TOPICS ON THE COHEN-MACAULAY PROPERTY OF REES ALGEBRAS

AND

THE GORENSTEIN LINKAGE CLASS OF A COMPLETE INTERSECTION

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For my mom; without her unconditional sacrifice, I would not be here.

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ABSTRACT

Dang, Tan Ph.D., Purdue University, August 2020. Topics on the Cohen-Macaulay Property of Rees Algebras and the Gorenstein Linkage Class of a Complete Intersection. Major Professor: Bernd Ulrich, Professor.

We study the Cohen-Macaulay property of Rees algebras of modules of Kähler differentials. When the module of differentials has projective dimension one, it is known that condition F_1 is sufficient for the Rees algebra to be Cohen-Macaulay. The converse was proved if the module of differentials is already F_0. We weaken the condition F_0 globally by assuming some homogeneity condition.

We are also interested in the defining ideal of the Rees algebra of a Jacobian module. If the Jacobian module is an ideal, we prove a formula for computing the defining ideal. Using the formula, we give an explicit description of the defining ideal in the monomial case. From there, we characterize the Cohen-Macaulay property of the Rees algebra.

In the last chapter, we study Gorenstein linkage mostly in the graded case. In particular, we give an explicit example of a class of monomial ideals that are in the homogeneous Gorenstein linkage class of a complete intersection. To do so, we prove a Gorenstein double linkage construction that is analogous to Gorenstein biliaison.

1. INTRODUCTION

There are two main topics in this thesis: the Cohen-Macaulay property of the Rees algebra of certain modules and the Gorenstein linkage class of complete intersections.

First, let us provide some motivation and summarize the results regarding the first topic. The Rees algebra of an ideal belongs to a class of rings collectively known as *blowup algebras*. Let R be a ring and I an R-ideal. The Rees ring of I is defined as

$$\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n.$$

The Rees algebra plays an important role in the study of desingularization, and classically it is the coordinate ring associated with blowing up a variety along a subvariety. Furthermore, as the definition suggests, the Rees algebra of ideal plays a central role in the theory of integral dependence of ideals and other asymptotic behaviors of power of ideals in general. This can be generalized to the case of modules.

The definition of Rees algebras of modules involves another example of blowup algebras, namely the *symmetric algebras*, denoted by $\mathcal{S}(E)$. Similar to the classical blowup,

$$\mathrm{Spec}(\mathcal{S}(E)) \longrightarrow \mathrm{Spec}(R)$$

is the fiberation of $\mathrm{Spec}(R)$ by a collection of planes. Now for a module E having a rank, the Rees algebra of E, denoted by $\mathcal{R}(E)$, can be defined as

$$\mathcal{R}(E) = \mathcal{S}(E)/\tau$$

where τ is the R-torsion submodule of $\mathcal{S}(E)$. In many situations, the symmetric algebras are the coordinate rings of certain objects in algebraic geometry. Hence, the Rees algebra is a natural object to study because removing undesirable components of $\mathrm{Spec}(\mathcal{S}(E))$ requires killing the torsion as one of the first steps. A particular example of this phenomenon is the main subject of Chapter 3, namely the module of differentials.

The module of Kähler differentials, or just the module of differentials, is a classical object that was first introduced by Erich Kähler in the 1930s. Let R be an affine algebra over a field k. The module of differentials of R over k is denoted by $\Omega_k(R) = \Omega_{R/k}$. Classically, the module of differentials characterizes the smoothness of R by means of the famous Jacobian criterion. We apply the symmetric algebra and the Rees algebra functors to $\Omega_k(R)$. When R is an affine algebra over an algebraically closed field k, the ring $\mathcal{S}(\Omega_{R/k})$ is called the *Zarisky tangent algebra* [17]. In the language of schemes, the $\mathrm{Spec}(\mathcal{S}(\Omega_{R/k}))$ is the first jet scheme of $\mathrm{Spec}(R)$. When R is regular, the Rees algebra of $\Omega_k(R)$ and the symmetric algebra of $\Omega_k(R)$ coincide, however they are generally quite different. The Rees algebra of $\Omega_k(R)$ is the coordinate ring for a correspondence in biprojective space [19]. In particular, when R is standard graded, $R/m \otimes_R \mathcal{R}(\Omega_{R/k})$ is the homogeneous coordinate ring of the tangential variety, which is the closure of the union of all tangent spaces at smooth points.

In Chapter 3, we consider the case where the projective dimension of $\Omega_k(R)$ is at most one. For instance, R is a reduced and locally a complete intersection. In general, when an R-module E has projective dimension at most one, E satisfies the condition F_0 if and only if $\mathcal{S}(E)$ is a complete intersection; furthermore, E satisfies F_1 if and only if $\mathcal{S}(E)$ is R-torsion free (see [1, Propositions 3 and 4], [10, 1.1], [18, 3.4]). Therefore, a sufficient condition for the Cohen-Macaulay property of $\mathcal{R}(E)$ is that E satisfies the condition F_1. The converse is not true in general. In [16, 3.1], it is shown that for $\Omega_k(R)$, the converse is true if $\Omega_k(R)$ also already satisfies F_0. Theorem 3.2.1 weakens the condition F_0 by requiring some homogeneity condition. A consequence of the main result of Chapter 3 is that the Cohen-Macaulay property of $\mathcal{R}(\Omega_{R/k})$ is highly restrictive, which in most cases will only occur if $\Omega_k(R)$ is of linear type and the symmetric algebra is a complete intersection.

We are also interested in the Rees algebra of a cousin of the module of differentials, namely the Jacobian module. The Jacobian module plays a role in the theory of subspace arrangements, in particular hypersurface arrangements, and free divisors. The Rees algebra of the Jacobian module is the coordinate ring of the conormal variety of X, which is the closure of the set of pairs (x, H) where x is a smooth point and H is a hyperplane tangent to X at x.

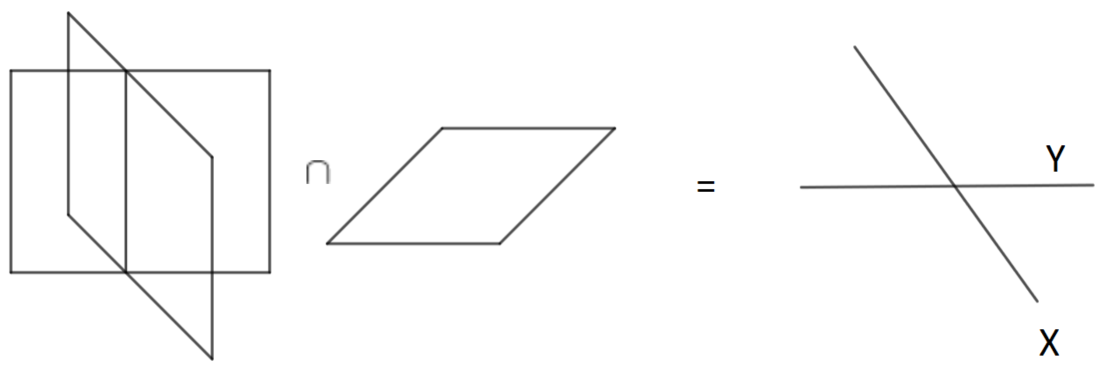
In Chapter 4, we study the case when the Jacobian module is an ideal. Ultimately, we want to characterize the Cohen-Macaulay property of the Rees algebra of the Jacobian module. There are different general frameworks for asserting the Cohen-Macaulay property of the Rees algebra of an ideal. In this thesis, we choose the most hands-on approach by first computing the defining ideal of the Rees algebra and then extracting the Cohen-Macaulay property from the explicit presentation.

Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring in n variables over a field k. Let $f \in S$ and write $R = S/(f)$. By $\operatorname{Jac}(f)$, we denote the Jacobian module of R over k. Write $\mathcal{J}(f)$ for the defining ideal of the Rees algebra of $\operatorname{Jac}(f)$. Proposition 4.1.1 allows us to compute $\mathcal{J}(f)$ from $\mathcal{J}(f_i)$ where f_i are the irreducible factors of f. The defining ideals of the Rees algebra of Jacobian ideals encode information about the syzygy of the Jacobian ideals. Hence by Proposition 4.1.1, information about the hypersurface arrangements can be deduced from the hypersurface arrangement of the irreducible factors. We apply this proposition to the case where f is a monomial, and observe that the Rees algebra of the Jacobian ideal is either Cohen-Macaulay or almost Cohen-Macaulay.

The other topic in this thesis is Gorenstein linkage (liaison), and in particular the Gorenstein linkage class of a complete intersection.

Gorenstein linkage is a generalization of complete intersection linkage, which is simply called linkage. The theory of linkage has been an active and fruitful area of research. It had yielded many results that are useful in the study of other algebraic objects. For example, it has produced classes of ideals that are strongly Cohen-Macaulay, which is an important property in the theory of Rees algebras [20, 4.2.4]. We will discuss in details the definition of (Gorenstein) linkage in Section 2.6.

Let X, Y be equidimensional subschemes of $\mathbb{P}^n$ without any common component. If $X \cup Y$ is a complete intersection, i.e. the corresponding ideal is a complete intersection, then X and Y are said to be directly geometrically linked. It follows that $I_X = I_{X\cup Y} : I_Y$ and $I_Y = I_{X\cup Y} : I_X$. For example, take $I_X = (x_1, x_2)$, and $I_Y = (x_1, x_3)$. In this case, $I_X \cap I_Y = (x_2x_3, x_1)$. The lines X and Y are directly linked by a complete intersection of a union of two planes and one plane.



As the theory developed, the condition of having no common component appeared too restrictive. The authors in [14] rephrased the definition of linkage in the language of quotient ideals, namely $I = K : J$ and $J = K : I$ where K is a complete intersection ideal. It is clear that the definition is symmetric, but rarely reflexive or transitive. However, this relation induces an equivalence relation by allowing linking in finitely many steps. A particularly interesting linkage class is the <u>li</u>nkage <u>c</u>lass of a <u>c</u>omplete <u>i</u>ntersection, abbreviated as licci.

In low codimensions, the linkage class of a complete intersection is well understood. However, in higher codimensions, the complete intersection condition of K is quite restrictive. We wish to use a larger class of ideals while retaining as many properties of linkage as possible. One direction to generalize complete intersection linkage is to use Gorenstein ideal, which leads to Gorenstein linkage.

A natural question to ask is finding classes of ideals that are in the <u>G</u>orenstein <u>li</u>nkage <u>c</u>lass of a <u>c</u>omplete <u>i</u>ntersection, abbreviated as glicci. Many classes of ideals have been shown to be glicci, for example standard determinantal ideals [11, 3.6] and residual intersections of licci ideals [8, 4.6]. In Section 5.2, our main result is another class of glicci ideals in the graded case. In particular, we are interested in ideals of finite colength. By Macaulcay's inverse systems (see Section 2.5), all Gorenstein ideals of finite colength can be obtained from the Matlis duals of cyclic modules. Using the framework provided by inverse systems, we introduce a Gorenstein double linkage construction that is analogous to Gorenstein basic double linkage, or Gorenstein biliaison. The latter is ubiquitous in the literature involving glicci ideals. Applying the new double linkage construction, we prove that a certain class of monomial ideals of finite colength is homogeneously glicci.

2. PRELIMINARIES

In this chapter, we review information that provides the background for the work in this thesis.

