AUTOMORPHISM GROUPS AND CHERN BOUNDS OF FIBRATIONS

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Dedicated to my wife and family.

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ABSTRACT

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In this thesis, I study two problems. First, I generalize a result by H-Y Chen [1] to show that if X is a smooth variety of general type and irregularity $q \geq 1$ that embeds into its Albanese variety as a smooth variety Y of general type with codimension one or two, then $|Aut(X)| \leq |Aut(F_{min})||Aut(Y)|$ where F_{min} is the minimal model of a general fiber. Then I describe a special type of fibration called a K-Fibration as a generalization to Kodaira Fibrations where we can compute its Chern numbers in dimensions 2 and 3. K-Fibrations act as an initial step in constructing examples of varieties that satisfy the generalization with the goal of computing their automorphism group explicitly.

1. INTRODUCTION

I work over $\mathbb{C}$. It is historical fact, proved by A. Hurwitz [2], that for a smooth curve C of genus g of at least 2, the cardinality of its automorphism group has an upper bound of $42(2g-2)$. In 1963, H. Matsumura showed that for a variety of general type, the automorphism group is finite [3] spurring a search for a bound. In the early 1990s, G. Xiao proved that for a minimal smooth surface S of general type (which for curves means genus at least two) that it has as upper bound of $(42K_S)^2$ where K_S is the canonical divisor of S [4,5]. Most recently in 2013, C.D. Hacon, J. McKernan, and C. Xu [6] proved the most general case that if X is a smooth variety then there exists a fixed constant C for each dimension with the automorphism group bounded by $CV(X)$ where $V(X)$ is the volume of the canonical divisor. Presently, the value of the constant for dimensions 3 and higher is unknown and is currently an ongoing search.

In chapter two, I prove the following result relating the automorphism group of a variety to that of its image in the Albanese.

Theorem 1. *Suppose that X is a smooth variety of general type, $Aut(X)$ fixes a point P_0 and irregularity $q(X) \geq 1$ such that its image Y in $Alb(X)$ is smooth and such that the general fiber F of $X \to Y$ has general type and dimension one or two. Let F_{min} be the minimal model of F. Then we have*

$$|Aut(X)| \leq |Aut(F_{min})||Aut(Y)| \tag{1.1}$$

A future goal is to determine how fine of an inequality this is for bounding the automorphism group, leading to a search for examples of n-folds X that satisfy the conditions of the above theorem and have ample canonical bundles such that we can compute their automorphism groups explicitly. In this case, using the above theorem, the automorphism group would be bounded by $cK_{F_min}^{\dim(F)}K_Y^{\dim(Y)}$, if K_Y is ample, with some constant c. This bound has a nice relation to the first Chern numbers of both F_{min} and Y. For a smooth variety V with K_V ample, the first Chern class of the tangent bundle is $c_1(T_V) = -K_V$ so that the bound can be written in terms of the first Chern number of the tangent bundle $c_1(T_V)^{\dim(V)} = (-K_V)^{\dim(V)}$. In fact, the automorphism group of a minimal variety can be viewed as a bound on the first Chern number.

Stemming from this search for an example of such a fibration, where we can explicitly compute its automorphism group, we develop K-Fibrations as a generalization to Kodaira Fibrations [7]. What is fascinating about K-Fibrations is, due to their inductive construction as certain r-cyclic covers, we can compute their Chern numbers and determine relations between the Chern numbers in special cases. Future work will focus on adjusting this construction to allow for the determination of their automorphism group and to use it as a basis to find an example of a variety that embeds into its Albanese such that the general fibers are curves or surfaces.

As a review, Chern classes c_i of a vector bundle are characteristic classes in cohomology. The first Chern class for line bundles, c_1 is a homomorphism from $Pic(X)$ to $H^2(X, \mathbb{Z})$, which gives $c_1(L_B) = c_1(\mathcal{O}_X(B)) = B$ where L_B is the line bundle associated with a divisor B. The intermediate Chern classes c_i land in $H^{2i}(X, \mathbb{Z})$ and their construction can be found in [8] or [9].

Chern numbers of a smooth variety are the intersection of the Chern classes $c_i(T_X)$, where T_X is the tangent bundle of X, that yield integer values. For example, curves have a single Chern number, $c_1(T_X) = -\deg(K_X) = 2 - 2g$ in $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ while surfaces have two $c_1^2(T_X)$ and $c_2(T_X)$ in $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ as $c_1(T_X) = -K_X$ is a divisor in $H^2(X, \mathbb{Z})$ thus not an integer. I will use $c_i(X) = c_i(T_X)$ for simplicity.

Chern numbers of the tangent bundle are isomorphism invariants defined by S.S. Chern [10], that is if $X \cong Y$, then the Chern numbers are the same. Thus if the numbers are different, then the spaces are not isomorphic. The converse does fail.

For example, if you take two elliptic curves with different j-invariants they are not isomorphic but their first Chern numbers are both zero.

