module of dimension d. Then H_M is of polynomial type of degree $d - 1$.*

Proof We prove the case $M = R/\mathfrak{p}$ for $\mathfrak{p}$ a graded prime ideal of R first. Proceed by induction on $\dim(R/\mathfrak{p}) = d$.

(Case $d = 0$**)** If $\dim R/\mathfrak{p} = 0$, then $\mathfrak{p} = \mathfrak{m}_0 \oplus R_+$ since this is the homogeneous maximal ideal. In particular $R/\mathfrak{p} \simeq k$ and hence $H_{R/\mathfrak{p}}(n) = 0$ for $n \geq 1$. In particular $H_{R/\mathfrak{p}}$ is of polynomial type of degree -1.

(Case $d > 0$**)** Since R is standard graded, we know that $R = R_0[R_1]$ and hence $R/\mathfrak{p} = k[\frac{R_1}{\mathfrak{p}\cap R_1}]$. If $R_1 = \mathfrak{p} \cap R_1$, then $\dim R/\mathfrak{p} = 0$, a contradiction. Hence we can pick $0 \neq x \in R/\mathfrak{p}$ homogeneous of degree 1. Consider the exact sequence:

$$0 \to R/\mathfrak{p}(-1) \xrightarrow{x} R/\mathfrak{p} \to R/(\mathfrak{p}, x) \to 0.$$

By additivity

$$\Delta H_{R/\mathfrak{p}}(n) = H_{R/\mathfrak{p}}(n) - H_{R/\mathfrak{p}}(n-1) = H_{R/(\mathfrak{p},x)}(n).$$

Note that $R/\mathfrak{p}$ is a Noetherian standard graded domain with $R_0 = k$ a field, and hence $\dim R/(\mathfrak{p}, x) = \dim R - 1$. By induction hypothesis $H_{R/(\mathfrak{p},x)}$ is of polynomial type of degree $d-2$. But this says that $\Delta H_{R/\mathfrak{p}}$ is of polynomial type of degree $d-2$, hence $H_{R/\mathfrak{p}}$ is of polynomial type of degree $d-1$, according to Lemma 2.3.2.

Now we prove the general case. Consider a graded prime filtration of M, i.e., a chain

$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_\ell = M$$

of graded submodules of M such that for each i, $N_i/N_{i-1} \simeq R/\mathfrak{p}_i$, $1 \leq i \leq \ell$, where $\mathfrak{p}_i$ are graded prime ideals. Considering the exact sequences:

$$0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0$$

it is easy to see by induction that:

$$H_M = \sum_{i=1}^{\ell} H_{N_i/N_{i-1}} = \sum_{i=1}^{\ell} H_{R/\mathfrak{p}_i}$$

We know that $H_{R/\mathfrak{p}_i}$ is of polynomial type of degree $\dim(R/\mathfrak{p}_i)$. It is straightforward that H_M is of polynomial type. We also know that the leading coefficient of the polynomial associated to $H_{R/\mathfrak{p}_i}$ must be positive, since $H_{R/\mathfrak{p}_i}(n) \geq 0$. In particular

$$\deg H_M = \max_{1 \leq i \leq \ell} \{\deg H_{R/\mathfrak{p}_i}\} = \max_{1 \leq i \leq \ell} \{\dim(R/\mathfrak{p}_i) - 1\}.$$

But all the minimal primes of M are homogeneous, and they must appear in every prime filtration of M. In particular:

$$d - 1 = \dim M - 1 = \max_{\mathrm{Min}(M)} \{\dim(R/\mathfrak{p}) - 1)\} \leq \max_{1 \leq i \leq \ell} \{\dim(R/\mathfrak{p}_i) - 1\} \leq d - 1.$$

■

2.5 Hilbert-Samuel Functions

Definition 2.5.1 (Ideal of definition) *Let R be a Noetherian semilocal ring. An **ideal of definition** is an ideal I with $\sqrt{I} = Rad(R)$, where $Rad(R)$ is the Jacobson radical of R, i.e., the intersection of all maximal ideals. Note that if Q is an ideal of definition, then R/Q is an Artinian ring.*

Definition 2.5.2 (Hilbert-Samuel function) *Let R be a semilocal Noetherian ring, I an ideal of definition and M a finitely generated R-module. Define the **Hilbert-Samuel function of** M **with respect to** I as follows:*

$$L_{I,M} : \mathbb{N}_0 \to \mathbb{N}_0$$

$$L_{I,M}(n) := \lambda_R(M/I^n M) = \sum_{j=0}^{n-1} H_{\mathrm{gr}_I(M)}(j) = \sum_{j=0}^{n-1} \lambda_R(I^j M/I^{j+1} M)$$

Note that $gr_I(R)$ is an Noetherian standard graded ring with $[gr_I(R)]_0 = R/I$ an Artinian ring. Also $gr_I(M)$ is a finite $gr_I(R)$-graded module. Hence the previous definition makes sense.

Proposition 2.5.1 (Existence of the Hilbert-Samuel polynomial) *Let R be a semilocal Noetherian ring, I an ideal of definition and M a finitely generated R-module. There is a unique polynomial $P_{I,M}(t) \in \mathbb{Q}[t]$, with $L_{I,M}(n) = P_{I,M}(n)$ for every $n \gg 0$. Furthermore*

$$P_{I,M}(t) = \sum_{i=0}^{d} (-1)^i e_i B_{d-j}(t-1)$$

with $e_i \in \mathbb{Z}$, $0 \leq i \leq d$, and $e_0 \neq 0$ if $M \neq 0$.

Proof If $M = 0$, then $P_{I,M}(t) = 0$. Hence wee say assume that $0 \neq M$. We know that $H_{gr_I(M)}$ is of polynomial type of degree $d-1$ for some d. If $M \neq 0$, then $gr_I(M) \neq 0$, and:

$$\Delta L_{I,M}(n) = \sum_{j=0}^{n-1} H_{gr_I(M)}(j) - \sum_{j=0}^{n-2} H_{gr_I(M)}(j) = H_{gr_I(M)}(n-1).$$

It follows that $L_{I,M}$ is of polynomial type of degree d. Let $L_{I,M} \sim P_{I,M}$. Write:

$$P_{I,M}(t) = \sum_{i=0}^{d} (-1)^i e_i B_{d-i}(t)$$

A priori, $e_i \in \mathbb{Q}$, but since $L_{I,M}$ is integer-valued, $e_i \in \mathbb{Z}$ and $e_0 \neq 0$ as wanted. ■

Definition 2.5.3 (Hilbert-Samuel polynomial) *Let R be a semilocal Noetherian ring, I an ideal of definition and M a finitely generated R-module. The polynomial $P_{I,M}$ is called the **Hilbert-Samuel polynomial of M with respect to I.***

Let R and M be as in Definition 2.5.3. Define

$$\delta(M) = \min \left\{ n \in \mathbb{N}_0 | \exists a_1, ..., a_n \in \mathrm{Rad}(R), \lambda\left(\frac{M}{(a_1, ..., a_n)M}\right) < \infty \right\}.$$

Theorem 2.5.1 (Fundamental Theorem of Dimension Theory) *Let R be a local ring and $M \neq 0$ a finite R-module. Then $\dim(M) = \delta(M)$. This is also equal to the degree of the Hilbert-Samuel polinomial with respect to any $\mathfrak{m}$-primary ideal.*

For a proof, see [17], Theorem 14.3.

Corollary 2.5.2 *Let R be a Noetherian local ring and I an ideal of definition. Let $0 \neq M$ be a finite R-module. Then* $\dim \mathrm{gr}_I(M) = \dim M$*.*

Proof In this case $[gr_I(R)]_0$ is an Artinian local ring. Hence by Theorem 2.4.3, $H_{gr_I((M)}$ is of polynomial type of degree $\dim(gr_I(M)) - 1$. On the other hand, by the proof of Theorem 2.5.1 we know that

$$\Delta L_{I,M}(n) = H_{gr_I(M)}(n-1),$$

and so $L_{I,M}$ must be of polynomial type of degree $\dim(gr_I(M))$. But by the fundamental theorem of dimension theory, Theorem 2.5.1, $L_{I,M}$ is of polynomial type of degree $\dim M$. We conclude that $\dim M = \dim gr_I(M)$. ■

Let R and M be as in Theorem 2.5.1. Let $P_{I,M}$ be the Hilbert-Samuel polynomial of M with respect to I. If $0 \neq M$ and $d = \dim M$, then

$$P_{I,M}(t) = \sum_{j=0}^{d} (-1)^j e_j \binom{t+d-j+1}{d-j}$$

where $e_j \in \mathbb{Z}$ and $e_0 \neq 0$. In this case, one can write:

$$P_{I,M}(n) = \frac{e_0 X^d}{d!} + O(X^{d-1})$$

Definition 2.5.4 (Hilbert-Samuel mutiplicity) *Let $(R, \mathfrak{m})$ be a Noetherian semilocal ring and M a finite R-module. Let I be an R-ideal of definition for M. Define the* ***Hilbert-Samuel multiplicity of*** *M as*

$$e(I; M) := \begin{cases} e_0 \text{ if } M \neq 0; \\ \\ 0 \text{ otherwise.} \end{cases}$$

Remark 2.5.3 *Assume $0 \neq M$. Since $L_{I,M}(n) = P_{I,M}(n)$ for $n \gg 0$ note that:*

$$\frac{e_0 \cdot n^d}{d!} + O(n^{d-1}) = \lambda_{R/I}(M/I^n M)$$

In particular, multiplying by $d!/n^d$ gives

$$\frac{e_0 d!}{d!} + O(n^{-1}) = \frac{d! \lambda_R(M/I^n M)}{n^d}$$

and so, taking limits as $n \to \infty$ we get

$$e(I;M) = e_0 = \lim_{n\to\infty} e_0 + O(n^{-1}) = \lim_{n\to\infty} \frac{d!\lambda_R(M/I^nM)}{n^d}.$$

It is also easy to see that in this case

$$e(I;M) = \Delta^d L_{I,M}(n) \text{ for } n \gg 0.$$

If $I = \mathfrak{m}$ we write $e(\mathfrak{m};M) =: e(M)$ and if $M = R$ we write $e(I) := e(I;R)$.