Section 2.1 lays out some basic notions about matrices. In Sections 2.2 and 2.3, we set up the foundation for Chapters 3 and 4. In Section 2.4, we review the Eagon-Northcott complex, a well-known complex of free modules that generalizes the Koszul complex. In Section 2.5, we recall a duality of Artinian ideals that will be helpful in Chapter 5. And in Section 2.6, we provide some background information on linkage and Gorenstein linkage.

Throughout this work, all rings are unital and commutative.

2.1 Matrix, minors, and rank

Let R be a ring and ψ be an n by m matrix with entries in R

$$\psi = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}.$$

Denote the ideal generated by all t by t minors of ψ as $I_t(\psi)$. The *determinantal rank of* ψ, or just *rank of* ψ, is defined as

$$\text{rank } (\psi) = \max \{t \mid I_t(\psi) \neq 0\}.$$

Let $M = \text{coker } (\psi)$. We define the *ith Fitting ideal of* M to be $\text{Fitt}_i(M) = I_{n-i}(\psi)$. The Fitting ideals only depend on M. Furthermore, by [5, 20.6]

$$V(\text{Fitt}_i(M)) = \{p \in \text{Spec}(R) \mid \mu_{R_p}(M_p) > i\}.$$

2.2 Rees algebras of modules with projective dimension one

We start with the definition of Rees algebras of ideals and modules in general.

Let R be a Noetherian ring, and $I = (x_1, \ldots, x_n)$ and R-ideal. We define the Rees algebra of I to be

$$\mathcal{R}(I) := \bigoplus_{n=0}^{\infty} I^n \cong R[It] = R[x_1 t, \ldots, x_n t] \subset R[t]$$

where $R[t]$ is a polynomial ring in one variable.

This definition does not lend itself to an easy generalization to the case of modules because it is not obvious what powers of modules should be. The inclusion $R[It] \subset R[t]$ suggests that we think of the Rees algebra of a module $E \subset R^r$ as the algebra generated by the image of the module inside the symmetric algebra of R^r. This approach has a disadvantage that it is dependent on the embedding $E \subset R^r$. A general definition of the Rees algebra of module that is independent of the embedding can be found in [4]. In this thesis, we will use a slightly more narrow, but very common definition of the Rees algebra of modules where we require the module to have a rank.

First, we say a module E over a Noetherian ring R has a rank r when $E \otimes_R \mathrm{Quot}(R) \cong \mathrm{Quot}(R)^e$, where $\mathrm{Quot}(R)$ denotes the total ring of fractions of R. This class of modules is quite common. For example, when R is a domain, $\mathrm{Quot}(R)$ is a field, and every R-module has rank. Or, when E admits a finite resolution by finitely generated free R-modules, then E has a rank.

Next, for an R-module E, we define the symmetric algebra of E as

$$\mathcal{S}(E) := \frac{\otimes E}{(x \otimes y - y \otimes x \mid x, y \in E)}$$

where $\otimes E$ is the tensor algebra of E. This algebra has a universal property: for every commutative R-algebra S and every R-linear map $\mu : E \to S$, there exists a unique R-linear map from $\mathcal{S}(E)$ to S so that it restricts to μ on E. If $E \cong R^n$, then the symmetric algebra of E is nothing but $R[T_1, \ldots, T_n]$, the polynomial in n variables over R.

Now let R be a Noetherian ring and $E = Rx_1 + \ldots + Rx_n$ a finite R-module. We can then map R^n surjectively onto E by sending the standard basis to the chosen generators of E. Then by applying the symmetric algebra functor, we obtain $R[T_1, \ldots, T_n] \twoheadrightarrow \mathcal{S}(E)$ and consequently

$$\mathcal{S}(E) \cong \frac{R[T_1, \ldots, T_n]}{\mathcal{L}}$$

where $\mathcal{L}$ is called the defining ideal of the symmetric algebra of E. We notice that $\mathcal{L}$ is generated by linear forms w in $R[T_1, \ldots, T_n]$ satisfying $w(x_1, \ldots, x_n) = 0$.

More concretely, if E admits a finite presentation

$$R^m \xrightarrow{\psi} R^n \to E \to 0 \ ,$$

then $\mathcal{L} = (l_1, \ldots, l_m)$ so that

$$[l_1 \ldots l_m] = [T_1 \ldots T_n] \cdot \psi \ .$$

We now define the Rees algebra of a module E having a rank.

Definition 2.2.1

$$\mathcal{R}(E) := \mathcal{S}(E)/\tau$$

where τ is the torsion R-submodule of $\mathcal{S}(E)$. If $\mathcal{R}(E) = \mathcal{S}(E)$, E is said to be of *linear type.*

This definition generalizes the definition of Rees algebra of ideals. Let I be an R-ideal with positive grade. From the natural embedding $I \subset R$, we obtain a homomorphism $\mathcal{S}(I) \to R[t]$. Its image is the graded R-algebra $R[It]$ whereas its kernel is precisely the torsion submodule of $\mathcal{S}(I)$.

Now with a working definition of $\mathcal{R}(E)$, we can define powers of E as $E^i := [\mathcal{R}(E)]_i$.

The Krull-dimension of the Rees algebra of a module is well-understood.

Proposition 2.2.1 ([15, 2.2]) *Let R be a Noetherian ring of dimension d and E a finitely generated R-module with rank r. Then*

$$\dim\ \mathcal{R}(E) = d + r.$$

For ideals with zero height, the formula above cannot be used. Instead, we use following formula.

Proposition 2.2.2 ([5, 13.8]) *Let R be a Noetherian ring. I an R-ideal. Then*

$$\dim \mathcal{R}(I) = \max \{\dim R, 1 + \dim R/p \mid p \in \mathrm{Min}(R) \backslash V(I)\}.$$

Next, we review the definition for condition F_k, and go over theorems about blow-up algebras of modules with project dimension one that will be needed in Chapter 3.

Definition 2.2.2 *Let E be a finite R-module with rank e. We say E satisfies condition F_k if*

$$\mu(E_p) \leq \dim R_p + e - k$$

for all prime ideals p so that E_p is not R_p-free.

Equivalently, when ψ is a presentation matrix of E, we can rephrase condition F_k as

$$\mathrm{ht}\ I_i(\psi) \geq \mathrm{rank}(\psi) - i + 1 + k, \text{ for } 1 \leq i \leq \mathrm{rank}(\psi).$$

We observe that this condition can be readily checked by a computer algebra system.

Condition F_k, in particular F_0 and F_1, characterize certain properties of the symmetric algebra.

Theorem 2.2.1 ([1], [10, 1.1], [18, 3.4]) *Let R be a local Cohen-Macaulay ring, and E a finite R-module with projective dimension at most one. Then*

- *E satisfies conditions F_1 if and only if $\mathcal{S}(E)$ is R-torsion free.*
- *E satisfies conditions F_0 if and only if $\mathcal{S}(E)$ is a complete intersection.*

Using Bourbaki ideals is a well-known technique to study properties of the Rees algebra of modules. In [15], the authors introduced generic Bourbaki ideals, and proved that some properties of the Rees algebra of a module can be inferred from those of the Rees algebra of its generic Bourbaki ideal.

Theorem 2.2.2 ([15, 3.5]) *Let R be a Noetherian local ring, E a finitely generated R-module with* rank $e > 0$. *Let I be the generic Bourbaki ideal of E. Then*

- *The Rees algebra $\mathcal{R}(E)$ is Cohen-Macaulay if and only if $\mathcal{R}(E)$ is Cohen-Macaulay.*
- *E is of linear type and* grade $\mathcal{R}(E)_+ \geq e$ *if and only if I is of linear type.*

As an application, the Cohen-Macaulay property of $\mathcal{R}(E)$ can be characterized by property regarding certain ideals of minors.

Theorem 2.2.3 ([15, 4.7]) *Let R be a Gorenstein local ring with infinite residue field, E a finite R-module with rank e, projective dimension at most one, and presentation matrix ϕ. Assume E satisfies condition F_1 locally on the punctured spectrum of R. The following are equivalent:*

- *$\mathcal{R}(E)$ is Cohen-Macaulay.*
- *After elementary row operations, $I_t(\phi)$ is generated by the maximal minors of the matrix consisting of the last t rows of ϕ, where $t = \mu(E) - \dim\ R - e + 1$.*

We want to remark that the row operations in theorem above are corresponding to a general change of generators of E. If R contains an infinite field k, the row operations and correspondingly the change of generators can be induced from an element of $\mathrm{GL}_n(k)$.

For a torsion-free R-module E, the authors in [15] also showed that Cohen-Macaulay property of $\mathcal{R}(E)$ is rather restrictive.

Theorem 2.2.4 ([15, 4.3]) *Let R be a Cohen-Macaulay ring and E a finitely generated torsion-free R-module having a rank. If $\mathcal{R}(E)$ is Cohen-Macaulay, then E is locally free at codimension* 1.

2.3 Module of differentials and Jacobian module

Let A be a ring, $S = W^{-1}A[\{X_i | i \in \mathcal{I}\}]$, and $R = S/I$ for some S-ideal $I = (f_j | j \in \mathcal{J})$. The module of differentials of R over A is defined as

Definition 2.3.1

$$\Omega_A(R) = \Omega_{R/A} := \bigoplus_{i \in \mathcal{I}} R dx_i / (\Sigma \overline{\frac{\partial f_j}{\partial X_i}} dx_i | j \in \mathcal{J})$$

where $\{dx_i \mid i \in \mathcal{I}\}$ is a basis, $\frac{\partial f_j}{\partial X_i}$ are the partial derivatives of f_j with respect to X_i and $^-$ denotes the image in R.

When the index sets $\mathcal{I}$ and $\mathcal{J}$ are finite, say $S = W^{-1}A[X_1, \ldots, X_n]$, $I = (f_1, \ldots, f_m)$, there is readily a presentation of $\Omega_A(R)$

$$R^m \xrightarrow{\theta} R^n \to \Omega_A(R) \to 0 \;\text{ where }\; \theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $\frac{\partial f_j}{\partial x_i}$ is the image of $\frac{\partial f_j}{\partial X_i}$ in R. Notice, the matrix θ is the image in R of the Jacobian matrix of $f_1, \ldots, f_m$.

Equivalently, the module of differentials together with the universal A-derivation $d_{R/A} : R \to \Omega_A(R)$ with $d_{R/A}(x_i) = dx_i$ can be defined via a universal property. Namely, for every A-derivation $\delta : R \to M$, there exists uniquely a R-linear map $\mu : \Omega_A(R) \to M$ with $\delta = \mu \circ d_{R/A}$. A map $\delta : R \to M$, where M is R-module, is called a A-*derivation* if it is a homomorphism of additive groups, and satisfies the product rule $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in R$, and $A \subset \ker(\delta)$. Hence, the definition of the module of differentials only depends on the A-algebra R and not on its presentation.

With the same R, S and I as before, we dualize the presentation into the ring R and obtain

$$0 \to \Omega^*_{R/A} \to (R^*)^n \xrightarrow{T(\theta)} (R^*)^m.$$

Now the image of $T(\theta)$ is called the *Jacobian module of R over A*.

By the well-known Jacobian criterion, the freeness of the module of differentials is related to the regularity of the ring. We will state a slightly different version of the criterion here.

Theorem 2.3.1 ([5, 16.22]) *Let (R, m) be a local k-algebra essentially of finite type over a field k. Assume either k has characteristic 0 or that R is reduced and k is perfect. Then*

$$R \text{ is regular} \Longleftrightarrow \Omega_k(R) \text{ is free as } R\text{-module.}$$

Additionally, if the equivalence holds, $\operatorname{rank}(\Omega_k(R)) = \dim\, R + \operatorname{trdeg}_k(R/m)$.