Chern numbers are not birational invariants of smooth varieties as seen by the blow-up $b : Y \to X$ of a smooth surface X at a single point. The canonical divisor of Y is $K_Y = b^*K_X + E$ where E is the exceptional divisor and hence has a Chern number

$$c_1^2(Y) = K_Y^2 = b^*c_1^2(X) + 2b^*c_1(X) \cdot E + E^2 \tag{1.2}$$

which is not $c_1^2(X)$.

In the third chapter, I review historical bounds on Chern numbers then use a result by Hunt [11] and Hirzebruch-Riemann-Roch to obtain new relations in dimensions 3 and 4.

In the fourth chapter, I describe a generalization to a construction described by Kodaira [7] called K-Fibrations and prove a few new properties of these varieties.

In the fifth chapter, I compute the Chern numbers of the K-Fibration in dimension 2 as done in [12] and in dimension 3 to get a new asymptotic relations and bounds on Chern numbers in a special case as described by Kas [13].

Lastly, I include an appendix outlining the relation between vector bundles and locally free sheaves along with the construction of the tautological section of a sheaf as review.

2. AUTOMORPHISMS OF FIBRATIONS EMBEDDED INTO THE ALBANESE

The question of finding the automorphism group or its order of a smooth variety is a very active one. One of the goals of bounding the automorphism group is to do go in terms of its volume $V(X)$ where

$$V(X) = \text{limsup}_m \frac{h^0(X, mK_X)}{m^{dim(X)}/dim(X)!}. \tag{2.1}$$

. Recall that X is of general type if its Kodaira dimension $\kappa(X) = \dim(X)$ and minimal if for any curve C in X, then $K_X \cdot C \geq 0$. When X is minimal and of general type, this can be written in terms of the canonical divisor or first Chern number: $V(X) = K_X^{\dim(X)} = (-1)^{\dim(X)} c_1(X)$. In 2011, H-Y. Chen proved the following result for a smooth projective threefold with a certain fibration property over its Albanese:

Theorem 2 (H-Y. Chen [1])**.** *Let X be a smooth projective threefold of general type, and let $alb_X : X \to A$ be the Albanese map of X. Suppose that the image of alb_X is a curve C with genus $g(C) \geq 2$. Then the order of the automorphism group of X is*

$$|Aut(X)| \leq \frac{1}{3} 42^3 V(X) \tag{2.2}$$

The goal of this section is to provide more detail of the proof that the map $\phi : Aut(X) \to Aut(alb_X(X))$ is well-defined as, unfortunately, there is a small gap in the proof and to generalize the result to a smooth projective n-fold with the codimension of the image of X in $alb(X)$ at most 2.

Let us quickly recall that the Albanese of X is defined to be

$$Alb(X) = H^0(X, \Omega^1)^\vee / H_1(X, \mathbb{Z}) \tag{2.3}$$

with the Albanese map $\alpha_{P_0} : X \to Alb(X)$ with base point P_0 in X taking

$$P \mapsto (\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_q) \tag{2.4}$$

where $q = \dim(H^0(X, \Omega^1_X))$ is the irregularity and $\omega_1, \ldots, \omega_q$ a basis for $H^0(X, \Omega_X)$. This map does depend on choice of base point $P_0 \in X$.

Lemma 1. *Suppose that X is a smooth variety with irregularity $q \geq 1$. Suppose that $Aut(X)$ is finite, fixes a point $P_0 \in X$, and that it embeds into its Albanese variety as a smooth variety $Y = \alpha_{P_0}(X)$, then there is a well-defined map $\phi : Aut(X) \to Aut(Y)$.*

Proof. Let $G = Aut(X)$ and $H = Aut(Y)$. Let $q = \dim(H^0(X, \Omega^1_X))$.

Suppose the action of G on X fixes a point P_0. Then define the Albanese map $\alpha_{P_0} : X \to Y$ by

$$\alpha_{P_0}(P) = \left(\int_{P_0}^{P} \omega_i \right)_{i=1,\ldots,q} \tag{2.5}$$

where the ω_i is a basis of $H^0(X, \Omega^1_X)$. Define for $g \in G$

$$\phi(g)(\alpha_{P_0}(P)) = \left(\int_{P_0}^{g(P)} g^*\omega_i \right)_{i=1,\ldots,q} \tag{2.6}$$

which is a group homomorphism as needed.

□

There is an issue if $Aut(X)$ does not fix a point of X. Without a fixed point, the universal property of the Albanese does not have the same image in $Alb(X)$ when permuted by the action of $Aut(X)$, though the images are isomorphic. Special care is needed when defining a map above or via the universal property from $Aut(X)$ to $Aut(Y)$ which is an area of future work. I do believe that the fix point condition can be dropped. One approach is to define a map $X \to Sym^n(X)$ where $n = |Aut(X)|$ that now presents fixed points of the induced action of $Aut(X)$ on $Sym^n(X)$. Additionally, this action is an embedding of $Aut(X)$ in $Aut(Sym^n(X))$ (In fact, $Aut(X) \cong Aut(Sym^d(X))$ if X is a curve of genus $g > 2$ and $d > 2g - 2$ was shown recently by Biswas and Gómez in [14]). Next, construct a map from $Sym^n(X) \to Alb(X)$ such that it sends a base point P_0 of the Albanese to the origin so that the composition $X \to Sym^n(X) \to Alb(X)$ factors through the Albanese map α_{P_0}. What is left is to define a map from $Aut(X)$ as acting on $Sym^n(X)$ to $Aut(Y)$. The issue issue is to define it in such that it is a group homomorphism.