3. ε-MULTIPLICITY

3.1 Some facts about length and local cohomology

One would like a notion of Hilbert-Samuel multiplicity in the case where I is not $\mathfrak{m}$-primary. The problem in this case is that the quotients R/I^n may not have finite length. A way to fix this is the section functor.

Proposition 3.1.1 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite R-module. $\mathfrak{m}^s M = 0$ for some $s \in \mathbb{N}$ if and only if $\lambda_R(M) < \infty$.*

Proof ($\Rightarrow$) Assume $\mathfrak{m}^s M = 0$ for some $s \in \mathbb{N}$. Consider the chain:

$$0 = \mathfrak{m}^s M \subseteq \mathfrak{m}^{s-1} M \subseteq \cdots \subsetneq \mathfrak{m} M \subseteq M$$

Note that

$$\lambda_R(M) = \lambda_R(M/\mathfrak{m}^s M) = \sum_{i=0}^{s-1} \lambda(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M) < \infty$$

since $\mathfrak{m}^i M/\mathfrak{m}^{i+1} M$ are a finite k-vector spaces and hence

$$\lambda_R(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M) = \dim_k(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M)$$

($\Leftarrow$) Assume $\mathfrak{m}^s M \neq 0$ for any $s \in \mathbb{N}$. By Nakayama's lemma, one obtains strict containments

$$\mathfrak{m}^s M \subsetneq \mathfrak{m}^{s-1} M \subsetneq \cdots \subsetneq \mathfrak{m} M \subsetneq M$$

For each s, this gives a chain of length $s+1$ and since s is arbitrary, $\lambda_R(M) = \infty$. ■

Definition 3.1.1 (Section functor) *Let R be a ring, I an R-ideal and M an R-module. The* ***section functor of M with respect to I, or 0-th local cohomology of M with respect to I*** *is*

$$\Gamma_I(M) = 0 :_M I^\infty = \bigcup_{j \geq 0} (0 :_M I^j) \simeq \bigcup_{j \geq 0} \operatorname{Hom}_R(R/I^j, M).$$

Remark 3.1.1 *Let R be a ring, I an R-ideal and S a graded R-algebra with $R = S_0$. Let $M = \bigoplus_{j\geq 0} M_j$ be a graded S-module. Then $\Gamma_I(M)$ is a graded S-submodule of M.*

Proposition 3.1.2 *Let $(R, \mathfrak{m}, k)$ be a local ring ring and M a finite R-module. Then*

(a) *$\Gamma_{\mathfrak{m}}(M)$ is the unique largest submodule of M with finite length;*

(b) *$\Gamma_{\mathfrak{m}}(M) = 0$ if and only if* $\operatorname{depth}(M) > 0$

(c) *Let $\overline{M} = M/\Gamma_{\mathfrak{m}}(M)$. Then $\Gamma_{\mathfrak{m}}(\overline{M}) = 0$, so* $\operatorname{depth}(\overline{M}) > 0$.

Proof Let N be a submodule of M with finite length. By Proposition 3.1.1 there is $s \in \mathbb{N}$ such that $\mathfrak{m}^s N = 0$, and so $N \subseteq \Gamma_{\mathfrak{m}}(M)$. On the other hand, since M is Noetherian, the chain

$$0 :_M \mathfrak{m} \subseteq 0 :_M \mathfrak{m}^2 \subseteq \cdots$$

must stabilize and so there is $s \in \mathbb{N}$ such that $\mathfrak{m}^s \Gamma_{\mathfrak{m}}(N) = 0$. In particular $\lambda_R(\Gamma_{\mathfrak{m}}(M)) < \infty$ by Proposition 3.1.1.

For (b), recall that if I is an R-ideal, then I contains a non-zero divisor of M (and so, $\operatorname{depth}_I(M) > 0$) if and only if $\operatorname{Hom}_R(R/I, M) = 0$. If $\Gamma_{\mathfrak{m}}(M) = 0$, then $0 :_M \mathfrak{m} = 0$. But $(0 :_M \mathfrak{m}) \simeq \operatorname{Hom}_R(k, M)$, hence $\operatorname{depth}_{\mathfrak{m}}(M) = \operatorname{depth}(M) > 0$. On the other hand, if $\operatorname{depth}(M) > 0$, then $\operatorname{depth}_{\mathfrak{m}^s}(M) > 0$ for $s \geq 1$, hence $0 :_M \mathfrak{m}^s = 0$. ∎

Lemma 3.1.2 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $I = (x_1, ..., x_n)$ a proper R-ideal. If $\lambda_R(M) < \infty$, then*

$$H^i_I(M) = H^i(\check{C}^{\bullet}(x_1, ..., x_n; M)) = 0 \quad \textit{for} \quad i > 0.$$

Proof Note that $M_{x_j} = 0$ for all j, hence $\check{C}^i(x_1, ..., x_n; M) = 0$ for $i > 0$. ∎

Proposition 3.1.3 (Subadditivity of $\lambda(\Gamma_{\mathfrak{m}}(-))$) *Let R be a Noetherian ring, I an R-ideal and consider an exact sequence of finite R-modules*

$$0 \to M' \to M \to M'' \to 0$$

(a) $\lambda_R(\Gamma_I(M)) \leq \lambda_R(\Gamma_I(M')) + \lambda_R(\Gamma_I(M''))$*;*

(b) Equality holds in (a) if $\lambda(M') < \infty$*.*

Proof Consider the longexact sequence

$$0 \to \Gamma_I(M)' \to \Gamma_I(M) \to \Gamma_{\mathfrak{m}}(M'') \to H^1_I(M') \to \cdots$$

and let $C = \mathrm{Coker}(\Gamma_I(M) \to \Gamma_{\mathfrak{m}}(M''))$. Then we have an exact sequence:

$$0 \to \Gamma_I(M)' \to \Gamma_I(M) \to \Gamma_{\mathfrak{m}}(M'') \to C \to 0$$

Using additivity of length, we have

$$\begin{aligned} \lambda_R(\Gamma_I(M)) &= \lambda_R(\Gamma_I(M')) + \lambda_R(\Gamma_I(M'')) - \lambda_R(C) \\ &\leq \lambda_R(\Gamma_I(M')) + \lambda_R(\Gamma_I(M'')) \end{aligned}$$

For (b) note that if $\lambda(M') < \infty$, then $H^i_I(M') = 0$ for $i > 0$ by Proposition 3.1.2. In particular, $H^i_I(M') = 0$ giving the short exact sequence

$$0 \to \Gamma_I(M)' \to \Gamma_I(M) \to \Gamma_{\mathfrak{m}}(M'') \to 0$$

and the result follows from additivity of λ. ■

3.2 j-multiplicity

In studying intersection theory, Achilles and Manaressi introduce a multiplicity associated to an R-ideal that is not $\mathfrak{m}$-primary, but has maximal analytic spread [1]. Today, this multiplicity is known as j-multiplicity, and it has been generalized to modules.

Let $R = \bigoplus_{n\geq 0} R_n$ a Noetherian standard graded ring with $(R_0, \mathfrak{m})$ a Noetherian local ring and $M = \bigoplus_{j=0}^{\infty} M_j$ a finite graded R-module. By Remark 3.1.1, $\Gamma_{\mathfrak{m}}(M)$ is a graded submodule of M.

By 3.1.2, $\Gamma_{\mathfrak{m}}(M)$ has finite length and so by Proposition 3.1.1 there is $s \in \mathbb{N}$ such that $\mathfrak{m}^s\Gamma_{\mathfrak{m}}(M) = 0$. In particular $\Gamma_{\mathfrak{m}}(M)$ is a finite graded $R/\mathfrak{m}^s R$-module. Note that $R/\mathfrak{m}^s R$ is a Noetherian standard graded ring, with $[R/\mathfrak{m}^s R]_0 = R_0/\mathfrak{m}^s$ an Artinian local ring.

With this, there is a well-defined Hilbert function for $\Gamma_{\mathfrak{m}}(M)$

$$H_{\Gamma_{\mathfrak{m}}(M)} : \mathbb{N}_0 \to \mathbb{N}_0$$

$$n \mapsto \lambda_{R_0}([\Gamma_{\mathfrak{m}}(M)]_n) = \lambda_{R_0}(\Gamma_{\mathfrak{m}}(M_n))$$

In this case, the Hilbert polynomial $P_{\Gamma_{\mathfrak{m}}(M)} \in \mathbb{Q}[x]$ has degree $\dim(\Gamma_{\mathfrak{m}}(M)) - 1$.

Definition 3.2.1 (j-multiplicity) *Let $R = \bigoplus_{n \geq 0} R_n$ a Noetherian standard graded ring with $(R_0, \mathfrak{m})$ a Noetherian local ring and $M = \bigoplus_{j=0}^{\infty} M_j$ a finite graded R-module. Let $d = \dim M$ and $\delta = \dim \Gamma_{\mathfrak{m}}(M)$. For $D \geq d$ define:*

$$j_D(M) = \begin{cases} 0 & \text{if } D > \delta; \\ \\ e(\Gamma_{\mathfrak{m}}(M)) & \text{if } D = \delta. \end{cases}$$

If $D = \dim M$, then we write $j(M) = j_D(M)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated d-dimensional R-module. Let I be an R-ideal (not necessarily primary) and $D \geq d$. Note that $\mathrm{gr}_I(R)$ is a standard graded ring with $R_0 = R/I$ Noetherian local, and $\mathrm{gr}_I(M)$ is a finite $\mathrm{gr}_I(R)$-module. Hence we are in the context of the graded j-multiplicity and so the following definition makes sense:

Definition 3.2.2 (j-multiplicity) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated d-dimensional R-module. Let I be an R-ideal and $D \geq d$. Then the* j***-multiplicity of*** I ***with respect to*** M *is*

$$j_D(I; M) := j_D(\mathrm{gr}_I(M))$$

where the latter j-multiplicity is computed considering $\mathrm{gr}_I(M)$ as an $\mathrm{gr}_I(R)$-module. We write $j(I; M) = j_d(I; M)$.