Noticing that both conditions localize, as a corollary, we can detect the singular locus of R by means of $\Omega_k(R)$. We define the *singular locus of R* as follow,

$$\operatorname{Sing}(R) = \{q \in \operatorname{Spec}(R) \mid R_q \text{ is not regular}\}.$$

Theorem 2.3.2 ([5, 16.20]) *Let k be a perfect field, W be a multiplicative subset of the polynomial ring $k[X_1, \ldots, X_n]$, $I \subset W^{-1}k[X_1, \ldots, X_n]$ an equicodimensional ideal of height g (i.e. every minimal prime of I has the same height). Let $D = n - g$, and $R = W^{-1}k[X_1, \ldots, X_n]/I$. Then*

$$\operatorname{Sing}(R) = V(\operatorname{Fitt}_D(\Omega_k(R))).$$

The $\operatorname{Fitt}_D(\Omega_k(R))$ is called the *Jacobian ideal* of R over k, and denoted by $\mathcal{J}(R/k)$. Notice that the right hand side only depends on the k-algebra R, and can be computed from a presentation of $\Omega_k(R)$, which is readily available.

The Jacobian module also displays similar properties with respect to the regularity of R by following theorem:

Theorem 2.3.3 ([12]) *Let k be a field of characteristic 0 and R a reduced local k-algebra essentially of finite type. The following are equivalent:*

1. *R is regular*

2. *$\Omega_k(R)/\tau$ is free where τ is the R-torsion submodule of $\Omega_k(R)$*

3. R is equidimensional and $\mathcal{J}(R/k)$ is principal

4. The Jacobian module of R is free.

2.4 Eagon-Northcott complex

The Eagon-Northcott complex plays an important role in Chapter 3. We will review its construction in details here (see [3]).

Let R be a ring and ψ be an n by m matrix with $n \leq m$ and entries in R

$$\psi = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{m2} & \dots & a_{nm} \end{bmatrix}.$$

Let K be the exterior algebra generated by m symbols $X_1, \dots, X_m$. For each row of ψ, there is an associated differentiation Δ_k, namely

$$\Delta_k(X_{i_1} \wedge \cdots \wedge X_{i_s}) = \sum_{p=i}^{s} (-1)^{p+1} a_{ki_p} X_{i_1} \wedge \cdots \wedge \hat{X}_{i_p} \wedge \cdots \wedge X_{i_s}.$$

We notice that these differentiations are alternating, i.e. for $k \neq h$,

$$\Delta_k \Delta_h + \Delta_h \Delta_k = 0.$$

Let $S = R[Y_1, \dots, Y_n]$ be a polynomial ring in n variables. Denote S_t the R-module consists of forms of degree t. Now the Eagon-Northcott complex can be described as follow

$$E_{.} : 0 \to E_{m-n+1} \xrightarrow{d} E_{m-n} \xrightarrow{d} \dots \xrightarrow{d} E_1 \xrightarrow{d} E_0$$

where

$$E_0 = R, \text{ and } E_{q+1} = K_{n+q} \otimes S_q \text{ for } q = 0, 1, \dots, m-n.$$

To describe the differentiation homomorphism d, we notice that each component of $E_{.}$ is a free R-module with a natural basis

$$X_{i_1} \wedge \cdots \wedge X_{i_{n+q}} \otimes Y_1^{\mu_i} \dots Y_n^{\mu_n} \text{ with } i_1, \dots, i_{n+q} \leq m \text{ and } \mu_1 + \cdots + \mu_n = q.$$

For $q \geq 0$, let d be defined on the basis elements by

$$d(X_{i_1} \wedge \cdots \wedge X_{i_{n+q}} \otimes Y_1^{\mu_i} \dots Y_n^{\mu_n}) = \sum_j \Delta_j(X_{i_1} \wedge \cdots \wedge X_{i_{n+q}}) \otimes Y_1^{\mu_1} \dots Y_j^{\mu_j - 1} \dots Y_n^{\mu_n}.$$

The summation only includes terms with $\mu_j - 1 \geq 0$.

For $q = 0$, we simply use

$$d(X_{i_1} \wedge \cdots \wedge X_{i_n} \otimes 1) = \det \begin{bmatrix} a_{1i_1} & a_{1i_2} & \dots & a_{1i_n} \\ a_{2i_1} & a_{2i_2} & \dots & a_{2i_n} \\ \vdots & \vdots & & \vdots \\ a_{ni_1} & a_{ni_2} & \dots & a_{ni_n} \end{bmatrix}.$$

For a concrete example, let ψ be a generic matrix

$$\psi = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{bmatrix}.$$

Then the Eagon-Northcott complex will be

$$0 \to R^3 \to R^8 \to R^6 \to R.$$

The entries of d_0 are nothing but the maximal minors of ψ. For the other maps d, we can write down their explicit formulas

$$d_1 = \begin{bmatrix} -x_3 & -x_7 & -x_4 & -x_8 & 0 & 0 & 0 & 0 \\ x_2 & x_6 & 0 & 0 & -x_4 & -x_8 & 0 & 0 \\ -x_1 & -x_5 & 0 & 0 & 0 & 0 & -x_4 & -x_8 \\ 0 & 0 & x_2 & x_6 & x_3 & x_7 & 0 & 0 \\ 0 & 0 & -x_1 & -x_5 & 0 & 0 & x_3 & x_7 \\ 0 & 0 & 0 & 0 & -x_1 & -x_5 & -x_2 & -x_6 \end{bmatrix},$$

and

$$d_2 = \begin{bmatrix} x_4 & x_8 & 0 \\ 0 & x_4 & x_8 \\ -x_3 & -x_7 & 0 \\ 0 & -x_3 & -x_7 \\ x_2 & x_6 & 0 \\ 0 & x_2 & x_6 \\ -x_1 & -x_5 & 0 \\ 0 & -x_1 & -x_5 \end{bmatrix}.$$

We observe that the Eagon-Northcott complex is a generalization of the Koszul complex. When $n = 1$, the Eagon-Northcott complex is the Koszul complex associated with the sequence $a_{11}, a_{12}, \ldots, a_{1m}$. They share similar properties on the exactness of the complex.

Theorem 2.4.1 ([14, 1]) *Let R be a Noetherian ring, and ψ an n by m matrix with $n \leq m$. Let I be the ideal generated by the maximal minors of ψ and $I \neq R$. Let $t = \max\{i \mid H_i(E_.) \neq 0\}$, then*

$$\operatorname{grade} I = m - n - t + 1.$$

In particular, $\operatorname{grade} I \leq m - n + 1$*.*

As an immediate corollary, if the ideal of maximal minors has maximum grade, then the Eagon-Northcott complex is acylic and it will provide a free resolution for the ideal of maximal minors.

Theorem 2.4.2 ([14, 2]) *Let R be a local Cohen-Macaulay ring, and ψ an n by m matrix with $n \leq m$. Let I be the ideal generated by the maximal minors of ψ and $I \neq R$. If the height of I is maximal, ie* $\operatorname{ht} I = m - n + 1$*, then*

$$E_. : 0 \to E_{m-n+1} \xrightarrow{d} E_{m-n} \xrightarrow{d} \ldots \xrightarrow{d} E_1 \xrightarrow{d} E_0$$

is acyclic and $H_0(E_.) = R/I$. Furthermore, if $I_1(\psi)$ is contained in the maximal ideal, then the projective dimension of R/I is precisely $m - n + 1$.

2.5 Macaulay Inverse Systems

In this section, we state the Matlis duality and review its applications to Macaulay's inverse systems, which is the main tool for Chapter 5.

Theorem 2.5.1 (Matlis Duality) *Let R be a complete Noetherian local ring with residue field k. Let E be the injective envelope of k as R-module. Denote $-' = Hom_R(-, E)$. Let M be a R-module.*

a) M' is Artinian if and only if M is Noetherian, and M' is Noetherian if and only if M is Artinian.

b) M'' is naturally isomorphic to M.

For an R-module M, the *injective envelope* or *injective hull* of M is an injective R-module E such that $M \subset E$ is an essential extension. In particular, with the assumptions of Theorem 2.5.1, the injective envelope of k is a faithful R-module E so that the extension $k \subset E$ is essential. We can now describe Macaulay inverse systems.

Let $R = k[[X_1, \ldots, X_n]]$ be a power series ring in n variables over a field k. Set $T = [X_1^{-1}, \ldots, X_n^{-1}]$, the set of the "inverse polynomials," and $A = k[X_1, \ldots, X_n]$, the polynomial ring. We define an A-module structure on T by defining a multiplication between the monomials of A and T as follow: For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \geq 0$,

$$X_1^{\alpha_1} \ldots X_n^{\alpha_n}\ X_1^{-\beta_1} \ldots X_n^{-\beta_n} = \begin{cases} X_1^{\alpha_1-\beta_1} \ldots X_n^{\alpha_n-\beta_n} & , \quad \text{if } \alpha_i - \beta_i \leq 0 \text{ for all } i \\ \qquad 0 & , \qquad \text{otherwise} \end{cases}.$$

Observe that since every element of T is annihilated by some powers of the ideal $(X_1, \ldots, X_n)A$, the multiplication induces an R-module structure on T. Furthermore, for every $i \geq 0$, $A_i \times T_{-i} \xrightarrow{\text{mult}} k$ is a nondegenerate bilinear form, and $(RA_{i+1})T_{-i} = 0$. Hence, $k \subset T$ is an essential extension and further T is a faithful R-module. Therefore, T is the injective envelope of k.

Applying Matlis Duality, we obtain a one-to-one correspondence between m-primary R-ideals and finitely generated R-submodules of T, which we denote by I and M respectively:

$$I \mapsto M = 0 :_T I \text{ and } M \mapsto I = 0 :_R M = \text{ann}_R M.$$

For a concrete example, let $R = k[[x, y, z]]$ and $T = [x^{-1}, y^{-1}, z^{-1}]$ the injective envelope of k. Let $I = (x^3, y^2, z^4, x^2y, yz^3, xz)$ be an m-primary ideal of R. Then $M = 0 :_T I = Rx^{-1}y^{-1} + Rx^{-2} + Rz^{-3} + Ry^{-1}z^{-2}$. Vice versa, we can check $I = 0 :_R M = (x^3, y^2, z^4, x^2y, yz^3, xz)$.

Furthermore, R/I is Gorenstein if and only if M is cyclic. Such ideal I is called a Gorenstein ideal. This duality gives an easy way to generate Gorenstein m-primary ideals in a ring of arbitrary dimension. Under the setting of the example above, we can choose any cyclic R-module, say $M = R(x^{-4} + x^{-2}y^{-1}z^{-1} + z^{-4})$. Then $I = 0 :_R M = (y^2, yz^2, xz^2, x^2y - z^3, x^3 - xyz)$ is a Gorenstein ideal of finite colength.

2.6 Linkage and Gorenstein linkage

Let R be a local Cohen-Macaulay ring. Two proper R-ideals I and J are said to be *directly linked* if there exists a regular sequence $x_1, \ldots, x_g$ so that $I = (x_1, \ldots, x_g) : J$ and $J = (x_1, \ldots, x_g) : I$. We denote this relation by $I \sim J$. This induces an equivalence relation: I and J are in the same linkage class if

$$I = I_0 \sim I_1 \sim \cdots \sim I_n = J, \text{ for some } n.$$

If we require n to be even, the equivalence class is called the *even linkage class.* Two R-ideals I and J are said to be *geometric linked* if in addition to being directly linked, $\text{ht}(I + J) > g$. Notice that $(x_1, \ldots, x_g) \subset I \cap J$ and ht $I =$ ht $J = g$.