Theorem 3. *Suppose that X is a smooth variety of general type, $Aut(X)$ fixes a point P_0, and X has irregularity $q \geq 1$ such that its image Y in $Alb(X)$ is smooth and such that the general fiber F of $X \to Y$ has general type and dimension one or two. Let F_{min} be the minimal model of F. Then we have*

$$|Aut(X)| \leq |Aut(F_{min})||Aut(Y)| \tag{2.7}$$

Proof. By Lemma 1 we have an exact sequence

$$0 \longrightarrow K \longrightarrow Aut(X) \xrightarrow{\phi} Aut(Y) \tag{2.8}$$

where K is the kernal of the group homomorphism ϕ.

Next, we need K injects into $Aut(F_{min})$. If F is a curve, this is clear. If F is a surface, then we will have an embedding of K into $Aut(F)$. But this embeds into the birational automorphism group, which can be identified with $Aut(F_{min})$. Hence $|K| \leq |Aut(F_{min})|$.

Thus by the exact sequence, we have

$$|Aut(X)| \leq |K||Aut(Y)| \leq |Aut(F_{min})||Aut(Y)|$$

. $\square$

Remark. In higher codimensions (i.e. the general fibers F have dimension 3 or higher), there are major issues concerning the uniqueness of a minimal model.

To relate the automorphism groups of F_{min} and Y to the volume of X, we use the following result of Y. Kawamata:

Theorem 4 (Y. Kawamata [15])**.** *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties with connected fibers. Assume that both Y and the general fiber F of f are varieties of general type. Then*

$$\frac{V(X)}{dim(X)!} \geq \frac{V(Y)}{dim(Y)!}\frac{V(F)}{dim(F)!} \tag{2.9}$$

We will apply this to theorem 3.

Corollary 1. *Suppose that X is a smooth variety of general type with the assumptions of theorem 3. Then we have*

$$|Aut(X)| \leq CV(X) \tag{2.10}$$

where if $\dim(X) = 3$ *then* $C = \frac{1}{3}42^3$*;* $C = \frac{1}{3}42^4$ *if both F and Y are surfaces, and C is a constant that only depends on* $\dim(X)$ *in all other cases.*

Proof. By Theorem 3, we know that

$$|Aut(X)| \leq |Aut(F_{min})||Aut(Y)|.$$

As a summary of known general bounds, we have

i. $|Aut(C)| \leq 42V(C)$ for C a curve [2]

ii. $|Aut(S)| \leq 42^2V(S)$ for S a surface [4, 5]

iii. $|Aut(X)| \leq cV(X)$ for X a n-fold $n > 2$ and $c > 0$ depending only on $\dim(X)$ [6]

If $\dim(X) = 3$, this was done in [1] for Y a curve. Likewise if Y is a surface and F a curve, we get the bound

$$|Aut(X)| \leq 42^3V(F)V(Y) \leq \frac{1}{3}42^3V(X).$$

If $\dim(X) = 4$, we have three cases (a) Y is a surface, (b) Y is a 3-fold, or (c) Y a curve.

For case (a), we have

$$|Aut(X)| \leq 42^4V(F)V(Y) \leq \frac{1}{3}42^4V(X).$$

For (b), we need to apply bound (iii) to get

$$|Aut(X)| \leq 42cV(F)V(Y) \leq \frac{1}{4}42cV(X).$$

For (c), we need to apply bound (iii) to $Aut(F_{min})$ to get

$$|Aut(X)| \leq 42cV(F)V(Y) \leq \frac{1}{4}42cV(X).$$

If $\dim(X) \geq 5$, if F is a curve we have

$$|Aut(X)| \leq \frac{42c}{\dim(X)} V(X)$$

and if F is a surface

$$|Aut(X)| \leq \frac{2 \cdot 42^2 c}{\dim(X)(\dim(X) - 1)} V(X).$$

$\square$

The following chapters stem from work with trying to construct a concrete example of a variety X with the conditions of Theorem 3 where we can explicitly compute its automorphism group to determine how firm the inequality is on the automorphism bound. It is known via the Hurwitz Formula that if $X = F \times Y$, for non-isomorphic curves F and Y, that $Aut(X) \cong Aut(F) \times Aut(Y)$. In this case, Theorem 3 in equality even if the conditions are not met. The challenge is to satisfy the conditions.

The underlying idea is if we can construct a smooth curve C of genus at least 2 in an Abelian variety A and then construct a fibration $X \to C$ such that C is the image of the Albanese from $\alpha : X \to Alb(X)$ for some base point via the universal property. I.e., if $f : X \to C \subset A$, then there is a map $h : Alb(X) \to A$ which gives a factoring $h \circ \alpha = h$ by the universal property of the Albanese. We now have an induced map $h : \alpha(X) \to C$ via this composition and if $\alpha(X)$ is a smooth curve, they are isomorphic. In this case, what would be left to determine is the contribution of the fibration on the automorphism group.