Lemma 3.2.1 *Let $R = \bigoplus_{n\geq 0} R_n$ a Noetherian standard graded ring with $(R_0, \mathfrak{m})$ a Noetherian local ring and $M = \bigoplus_{j=0}^{\infty} M_j$ a finite d-dimensional graded R-module. If $\mathfrak{q} \in Min(M) \cap V(\mathfrak{m}R)$, then $\Gamma_{\mathfrak{m}}(M)_{\mathfrak{q}} = N_{\mathfrak{q}}$.*

Proof If $\mathfrak{q} \in \mathrm{Min}_R(M)$, then $\dim M_{\mathfrak{q}} = 0$, or equivalently $R_{\mathfrak{q}}/\mathrm{ann}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ is an Artinian local ring and so there is k such that $\mathfrak{q}^k M_{\mathfrak{q}} = 0$. But then $\mathfrak{m}^k M_{\mathfrak{q}} \subseteq \mathfrak{q}^k M_{\mathfrak{q}} = 0$ and hence $M_{\mathfrak{q}} = \Gamma_{\mathfrak{m}}(M_{\mathfrak{q}}) = \Gamma_{\mathfrak{m}}(M)_{\mathfrak{q}}$. ■

Lemma 3.2.2 *Let $R = \bigoplus_{n\geq 0} R_n$ a Noetherian standard graded ring with $(R_0, \mathfrak{m})$ a Noetherian local ring and $M = \bigoplus_{j=0}^{\infty} M_j$ a finite d-dimensional graded R-module. Let $D > d$. The following are equivalent:*

(a) $\dim(M/\mathfrak{m}M) < D$;

(b) $\dim \Gamma_{\mathfrak{m}}(M) < D$.

Proof First notice that a power of $\mathfrak{m}R$ annihilates $\Gamma_{\mathfrak{m}}(M)$. In particular

$$\mathrm{Supp}_R(\Gamma_{\mathfrak{m}}(M)) \subseteq \mathrm{Supp}_R(M) \cap V(\mathfrak{m}R) = \mathrm{Supp}_R(M/\mathfrak{m}M)$$

and so $\dim \Gamma_{\mathfrak{m}}(M) \leq \dim M/\mathfrak{m}M$. This shows (a)$\Rightarrow$ (b).

(b)$\Rightarrow$ (a) Assume that $\dim M/\mathfrak{m}M = D$. Then there is $\mathfrak{q} \in \mathrm{Supp}_R(M)$ with $\mathfrak{m}R \subseteq \mathfrak{q}$ and $\dim R/\mathfrak{q} = D$. Since $D \geq \dim M$, such a $\mathfrak{q}$ is minimal in $\mathrm{Supp}_R(M)$ and so by Lemma 3.2.1, $\Gamma_{\mathfrak{m}}(M)_{\mathfrak{q}} = M_{\mathfrak{q}} \neq 0$. Hence

$$D = \dim(M_{\mathfrak{q}}/\mathfrak{m}M_{\mathfrak{q}}) \leq \dim M_{\mathfrak{q}} = \dim \Gamma_{\mathfrak{m}}(N)_{\mathfrak{q}} \leq \dim \Gamma_{\mathfrak{m}}(N) \leq D,$$

and so equality holds throughout. ■

Remark 3.2.3 *By Lemma 3.2.2 we have that*

$$j_D(M) = \begin{cases} 0 & \text{if } \dim(M/\mathfrak{m}M) < D; \\ \\ e(\Gamma_{\mathfrak{m}}(M)) & \text{if } \dim(M/\mathfrak{m}M) = D. \end{cases}$$

and that $j(M) \neq 0$ if and only if $\dim(M/\mathfrak{m}M) = \dim(M)$.

Remark 3.2.4 *Let $(R_0, \mathfrak{m})$ be an Artinian local ring and R a standard graded Noetherian ring. Let M be a finite d-dimensional graded R-module. Since R_0 is Artinian, there is $k \in \mathbb{N}$ such that $\mathfrak{m}^k = 0$ and so $\Gamma_{\mathfrak{m}}(M_n) = M_n$, $\Gamma_{\mathfrak{m}}(M) = M$ and $\delta = d$. If $d \geq 1$, then*

$$j_D(M) = \lim_{n\to\infty} \frac{(D-1)!\lambda(\Gamma_{\mathfrak{m}}(M))}{n^{D-1}} = \lim_{n\to\infty} \frac{(D-1)!\lambda(M_n)}{n^{D-1}} = e(M)$$

so the j multiplicity generalizes Hilbert's multiplicity.

Definition 3.2.3 (Internal grading, [25]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and S a standard graded R-algebra. Let I be an S-ideal generated by linear forms. The internal grading on $S[t, t^{-1}]$ is the grading obtained by setting $\deg t = 0$. Restricting this grading to the extended Rees algebra of I gives the* ***internal grading of the extended Rees algebra of*** *S.*

Note that $\mathscr{R}_I^+(S) \subseteq S[t, t^{-1}]$ are Noetherian standard graded rings with the internal grading. In fact, note that $S[t, t^{-1}]_0 = R[t, t^{-1}]$, hence

$$S[t, t^{-1}] = (R[t, t^{-1}])[S_1] = S[t, t^{-1}]_0[S]_1,$$

and since $\mathscr{R}_I^+(S)_0 = R[t^{-1}]$, we see that

$$\mathscr{R}_I^+(S) = R[t^{-1}][S_1, It] \subseteq \mathscr{R}_I^+(S)_0[\mathscr{R}_I^+(S)]_1.$$

Given a graded S-module M, the modules $\mathscr{R}_I^+(M)$ and $M[t, t^{-1}] = M \otimes_S S[t, t^{-1}] = M \otimes_R R[t, t^{-1}]$ are finite graded modules over $\mathscr{R}_I^+(S)$ and $S[t, t^{-1}]$ respectively. Factoring out the homogeneous element t^{-1} we see that $\mathrm{gr}_I(S)$ is a Noetherian standard graded ring and $[\mathrm{gr}_I(S)]_0 = R$. Also notice that $\mathrm{gr}_I(M)$ is a finitely generated $\mathrm{gr}_I(S)$-module. Furthermore

$$[\mathrm{gr}_I(M)]_n = \bigoplus_{i=0}^{\infty} \left[\frac{I^i M}{I^{i+1} M}\right]_n.$$

Note that only finitely many of these direct summands are non-zero since I is an ideal generated by linear forms.

Definition 3.2.4 (j-multiplicity of modules [25], Definition 4.1) *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $d = \dim R$, $E \subseteq F \simeq R^r$ be finite R-modules. Set $S = \mathrm{Sym}_R(F)$. Set $D = d + r$ and $I = E \cdot S$. Define the j-multiplicity of E as*

$$j(E) := j_D(F \cdot \mathrm{gr}_I(S))$$

where $F \cdot \mathrm{gr}_I(S)$ is a $\mathrm{gr}_I(S)$-module and $\mathrm{gr}_I(S)$ has the internal grading.

Define a function

$$\Sigma_E(n) = \lambda_R([\Gamma_{\mathfrak{m}}(F \cdot \mathrm{gr}_I(S))]_n) = \sum_{i=0}^{n-1} \lambda_R\left(\Gamma_{\mathfrak{m}}\left(\frac{E^i F^{n-i}}{E^{i+1}F^{n-i-1}}\right)\right).$$

Note that if $D = d + r > 0$, then

$$j_D(E) = \lim_{n\to\infty} \frac{(D-1)!\Sigma_E(n)}{n^{D-1}} = \lim_{n\to\infty} \frac{(d+r-1)!\Sigma_E(n)}{n^{d+r-1}} \in \mathbb{N}_0.$$

3.3 ε-multiplicity

Definition 3.3.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E \subseteq F \subseteq R^r$. Assume E and F have a rank. Define*

$$\Lambda_{E|F}(n) = \lambda_R\left(\Gamma_{\mathfrak{m}}\left({}^{F^n}/{}_{E^n}\right)\right).$$

Lemma 3.3.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $E \subseteq F = R^r$. Assume that E has a rank.*

(a) $\Lambda_{E|F}(n) \leq \Sigma_{E|F}(n)$;

(b) Equality holds in (a) if $\lambda(F/E) < \infty$.

Proof Fix n and consider the filtration

$$E^n \subseteq E^{n-1}F \subseteq E^{n-2}F^2 \subseteq \cdots \subseteq EF^{n-1} \subseteq F^n$$

Using induction and exact sequences of the form:

$$0 \to \frac{E^{i+1}F^{n-i-1}}{E^{i+2}F^{n-i-2}} \to \frac{E^i F^{n-i}}{E^{i+2}F^{n-i-2}} \to \frac{E^i F^{n-i}}{E^{i+1}F^{n-i-1}} \to 0$$

together with the subadditivity of $\lambda_R(\Gamma_{\mathfrak{m}}(-))$ (see Prop. 3.1.3), one obtains

$$\Gamma_{E|F}(n) = \lambda_R(\Gamma_{\mathfrak{m}}(F^n/E^n)) \leq \sum_{i=0}^{n-1} \lambda_R\left(\Gamma_{\mathfrak{m}}\left(\frac{E^iF^{n-i}}{E^{i+1}F^{n-i-1}}\right)\right) = \Sigma_{E|F}(n).$$

Furthermore, by Proposition 3.1.3, additivity holds if $\lambda(F/E) < \infty$ and so we recover equality. ∎

Lemma 3.3.2 *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let $E \subseteq F \subseteq R^r$ be R-modules having a rank. Let U be a submodule of E having a rank. Then*

$$\Lambda_{U|E}(n) \leq \Lambda_{U|F}(n) \leq \Lambda_{U|E}(n) + \Lambda_{E|F}(n),$$

and the second inequality is an equality if $\lambda_R(E/U) < \infty$.

Proof Consider the exact sequence

$$0 \to E^n/U^n \to F^n/U^n \to F^n/E^n \to 0$$

The result follows by subadditivity of $\lambda_R(\Gamma_{\mathfrak{m}}(-))$. Furthermore, in short exact sequences, additivity holds if the first module has finite length. In particular, if $\lambda_R(E/U) < \infty$, then equality holds. ∎

There are examples of modules for which the Γ function is not of polynomial type. We will provide examples in the next section. We will show that if the underlying ring has positive depth, then Γ is bounded above by a polynomial of degree depending only on the dimension of the ring and the rank of the module E. The following results are from [23]:

Theorem 3.3.3 (Kleiman,Ulrich,Validashti, [23]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $E \subseteq F \subseteq R^r$. Assume rank$(E) = e$ and rank$(F) = f$. If E is free and* $\operatorname{depth}(R) > 0$*, then $\Lambda_{E|F}(n)$ is bounded above by a polynomial of degree e.*

Proof (**Case** $e=0$) If $e=0$, since E is free, $E=0$. In this case $F^n/E^n = F^n \subseteq R^{rn}$. Note that $\Gamma_{\mathfrak{m}}(F^n)$ is a submodule of $\Gamma_{\mathfrak{m}}(R^{rn})$. Note that by Proposition 3.1.2(b) the latter is zero, given that $\operatorname{depth}(R) > 0$.