Under suitable conditions, directly linked ideals can be readily produced.

Theorem 2.6.1 ([14]) *Let R be a local Gorenstein ring and I be an unmixed ideal of height g. Let $K \neq I$ be an ideal generated by a regular sequence $x_1, \ldots, x_g$ in I. If $J = K : I$, then $I = K : J$. In other words, $I \sim J$.*

Many properties of I can be transferred or are at least closely related to those of the ideals in the same linkage or even linkage class. Those properties include but are not limited to: the Cohen-Macaulay property and its related conditions, like sliding depth and strong Cohen-Macaulayness, as well as other homological invariants. Finding what properties are preserved under linkage is a general problem, usually called finding *linkage invariants*. We state the most basic invariant.

Proposition 2.6.1 ([14]) *With the hypothesis if Theorem 2.6.1, if I is a Cohen-Macaulay ideal, then $J = K : I$ is also Cohen-Macaulay.*

Since ideals in the same (even) linkage class share many properties, it is not surprising that the linkage class of a complete intersection is a well-studied class of ideals that provides many examples and counter examples to various questions. In particular, in low codimensions, it is well-understood.

Theorem 2.6.2 ([6]) *Let R be a Gorenstein local ring, and I be a perfect ideal of codimension two. Then I is in the linkage class of a complete intersection.*

Theorem 2.6.3 ([22]) *Let R be a Gorenstein local ring, and I be a perfect Gorenstein ideal of codimension three. Then I is in the linkage class of a complete intersection.*

Perhaps, it is a strength and also a weakness that many strong properties are preserved under even linkage. This prevents many classes of ideals to be licci. To overcome this disadvantage, the definition of direct linkage can be generalized. One way to generalize is to replace complete intersection ideals by Gorenstein ideals.

Definition 2.6.1 *Let R be a regular local ring, and I and J be R-ideals. Two proper R-ideals I and J are said to be Gorenstein directly linked if there exists a Gorenstein ideal K so that $I = K : J$ and $J = K : I$.*

Notice, $K \in I \cap J$ and K, I, J are of the same height. Theorem 2.6.1 and Proposition 2.6.1 can be generalized to this setting.

Theorem 2.6.4 ([14]) *Let R be a regular local ring. Let I be an unmixed ideal of height g. Let $K \subsetneq I$ be a Gorenstein ideal of height g. If $J = K : I$, then $I = K : J$. In addition, if I is Cohen-Macaulay, then so is J.*

In low codimensions, Gorenstein ideals are complete intersections, hence the two definitions coincide. In particular, let R be a regular local ring and K an unmixed ideal of height one. Since R is a unique factorization domain, K is principal, hence a complete intersection.

If K is a Gorenstein of height 2, then K must have a resolution of the form

$$0 \to R \to R^{\mu(K)} \to R \to R/K \to 0.$$

Computing ranks along the exact sequence, we observe that $\mu(K) = 2$, hence K is a complete intersection.

In codimension three, Gorenstein ideals are not necessarily complete intersections.

Theorem 2.6.5 ([2, 4]) *Let R be a regular local ring, and I be an ideal of height 3. Then I is Gorenstein if and only if I is generated by the $2n$-order Pfaffians of a $(2n+1)$ by $(2n+1)$ alternating matrix.*

For example, in $R = k[X, Y, Z]$, let

$$K = (XY, YZ, XZ, X^2 - Y^2, Y^2 - Z^2).$$

The ideal K has height 3 and is minimally generated by 5 elements, hence not a complete intersection. Its generators are the 4 by 4 Pfaffians of the following alternating matrix

$$\begin{bmatrix} 0 & -Z & 0 & Y & 0 \\ Z & 0 & 0 & -X & Y \\ 0 & 0 & 0 & Y & -X \\ -Y & X & -Y & 0 & -Z \\ 0 & -Y & X & Z & 0 \end{bmatrix}.$$

In codimension three or higher, the Gorenstein linkage class of a complete intersection is different from the (complete intersection) linkage class of a complete intersection. Many classes of ideals have been proven to be glicci (and not licci), and we will review some of the results in Section 5.1.

3. REES ALGEBRAS OF MODULES OF DIFFERENTIALS

In this chapter, we are interested in determining necessary conditions for the Rees algebra of a module of differentials to be Cohen-Macaulay. The module of differentials has a well-understood presentation. We are interested in reduced complete intersection rings because their modules of differentials have projective dimension one. In this setting, many results are known about the Cohen-Macaulay property of the Rees algebra of such modules (see Chapter 2, Section 2). Theorem 3.2.1 says under suitable hypothesis, if the Rees algebra of the module of differentials is Cohen-Macaulay, then the module of differential is F_1 if it is F_0 locally on the punctured spectrum.

3.1 Background

First, recall a presentation for $\Omega_A(R)$. Let $R = S/I$ where $S = W^{-1}A[X_1, \ldots, X_n]$, $I = (f_1, \ldots, f_m)$,

$$R^m \xrightarrow{\theta} R^n \to \Omega_A(R) \to 0 \quad \text{with} \quad \theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix},$$

where $\frac{\partial f_j}{\partial x_i}$ is the image of $\frac{\partial f_j}{\partial X_i}$ in R. Associated with the module of differentials there is a very useful exact sequence, called the *conormal sequence.*

Proposition 3.1.1 ([5, 16.3]) *Let $R = S/I$ where S is an algebra over a ring A. There exists an exact sequence of R-modules*

$$I/I^2 \xrightarrow{d} R \otimes_S \Omega_A(S) \to \Omega_A(R) \to 0,$$

where d maps the class of f to $1 \otimes df$.

A necessary and sufficient condition for the map d to be split-injective is known [5, 16.12]. We are interested in a class of rings where the map d is injective.

Proposition 3.1.2 ([21]) *Let S be a Noetherian ring and $I \neq S$ an S-ideal. If I is a complete intersection, then I/I^2 is a free S/I-module.*

As a consequence of Proposition 3.1.1 and Proposition 3.1.2, if $R = S/I$ is a reduced k-algebra essentially of finite type over a field k, which is locally also a complete intersection, then I/I^2 is a locally free R-module, and d is injective. Therefore, $\Omega_k(R)$ has projective dimension at most one (also see [5, 17.12]).

In this case, as an immediate observation from Theorem 2.2.1, if we assume $\Omega_k(R)$ satisfies condition F_1, then $\mathcal{S}(\Omega_k(R))$ is a torsion-free Cohen-Macaulay algebra. Consequently, the Rees algebra is Cohen-Macaulay. In [16, 3.1], the authors explored the converse and found a positive answer if the module satisfies condition F_0 beforehand.

Theorem 3.1.1 ([16, 3.1]) *Let k be a field of characteristic zero, and R a k-algebra essentially of finite type which is also locally a complete intersection. Assume the following conditions:*

1. *$\mathcal{R}(\Omega_{R/k})$ is Cohen-Macaulay.*

2. edim $R_p \leq 2$ dim R_p *for every prime ideal $p \in \operatorname{Spec}(R)$.*

Then edim $R_p \leq 2$ dim $R_p - 1$ *for every non-minimal prime ideal $p \in \operatorname{Spec}(R)$.*

In the same paper, the authors further posed the question whether assumption (2) in Theorem 3.1.1 can be dropped. The next section attempts to answer the question by weakening assumption (2). We can translate assumption (2) and the conclusion of Theorem 3.1.1 into condition F_k by Proposition 3.1.4 (also see [17, proof of 2.3]). Before stating the proposition, we review a formula for the rank of $\Omega_k(R)$, which is well-known. For a lack of reference, we give a quick proof.

Proposition 3.1.3 *Let k be a field of characteristic 0, and (R, m) be a local k-algebra essentially of finite type. Assume R is reduced and equidimensional. Then*

$$\operatorname{rank} \Omega_k(R) = \dim R + \operatorname{trdeg}_k(R/m).$$

Proof Since R is reduced, R_p is regular for every prime ideal $p \in \text{Ass}(R) = \text{Min}(R)$. Thus by Theorem 2.3.1, $\Omega_k(R_p)$ is free as R_p-module with rank dim $R_p + \text{trdeg}_k(R_p/pR_p) = \text{trdeg}_k(R_p/pR_p) = \dim R/p + \text{trdeg}_k(R/m)$. But since R is equidimensional, for every $p \in \text{Min}(R)$, $\dim R/p = \dim R$. ■

Proposition 3.1.4 *Let k be a field of characteristic 0 and R a local reduced k-algebra essentially of finite type, which is also equidimensional. Then*

$\Omega_k(R)$ satisfies condition F_k if and only if $\text{edim } R_p \leq 2 \dim R_p - k$ *for every prime ideal $p \in \text{Sing}(R)$.*

Proof Replace R by R_p where p is a prime ideal in $\text{Sing}(R)$, we need to show

$$\mu(\Omega_k(R)) \leq \dim R + \text{rank}(\Omega_k(R)) - k \text{ if and only if } \text{edim } R \leq 2 \dim R - k.$$

By Proposition 3.1.3, $\text{rank}(\Omega_k(R)) = \dim R + \text{trdeg}_k(R/m)$, and quite generally, by [7], we have $\mu(\Omega_k(R)) = \text{edim } R + \text{trdeg}_k(R/m)$. Thus, the left hand side can be written as

$$\text{edim } R + \text{trdeg}_k(R/m) \leq 2 \dim R + \text{trdeg}_k(R/m) - k,$$

which is equivalent to the right hand side. ■

3.2 Main result

First, we recall the Euler relations for partial derivatives, then we prove a technical proposition that plays an important role in our proof.

Let f be a homogeneous form in $k[X_1, \ldots, X_n]$, a positively graded polynomial ring in n variables with $\deg X_i = \delta_i$. Then the partial derivatives of f, namely $\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}$, satisfy the following Euler relation

$$\delta_1 X_1 \frac{\partial f}{\partial X_1} + \delta_2 X_x \frac{\partial f}{\partial X_2} + \cdots + \delta_n X_n \frac{\partial f}{\partial X_n} = (\deg f) f.$$

Proposition 3.2.1 *Let $R = k[X_1, \ldots, X_n]/I$ be a positively graded algebra over a field k with* $\text{char } k = 0$*. Let $d = \dim R \geq 2$. Let I be generated by a homogeneous* *regular sequence* $f_1, f_2, \ldots, f_{n-d}$ *contained in* $(X_1, \ldots, X_n)^2$. *Consider the* n *by* $n-d$ *matrix*

$$\theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_{n-d}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_{n-d}}{\partial x_n} \end{bmatrix}$$

where $\frac{\partial f_i}{\partial x_j}$ *is the image in* R *of the partial derivative of* f_i *with respect to* X_j. *Write* $t = n - 2d + 1 \geq 1$. *Let* θ' *be the submatrix of* θ *consisting of the last* t *rows. If* $I_t(\theta) = I_t(\theta')$, *then* $\operatorname{ht} I_t(\theta) < d$.