This principle has led to the following construction and discussion of K-Fibrations which were meant as a potential means to satisfy these conditions with future work of computing their automorphism group.

3. RELATIONSHIPS BETWEEN CHERN NUMBERS

Before going into K-Fibrations, we review Chern numbers of smooth varieties.

Chern numbers are the intersection of an appropriate amount of Chern classes of the tangent bundle T_X of a smooth variety X. To be an integer, if $n = \dim(X)$ and N_i such that $n = \sum_{i=1}^{n} iN_i$, then the intersection $c_1^{N_1} \cdot \ldots \cdot c_n^{N_n}(T_X)$ is an integer. For simplicity, we write $c_1^{N_1} \cdot \ldots \cdot c_n^{N_n}(X)$ instead of with the tangent bundle.

Chern numbers are related by a variety of inequalities. The most famous being the [16] where Yau show that for a variety X of dimension $n \geq 2$ and ample canonical divisor K_X that

$$(-1)^n c_1^n(X) \leq (-1)^n \frac{2(n+1)}{n} c_1^{n-2} c_2(X) \tag{3.1}$$

with equality if and only if X is a compact quotient of the n-ball.

For surfaces, the Chern numbers for a minimal surface X of general type satisfy the following inequalities [12]:

i. $c_1^2(X) > 0$

ii. $c_2(X) > 0$

iii. $c_1^2(X) \leq 3c_2(X)$

iv. $c_1^2(X) + c_2(X) = 12\chi(X)$

v. $c_1^2 \geq c_2(X)/5 - 36/5$ if $c_1^2(X)$ is even

vi. $c_1^2(X) \geq c_2(X)/5 - 6$ if $c_1^2(X)$ is odd

with the last two being related to Noether's inequality.

For threefolds X, less is known. For when K_X is ample we have by [11] that

i. $c_1^3(X) < 0$,

ii. $c_1c_2(X) < 0$,

iii. $c_1c_2(X) = 24\chi(X)$,

iv. $-c_1^3(X) \leq \frac{8}{3}(-c_1c_2(X))$.

To get an inequality regarding $c_3(X)$, Hunt in his paper [11], at a suggestion of Ven de Ven, uses the following procedure which I generalize for $\dim(X) = n$.

Assuming that K_X is very ample, let $i : X \to \mathbb{P}^N$ be the canonical embedding with $K_X = i^*\mathcal{O}_{\mathbb{P}^N}(1)$ and let $f : X \to G(n+1, N+1)$ taking x to the tangent plane to X at x be the Gauss map.

There is a bundle sequence on $G(n+1, N+1)$ with S be universal bundle [8]

$$0 \longrightarrow S \longrightarrow \mathbb{C}^{N+1} \longrightarrow Q \longrightarrow 0$$

Pulling it back to a sequence on X via f

$$0 \longrightarrow f^*S \longrightarrow f^*\mathbb{C}^{N+1} \longrightarrow f^*Q \longrightarrow 0.$$

Now on $\mathbb{P}^N$, we have the standard sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow (\mathcal{O}_{\mathbb{P}^N}(1))^{N+1} \longrightarrow T_{\mathbb{P}^N} \longrightarrow 0$$

so twisting gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow (\mathcal{O}_{\mathbb{P}^N})^{N+1} \longrightarrow T_{\mathbb{P}^N}(-1) \longrightarrow 0$$

and going back to X

$$0 \longrightarrow i^*\mathcal{O}_{\mathbb{P}^N}(-1) \longrightarrow i^*(\mathcal{O}_{\mathbb{P}^N})^{N+1} \longrightarrow i^*T_{\mathbb{P}^N}(-1) \longrightarrow 0.$$

Now, using i and adjunction, we have on X the sequence

$$0 \longrightarrow T_X \longrightarrow i^*T_{\mathbb{P}^N} \longrightarrow N_{\mathbb{P}^n/X} \longrightarrow 0$$

so twisting by -1 to align the $T_{\mathbb{P}^N}(-1)$ gives

$$0 \longrightarrow T_X(-1) \longrightarrow i^*T_{\mathbb{P}^N}(-1) \longrightarrow N_{\mathbb{P}^n/X}(-1) \longrightarrow 0.$$

Piecing these sequences together, we get the following diagram on X:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & f^*S \arrow[r, "(2)"] \arrow[d] & T_X(-1) \arrow[d] \arrow[dr, dashed, "(3)"] & \\
0 \arrow[r] & i^*\mathcal{O}_{\mathbb{P}^N}(-1) \arrow[r] \arrow[ur, dashed, "(1)"] & i^*(\mathcal{O}_{\mathbb{P}^N})^{N+1} \arrow[r] \arrow[d] & i^*T_{\mathbb{P}^N}(-1) \arrow[r] \arrow[d] & 0 \\
 & & f^*Q \arrow[d] & N_{\mathbb{P}^N/X}(-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

with (1) by construction of S (fibers are planes), (2) coming from the natural inclusion of $\mathbb{C}$ in $\mathcal{O}_{\mathbb{P}^N}$, and composition of several above sequences giving (3). Thus along this new map we get the sequence