We may assume $e > 0$. Completing we may also assume $R = \widehat{R}$. By Cohen's Structure Theorem, there exists a complete local ring T with $R \simeq T/\mathfrak{a}$. Let $\mathbf{x}$ be a maximal T-regular sequence inside $\mathfrak{a}$ and write $S = T/(\mathbf{x})$. Then $S \twoheadrightarrow R$ and $(S, \mathfrak{n})$ is a complete local Gorenstein ring with $\dim S = \dim R = d$. Write $W = \operatorname{Ext}_S^{d-1}(R, S)$.

Since S is Gorenstein, S is Cohen-Macauley and $\omega_S \simeq S$ so$\dim W \leq d-(d-1) = 1$. Recall there is a fixed $s \in \mathbb{N}$ such that

$$\mathfrak{m}^s \Gamma_{\mathfrak{m}}(F \cdot \operatorname{gr}_I(A)) = 0.$$

The graded components of $\Gamma_{\mathfrak{m}}(F \cdot \operatorname{gr}_I(A))$ are of the form

$$\bigoplus_{i=0}^{n-1} \Gamma_{\mathfrak{m}}(E^i F^{n-i}/E^{i+1}F^{n-i-1}),$$

hence $\mathfrak{m}^s \Gamma_{\mathfrak{m}}(E^i F^{n-i}/E^{i+1}F^{n-i-1}) = 0$ for all n and all i. Now F^n/E^n has a filtration with n factors of the form $E^i F^{n-i}/E^{i+1}F^{n-i-1}$. Hence, by left-exactness of $\Gamma_{\mathfrak{m}}$, $\Gamma_{\mathfrak{m}}(F^n/E^n)$ has a filtration with n factors that are submodules of $\Gamma_{\mathfrak{m}}(E^i F^{n-i}/E^{i+1}F^{n-i-1})$, hence are annihilated by $\mathfrak{m}^s$. In particular for all $n \in \mathbb{N}$

$$\mathfrak{m}^{sn} \Gamma_{\mathfrak{m}}(F^n/E^n) = 0.$$

The exact sequence

$$0 \to E^n \to F^n \to F^n/E^n \to 0$$

induces an exact sequence

$$\operatorname{Ext}_S^{d-1}(E^n, S) \to \operatorname{Ext}_S^{d}(F^n/E^n, S) \to \operatorname{Ext}_S^{d}(F^n, S).$$

Recall that S is a complete local Gorenstein ring of dimension d, hence $\operatorname{Ext}_S^d(-,S) \simeq \Gamma_{\mathfrak{n}}(-)^+$, where $-^+ = \operatorname{Hom}_S(-, E_S(S/\mathfrak{n}))$ denotes the Matlis dual. Also, for R-modules $\Gamma_{\mathfrak{n}}(-) \simeq \Gamma_{\mathfrak{m}}(-)$. Hence

$$\operatorname{Ext}_S^d(F,S) \simeq \Gamma_{\mathfrak{m}}(F)^+.$$

Since F is R-torsionfree and $\operatorname{depth}(R) > 0$, $\Gamma_{\mathfrak{m}}(F) = 0$. Hence $\operatorname{Ext}_S^d(F,S) = 0$. Also $\operatorname{Ext}_S^d(F^n/E^n, S) \simeq \Gamma_{\mathfrak{m}}(F^n/E^n)^+$. With this we obtain

$$\lambda_R(\Gamma_{\mathfrak{m}}(F^n/E^n)) = \lambda_S(\Gamma_{\mathfrak{m}}(F^n/E^n)) = \lambda_S(\operatorname{Ext}_S^d(F^n/E^n,S)) = \lambda_R(\operatorname{Ext}_S^d(F^n/E^n,S)).$$

Now we have seen that

$$R/\mathfrak{m}^{sn} \otimes_R \operatorname{Ext}_S^{d-1}(E^n,S) \twoheadrightarrow \operatorname{Ext}_S^d(F^n/E^n,S),$$

where $\lambda_R(\operatorname{Ext}_S^d(F^n/E^n,S)) = \Lambda_{E|F}(n)$. Since $E \simeq R^e$, $\mathscr{R}(E) \simeq \operatorname{Sym}(E)$, hence $E^n \simeq S_n(E) \simeq R^k$, where $k = \binom{n+e-1}{e-1}$. In particular

$$\begin{aligned} R/\mathfrak{m}^{sn} \otimes_R \operatorname{Ext}_S^{d-1}(E^n,S) &\simeq (R/\mathfrak{m}^{sn} \otimes_R \operatorname{Ext}_R^{d-1}(R,S))^{\bigoplus \binom{n+e-1}{e-1}} \\ &= (W/\mathfrak{m}^{sn}W)^{\bigoplus \binom{n+e-1}{e-1}}. \end{aligned}$$

So $\lambda_R(R/\mathfrak{m}^{sn} \otimes_R \operatorname{Ext}_S^{d-1}(E^n,S)) = \binom{n+e-1}{e-1}\lambda_R(W/\mathfrak{m}^{sn}W)$.

Since $\dim_R W \leq 1$, $\lambda_R(W/\mathfrak{m}^{sn}W)$ is bounded by a linear polynomial. Hence $\Lambda_{E|F}(n)$ is bounded above by a polynomial of degree $(e-1)+1 = e$. ■

Theorem 3.3.4 (Kleiman, Ulrich, Validashti[23]) *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $\dim R = d$ and $\operatorname{depth}(R) > 0$. Let $E \subseteq F$ be finite modules of ranks e and f embedded in some free module. Then $\Lambda_{E|F}(n)$ is bounded above by a polynomial of degree $d+e-1$ (independent of f.)*

Proof Let $K = \operatorname{Quot}(R)$. Then $K \otimes_R E \simeq K^e$ has a K-basis $x_1, ..., x_e$ so that after dividing by a non-zero divisor in R

$$E \subseteq E' = \sum_{i=1}^{e} Rx_i \simeq R^e.$$

There exists a non-zero divisor a on R so that $aE' \subseteq E$. Let $F \hookrightarrow R^r$. In K^r we have R-submodules $E' \subseteq a^{-1}E \subseteq F' := a^{-1}F \hookrightarrow a^{-1}R^r \simeq R^r$. Thus $E' \subseteq F' \hookrightarrow R^r$, where $E' \simeq R^e$ and F' is a finite R-module having a rank, since $F' \otimes_R K = F \otimes_R K$. In particular by Lemema 3.3.2

$$\Lambda_{E|F}(n) \leq \Lambda_{E|F'}(n) \leq \Lambda_{E|E'}(n) + \Lambda_{E'|F'}(n)$$

for every n. Since E' is free of rank e, $\Lambda_{E|E'}(n)$ is bounded above by a polynomial of degree $d+e-1$ by Definition 3.2.4 and Lemma 3.3.1(a). Since $E' \simeq R^e$, $\Lambda_{E'|F'}(n)$ is bounded above by a polynomial of degree $e \leq d+e-1$ by Theorem 3.3.3. ■

Definition 3.3.2 (ε-multiplicity) *Let $(R, \mathfrak{m})$ be a Noetherian local ring with $\dim R = d$ and depth$(R) > 0$. Let $E \subseteq F$ be finite modules of ranks e and f embedded in some free module. Define the **ε-multiplicity** of $E \subseteq F$ as*

$$\varepsilon(E|F) = (d+e-1)! \limsup_{n\to\infty} \frac{\Lambda_{E|F}(n)}{n^{d+e-1}}.$$

If F is a fixed free module, we write $\varepsilon(E) = \varepsilon(E|F)$.

Remark 3.3.5 *Under the hypothesis of Definition3.3.2, notice that by Theorem 3.3.4, $\varepsilon(E|F) \in \mathbb{R}_+$*

Remark 3.3.6 *Cutkosky [5] has constructed examples of ideals $I \subseteq R$ where $\varepsilon(I) \notin \mathbb{Q}$, in more details if $R = \mathbb{C}[x_1, ..., x_4]$ and $\mathfrak{m} = (x_1, ..., x_4)$, Cutkosky has proved that there is a nonsingular projective curve $\mathscr{C} \subseteq \mathbb{P}^3_{\mathbb{C}}$ with defining R-ideal I and*

$$\lim_{n\to\infty} \frac{\lambda_R(\Gamma_{\mathfrak{m}}(R/I^n))}{n^4} \notin \mathbb{Q}$$

Theorem 3.3.7 (Cutkosky, [?], Theorem 3.2.) *Let $(R, \mathfrak{m})$ be an analytically unramified Noetherian local ring. Let $E \subseteq F$ be finite modules with E having a rank e and F free. Then*

$$\varepsilon(E|F) = \lim_{n\to\infty} \frac{(d+e-1)!\Lambda_{E|F}(n)}{n^{d+e-1}}.$$

Definition 3.3.3 (Fiber cone, analytic spread) *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. For a finite R-module E having a rank, one defines*

$$\mathscr{F}(E) := \frac{\mathscr{R}(E)}{\mathfrak{m}\mathscr{R}(E)} = \bigoplus_{n=0}^{\infty} \frac{E^i}{\mathfrak{m}E^i}.$$

The dimension of the fiber cone is called the ***analytic spread of*** *E and denoted by $\ell(E)$.*

Remark 3.3.8 *Whenever we have an inclusion of modules $E \subseteq F \simeq R^e$ and rank$(E) = e$, we have $\varepsilon(E) \leq j(E)$. This follows directly from Lemma 3.3.1.*

Proposition 3.3.1 (Ulrich, Validashti [24]) *Let $(R, \mathfrak{m})$ be a equidimensional, universally catenary local ring of dimension d, and $E \subsetneq F$ R-modules with $F \simeq R^e$. Assume E has a rank e. The following are equivalent:*

(a) $\varepsilon(E) > 0$;

(b) $\ell(E) = d + e - 1$, i.e., E has maximal analytic spread.