Proof Write $\theta' = \begin{pmatrix} \theta_1 & | & \theta_2 \end{pmatrix}$ where θ_1 is a square t by t matrix and θ_2 is a t by $d-1$ matrix. Let Θ be the transpose of the Jacobian matrix associated with $f_1, \ldots, f_{n-d}$. Let Θ' be the submatrix of Θ consisting of the last t rows, and we write $\Theta' = \begin{pmatrix} \Theta_1 & | & \Theta_2 \end{pmatrix}$ where Θ_1 is a square t by t matrix.

We claim that we may assume $I_1(\Theta_2) \subset (X_1, \ldots, X_{n-1}, X_n^2)$. This is because after factoring out by $(X_1, \ldots, X_{n-1})$, we are in the ring $\overline{k[X_1, \ldots, X_n]} \cong k[X_n]$. Since Θ_1 has t columns and Θ_2 has t rows, we can perform elementary column operations which do not change I, $I_t(\theta)$, and $I_t(\theta')$, until $I_1(\overline{\Theta_2}) \subset (\overline{X_n}^2)$. Note that we only need to perform k-linear combination involving columns of the same degree, hence preserving the homogeneity property. Thus, the claim is proved.

Now, let $\Delta_{[i_1,\ldots,i_t]\times[j_1,\ldots,j_t]}$ be the determinant of the submatrix of θ consisting of entries from rows $i_1, \ldots, i_t$ and columns $j_1, \ldots, j_t$. Write $\Delta_{[i_1,\ldots,i_t]} = \Delta_{[i_1,\ldots,i_t]\times[1,\ldots,t]}$, $\partial f_i / \partial x_j = a_{ji}$ and $[1, \ldots, \hat{j}, \ldots, t]$ for the tuple of integers from 1 to t with j removed. By the Laplace expansion for determinants, we have

$$\Delta_{[2d,\ldots,n]} = \sum_{j=1}^{t} (-1)^{j+t} a_{nj} \Delta_{[2d,\ldots,n-1]\times[1,\ldots,\hat{j},\ldots,t]}. \tag{3.1}$$

Denote $\deg X_i = \delta_i$. Since all f_j are homogeneous, there exists the following Euler relations in R. For all $1 \leq j \leq n-d$,

$$\sum_{i=1}^{n} \delta_i x_i a_{ij} = 0$$

where x_i is the image of X_i in R.

Solving for $\delta_n x_n a_{nj}$ and substituting into the equation (3.1) above, we obtain

$$\begin{aligned}\delta_n x_n \Delta_{[2d,\ldots,n]} &= \sum_{j=1}^{t}(-1)^{j+t}\delta_n x_n a_{nj}\Delta_{[2d,\ldots,n-1]\times[1,\ldots,\hat{j},\ldots,t]} \\ &= \sum_{j=1}^{t}\sum_{i=1}^{n-1}(-1)^{j+t+1}\delta_i x_i a_{ij}\Delta_{[2d,\ldots,n-1]\times[1,\ldots,\hat{j},\ldots,t]} \\ &= \sum_{i=1}^{n-1}\delta_i x_i \sum_{j=1}^{t}(-1)^{j+t+1}a_{ij}\Delta_{[2d,\ldots,n-1]\times[1,\ldots,\hat{j},\ldots,t]} \\ &= \sum_{i=1}^{n-1}(-1)^i\delta_i x_i \Delta_{[i,2d,\ldots,n-1]\times[1,\ldots,t]}\ .\end{aligned} \tag{3.2}$$

Since $I_t(\theta) = I_t(\theta')$,

$$\Delta_{[i,2d,\ldots,n-1]\times[1,\ldots,t]} = \sum_{1\leq\lambda_1<\cdots<\lambda_t\leq n-d} h_{i,\lambda_i,\ldots,\lambda_t}\Delta_{[2d,\ldots,n]\times[\lambda_1,\ldots,\lambda_t]},$$

for $h_{i,\lambda_i,\ldots,\lambda_t} \in R$. Thus we can rearrange the equation (3.2), and obtain

$$\Delta_{[2d,\ldots,n]}(\delta_n x_n - \sum_{i=1}^{n-1}(-1)^i\delta_i x_i h_{i,1,\ldots,t}) = \sum_{i=1}^{n-1}\sum_{[\lambda_i,\ldots,\lambda_t]\neq[1,\ldots,t]}(-1)^i\delta_i x_i h_{i,\lambda_1,\ldots,\lambda_t}\Delta_{[2d,\ldots,n]\times[\lambda_i,\ldots,\lambda_t]}. \tag{3.3}$$

Now suppose that ht $I_t(\theta')$ = ht $I_t(\theta) = d$. We notice that this is the maximal possible height of the ideal of minors of size t of θ' because $(n-d)-t+1 = n-d-(n-2d+1)+1 = d$. By Theorem 2.4.2, the Eagon-Northcott complex associated to θ'

$$E_. : 0 \to E_{n-m+1} \to E_{n-m} \to \ldots \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0$$

is acyclic. Hence ker d_1 = im d_2. From equation (3.3), we have

$$(\delta_n x_n - \sum_{i=1}^{n-1} x_i g_i, g_2', \ldots, g_k') \in \ker\, d_1 = \mathrm{im}\, d_2$$

where g_i and g_i' are elements in R.

By the construction of the Eagon-Northcott complex (Section 2.4), the entries in the first row of d_2 are in the ideal generated the entries of θ_2. Hence,

$$\delta_n x_n - \sum_{i=1}^{n-1} x_i g_i \in I_1(\theta_2).$$

Hence, back in $k[X_1, \ldots, X_n]$, we have

$$\delta_n X_n - \sum_{i=1}^{n-1} X_i G_i \in I_1(\Theta_2) + I$$

where G_i is the preimage of g_i in $k[X_1, \ldots, X_n]$.

But this is impossible because δ_n is a unit, $I_1(\Theta_2) \subset (X_1, \ldots, X_{n-1}, X_n^2)$ and $I \subset (X_1, \ldots, X_n)^2$. ■

We are now ready to state the main theorem of this chapter.

Theorem 3.2.1 *Let $R = k[X_1, \ldots, X_n]/I$ be a standard graded k-algebra with char $k = 0$. Let I be generated by a homogeneous regular sequence. Assume* edim $R_p \leq 2$ dim R_p *for every prime ideal $p \in \mathrm{Spec}(R) \backslash V(R_+)$. Then*

$\mathcal{R}(\Omega_k(R))$ is Cohen-Macaulay if and only if edim $R_p < 2$ dim R_p *for every non-minimal homogeneous prime ideal $p \in \mathrm{Spec}(R)$.*

Proof The backward direction [$\Leftarrow$] is obvious by Theorem 2.2.1 and Proposition 3.1.4. We want to note that the conditions F_k can be rephrased as conditions of the heights of Fitting ideals. Since the ring is graded, it is enough to check the conditions locally at homogeneous ideals. We prove the forward direction [$\Rightarrow$].

Let $d = \dim R \geq 1$. By the hypothesis, locally at codimension $d-1$, $\Omega_k(R)$ satisfies condition F_0 hence by Theorem 3.1.1, also condition F_1. Now localize at the homogeneous maximal ideal and we may also assume $I \subset (X_1, \ldots, X_n)^2$. Let I be generated by a homogeneous regular sequence $f_1, f_2, \ldots, f_{n-d}$. Write $n = \text{edim } R$. Assume to the contrary that $n \geq 2d$. If $d = 1$, then the Cohen-Macaulayness of $\mathcal{R}(\Omega_{R/k})$ implies that $\Omega_{R/k}$ modulo its torsion is free by Theorem 2.2.4. Thus by Theorem 2.3.3, R is regular, hence $n = d$, which contradicts $n \geq 2d = 2$. We now assume $d \geq 2$.

We want to reduce the case where the residue field extension $k \subset R/m$ is algebraic. Let $r = \mathrm{trdeg}_k(R/m) \geq 1$. Write $R = k[x_1, \ldots, x_n]_m$ and pick r general linear k-combination $y_1, \ldots, y_r$ so that their residues yield a transcendence basis for R/m

over k. Observe that $L = k(y_1, \ldots, y_r)$ is a subfield of R. Furthermore, $\Omega_L(R) = \Omega_k(R)/(Rdy_1 + \ldots + Rdy_r)$ where $d : R \to \Omega_k(R)$ is the universal derivation of R over k. By Proposition 3.1.3, rank $\Omega_k(R) > r$. Now we have an isomorphism

$$\mathcal{R}(\Omega_L(R)) \cong \mathcal{R}(\Omega_k(R))/(dy_1 + \ldots + dy_r),$$

which shows $\mathcal{R}(\Omega_L(R))$ is Cohen-Macaulay because $\mathcal{R}(\Omega_k(R))$ is Cohen-Macaulay. Hence, we can replace k by L to assume the residue field extension is algebraic.

By Proposition 3.1.1 and Proposition 3.1.2, $\Omega_k(R)$ has projective dimension at most one. If it is zero, then $\Omega_k(R)$ is free, hence by Theorem 2.3.1, R is regular, hence $n = d$, which is a contradiction.

Now consider a presentation of $\Omega_k(R)$ given by the n by $n - d$ matrix

$$\theta = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_{n-d}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_{n-d}}{\partial x_n} \end{bmatrix},$$

where $\frac{\partial f_i}{\partial x_j}$ is the image in R of the partial derivative of f_i with respect to X_j. Write

$$t = \mu(\Omega_{R/k}) - \dim R - \operatorname{rank} \Omega_{R/k} + 1 = n - 2d + 1 \geq 1.$$

We can assume ht $I_t(\theta) = d$. Otherwise we choose a prime q of height at most $d-1$ containing $I_t(\theta)$, $(\Omega_k(R))_q$ satisfies F_0 by hypothesis. Hence ht $I_t(\theta)_q \geq n-d-t+1 = n - d - (n - 2d + 1) + 1 = d$, which is absurd.

Since $\mathcal{R}(\Omega_{R/k})$ is Cohen-Macaulay and locally satisfies condition F_1 on the punctured spectrum, by Theorem 2.2.3, after elementary row operations over k, $I_t(\theta) = I_t(\theta')$ where θ' is the submatrix of θ consisting of the last t rows. Notice this change of variables corresponds to a general change of generators $x_1, \ldots, x_n$.

However, by Proposition 3.2.1, ht $I_t(\theta) < d$, which is a contradiction. ■

Comparing to [16, 3.1], Theorem 3.2.1 weakens the condition F_0 globally to F_0 locally on the punctured spectrum, but the ring needs to be homogeneous. As a corollary, for a smooth non-degenerate complete intersection subvariety, the Cohen-Macaulay property of the Rees algebra of $\Omega_{R/k}$ corresponds to a rather restrictive condition on the dimension of the subvariety.

Corollary 3.2.2 *Let k be a field of characteristic 0. Let $V \subset \mathbb{P}^n$ be a smooth non-degenerate complete intersection subvariety of dimension d. Let R be the homogeneous coordinate ring of V. Then $\mathcal{R}(\Omega_{R/k})$ is Cohen-Macaulay if and only if $n \leq 2d$.*

Proof It is immediate from Theorem 3.2.1 after we notice that edim $R = n + 1$ and we only need to check the inequality at the homogeneous maximal ideal. ■

4. REES ALGEBRAS OF JACOBIAN MODULES

In this chapter, we compute the defining ideal of the Rees algebra of a Jacobian module, then characterize its Cohen-Macaulay property.