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow f^*S \longrightarrow T_X(-K_X) \longrightarrow 0.$$

Then applying the Chern character gives

$$ch(f^*S) = ch(\mathcal{O}_X(-K_X))ch(T_X(-K_X)). \tag{3.2}$$

Using Gauss-Bonnet Theorem I [8], we have that $c_n(f^*S) = f^*c_n(S) = (-1)^n \cdot f^*\sigma^*_{1,\dots,1}$ where $\sigma^*_{1,\dots,1}$ is the Poincaré dual of the Schubert cycle $\sigma_{1,\dots,1}$. Coupled with knowing in the intersection pairing in homology of $G(n+1, N+1)$ that $\sigma_{1,\dots,1} \cdot \sigma_a \geq 0$ for any Schubert cycle σ_a, we find that $c_n(f^*S) \geq 0$ if n is even and $c_n(f^*S) \leq 0$ if n is odd. By taking the n-graded piece on the right side of 3.2 and knowing the sign of $c_n(f^*S)$, we can generate new relations on the Chern numbers.

Thus we get in dimension 3,

$$c_3(f^*S) = c_3(T_X(-K_X)) - K_X \cdot c_2(T_X(-K_X)) \leq 0$$

so that we have relation

$$c_3(X) + 2c_1c_2(X) + 7c_1^3(X) \leq 0.$$

In dimension 4, we obtain

$$c_4(f^*S) = -K_X c_3(T_X(-K_X)) + c_4(T_X(-K_X))$$

giving

$$9c_1^4(X) + 3c_1^2c_2(X) + 2c_1c_3(X) + c_4(X) \geq 0$$

which is missing a term containing c_2^2.

Application of Hirzebruch-Riemann-Roch

To obtain more relations, we note that as K_X is ample, then by Kodaira Vanishing (or more generally if K_X is numerically effective (NEF) and big by Kawamata-Viehweg Vanishing) we have that $0 < \chi(mK_x)$ for $m > 1$. Thus Hirzebruch-Riemann-Roch [9] gives

$$0 < \chi(mK_X) = \int_X ch(mK_X)td(T_X).$$

This yields in dimension 2,

$$(1 + 6m^2)c_1^2(X) + c_2(X) > 0, \tag{3.3}$$

in dimension 3 we have

$$(1 - 2m)c_1c_2(X) + 2(-m + 3m^2 - 2m^3)c_1^3(X) > 0, \tag{3.4}$$

and

$$\begin{aligned}(-1 + 30m^2 - 60m^3 + 30m^4)c_1^4(X) + (4 - 30m + 30m^2)c_1^2c_2(X)\\ + 3c_2^2(X) + c_1c_3(X) - c_4(X) > 0\end{aligned} \tag{3.5}$$

in dimension 4. Fortunately, we have obtained a relation involving all five Chern Numbers in dimension 4. A quick analysis yeilds that the minimum of the coefficients occurs when $m = 2$ in dimensions 2 and 4 and maximum at $m = 2$ in dimension 3.

4. K-FIBRATIONS

We will build a fibration $X \to C$ where the canonical divisor can be computed along each step for X of dimension 2 or larger. As the techniques are based upon the construction of Kodaira Fibrations [12] which already refer to a few mathematical objects, I will name these **K-Fibrations with level** n where $n = \dim(X) - 1$. These will be defined inductively. A K-Fibration of level n will be a family of smooth curves over a K-Fibration of level $n-1$.

The aim of this construction is to provide a scaffolding to compute Chern numbers in higher dimension in addition to a potential means to construct an example of a fibration over the Albanese as discussed in chapter 1.

4.1 THE CONSTRUCTION

We follow the initial construction in [12]. Let D_0 be a complex smooth connected curve of genus at least 2. Take a covering map $D \to D_0$ with covering group G of order kr for some $k > 0$. Note the action of G on D is fixed-point-free as G is the group of deck transformations on the covering map. To construct $h : C \to D$ with the property that all elements of $h^*H^1(D, \mathbb{Z})$ are r-divisible in $H^1(C, \mathbb{Z})$, we pass to topology.

We have surjections $\pi_1(D) \to H_1(D, \mathbb{Z})$ and $H_1(D, \mathbb{Z}) \to H_1(D, \mathbb{Z}/r\mathbb{Z})$ allowing us to treat $H_1(D, \mathbb{Z}/r\mathbb{Z})$ as a quotient of $\pi_1(D)$. Thus take $h : C \to D$ to be the unbranched cover with covering group $H_1(X, \mathbb{Z}/r\mathbb{Z})$. We then have the following exact sequence

$$0 \longrightarrow \pi_1(C) \longrightarrow \pi_1(D) \longrightarrow H_1(X, \mathbb{Z}/r\mathbb{Z}) \longrightarrow 0 \tag{4.1}$$

and with applying the right-exact abelianization functor we get $h_*(H_1(C,\mathbb{Z}))$ is r-divisible in $H_1(D,\mathbb{Z})$. By duality, $h^*H^1(D,\mathbb{Z})$ is r-divisible in $H^1(C,\mathbb{Z})$. Another way to obtain r-divisibility is if we have a r-torsion group H (all elements have order r) with a surjective map $H_1(X,\mathbb{Z}) \to H$, then we can take the unbranched covering space associated to this quotient with covering group H and $h_*H_1(C,\mathbb{Z})$ having the property that $\sum_{g\in G}(gh)_*\alpha = 0 \bmod r$ for all $\alpha \in H_1(C,\mathbb{Z})$. This r-divisibility condition is needed momentarily to construct a needed line bundle and collections $(h : C \to D, G, r)$ that satisfy it are called admissible [13].