Proof See [24], Theorem 4.4 ∎

Remark 3.3.9 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. If I is an $\mathfrak{m}$-primary ideal, then*

$$e(I) = j(I) = \varepsilon(I)$$

Definition 3.3.4 *[24] Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $U \subseteq E$ be submodules of a free module F. Assume that $U_{\mathfrak{p}} = E_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Min}(R)$. Consider the inclusion of R-algebras*

$$\mathscr{R}(U) \subseteq \mathscr{R}(E) \subseteq \mathrm{Sym}(F).$$

We say that U is a reduction of E (or that E is integral over U) if $\mathscr{R}(U) \subseteq \mathscr{R}(E)$ is an integral extension of rings.

The following theorem, is an application of the ε-multiplicity, giving a criterion for the integral dependence of modules, based on the ε-multiplicity, just as Rees' theorem does in the $\mathfrak{m}$-primary case.

Theorem 3.3.10 (Ulrich, Validashti [24], Theorem 3.1.) *Let R be a locally equidimensional, universally catenary Noetherian ring. Let $U \subseteq E$ be submodules of a finite free module F and assume that both U and E have a rank e. The following are equivalent:*

(a) *E is integral over U;*

(b) $\varepsilon(U_{\mathfrak{p}}) = \varepsilon(E_{\mathfrak{p}})$ *for every* $\mathfrak{p} \in Spec(R)$*;*

(c) $\varepsilon(U_{\mathfrak{p}}) \leq \varepsilon(E_{\mathfrak{p}})$ *for every* $\mathfrak{p} \in \mathrm{Supp}_R(E/U)$ *with* $\ell(U_{\mathfrak{p}}) = \dim R_{\mathfrak{p}} + e - 1$.

The previous theorem motivates the need for computational tools of ε multiplicity.

4. ε-MULTIPLICITY AND THE SATURATION REES ALGEBRA

4.1 Background

When studying the ε-multiplicity of modules, one graded algebra of interest appears. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $E \subseteq F = R^r$ be R-modules with a rank. Let $S = \operatorname{Sym}(F)$. Note that

$$\Lambda_{E|F}(n) = \lambda_R(\Gamma_{\mathfrak{m}}(F^n/E^n)) = \lambda_R\left(\frac{E^n :_{F^n} \mathfrak{m}^\infty}{E^n}\right) = \lambda_R\left(\frac{E^n :_{S_n} \mathfrak{m}^\infty}{E^n}\right)$$

Definition 4.1.1 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and $E \subseteq F = R^r$ be R-modules with a rank. Let $S = Sym(F)$. The R-algebra*

$$\mathscr{R}_{sat}(E) = \mathscr{R}(E) :_S \mathfrak{m}^\infty = \bigoplus_{i=0}^{\infty} E^n :_{S_n} \mathfrak{m}^\infty \subseteq S$$

*is called the **saturation Rees algebra of** E.*

We show that the Noetherianness of this algebra implies the rationality of $\varepsilon(E)$. As a matter of fact, in this case we can express the epsilon multiplicity as the Hilbert multiplicity of some graded module. To prove this, we need a criterion for the Noetherianness of certain algebras.

Definition 4.1.2 (Veronese subalgebra) *Let S be a Noetherian ring and $\mathscr{A} = \bigoplus_{n=0}^{\infty} A_n$ a graded S-algebra. The S-subalgebra $\bigoplus_{n=0}^{\infty} A_{rn}$ is called the r**-th Veronese subalgebra** of $\mathscr{A}$ and denoted $\mathscr{A}^{(r)}$.*

Theorem 4.1.1 *Let S be a Noetherian ring and $\mathscr{A} = \bigoplus_{i=0}^{\infty} A_i$ be a positively graded S-algebra with $A_0 = S$. If $\mathscr{A}$ is a Noetherian S-algebra; then there exists $r \in \mathbb{N}$ such that the r-th Veronese subalgebra of $\mathscr{A}$ is standard graded, i.e. $\mathscr{A}^{(r)} = S[A_r]$.*

Proof Assume $\mathscr{A}$ is Noetherian, in which case $\mathscr{A}$ is finitely generated. Write $\mathscr{A} = S[f_1, ..., f_n]$, where $f_i \in A_{e_i}$, $1 \leq i \leq n$. Let $e = \mathrm{lcm}(e_1, ..., e_n)$. Now set $g_i = f_i^{e/e_i}$. Note that $g_i \in A_e$, hence the S-subalgebra of $\mathscr{A}$, $B := S[g_1, ..., g_n]$ is generated over S by homogeneous elements of degree e, in particular $B^{(e)}$ is standard graded. Now note that f_i is integral over B, $1 \leq i \leq n$, given that they are a root of the polynomial $t^{e/e_i} - g_i \in B[t]$. In particular, A is a finitely generated B-module, and so is the A-submodule $A^{(e)}$. Write

$$A^{(e)} = \sum_{j=0}^{\ell} BA_{je}$$

and define $r = e\ell$. We will prove that $A^{(r)}$ is standard graded, i.e., $A_{ri} = A_r^i$ for $i \geq 0$. Proceed by induction on i. If $i = 0$, then $A_0 = R = A_r^0$. Assume $i \geq 1$. Since $A_r^i \subseteq A_{ri}$ for all i, it is enough to show that $A_{ri} \subseteq A_r^i$.

We proof that $A_{es} = B_e(s - \ell)A_{e\ell}$ for $s \leq \ell$. To do this, note that

$$A^{(e)} = \sum_{j=0}^{\ell} BA_{ej}$$

so for any $s \geq \ell$ we have

$$A_{es} = \sum_{j=0}^{\ell} B_{e(s-j)}A_{ej}$$

Note that since $B^{(e)}$ is standard graded, $B_{ej} = B_{e(j-1)}B_e \subseteq B_{e(j-1)}A_e$ for any $j \geq 1$. In particular we have a filtration

$$B_{es}A_0 \subseteq B_{e(s-1)}A_e \subseteq \cdots \subseteq B_{e(s-\ell)}A_{e\ell},$$

so $A_{es} = B_{e(s-\ell)}A_{e\ell}$.

Now by the claim and the induction hypothesis:

$$A_{ir} = A_{e\ell i} = B_{e\ell(i-1)}A_{e\ell} \subseteq A_{r(i-1)}A_r = A_r^i$$

■

Remark 4.1.2 *In the context of theorem 4.1.1, if $\mathscr{A} = S[A_1, ..., A_d]$, and $r = d \cdot d!$, then $\mathscr{A}^{(r)} = S[A_r]$. For Example see the proof of Lemma 5.2. in [10].*

Proposition 4.1.1 *In addition to the assumptions of Theorem 4.1.1, assume there is $x \in A_1$ which is a non-zero divisor on $\mathscr{A}$. The following are equivalent:*

(a) $\mathscr{A}$ is a Noetherian S-algebra;

(b) there is $r \in \mathbb{N}$ such that $\mathscr{A}^{(r)}$ is astandard graded Noetherian S-algebra;

(c) there is $r \in \mathbb{N}$ such that $\mathscr{A}^{(r)}$ is finitely generated a Noetherian S-algebra.

Proof We already proved (a)$\Rightarrow$ (b), and (b)$\Rightarrow$ (c) is trivial. We prove that (c)$\Rightarrow$ (a). For $0 \leq j \leq r-1$ define

$$\mathscr{A}^{(r;j)} := \bigoplus_{i=0}^{\infty} A_{ri+j}$$

which is an $\mathscr{A}^{(r)}$-submodule of $\mathscr{A}$. Note that $\mathscr{A} = \bigoplus_{j=0}^{r-1} \mathscr{A}^{(r;j)}$, so it is enough to show that each $\mathscr{A}^{(r;j)}$ is finitely generated as a module over $\mathscr{A}^{(r)}$. Since $x \in A_1$ is a non-zero divisor, so is $x^{r-j} \in A_{r-j}$. Note that

$$x^{r-j}\mathscr{A}^{(r;j)} \subseteq \bigoplus_{i\geq 0} A_{r-j}A_{ri+j} \subseteq \bigoplus_{i\geq 0} A_{r(i+1)} \subseteq \mathscr{A}^{(r)}.$$

Since x^{r-j} is a non-zero divisor, the $\mathscr{A}^{(r)}$-linear map $\mu_x : \mathscr{A}^{(r;j)} \to \mathscr{A}^{(r)}$ is injective, hence $\mathscr{A}^{(r;j)}$ must be isomorphic to an ideal of $\mathscr{A}^{(r)}$. Since $\mathscr{A}^{(r)}$ is finitely generated as an S-algebra and S is Noetherian, $\mathscr{A}^{(r)}$ is a Noetherian ring. Therefore $\mathscr{A}^{(r;j)}$ must be finitely generated as a module over $\mathscr{A}^{(r)}$. But since $\mathscr{A}$ is a finite direct sum of the modules $\mathscr{A}^{(r)}$, $\mathscr{A}$ must be finitely generated as a module over $\mathscr{A}^{(r)}$. In particular, $\mathscr{A}$ is a Noetherian ring. ■

Theorem 4.1.3 (Simis, Ulrich, Vasconcelos, [22], Proposition 3.2) *Let R be a Noetherian local ring and let $A \subseteq B$ be a homogeneous inclusion of standard graded Noetherian R-algebras and $R = A_0 = B_0$ with $\lambda_R(B_1/A_1) < \infty$. Write $d = \dim B$ and $\mathfrak{m}$ for the maximal ideal of R. Let $G = \mathrm{gr}_{A_1B}(B)$, with internal grading.*

(a) For $n \geq 0$, $\lambda_R(B_n/A_n) = \lambda([B_1G]_n)$;

(b) For $n \gg 0$, $\lambda(B_n/A_n)$ is of polynomial type of degree $\dim B_1G - 1 \leq d-1$*;*

(c) If $P_{A|B}(X) \in \mathbb{Q}[X]$ is the polynomial associated to $\lambda(B_n/A_n)$, then:

$$P_{A|B}(X) = \frac{e(A|B)}{(d-1)!}\ X^{d-1} + O(X^{d-2})$$

where

$$e(A|B) = \begin{cases} 0 & \textit{if } \dim(B_1G) < d; \\ \\ e(B_1G) & \textit{if } \dim(B_1G) = d \end{cases}$$

.