First, recall the Jacobian module. Let $R = S/I$ where $S = W^{-1}A[X_1, \dots, X_n]$, $I = (f_1, \dots, f_m)$, and

$$S^n \xrightarrow{\phi} S^m \quad \text{with} \quad \phi = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_n} \end{bmatrix}.$$

The matrix ϕ is the Jacobian matrix. If we denote $^-$ for the images in R, then the image of $\bar{\phi}$ in R^m is the *Jacobian module of* R *over* A. If $m = 1$, then the Jacobian module is the Jacobian ideal, denoted by $\mathrm{Jac}(I)$ or $\mathrm{Jac}(f)$,

$$\mathrm{Jac}(I) = \mathrm{Jac}(f) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$$

where $I = (f)$ and $\frac{\partial f}{\partial x_i} = \overline{\frac{\partial f}{\partial X_i}}$. For simplicity of notation, we write $f_{X_i} = \frac{\partial f}{\partial X_i}$.

Notice that $\mathcal{R}(\mathrm{Jac}(I)) = R[\frac{\partial f}{\partial x_1}t, \dots, \frac{\partial f}{\partial x_n}t]$. We can map $R[T_1, \dots, T_n]$ surjectively onto the Rees algebra of $\mathrm{Jac}(f)$ by sending T_i to $\frac{\partial f}{\partial x_i}t$. The kernel, denoted by $\mathcal{J}(f)$, is the defining ideal of $\mathcal{R}(\mathrm{Jac}(I))$, i.e.

$$\mathcal{R}(\mathrm{Jac}(I)) = \frac{R[T_1, \dots, T_n]}{\mathcal{J}(f)}.$$

4.1 Main result

We start with a theorem that will be a helpful tool in computing the defining ideals.

Theorem 4.1.1 *Let A be a Noetherian unique factorization domain and W a multiplicative subset of A. Let $R = S/I$ where $S = W^{-1}A[X_1, \dots, X_n]$ and $I = (f) = (gh)$.*

Denote $\mathcal{J}(-)$ for the defining ideal of the Rees algebra of a Jacobian ideal as above. Write $-'$ for the preimage in $S[T_1, \ldots, T_n]$. If g, h generate an ideal of positive height in R, then $\mathcal{J}(f)' = \mathcal{J}(g)' \cap \mathcal{J}(h)'$.

Proof Recall $\mathcal{J}(f) = (w \text{ homogeneous form in } R[T_1, \ldots, T_n] \mid w(f_{x_1}, \ldots, f_{x_n}) = 0)$. Let w' be the preimage of w in $S[T_1, \ldots, T_n]$. We have $w'(f_{X_1}, \ldots, f_{X_n}) \in (f)$. Similarly for $\mathcal{J}(g)$ and $\mathcal{J}(h)$.

Claim: For any homogeneous form $L \in S[T_1, \ldots, T_n]$,

$$L(f_{X_1}, \ldots, f_{X_n}) = L(g_{X_1}, \ldots, g_{X_n})h^{\deg L} + ug,$$

for some $u \in S$. It is enough to show the claim for L a monomial.

We induct on the degree of L. Notice that the base case deg $L = 0$ is clear. For the inductive step, without loss of generality, we write $L = T_1 \cdot L'$ where L' is a monomial with deg $L' =$ deg $L - 1$. By the induction hypothesis, we obtain

$$\begin{aligned} L(f_{X_1}, \ldots, f_{X_n}) &= f_{X_1} \cdot L'(f_{X_1}, \ldots, f_{X_n}) \\ &= (g_{X_1}h + gh_{X_1})(L'(g_{X_1}, \ldots, g_{X_n})h^{\deg L'} + ug) \\ &= g_{X_1}L'(g_{X_1}, \ldots, g_{X_n})h^{\deg L} + u'g \\ &= L(g_{X_1}, \ldots, g_{X_n})h^{\deg L} + u'g, \end{aligned} \tag{4.1}$$

for some $u' \in S$. Hence, the inductive step is complete, and the claim is proved.

We now prove $\mathcal{J}(f)' \subset \mathcal{J}(g)' \cap \mathcal{J}(h)'$. Since g and h play the same role, it is enough to prove $\mathcal{J}(f)' \subset \mathcal{J}(g)'$. Let $w' \in \mathcal{J}(f)'$. Hence, $w'(f_{X_1}, \ldots, f_{X_n}) \in (f) \subset (g)$. By the claim, we have

$$w'(f_{X_1}, \ldots, f_{X_n}) = w'(g_{X_1}, \ldots, g_{X_n})h^{\deg w'} + ug \in (g).$$

Thus,

$$w'(g_{X_1}, \ldots, g_{X_n})h^{\deg w'} \in (g).$$

Since ht $(g, h)S = 2$, $(g, h)S$ is a complete intersection ideal in S , we obtain $w'(g_{X_1}, \ldots, g_{X_n}) \in (g)$. Hence, $w' \in \mathrm{Jac}(g)'$. Similarly for $\mathrm{Jac}(h)'$.

Finally, we prove $\mathcal{J}(g)' \cap \mathcal{J}(h)' \subset \mathcal{J}(f)'$. Let $w' \in \mathcal{J}(g)' \cap \mathcal{J}(h)'$. Hence, $w'(g_{X_1}, \ldots, g_{X_n}) \subset (g)$, and $w'(h_{X_1}, \ldots, h_{X_n}) \subset (h)$. By the claim, we have

$$w'(f_{X_1}, \ldots, f_{X_n}) = w'(g_{X_1}, \ldots, g_{X_n})h^{\deg w'} + ug.$$

Observe that the right hand side is in (g). Hence $w'(f_{X_1}, \ldots, f_{X_n}) \in (g)$. The same argument for h shows $w'(f_{X_1}, \ldots, f_{X_n}) \in (h)$. Thus, $w'(f_{X_1}, \ldots, f_{X_n}) \in (g) \cap (h) = (gh) = (f)$ because $(g,h)S$ is a complete intersection ideal in S. ■

From this theorem, notice that if we have the defining ideals of the Rees algebras of $\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_s)$ where any f_i, f_j with $i \neq j$ generate an ideal of positive height in R, then we can compute the defining ideal of the Rees algebra of $\mathrm{Jac}(f_1 \ldots f_s)$. The next proposition gives explicit formulas for the defining ideals of the Rees algebra for some classes of ideals.

Proposition 4.1.1 *Let $R = S/I$ with $S = k[X_1, \ldots, X_n]$, k a field, and $I = (f)$. Consider a presentation of the Rees algebra of* $\mathrm{Jac}(f)$

$$\mathcal{R}(\mathrm{Jac}(I)) = \frac{R[T_1, \ldots, T_n]}{\mathcal{J}(f)}.$$

1. *If $f = X_i^\delta$, then*

$$\mathcal{J}(f) = \begin{cases} (T_1, \ldots, \widehat{T_i}, \ldots, T_n) & , \text{ if } \delta = 1 \\ (x_iT_i, T_1, \ldots, T_i^2, \ldots, T_n) & , \text{ if } \delta \geq 2 \end{cases}.$$

2. *If $f = X_1^{\delta_1} \ldots X_n^{\delta_n}$, $\mathcal{I}_1 = \{\delta_i \mid \delta_i \geq 1\}$, and $\mathcal{I}_2 = \{\delta_i \mid \delta_i > 1\}$, then*

$$\mathcal{J}(f) = (\{x_iT_i,\ T_iT_j \mid i,j \text{ in } \mathcal{I}_1 \text{ and } i \neq j\}) + (\{T_i^2 \mid i \in \mathcal{I}_2\}) + (\{T_i \mid i \notin \mathcal{I}_1\}).$$

Proof Part (1) is clear if $\delta = 1$. Now assume $\delta \geq 2$, we notice that $f_{X_\gamma} = 0$ for all $\gamma \neq i$; therefore $T_1, \ldots, \widehat{T_i}, \ldots, T_n \in \mathcal{J}(f)$. Factoring out these variables $T_1, \ldots, \widehat{T_i}, \ldots, T_n$, we are in the ring $\overline{R[T_1, \ldots, T_n]} \cong R[T_i]$. Since $f_{X_i} = \delta X_i^{\delta-1}$, the forms $\overline{x_iT_i}$ and $\overline{T_i^2}$ generate the whole defining ideal in $\overline{R[T_1, \ldots, T_n]}$. Thus, in $R[T_1, \ldots, T_n]$, $\mathcal{J}(f) = (x_iT_i, T_1, \ldots, T_i^2, \ldots, T_n)$.

We now prove part (2). We can assume $\mathcal{I}_1 = \{1, \ldots, n\}$, i.e. $\delta_i \geq 1$ for all i since otherwise, we can just drop the variables with $\delta_i = 0$. Let $J = \mathcal{J}(f)$, and $J_i = \mathcal{J}(X_i^{\delta_i})$. Denote $-'$ the preimage in $S[T_1, \ldots, T_n]$. By Theorem 4.1.1 and induction, we have

$$J' = J_1' \cap \ldots \cap J_n'.$$

By part (1) of this theorem, we obtain

$$J' = \bigcap_{i \notin \mathcal{I}_2} (X_i^{\delta_i}, T_1, \ldots, \widehat{T_i}, \ldots, T_n)) \cap \bigcap_{i \in \mathcal{I}_2} (X_i^{\delta_i}, X_i T_i, T_1, \ldots, T_i^2, \ldots, T_n). \tag{4.2}$$

All ideals in the equation (4.2) are monomial ideals, and there are explicit formula for intersections of monomial ideals. We compute the first intersection. Notice, if $i \notin \mathcal{I}_2$, then $\delta_i = 1$.

$$\begin{aligned} \bigcap_{i \notin \mathcal{I}_2} (X_i, T_1, \ldots, \widehat{T_i}, \ldots, T_n) &= (\Pi_{i \notin \mathcal{I}_2} X_i) + (\{T_l T_j \mid l \neq j\}) \\ &\quad + (\{T_j \mid j \in \mathcal{I}_2\}) + (\{X_i T_i \mid i \notin \mathcal{I}_2\}). \end{aligned}$$

We want to point out that all monomial generators on the right hand side are expected because they are the obvious least common multiples of the generators on the left hand side. However, perhaps it is curious that the mixed products of the form $X_i T_j$ do not appear on the right hand side. In order for $X_i T_j$ to appear, $i \notin \mathcal{I}_2$. If $j \in \mathcal{I}_2$, then $X_i T_j$ is already in the ideal generated by $\{T_i \mid i \in \mathcal{I}_2\}$. If $j \notin \mathcal{I}_2$, then $X_i T_j \notin (X_j, T_1, \ldots, \widehat{T_j}, \ldots, T_n)$ with $i \neq j$.