Let B_g be the graph $\Gamma(g\circ h) \subset C\times D$ for $g \in G$ and set $B = \cup_{g\in G}B_g$. Then B is a smooth curve in $C\times D$ with $|G|$ components. By the following Lemma, there is a line bundle $\mathcal{L}$ in $\mathrm{Pic}(C\times D)$ with $\mathcal{O}_{C\times D}(B) = \mathcal{L}^{\otimes r}$. We construct a r-fold cyclic covering $k_1 : X_1 \to C\times D$ that is ramified over B. Let $p : L \to C\times D$ be the projection from the total space L of $\mathcal{L}$, $s \in \Gamma(C\times D, \mathcal{O}_{C\times D}(B))$ and $t \in \Gamma(L, p^*\mathcal{L})$ the tautological section. We get the map $k_1 = p|_{X_1} : X_1 \to C\times D$ where X_1 is the analytic space $p^*s - t^r$ in L.

Lemma 2 ([12])**.** *$\mathcal{O}_{C\times D}(B)$ is r-divisible in $Pic(C\times D)$.*

Proof. As $Pic^0(C\times D)$ is an abelian variety, if $\mathcal{O}_{C\times D}(B)$ is in it, then $\mathcal{O}_{C\times D}(B)$ is r-divisible. This means we just need to check if $c_1(\mathcal{O}_{C\times D}(B))$ is r-divisible in $H^2(C\times D,\mathbb{Z})$.

The cup product $<,>$ on $H^2(C\times D,\mathbb{Z})\times H^{4-2}(C\times D,\mathbb{Z}) \to \mathbb{Z}$ is a perfect pairing, so it is enough to prove the intersection

$$< c_1(\mathcal{O}(B)), \alpha > \equiv 0 \bmod r$$

for all $\alpha \in H^2(C\times D,\mathbb{Z})$. By the Künneth formula, we have three cases: (1) $\alpha \in p_C^*H^2(C,\mathbb{Z})$, (2) $\alpha \in P_D^*H^2(D,\mathbb{Z})$, and (3) $\alpha \in p_C^*H^1(C,\mathbb{Z}) \otimes p_D^*H^1(D,\mathbb{Z})$. In the cases (1) and (2), the intersection of B with divisors of the forms $P\times D$ for $P \in C$ and $C\times Q$ for $Q \in D$ (and hence linear combinations of them over $\mathbb{Z}$) are divisible by $|G|$ and hence by r. For case (3), let $\alpha = p_C^*(\gamma)\cdot p_D^*(\beta)$. Then we have

$$
\begin{aligned}
< c_1(\mathcal{O}(B)), \alpha > &= \sum_{g \in G} < c_1(\mathcal{O}(B_g)), p_C^*(\gamma) \cdot p_D^*(\beta) > \\
&= \sum_{g \in G} < \gamma, p_{C*}(c_1(\mathcal{O}(B_g)) \cdot p_D^*(\beta)) > \\
&= \sum_{g \in G} < \gamma, h^* g^* \beta > \equiv 0 \bmod r
\end{aligned}
$$

by the projection formula of Chern classes and $h^* H^1(D, \mathbb{Z})$ is r-divisible in $H^1(C, \mathbb{Z})$.

$\square$

Let $p_C : C \times D \to C$ be the projection. Then we say that $f_1 = p_C \circ k_1 : X_1 \to C$ is a **K-Fibration of level 1**. As a Diagram, we have

$$
\begin{tikzcd}
X_1 \arrow[r, "k_1"] \arrow[dr, bend right, "f_1"'] & C \times D \arrow[r, dash] \arrow[d, "p_C"] & B \\
 & C \arrow[r, "h"] & D
\end{tikzcd} \tag{4.2}
$$

where the dash indicates that it is branched over B.

Let $d = deg(h)$ and $g(C)$ and $g(D)$ denote the genus of C and D. Then $deg(f) = d$ by this construction.

We build the next level inductively. Let X_n be a level n **K-Fibration** with $f_n : X_n \to C$. Then we have a map $f'_n = h \circ f_n : X_n \to D$ with the same G acting on D without fixed points. Then, we mirror the same construction. Let $B_g = \Gamma(g \circ f'_n)$ and $B_n = \cup_{g \in G} B_g$ be the union of graphs in $X_n \times D$. We then take a r-fold ramified cover, which exists by the following Lemma, of $X_n \times D$ ramified over B_n to get a map $k_{n+1} : X_{n+1} \to X_n \times D$. Then $f_{n+1} = f_n \circ p_{X_n} \circ k_{n+1} : X_{n+1} \to C$ is a K-Fibration of level $n+1$. We treat $X_0 = C$ and $f_0 = \mathrm{id}_C$ in this notation. Again, we have the diagram

$$
\begin{tikzcd}
X_{n+1} \arrow[r, "k_{n+1}"] \arrow[ddr, bend right, "f_{n+1}"'] & X_n \times D \arrow[r, dash] \arrow[d, "p_{X_n}"] & B_n \\
 & X_n \arrow[r, "h \circ f_n"] \arrow[d, "f_n"] & D \\
 & C &
\end{tikzcd} \tag{4.3}
$$

The map k_{n+1} exists due to the following Lemma.