Proof Let $G = \mathrm{gr}_{A_1B}(B)$. Note that

$$G_n = \bigoplus_{i=0}^{\infty} \left[\frac{A_1^iB}{A_1^{i+1}B}\right]_n .$$

We consider the module B_1G. With this grading

$$[B_1G]_n = \bigoplus_{i=1}^{n} \frac{A_{i-1}B_{n-i+1}}{A_iB_{n-i}},$$

in particular one has

$$[0 :_G B_1G]_0 = [0 :_R B_1G] = \mathrm{ann}_R(B_1/A_1).$$

Note that $R/\mathrm{ann}(B_1/A_1)$ is an Artinian ring, and so we are in the context where B_1G is a finitely generated $G/(0 :_G B_1G)$-module, the latter being a standard graded ring with 0-th graded piece an Artinian local ring.

Recall that $[B_1G]_n = \bigoplus_{i=1}^{n} \frac{A_{i-1}B_{n-i+1}}{A_iB_{n-i}}$. For $i \geq 1$ and all j we have $A_iB_j = A_1^iB_j = A_1^{i-1}(A_1B_j) \subseteq A_{i-1}B_{j+1}$, so for $n \geq 0$ we have a filtration

$$A_n = A_nB_0 \subseteq A_{n-1}B_1 \subseteq \cdots \subseteq A_0B_n = B_n.$$

By additivity of length we see that:

$$\lambda_R\left(\frac{B_n}{A_n}\right) = \sum_{i=1}^{n} \lambda_R\left(\frac{A_{i-1}B_{n-i+1}}{A_iB_{n-i}}\right) = \lambda_R([B_1G]_n)$$

Statements (b) and (c) follow from the theory of Hilbert functions for graded modules over a standard graded ring with Artinian local zero-th graded component. ■

4.2 Rationality of the ε-multiplicity and the Noetherianness of the saturated Rees algebra

From now on, when writing $\lim_{n\to\infty} f(n) \in \mathbb{Q}$, we mean the limit exists and is rational.

Theorem 4.2.1 *Let $(R, \mathfrak{m}, k)$ be an analytically unramified Noetherian local ring of dimension $d > 0$. Consider $E \subseteq F := R^t$ an R-module with rank e. If $\mathscr{R}_{sat}(E)$ is an Noetherian R-algebra, then*

$$\varepsilon(E) = \lim_{n\to\infty} \frac{(d+e-1)!\Lambda_{E|F}(n)}{n^{d+e-1}} \in \mathbb{Q}.$$

Proof Rewrite $\Lambda_{E|F}(n)$ as follows:

$$\Lambda_{E|F}(n) = \lambda_R(\Gamma_{\mathfrak{m}}(F^n/E^n)) = \lambda_R\left(0 :_{F^n/E^n} \mathfrak{m}^\infty\right) = \lambda_R\left(\frac{E^n :_{F^n} \mathfrak{m}^\infty}{E^n}\right)$$

By Theorem 4.1.1, given that the R-algebra $\mathscr{R}_{\text{sat}}(E)$ is Noetherian, there is $r \in \mathbb{N}$ such that we know there is $r \in \mathbb{N}$ such that

$$\mathscr{R}_{\text{sat}}(E)^{(r)} = \bigoplus_{n=0}^{\infty} E^{rn} :_{F^{rn}} \mathfrak{m}^\infty$$

is standard graded, i.e., $E^{rn} :_{F^{rn}} \mathfrak{m}^\infty = (E^r :_{F^r} \mathfrak{m}^\infty)^n$ for $n \geq 0$ (not that this is not the n-th power of the module $E^r :_{F^r} \mathfrak{m}^\infty$, but the power inside S). Consider the R-standard graded algebras

$$A := R \oplus E^r \oplus E^{2r} \oplus \cdots = \mathscr{R}(E)^{(r)} \subseteq \mathscr{R}_{\text{sat}}(E)^{(r)} =: B.$$

We have an inclusion of standard graded R-algebras $A \subseteq B$, $A_0 = B_0 = R$ and:

$$\lambda_R(B_1/A_1) = \lambda_R(E^r :_{F^r} \mathfrak{m}^\infty/E^r) = \lambda_R(\Gamma_{\mathfrak{m}}(F^r/E^r)) < \infty$$

Now, we can apply Theorem 4.1.3 to the R-algebras $A \subseteq B$ to get that the numerical function

$$\Lambda_{E|F}(nr) = \lambda_R\left(\frac{E^{nr} :_{F^n} \mathfrak{m}^\infty}{E^{nr}}\right) = \lambda_R\left(\frac{A_n}{B_n}\right)$$

is of polynomial type of degree $\dim B_1 G - 1 := \delta$ where $G = \mathrm{gr}_{A_1 B}(B)$, i.e., there is a polynomial $P(X) \in \mathbb{Q}[X]$ with

$$P(X) = \frac{e(A|B)}{\delta!}X^\delta + O(X^{\delta-1})$$

with $P(n) = \Lambda_{E|F}(nr)$ for $n \gg 0$ and $e(A|B) \in \mathbb{Q}$. Since R is analytically unramified, by Theorem 3.3.7 the ε mutliplicity of E exists as a limit. In particular, any subsequence of $(d+e-1)!\Lambda_{E|F}(n)/n^{d+e-1}$ has as limit $\varepsilon(E)$. Putting this together gives:

$$\varepsilon(E|F) = \lim_{n\to\infty} \frac{(d+e-1)!\Lambda_{E|F}(nr)}{(nr)^{d+e-1}} = \lim_{n\to\infty} \frac{(d+e-1)!P(n)}{(nr)^{d+e-1}}$$

By Cutkosky's theorem, this limit exists and is finite hence $\delta \leq d+e-1$. If $\delta < d+e-1$, then $\varepsilon(E) = 0$. If $\delta = d+e-1$, then:

$$\lim_{n\to\infty} \frac{(d+e-1)!P(n)}{(nr)^{d+e-1}} = \frac{e(A|B)}{r^{d+e-1}} \in \frac{\mathbb{N}_0}{r^{d+e-1}}$$

■

The hypothesis that $\mathscr{R}_{\mathrm{sat}}(E)$ is Noetherian is not always satisfied. For example if $E = I \subseteq F = R$ for an R-ideal I this reduces to the Noetherianess of the saturated Rees algebra. As we mentioned in Remark 3.3.6, [5], Cutkosky has produced an example of an ideal $I \subseteq R$, where $\varepsilon(I)$ is not rational. By the previous result, the associated saturation Rees algebra cannot be finitely generated.

Other examples of ideals for which the saturation Rees algebra may not be Noetherian include monomial curves in $\mathbb{A}^3$. We will discuss more of this case in Chapter 5. Now we explore a case that we will generalize in the following section.

Definition 4.2.1 (Saturation Rees algebra) *Let S be a Noetherian ring, I and J S-ideals. The **saturation Rees algebra of I with respect to J** is*

$$\mathscr{R}_{sat}(I|J) := \bigoplus_{n\geq 0}(I^n :_S J^\infty)\tau^n \subseteq S[\tau].$$

The following is a result from Herzog, Hibi and Trung:

Theorem 4.2.2 (Herzog, Hibi, Trung[12], Theorem 3.2) *Let $R = k[x_1, ..., x_d]$ be a polynomial ring and I, J monomial ideals. Then $\mathscr{R}_{sat}(I, J)$ is Noetherian.*

Corollary 4.2.3 *Let $R = k[x_1, ..., x_d]$ be a polynomial ring and I a monomial R-ideal. Then $\varepsilon(I) \in \mathbb{Q}$.*

Corollary 4.2.3 has been proved using different techniques. For example, Herzog, Puthenpurakal and Verma have studied the nature of the funcion $\Lambda_I(n)$ for monomial ideals.

Definition 4.2.2 (Quasipolynomial type) *Let $F : \mathbb{N} \to \mathbb{N}$. We say that F has quasipolynomial type if there are polynomials $P_0, ..., P_{g-1} \in \mathbb{Q}[X]$ such that for $n \gg 0$:*

$$F(n) = P_i(n) \qquad \text{if } n = i (mod\ g)$$

The following result by Herzog, Puthenpurakal and Verma [13] is proved with much more generality than the following version:

Proposition 4.2.1 ([13]) *Let $R = k[x_1, ..., x_d]$ and I a monomial R-ideal.*

(a) The numerical function $\lambda_R((I^n :_R \mathfrak{m}^\infty)/I^n)$ is of polynomial type.

(a) Let $P_0, ..., P_{g-1}$ be the polynomials with $\Lambda_I(ng + i) = P_i(n)$ for $n \gg 0$. Then all these polynomials have the same degree and same leading coefficient.

This shows that $\varepsilon(I)$ exists and it is rational, being the normalized leading coefficient of any P_i, $0 \leq i \leq g - 1$. But one can say even more about the ε-multiplicity of monomial ideals. In fact, Jeffries and Montaño have explicit formulas for what these multiplicites should be.

Theorem 4.2.4 (Jeffries,Montaño [16], Theorem 5.1.) *Let $R = k[x_1, ..., x_d]$ and I a monomial R-ideal. Then*

$$\varepsilon(I) = d!\mathrm{vol}(\mathrm{out}(I))$$

which is a rational number.

4.3 The saturation algebra of monomial modules

Definition 4.3.1 (Monomial module) *Let k be a field and $R = k[x_1, ..., x_d]$ a polynomial ring in d-variables. Let F be a finite free R-module with basis $\{e_i\}_{i=1}^m$.*

*(a) A **monomial** in F is an element of the form $x_1^{\alpha_1} \cdots x_d^{\alpha_d} e_i$, $1 \leq i \leq m$;*

*(b) A **monomial module** is an R-submodule of F generated by monomials.*

Let E be a monomial submodule of F. Notice that E has a rank. Consider the inclusion $i : E \hookrightarrow F$ to define the ε-multiplicity of E. We show that the saturation Rees algebra of E with respect to any monomial ideal is Noetherian.