Next, we compute the other intersection,

$$\begin{aligned} \bigcap_{i \in \mathcal{I}_2} (X_i^{\delta_i}, X_i T_i, T_1, \ldots, T_i^2, \ldots, T_n) &= (\Pi_{i \in \mathcal{I}_2} X_i^{\delta_i}) + (\{T_i T_j \mid i \neq j\}) + (\{T_i \mid i \notin \mathcal{I}_2\}) \\ &\quad + (\{T_i^2 \mid i \in \mathcal{I}_2\}) + (\{X_i T_i \mid i \in \mathcal{I}_2\}). \end{aligned}$$

Similarly, all monomial generators on the right hand side are expected, and the mixed products of the form $X_i^{\delta_i}T_j$ do not appear. Now we can compute J',

$$
\begin{aligned}
J' = &\Big((\Pi_{i\notin\mathcal{I}_2}X_i) + (\{T_iT_j \mid i\neq j\}) + (\{T_i \mid i\in\mathcal{I}_2\}) + (\{X_iT_i \mid i\notin\mathcal{I}_2\})\Big) \quad \bigcap \\
&\Big((\Pi_{i\in\mathcal{I}_2}X_i^{\delta_i}) + (\{T_iT_j \mid i\neq j\}) + (\{T_i \mid i\notin\mathcal{I}_2\}) + (\{T_i^2 \mid i\in\mathcal{I}_2\}) + (\{X_iT_i \mid i\in\mathcal{I}_2\})\Big) \\
= &\ (\Pi X_i^{\delta_i}) + (\{T_iT_j \mid i\neq j\}) + (\{T_i^2 \mid i\in\mathcal{I}_2\}) + (\{X_iT_i \mid i\in\mathcal{I}_2\}) + (\{X_iT_i \mid i\notin\mathcal{I}_2\}) \\
= &\ (\Pi X_i^{\delta_i}) + (\{T_iT_j \mid i\neq j\}) + (\{T_i^2 \mid i\in\mathcal{I}_2\}) + (\{X_iT_i,\ \forall\, i\}) \\
= &\ (\{f,\ X_iT_i,\ T_iT_j \mid i\neq j\}) \ + (\{T_i^2 \mid i\in\mathcal{I}_2\}).
\end{aligned}
$$

This completes the proof.

■

Before we characterize the Cohen-Macaulay property of the Rees algebras of these Jacobian ideals, we first compute the Krull-dimension.

Theorem 4.1.2 *Let $R = S/I$ with $S = k[X_1,\ldots,X_n]$ a polynomial ring over a field k, and $I = (f)$. Write $f = X_1^{\delta_i}\ldots X_n^{\delta_n}$. Without loss of generality, assume $\delta_i \geq 1$ for all i. Then*

$$
\dim \mathcal{R}(\mathrm{Jac}(f)) = \begin{cases} n & , \quad \text{if } \ \delta_i = 1 \text{ for some } i \\ n-1 & , \quad \text{otherwise} \end{cases}.
$$

Proof We first observe that Proposition 2.1.1 cannot be used because in most cases, $\mathrm{ht}\ \mathrm{Jac}(I) = 0$. However, by Proposition 2.1.2, we have

$$
\dim \mathcal{R}(\mathrm{Jac}(f)) = \max\ \{\dim R, 1+\dim R/p \mid p \in \mathrm{Min}(R)\backslash V(\mathrm{Jac}(f))\}.
$$

We can detect all minimal prime ideals of R from their preimages in S. Let $q\in\mathrm{Spec}(S)$ such that q is a minimal prime of $(X_1^{\delta_i}\ldots X_n^{\delta_n})$. Then $q = (X_i)$ for some i.

Let $^-$ denote the image in R, and $\mathcal{I} = \{i \mid \delta_i = 1\}$. For $i\in\mathcal{I}$, $\bar{q} = \overline{(X_i)}$ is a minimal prime that does not contain $\mathrm{Jac}(I)$. Therefore, if $\mathcal{I}\neq\emptyset$,

$$
\max\ \{1+\dim R/\bar{q} \mid \bar{q}\in\mathrm{Min}(R)\backslash V(\mathrm{Jac}(f))\} = 1+(n-1) = n.
$$

On the other hand, if $\mathcal{I} = \emptyset$, then

$$\dim \mathcal{R}(\mathrm{Jac}(f)) = \dim R = n - 1.$$

■

We recall a remark about monomial ideals. Let $R = k[X_1, \ldots, X_n]$ be a polynomial in n variables over a field k. Let I be a monomial ideal, h an element of R. We write $h = h_1 + \cdots + h_t$ where h_i are distinct monomials. If $h \in I$, then $h_i \in I$ for all i.

Proposition 4.1.2 *Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring in n variables over a field k. Let $R = S/I$ where I is a monomial ideal so that $(I :_S X_1) \cap (I :_S X_2) \subset I$. Then $X_1 - X_2$ is a non zero-divisor on R.*

Proof By the hypothesis, we have $I : (X_1, X_2) = (I : X_1) \cap (I : X_2) \subset I$. Thus, (X_1, X_2) is not contained in any associated prime of I. However, since all associated prime ideals of I are monomial ideals, $X_1 - X_2$ is not in any associated prime of I. Hence, $X_1 - X_2$ is a non zero-divisor on R. ■

Theorem 4.1.3 *Let $R = S/I$ with $S = k[X_1, \ldots, X_n]$ a polynomial ring over a field, and $I = (f)$. Write $f = X_1^{\delta_i} \ldots X_n^{\delta_n}$. Without loss of generality, assume $\delta_i \geq 1$ for all i.*

1. *If $n = 1$, then the Rees algebra $\mathcal{R}(\mathrm{Jac}(f))$ is Cohen-Macaulay.*
2. *If $n \geq 2$, then the Rees algebra $\mathcal{R}(\mathrm{Jac}(f))$ is Cohen-Macaulay if and only if $\delta_i \geq 2$ for all i.*

Proof Part (1) follows easily from Proposition 4.1.1, part (1).

Assume $n \geq 2$. We first prove the backward direction [$\Leftarrow$]: if $\delta_i \geq 2$ for all i, then the Rees algebra $\mathcal{R}(\mathrm{Jac}(f))$ is Cohen-Macaulay. By Proposition 4.1.1, part (2), we can write

$$\mathcal{R}(\mathrm{Jac}(f)) = \frac{S[T_1, \ldots, T_n]}{J'}$$

where $S[T_1,\ldots,T_n]$ is a polynomial in the variables $X_1,\ldots,X_n,T_1,\ldots,T_n$ over the field k and

$$J' = (f, X_1T_1,\ldots,X_nT_n,T_1^2,\ldots,T_n^2) + (\{T_iT_j \mid i\neq j\}).$$

By Theorem 4.1.2, dim $\mathcal{R}(\mathrm{Jac}(f)) = n-1$. We claim that $X_1-X_2, X_1-X_3,\ldots,X_1-X_n$ is a regular sequence of length $n-1$ on $\mathcal{R}(\mathrm{Jac}(f))$. With our hypothesis, $\mathcal{R}(\mathrm{Jac}(f))$ is standard graded k-algebra. Hence once the claim is proved, we are done with this direction.

Induct on the length r of the sequence. For the base case $r=1$, we can apply Proposition 4.1.2 after verifying that

$$(J' : X_1)\cap(J' : X_2)\subset J'.$$

We compute the quotient ideals on the left hand side.

$$J' : X_1 = (\frac{f}{X_1}, T_1) + J', \qquad J' : X_2 = (\frac{f}{X_2}, T_2) + J'.$$

Hence, we obtain

$$(J' : X_1)\cap(J' : X_2) = \Big((\tfrac{f}{X_1}, T_1) + J'\Big)\cap\Big((\tfrac{f}{X_2}, T_2) + J'\Big)\subset J'.$$

For the inductive step, we want to show $X_1 - X_{r+1}$ is a non zero-divisor on

$$\frac{S[T_1,\ldots,T_n]}{J' + (X_1-X_2,\ldots,X_1-X_r)} \cong$$
$$\frac{k[X_1,X_{r+1},\ldots,X_n][T_1,\ldots,T_n]}{(X_1^{\delta_1+\cdots+\delta_r}X_{r+1}^{\delta_{r+1}}\ldots X_n^{\delta_n},\ X_1T_1,\ldots,X_1T_r,X_{r+1}T_{r+1},\ldots,X_nT_n,\ T_1^2,\ldots,T_n^2) + (\{T_iT_j \mid i\neq j\})}$$
$$=: A/L.$$

Again, we apply Proposition 4.1.2. It remains to show

$$(L : X_1)\cap(L : X_{r+1})\subset L.$$

Write $g = X_1^{\delta_1+\cdots+\delta_r}X_{r+1}^{\delta_{r+1}}\ldots X_n^{\delta_n}$. We compute the quotient ideals on the left hand side and obtain

$$L : X_1 = (\frac{g}{X_1}, T_1,\ldots,T_r) + L,$$
$$L : X_{r+1} = (\frac{g}{X_{r+1}}, T_{r+1}) + L.$$

The intersection computation follows similarly as above

$$(L : X_1) \cap (L : X_{r+1}) = \Big((\tfrac{g}{X_1}, T_1, \ldots, T_r) + L \Big) \cap \Big((\tfrac{g}{X_{r+1}}, T_{r+1}) + L \Big) \subset L.$$

This completes the backward direction.

We now prove the forward direction [$\Rightarrow$] by proving the contrapositive. If $\delta_i = 1$ for some i, then the Rees algebra $\mathcal{R}(\mathrm{Jac}(f))$ is not Cohen-Macaulay. By Proposition 4.1.1, part (2), we can write

$$\mathcal{R}(\mathrm{Jac}(f)) = \frac{S[T_1, \ldots, T_n]}{J'},$$

with

$$J' = (f, X_1T_1, \ldots, X_nT_n) + (\{T_iT_j \mid i \neq j\}) + (\{T_i^2 \mid i \notin \mathcal{I}\}),$$

where $\mathcal{I} = \{i \mid \delta_i = 1\} \neq \emptyset$.

By Theorem 4.1.2, dim $\mathcal{R}(\mathrm{Jac}(R)) = n$. Notice that $X_1 - X_2, \ldots, X_1 - X_n$ is still a regular sequence because the induction proof above follows through the same way. We only need to show

$$\operatorname{depth} \frac{S[T_1, \ldots, T_n]}{\mathcal{J}' + (X_1 - X_2, \ldots, X_1 - X_n)} = 0.$$

Notice that

$$\begin{aligned}&\frac{S[T_1, \ldots, T_n]}{\mathcal{J}' + (X_1 - X_2, \ldots, X_1 - X_n)} \cong \\ &\frac{k[X_1][T_1, \ldots, T_n]}{(X_1^{\deg f}, X_1T_1, \ldots, X_1T_n) + (\{T_iT_j \mid i \neq j\}) + (\{T_i^2 \mid i \notin \mathcal{I}\})}.\end{aligned}$$

Since $n \geq 2$, the degree of f is at least 2. Hence,

$$X_1^{\deg f - 1}(X_1, T_1, \ldots, T_n) \subset (X_1^{\deg f}, X_1T_1, \ldots, X_1T_n).$$

Therefore, the depth of the quotient ring is 0, which completes the proof. ■

As the final remark for this chapter, we observe from the proof above that in the setting of the theorem, the Rees algebra $\mathcal{R}(\mathrm{Jac}(f))$ is either Cohen-Macaulay or almost Cohen-Macaulay.

5. THE GORENSTEIN LINKAGE CLASS OF A COMPLETE INTERSECTION

In this chapter, we are interested in monomial ideals of finite colength. In Section 5.1, we review some known results on monomial ideals in the Gorenstein linkage class of a complete intersection. In Section 5.2, using Macaulay inverse systems, we introduce a Gorenstein double linkage construction that is analogous to Gorenstein basic double linkage, and we use it to identify classes of monomial ideals that are homogeneously glicci.

5.1 Background

We recall the definition of Gorenstein linkage.