Lemma 3. *$\mathcal{O}_{X_n \times D}(B_n)$ is r-divisible in Pic$(X_n \times D)$.*

Proof. As in the prior proof, using the cup product we need to check that

$$c_1(\mathcal{O}_{X_i \times D}(B_i)) \frown \alpha \equiv 0 \bmod r \tag{4.4}$$

for $\alpha \in H^{2(i+1)}(X_i \times D, \mathbb{Z})$. As D is a complex curve so that $H^i(D, \mathbb{Z}) = 0$ for $i > 2$, by the Künneth formula we have three cases: (1) $\alpha \in p_{X_i}^* H^{2(i+1)}(X_i, \mathbb{Z})$, (2) $\alpha \in p_{X_i}^* H^{2(i+1)-1}(X_i, \mathbb{Z}) \otimes p_D^* H^1(D, \mathbb{Z})$, and (3) $\alpha \in p_{X_i}^* H^{2(i+1)-2}(X_i, \mathbb{Z}) \otimes p_D^* H^2(D, \mathbb{Z})$.

In case (1), $p_{X_i}^*(\Delta)$ intersects B_i $|G|$ times and hence by r.

For case (3), by commutativity of intersections, the intersection class of $p_D^*(\beta)$ with B_i is divisible by $|G|$ thus by r.

Lastly, case (2), the harder case. As $h^* H^1(D, \mathbb{Z})$ is r-divisible in $H^1(C, \mathbb{Z})$, then under the pull-back f_i to X_i we also have $f_i^* h^* H^1(D, \mathbb{Z})$ is r-divisible in $H^1(X_i, \mathbb{Z})$. Let $\alpha = p_{X_i}^*(\Delta) \cdot p_D^*(\gamma)$ in $p_{X_i}^* H^{2(i+1)-1}(X_i, \mathbb{Z}) \otimes p_D^* H^1(D, \mathbb{Z})$. We then have

$$< c_1(\mathcal{O}(B_i)), \alpha > = \sum_{g \in G} < c_1(\mathcal{O}(B_{ig})), p_{X_i}^*(\Delta) \cdot p_D^*(\gamma) > \tag{4.5}$$

$$= \sum_{g \in G} < \Delta, p_{X_i *}(c_1(\mathcal{O}(B_{ig})) \cdot p_D^*(\gamma)) > \tag{4.6}$$

$$= \sum_{g \in G} < \Delta, f_i^* h^* g^* \gamma > \equiv 0 \bmod r \tag{4.7}$$

□

The K-Fibration construction allows us to have intersection calculations on X reduced to fibration calculations with $h : C \to D$ and numbers $|G|$ and r.

In [13], A. Kas describes an example construction which we will discuss in Chapter 5.

4.2 PROPERTIES OF K-FIBRATIONS

We have the following property of cyclic coverings that is highly useful for computations.

Proposition 1 (Lemma I.17.1 of [12])**.** *Let $f : X \to Y$ be an n-cyclic covering branched along a smooth effective divisor B in Y and with $\mathcal{L}^n = \mathcal{O}_Y(B)$ and B_1 the reduced divisor of $f^{-1}(B)$ in X. Then*

(i) $\mathcal{O}_X(B_1) = f^*\mathcal{L}$

(ii) $f^*B = nB_1$

(iii) $\mathcal{K}_X = f^*(\mathcal{K}_Y \otimes \mathcal{L}^{\otimes(n-1)})$

Proof. See [12] Lemma I.17.1. □

Proposition 2. *If $f : X \to C$ is a K-Fibration, then K_X is ample.*

Proof. Suppose this is a level n K-Fibration. Then we show this by induction as $K_{X_n} = k^*(K_{X_{n-1}} \otimes \mathcal{L}^r)$ and k is a finite surjective map. Thus it is left to show that $K_{C\times D}$ is ample or that $\mathcal{L}$ is ample. But these hold as C and D have genus at least two and $\mathcal{L}^r = \mathcal{O}(B)$. □

Proposition 3. *If $f : X \to C$ is a K-Fibration, then X is of general type.*

Proof. K_X is an ample line bundle we have that for some m large that K_X^m is very ample. Thus the map induced by the linear system $|K_X^m|$ is an embedding having dimension $\dim(X) = k(X)$. □

5. CHERN NUMBERS OF K-FIBRATIONS

Algebraic Geography is the study of the distribution of the Chern numbers of a variety where you can take X as a point $[c_1^n(X), \ldots, c_n(X)]$ in $\mathbb{P}^{n-1}$. For a surface, this corresponds to the Enrique-Kodaira Classification [12]. For higher dimensions, not much is known in general about regions [11].