Remark 4.3.1 *Let R be a Noetherian ring and $E \subseteq F$ finite R-modules with a rank and $F = R^r$. Let $S = Sym(F)$ and $I = E \cdot S$. Then $E^n = [I^n]_n$.*

Theorem 4.3.2 *Let k be a field and $R = k[x_1, ..., x_d]$ a polynomial ring. Let F be a finite free R-module and $E \subseteq F$ a monomial module. Let $S = \mathrm{Sym}(F)$. Given a monomial R-ideal J, the R-algebra:*

$$\mathscr{R}_{sat}(E, J) := \mathscr{R}(E) :_S J^\infty = \bigoplus_{n \geq 0} E^n :_{F^n} J^\infty \subseteq S$$

is Noetherian.

Proof Let $\{e_i\}_{i=1}^m$ be a basis for F. Write $S = R[t_1, ..., t_m]$. Consider monomial generators for E, say $E = \sum_{j=1}^e Rm_j$, with $m_j = \mathbf{x}^{\alpha_j} e_{i_j}$. Define the S-ideal $I := E \cdot S$. Let $\mu_j = \mathrm{Sym}(i)(m_j) = \mathbf{x}^{\alpha_j} t^{i_j}$. Note that $I = (\mu_1, ..., \mu_e)$ is a monomial S-ideal. Hence, by Theorem 4.2.2, the saturation Rees algebra

$$\mathscr{R}_{\mathrm{sat}}(I, J) = \bigoplus_{n=0}^{\infty} (I^n :_S J^\infty) \tau^n \subseteq S[\tau]$$

is Noetherian.

Now note that $\mathscr{R}_{\text{sat}}(E,J) = \bigoplus_{n=0}^{\infty}(E^n :_{F^n} J^\infty) = \bigoplus_{n=0}^{\infty}[I^n :_S J^\infty]_n$. In particular

$$\begin{aligned}\mathscr{R}_{\text{sat}}(I,J) &= \bigoplus_{n=0}^{\infty}(I^n :_S J^\infty)\tau^n = \bigoplus_{n=0}^{\infty}\bigoplus_{j=0}^{\infty}[I^n :_S J^\infty]_j\tau^n \\ &= \mathscr{R}_{\text{sat}}(E,J) \oplus \bigoplus_{n=0}^{\infty}\bigoplus_{j\neq n}[I^n :_S J^\infty]_j\tau^n,\end{aligned}$$

thus $\mathscr{R}_{\text{sat}}(E,J)$ is a direct summand of $\mathscr{R}_{\text{sat}}(I,J)$ as a module over $\mathscr{R}_{\text{sat}}(E,J)$. It follows that $\mathscr{R}_{\text{sat}}(E,J)$ is Noetherian. ■

Corollary 4.3.3 *Let k be a field and $R = k[x_1,...,x_d]$ a polynomial ring. Let F be a finite free R-module and $E \subseteq F$ a monomial module. Then $\varepsilon(E) \in \mathbb{Q}$.*

Proof By Theorem 4.3.2, $\mathscr{R}_{\text{sat}}(E) = \mathscr{R}_{\text{sat}}(E,\mathfrak{m})$ is Noetherian. Hence by Theorem 4.2.1, it follows that $\varepsilon(E)$ is rational. ■

5. ε-MULTIPLICITY OF SOME MONOMIAL CURVES

5.1 ε-multiplicity of monomial curves in $\mathbb{A}^3$.

Definition 5.1.1 (Monomial curve in $\mathbb{A}^3$) *Let k be a field and consider the k-algebra map*

$$\varphi : k[[X,Y,Z]] \to k[[t]]$$

$$X \mapsto t^\ell, Y \mapsto t^m, Z \mapsto t^n$$

*where $\gcd(\ell,m,n) = 1$. The algebra $im(\varphi)$ is called the **coordinate ring of the monomial curve of** ℓ, m, n. The image of the map $\lambda \mapsto (\lambda^\ell, \lambda^m, \lambda^n)$ is called a monomial curve of $\mathbb{A}^3(k)$. The kernel of this map is called the **defining ideal associated to the monomial curve**.*

Remark 5.1.1 *If $\mathfrak{p}$ is the defining ideal of a monomial curve in $\mathbb{A}^3(k)$, then $\mathfrak{p}$ is a perfect ideal of height 2. As a matter of fact, Herzog proved in [11], Proposition 3.3, that there are $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{N}$ such that*

$$\mathfrak{p} := \ker(\varphi) = I_2\left(\begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix}\right).$$

It is also easy to see that if $\mathfrak{p}$ is the ideal of maximal minors of a matrix as in Remark 5.1.1, and $\mathfrak{p} = \mathfrak{p}(\ell, m, n)$, then

$$\begin{cases} \ell = \beta\gamma + \beta\gamma' + \beta'\gamma' \\ m = \gamma\alpha + \gamma\alpha' + \gamma'\alpha \\ n = \alpha\beta' + \alpha'\beta + \alpha'\beta' \end{cases}$$

Not every matrix of the form

$$M = \begin{bmatrix} X^\alpha & Y^{\beta'} & X^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix}$$

will have maximal minors corresponding to the defining ideal of a monomial space curve. For example, in [8], Lemma 2.1. it is shown that starting with a matrix of this form, if $\mathfrak{p} = \mathfrak{p}(\ell, m, n)$ for some relatively prime integers, then $\alpha \neq 2\alpha'$, $2\beta \neq \beta'$ or $2\gamma \neq \gamma'$.

Definition 5.1.2 *Let R be a Noetherian ring, and I an R-ideal. Recall that the n-th symbolic power of I is defined as*

$$I^{(n)} := \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} I^n R_{\mathfrak{p}} \cap R$$

It is clear that $I^n \subseteq I^{(n)}$ for $n \geq 0$. If $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.

For an R-ideal I, define

$$\mathscr{A}(I) = \bigcup_{n \geq 1} \mathrm{Ass}_R(R/I^n)$$

Proposition 5.1.1 *Let R be a Noetherian ring and I an R-ideal with no embedded primes. Let $\Omega \subseteq V(I)$ be a set such that*

(a) Ω is finite;

(b) $\mathrm{Min}(I) \cap \Omega = \varnothing$;

(c) If $\mathfrak{q} \in \mathscr{A}(I)$ is not minimal in $\mathscr{A}(I)$, then $\mathfrak{q} \in \Omega$.

Then

$$I^{(n)} = I^n :_R \left(\bigcap_{\mathfrak{p} \in \Omega} \mathfrak{p} \right)^{\infty}.$$

For a proof, see [10], Lemma 1.30. In particular, note that if $(R, \mathfrak{m})$ is a Noetherian local d-dimensional ring and $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\mathrm{ht}(\mathfrak{p}) = d - 1$, then one can take $\Omega = \{\mathfrak{m}\}$. Clearly Ω is finite, it does not contain any minimal prime of I, since $\mathrm{Min}(\mathfrak{p}) = \{\mathfrak{p}\}$. Finally note that $\mathfrak{m}$ is the only potential embedded prime of any power of $\mathfrak{p}$. With this one obtains:

Corollary 5.1.2 *Let $(R,\mathfrak{m})$ be a d-dimensional Noetherian local ring and $\mathfrak{p} \in Spec(R)$ of height $d-1$. Then*

$$\Gamma_{\mathfrak{m}}(R/\mathfrak{p}^n) \simeq \mathfrak{p}^{(n)}/\mathfrak{p}^n.$$

Definition 5.1.3 *Let $(R,\mathfrak{m})$ be a Noetherian local ring, $\mathfrak{p}$ a prime ideal. The* ***symbolic Rees algebra of*** $\mathfrak{p}$ *is the graded R-algebra*

$$\mathscr{R}_s(\mathfrak{p}) = \bigoplus_{n\geq 0} \mathfrak{p}^{(n)} t^n \subseteq R[t]$$

Let $\mathfrak{p}$ be the defining ideal of a monomial curve. By Theorem 4.2.1, if the symbolic Rees algebra of $\mathfrak{p}$ is Noetherian, then $\varepsilon(\mathfrak{p}) \in \mathbb{Q}$, but there are examples of families of monomial curves in $\mathbb{A}^3(k)$ where the symbolic Rees algebra is not Noetherian, due to the work of Goto, Nishida and Watanabe:

Remark 5.1.3 *Let k be a field of characteristic 0, and for $n \geq 4$, an integer not divisible by 3, consider the monomial curves in $\mathbb{A}^3(k)$ with defining ideals*

$$\mathfrak{p} = \mathfrak{p}(7n-3, (5n-2)n, 8n-3).$$

These ideals do not have Noetherian symbolic Rees algebras as proved in [9], Corollary 1.2.

The j-multiplicity for monomial curves in $\mathbb{A}^3(k)$ has been completely described by Nishida and Ulrich in [19], Example 4.5.

Theorem 5.1.4 (Nishida, Ulrich[19], Ex. 4.5.) *Let k be an infinite field, $\mathfrak{p} = \mathfrak{p}(\ell, m, n)$ be the defining ideal of a monomial curve in $\mathbb{A}^3(k)$. If we write*

$$\mathfrak{p} := \ker(\varphi) = I_2\left(\begin{bmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{bmatrix}\right)$$

By replacing the variables X, Y and Z suitably, one may assume

$$\ell\alpha = \min\{\ell\alpha, n\beta, n\gamma, \ell\alpha', m\beta', n\gamma'\}.$$

Then $j(\mathfrak{p}) = \alpha\beta(\gamma+\gamma')$.

An interesting question to ask is whether there is an analoguous formula for the ε multiplicity in this class of ideals. Since the Noetherian property of the symbolic Rees algebra of $\mathfrak{p}$ guarantees the rationality of the ε-multiplicity, a more approachable question would be: given a monomial curve with defining ideal $\mathfrak{p}$ such that $\mathscr{R}_{\text{sat}}(\mathfrak{p})$ is Noetherian, is there a formula depending on $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ describing the ε-multiplicity?

We have studied numerically a particular case of this problem. Herzog and Ulrich have characterized the monomial curves in $\mathbb{A}^3(k)$ for which $\mathscr{R}_{sat}(\mathfrak{p}) = R[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$ (see [14], Corollary 2.12.).