In a regular local ring R, we say two proper ideals I, J are Gorenstein directly linked if there exists a Gorenstein ideal K so that $I = K : J$ and $J = K : I$. By Theorem 2.6.4, if I is unmixed of height g and $K \subsetneq I$ is a Gorenstein ideal of the height g, then $J = K : I$ is Gorenstein directly linked to I. We denote $I \sim_G J$. We say I and J are in the same Gorenstein linkage class if $I = I_0 \sim_G I_1 \sim_G \cdots \sim_G I_n = J$. Analogously, when the ring R is a graded regular ring, in addition, if $I,\ J$ and K are homogeneous ideals, we say I, J are homogeneously Gorenstein directly linked. We say an ideal is homogeneously glicci if it is in the homogeneous Gorenstein linkage class of a complete intersection.

Gorenstein basic double linkage (Gorenstein biliaison) is a common tool in proving a class of ideals to be glicci. The general idea is that after a double link, the linked ideal is "simpler" than the original ideal. Hence after finitely many steps, we may arrive at a complete intersection. We recall the definition of Gorenstein basic double linkage.

Proposition 5.1.1 ([9, 4.1]) *Let (R, m) be a local Gorenstein ring, K and H proper ideals of R, and $x \in m$. Assume that R/K is Cohen-Macaulay and generically Gorenstein, and that $K+xH$ is unmixed of height greater than the height of K. Then $K+xH$ and $K+H$ are Gorenstein linked in two steps.*

The graded case can be found in [11, 5.10]. The following theorems describe some classes of glicci ideals, and they use Gorenstein basic double linkage in an essential way in their proofs.

Theorem 5.1.1 ([13, 3.1]) *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over an infinite field k, and I a monomial ideal of finite colength. Then I can be deformed, via the procedure described in* [13, 2.1]*, to an ideal $J \subset R[T_1, \ldots, T_s]$ so that J is homogeneously glicci.*

Theorem 5.1.2 ([13, 3.1]) *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k of characteristic* 0*, and I a monomial ideal. If I is Borel-fixed, then I is homogeneously glicci.*

A monomial ideal I is said to be Borel-fixed if for any monomial $w \in I$, if $X_i \mid w$, then $\frac{X_j}{X_i} w \in I$ for all $1 \leq j < i \leq n$.

Theorem 5.1.3 ([9, 4.2]) *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over an infinite field k, m the homogeneous maximal ideal, and I an m-primary monomial ideal. Then I_m is glicci in R_m.*

The main difference between Theorem 5.1.1 and Theorem 5.1.3 is that in Theorem 5.1.3, the Gorenstein links are not necessarily homogeneous, and in Theorem 5.1.1, the Gorenstein linkage does not occur in the base ring. In the next section, we identify a class of monomial ideals of finite colength that is homogeneously glicci in the base ring.

We recall an explicit (complete intersection) double linkage that is used in [9]. We generalize this result to Theorem 5.2.1.

Proposition 5.1.2 ([9, 2.5]) *Let $R = k[X_1, \ldots, X_n]$, $m = (X_1, \ldots, X_n)$, and I a monomial m-primary ideal. If $I = (X^{d_1}, \ldots, X^{d_n}) + X_1^{a_1} \ldots X_n^{a_n} H$ for some monomial ideal $H \neq R$, then I is homogeneously doubly linked to*

$$J = (X^{d_1 - a_1}, \ldots, X^{d_n - a_n}) + H.$$

5.2 Main result

We use Macaulay inverse systems heavily in this section to prove the main theorem about Gorenstein double linkage.

Let $R = k[X_1, \ldots, X_n]$ and I a homogeneous ideal of finite colength. Let $T = k[X_1^{-1}, \ldots, X_n^{-1}]$ with an R-module structure as defined in Section 2.5. Write $M_I = 0 :_T I$, which is the Matlis dual of R/I. Since the duality is inclusion-reversing, we need to find a homogeneous element $F \in T$ so that $M_I \subset R \cdot F$. Then $K = 0 :_R F$ is a homogeneous Gorenstein ideal of finite colength contained in I. All homogeneous Gorenstein ideals of finite colength in I can be obtained this way. Then $J = K : I$ is homogeneously Gorenstein directly linked to I.

Furthermore, inverse systems give a framework to compute linkage.

Proposition 5.2.1 *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k, and $T = k[X_1^{-1}, \ldots, X_n^{-1}]$ the set of "inverse polynomials" with an R-module structure as defined Section 2.5.*

If two proper homogeneous ideals I and J of finite colength are homogeneously Gorenstein directly linked by an ideal K, then $0 :_T J = I(0 :_T K)$.

Proof We only need to show the duals of both sides are the same, namely $J = 0 :_R (I(0 :_T K))$. But it follows from the associativity property of colon ideals

$$0 :_R (I(0 :_T K)) = 0 :_R ((0 :_T K)I) = (0 :_R (0 :_T K)) :_R I = K :_R I = J.$$

■

For example, let $R = k[X,Y,Z]$, $T = k[X^{-1}, Y^{-1}, Z^{-1}]$, and $I = (X^5, Y^4, Z^6, XY)$. Then dual of I is $0 :_T I = RX^{-4}Z^{-5} + RY^{-3}Z^{-5}$. Let $F = X^{-4}Y^{-3}Z^{-5}$. Write $K = 0 :_R F$. Then the ideal $J = K : I$ and its dual can be computed as

$$0 :_T J = I \cdot F = RX^{-3}Y^{-2}Z^{-5} \text{ and } J = (X^4, Y^3, Z^6).$$

We now state a theorem about a novel Gorenstein double linkage construction.

Theorem 5.2.1 *Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k, and I a homogeneous R-ideal of finite colength. If there exists a homogeneous Gorenstein ideal K of finite colength so that $K \subsetneq I \subset K + (g)$ for some homogeneous element $g \in R$, then I is homogeneously Gorenstein linked in two steps to $J = I : g$.*

Proof Let T be as in Proposition 5.2.1. Write $RF = 0 :_T K$, and $L = K :_R I$. To show $I \sim_G L \sim_G J$, we prove

1. $0 :_R RgF \subset J$, and
2. $L = (0 :_R gF) :_R J$.

The inclusion (1) is clear because

$$0 :_R RgF = (0 :_R RF) :_R g = K :_R g \subset I :_R g = J.$$

For the equality (2), we claim that $gJF = IF$. Once the claim is proved, we have

$$(0 :_R gF) :_R J = 0 :_R gJF = 0 :_R IF = (0 :_R F) :_R I = K :_R I = L.$$

For the claim, since $K \subset I \subset K + (g)$, we have $I = (K + (g)) \cap I = K + (g) \cap I = K + g(I : g) = K + gJ$. Hence $IF = gJF$ since $KF = 0$. ■

Proposition 5.1.2 is a special case of Theorem 5.2.1 where $K = (X^{d_1}, \ldots, X^{d_n})$ and $g = X_1^{a_1} \ldots X_n^{a_n}$. Using Theorem 5.2.1, we identify a class of monomial ideals of finite colength that are homogeneously glicci.

Theorem 5.2.2 *Let $R = k[X,Y,Z]$ be a polynomial ring over a field k. Let I be a monomial ideal of finite colength.*

1. *If $\mu(I) \leq 5$, then I is homogeneously licci.*

2. *Assume $I = (X^\alpha, Y^\beta, Z^\gamma, X^{\alpha_1}Y^{\beta_1}, Y^{\beta_2}Z^{\gamma_1}, X^{\alpha_2}Z^{\gamma_2})$ and either one of the following conditions holds*

   . $\alpha \leq \beta_2 + \gamma_1$,

   . $\beta \leq \alpha_2 + \gamma_2$,

   . $\gamma \leq \beta_1 + \gamma_1$.

   Then I is homogeneously glicci.

Proof For part (1), we induct on the sum of the degrees of a minimal generating set of I. If this sum is 3, then I is a complete intersection, and we are done. Assume this sum is greater than 3. We may write $I = (X^\alpha, Y^\beta, Z^\gamma) + L$, where L a monomial ideal and $\mu(L) = \mu(I) - 3$. Thus, $\mu(L) \leq 2$. Observe that L cannot be generated by a single variable since $\mu(L) = \mu(I) - 3$. Also, L cannot be generated by two relatively prime monomials because otherwise one of would have to be a power of a variable, contradicting the fact that $\mu(L) = \mu(I) - 3$.

If $L = 0$, then I is a complete intersection, and we are done. Otherwise, since I is generated by at most 2 monomials but cannot be generated by 2 relatively prime monomials or a single variable, we can write

$$L = X^{\alpha_1}Y^{\beta_1}Z^{\gamma_1}L_1,$$

where $\alpha_1 + \beta_1 + \gamma_1 > 0$ and $L_1 \neq R$. Notice L_1 is a monomial ideal and $\mu(L_1) = \mu(L)$. By Proposition 5.1.2, I is homogeneously doubly linked to

$$I_1 = (X^{\alpha-\alpha_1}, Y^{\beta-\beta_1}, Z^{\gamma-\gamma_1}) + L_1.$$

Now apply induction hypothesis to I_1, and we are done.

For part (2), without loss of generality, we assume $\alpha \leq \beta_2 + \gamma_1$. We claim it suffices to show that there exists a homogeneous Gorenstein ideal K so that $K \subsetneq I \subset K + (X)$. It is because by Theorem 5.2.1, I will be homogeneously Gorenstein doubly linked to

$I : X = (X^{\alpha-1}, Y^\beta, Z^\gamma, X^{\alpha_1-1}Y^{\beta_1}, Y^{\beta_2}Z^{\gamma_1}, X^{\alpha_2-1}Z^{\gamma_2})$. Since $I : X$ again satisfies the hypothesis, we can repeat this process inductively until $\alpha_1 = 0$ or $\alpha_2 = 0$. Then the linked ideal satisfies the assumption of part (1), hence I is homogeneously glicci.

Consider a homogeneous ideal $K = (Y^\beta, Z^\gamma, X^\alpha Y^{\beta-\beta_2}, X^\alpha Z^{\gamma-\gamma_1}, X^{\beta_2+\gamma_1} - Y^{\beta_2}Z^{\gamma_1})$. By the hypothesis, it is clear $K \subsetneq I \subset K + (X)$. Notice K has height 3. It remains to show K is a Gorenstein ideal. By Theorem 2.6.5, this follows easily from computing the 4 by 4 Pfaffians of the following alternating matrix

$$\psi = \begin{bmatrix} 0 & 0 & X^\alpha & 0 & Y^{\beta_2} \\ 0 & 0 & 0 & Y^{\beta-\beta_2} & Z^{\gamma-\gamma_1} \\ -X^\alpha & 0 & 0 & Z^{\gamma_1} & 0 \\ 0 & -Y^{\beta-\beta_2} & -Z^{\gamma_1} & 0 & -X^{\beta_2+\gamma_1-\alpha} \\ -Y^{\beta_2} & -Z^{\gamma-\gamma_1} & 0 & X^{\beta_2+\gamma_1-\alpha} & 0 \end{bmatrix}.$$

Let ψ_i be the submatrix of ψ with the ith row and the ith column removed. Then

$$\begin{aligned} \det \psi_1 &= Z^{2\gamma}, \\ \det \psi_2 &= (X^{\beta_2+\gamma_1} - Y^{\beta_2}Z^{\gamma_1})^2, \\ \det \psi_3 &= Y^{2\beta}, \\ \det \psi_4 &= X^{2\alpha}Z^{2(\gamma-\gamma_1)}, \\ \det \psi_5 &= X^{2\alpha}Y^{2(\beta-\beta_2)}. \end{aligned}$$

■

Finally, we want to remark the class of ideals in part (2) of Theorem 5.2.2 is not (locally) licci by [9, 2.6].

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VITA

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