Theorem 5.1.5 (Herzog, Ulrich, [14] Corollary 2.12, [8], Corollary 4.3.) *Let R be a regular local ring of dimension 3, X, Y and Z a regular system of parameters and*

$$\mathfrak{p} = I_2\left(\begin{bmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{bmatrix}\right)$$

a prime ideal of codimension one. Let M be the matrix defining $\mathfrak{p}$. After suitable permutations of the rows and columns of M, one may assume that either (i) $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\gamma \leq \gamma'$ or that (ii) $\alpha > \alpha', \beta < \beta'$ and $\gamma < \gamma'$. The following statements are equivalent:

(a) $\mathscr{R}_s(I) = R[\mathfrak{p}t, \mathfrak{p}^{(2)}t^2]$;

(b) We have the following conditions:

- *(a) the matrix M satisfies (i);*
- *(b) $\beta = \beta'$ or $\alpha = \alpha'$ and $\gamma = \gamma'$.*

In this case, $\mathscr{R}_s(I)$ is Gorenstein, and $(\mathfrak{p}^{(2)})^n = \mathfrak{p}^{(2n)}$ for $n \geq 1$.

A lot is known about the function $\Lambda_{\mathfrak{p}}$ in this case. We have the following behavior of the symbolic powers:

$$\mathfrak{p}^{2n} :_R \mathfrak{m}^{\infty} = \mathfrak{p}^{(2n)} = (\mathfrak{p}^{(2)})^n$$

$$\mathfrak{p}^{2n+1} :_R \mathfrak{m}^\infty = \mathfrak{p}^{(2n+1)} = \mathfrak{p}(\mathfrak{p}^{(2)})^n$$

for all $n \geq 0$. In particular the functions $\Lambda_{\mathfrak{p}}(2n)$ and $\Lambda_{\mathfrak{p}}(2n+1)$ are of polynomial type of degree 3 ([13], Proposition 5.5) Huneke has shown that the R-module $\mathfrak{p}^{(2)}/\mathfrak{p}^2$ is cyclic [15], and Schenzel has fully described its generator [20]. In this case, using Theorem 4.2.1, one can write

$$\varepsilon(\mathfrak{p}) = \frac{3!}{2^3} \lim_{n\to\infty} \frac{\lambda_R((\mathfrak{p}^{(2)})^n/\mathfrak{p}^{2n})}{n^3} = \frac{e(\mathfrak{p}^{(2)}|\mathfrak{p}^2)}{8}$$

where $e(\mathfrak{p}^{(2)}|\mathfrak{p}^2)$ denotes the relative multiplicity associated to the homogeneous inclusion of standard Noetherian graded R-algebras $\mathscr{R}(\mathfrak{p}^2) \subseteq \mathscr{R}(\mathfrak{p}^{(2)})$.

Since we know that $\Lambda_{\mathfrak{p}}(2n)$ is of polynomial type of degree 3, we can use this to generate numerical evidence on what the ε-multiplicity of these monomial curves should be. For example, consider the family ideals:

$$\mathfrak{p}(\gamma') = I_2\left(\begin{bmatrix} X & Y & Z^{\gamma'} \\ Y & Z & X \end{bmatrix}\right)$$

By declaring $\deg X = 2\gamma' + 1$, $\deg Y = \gamma' + 2$ and $\deg Z = 3$, we may assume that $\mathfrak{p}$ is homogeneous and we compute:

$$\lambda_R(\mathfrak{p}^2 :_R (X,Y,Z)^\infty)^n/\mathfrak{p}^{2n}) = \lambda_R(\mathfrak{p}^{(2n)}/\mathfrak{p}^{2n}) = \lambda(\Gamma_{(X,Y,Z)}(R/\mathfrak{p}^{2n}))$$

for $\gamma' = 2, ..., 10$ and use this to get a polynomial of degree 3. The results obtained are summarized in Table 5.1.

There are two interesting facts about this data. The first one is that the estimation for the ε multiplicity is:

$$\varepsilon(\mathfrak{p}(\gamma')) = \frac{3(2\gamma'+1)(\gamma'+2)}{2} = \frac{\deg X \deg Y \deg Z}{2}$$

The second one is that for this family

$$\lambda_{\mathfrak{p}}(2n) = \varepsilon(\mathfrak{p})\left(\frac{4}{3}n^3 + n^2 - \frac{n}{3}\right).$$

so there is a *generic* polynomial giving the ε-multiplicity of the family.

Table 5.1.: Prediction of the ε-multiplicity for the family $\mathfrak{p}(\gamma')$

γ'	(ℓ, m, n)	Predicted $\Lambda_{\mathfrak{p}}(2n)$	Predicted $\varepsilon(\mathfrak{p})$
2	$(5,4,3)$	$40n^3 + 30n - 10n$	30
3	$(7,5,3)$	$70n^3 + \frac{105}{2}n^2 - \frac{35}{2}n$	$\frac{105}{2}$
5	$(11,7,3)$	$154n^3 + \frac{231}{2}n^2 - \frac{77}{2}n$	$\frac{231}{2}$
6	$(13,8,3)$	$208n^3 + 156n^2 - 52n$	156
8	$(17,10,3)$	$340n^3 + 255n^2 - 85n$	255
9	$(19,11,3)$	$418n^3 + \frac{627}{2}n^2 - \frac{209}{2}n$	$\frac{627}{2}$

5.2 Some conjectures for relative multiplicity

A possible approach for the computation of the ε-multiplicity of monomial curves, following the ideas of Ulrich and Nishida for the j-multiplicity, is to reduce the dimension of the ring. Using the notation for relative multiplicity introduced in Chapter 4, given two R-ideals $J \subseteq I$ in a local ring with $\lambda_R(I/J) < \infty$, denote $e(J|I) = e(\mathscr{R}(J)|\mathscr{R}(I))$. This function is the normalized leading coefficient of the polynomial that behaves like the numerical function $\lambda_R(I^n/J^n)$.

Following the ideas of Ulrich and Nishida in [19], a first step in proving that the formula for ε-multiplicity of the family $\mathfrak{p}(\gamma')$ holds, is to show the following conjecture: let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, $J \subseteq I$ R-ideal such that $\lambda_R(I/J) < \infty$. Assume that $\operatorname{grade}(J) > 0$. Is it true that for $x \in J$ general

$$e(J/(x))|I/(x)) = e(J|I)?$$

Note that $\lambda_R(I^n/J^n)$ is of polynomial type of degree $d+1 = \dim \mathscr{R}(I)$, while $\lambda_R(I^n + (x)/(x)/J^n + (x)/(x))$ has degree $\dim(R/(x)) + 1 = d$.

In particular, if this holds one may assume that $\mathrm{ht}(I) = \mathrm{ht}(J) = 0$. We studied some numerical examples to see whether there is enough evidence to assume this result holds. Since $\lambda_R(I^n/J^n)$ is of polynomial type, we compute it for large values of n in Macauley2, and extract the information about the relative multiplicity. In $k[X, Y, Z]$ the experiment was run for over 90 monomial ideals with finite colength and the result was positive for all of them, i.e., relative multiplicity carried over after going module general elements.

Another interesting fact for this family is that the relative multiplicity was bounded above by the colength. It is not true in general that $e(I|J) \leq \lambda_R(I/J)$, even for the primary case. For example take $R = k[[X, Y]]$. For $n > 2$ set $I = (X^n, Y^n)$ and $J = (X^{n+1}, X^nY, Y^n)$. Note that for all of these ideals, $\lambda_R(I/J) = 1$, yet $e(J|I) = e(J) - e(I) = n(n + 1) - n^2 = n$. This shows that the relative multiplicity can be made arbitrarily large without changing the relative colength.

Finally, after analizing some of the data, in particular when the colength and multiplicity match, one sees a patern. This data appears in Table 5.2. It leads to the following conjecture:

Conjecture. Let R be a regular local ring of dimension d. Let $J \subseteq I$ be R-ideals with $\lambda_R(I/J) < \infty$. If I is a complete intersection of height $d - 1$ and J is an almost complete intersection, then $e(J|I) = \lambda_R(I/J)$.

Table 5.2.: Colength and relative multiplicity of some monomial ideals in $\mathbb{Q}[X,Y,Z]$

I	J	$\lambda_R(I^n/J^n), n \gg 0$	$e(J\|I)$	$\lambda_R(I/J)$
(X^2Z^4, Y^9)	(X^2Z^7, X^5Z^4, Y^9)	$\frac{27}{2}n^3 + \frac{81}{2}n^2 + 27n$	81	81
(XZ, Y^3)	(XZ^22, X^2Z, Y^3)	$\frac{1}{2}n^3 + \frac{3}{2}n^2 + n$	3	3
(XY^7, Z^9)	(Z^9, XY^8, X^2Y^7)	$\frac{3}{2}n^3 + \frac{9}{2}n^2 + 3n$	9	9
(X^3, Y^4)	(X^3Z, Y^4, X^4)	$\frac{2}{3}n^3 + 2n^2 + \frac{4}{3}n$	4	4
(XZ^2, Y^6)	(XZ^5, X^4Z^2, Y^6)	$9n^3 + 27n^2 + 18n$	54	54
(Y, Z^2)	(Z^2, Y^2, XY)	$\frac{1}{3}n^3 + n^2 + \frac{2}{3}n$	2	2
(Y^2Z, X^5)	(Y^2Z^3, Y^4Z, X^5)	$\frac{10}{3}n^3 + 10n^2 + \frac{20}{3}n$	20	20
(Y^5, Z^7)	(Z^7, Y^7, X^2Y^5)	$\frac{14}{3}n^3 + 14n^2 + \frac{28}{3}n$	28	28
(Z, Y^5)	(Z^5, X^4Z, Y^5)	$\frac{40}{3}n^3 + 40n^2 + \frac{80}{3}n$	80	80
(X^2Z^5, Y^8)	(X^2Z^6, X^3Z^5, Y^8)	$\frac{4}{3}n^3 + 4n^2 + \frac{8}{3}n$	8	8
(X^4Z, Y^5)	(X^3Z^2, X^4Z, Y^5)	$\frac{5}{6}n^3 + \frac{5}{2}n^2 + \frac{5}{3}n$	5	5
(Z, Y^3)	(Z^3, X^2Z, Y^3)	$2n^3 + 6n^2 + 4n$	12	12
(Y^4, Z^8)	(Z^8, Y^8, X^5Y^5)	$\frac{64}{3}n^3 + 64n^2 + \frac{128}{3}n$	128	128
(X^4Z^6, Y^{11})	(X^4Z^7, X^5Z^6, Y^{11})	$\frac{11}{6}n^3 + \frac{11}{2}n^2 + \frac{11}{3}n$	11	11

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