Using the structure of K-Fibrations, we are able to compute the Chern numbers in dimensions 2 and 3, that is, K-Fibrations of levels 1 and 2 using results of T. Izawa on Chern Numbers for ramified coverings [17]. For a K-Fibration of level 3, new intersection computations appear involving the intermediate Chern classes of the branch locus B which pose new challenges not addressed in this thesis.

We know $c_n(X) = \chi(X)$ is the topological Euler characteristic by Gauss-Bonnet [8] or Hirzebruch–Riemann–Roch [9]. Additionally, as $c_1(X) = -K_X$, we can compute $c_1^n(X) = (-K_X)^n$ if the form of K_X is known. The more challenging intersections are the products $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X)$ for $i_1 + \ldots + i_k = n$. For these, T. Izawa proved the following theorem.

Theorem 5 (Izawa [17]). *Let $f : Y \to X$ be a ramified covering with multiplicity μ between complex manifolds of dimension n, $R_f = \sum_i r_i R_i$ the ramification divisor of f, and $B_f = \sum_i b_i B_i$ the branch locus of f. We set $f^* B_i = \sum_t n_{i_t} R_{i_t}$ where n_{i_t} denotes the mapping degree of the induced map $f|_{R_{i_t}} : R_{i_t} \to B_i$ with $b_i = \sum_t n_{i_t} r_{i_t}$. We assume that the ramification divisor and the irreducible components B_i of the branch locus B_f are all non-singular, and suppose that $n = \sum_{i+1}^n iN_i$. Then:*

$$c_1^{N_1} \cdots c_n^{N_n}(Y) - \mu c_1^{N_1} \cdots c_n^{N_n}(X) =$$

$$\sum_i \sum_{\alpha=0}^{n-1} \left(\sum_t \frac{n_{i_t}(1 - (r_{i_t} + 1)^{\alpha+1})}{(r_{i_t} + 1)^\alpha} \right) P_\alpha(c_1(B_i) \cdots c_{n-1}(B_i)) \cdot c_1(L_{B_i})^\alpha \frown [B_i], \quad (5.1)$$

where we set

$$\sum_{\alpha=0}^{n-1} P_\alpha(c_1 \cdots c_{n-1}) l^\alpha = l^{-1} \left(\prod_{i=1}^{n} (c_i(B) + c_{i-1}(B)l)^{N_i} - c_1^{N_1} \cdots c_n^{N_n}(B) \right). \tag{5.2}$$

L_{B_i} is the line bundle corresponding to B_i, $\frown$ is the cap product, and the formula for P_α should be interpreted formally as the cap product reduces to the intersection on X with B_i. In our K-Fibration setting, this reduces to the following corollary using the properties of a cyclic covering 1.

Corollary 2. *Let $k_{n+1} : X_{n+1} \to X_n \times D$ be a ramified cyclic covering with multiplicity r, $R_k = \sum_g (r-1) D_g$ the ramification divisor of k_{n+1}, and $B_n = \sum_{g \in G} B_g$ the branch locus of k_{n+1}. Then $k^* B_g = r D_g$. We assume that the ramification divisor and the irreducible components B_g of the branch locus B_k are all non-singular, and suppose that $n = \sum_{i+1}^{n} i N_i$. Then:*

$$\begin{aligned} &c_1^{N_1} \cdots c_n^{N_n}(Y) - r c_1^{N_1} \cdots c_n^{N_n}(X) = \\ &\sum_{g \in G} \sum_{\alpha=0}^{n-1} \left(\sum_t \frac{(1 - r^{\alpha+1})}{r^\alpha} \right) P_\alpha(c_1(B_g) \cdots c_{n-1}(B_g)) \cdot c_1(L_{B_g})^\alpha \frown [B_g], \end{aligned} \tag{5.3}$$

where we set

$$\sum_{\alpha=0}^{n-1} P_\alpha(c_1 \cdots c_{n-1}) l^\alpha = l^{-1} \left(\prod_{i=1}^{n} (c_i(B) + c_{i-1}(B)l)^{N_i} - c_1^{N_1} \cdots c_n^{N_n}(B) \right). \tag{5.4}$$

5.1 LEVEL 1 K-FIBRATION

I start with the construction in [12] about Kodaira Fibrations. We will review the computations of $c_2(X)$ and $c_1^2(X)$ as done in [12]. Starting with

$$K_X = k^*(K_{C \times D} \otimes \mathcal{L}^{\otimes(r-1)}) \tag{5.5}$$

which gives that for $C_0 = c_0 \times D$ and $D_0 = C \times d_0$ and using

$$K_{C \times D} = p_C^* K_C + p_D^* K_D = (2g(C) - 2)C_0 + (2g(D) - 2)D_0 \tag{5.6}$$

we have that

$$c_1^2(X) = K_X^2 = r\left(K_{C\times D} + \left(\frac{r-1}{r}\right)B\right)^2$$
$$= 2r\chi(C)\chi(D) - 2(r-1)|G|\chi(C) - \left(\frac{r^2-1}{r}\right)deg(h)|G|\chi(D) \quad (5.7)$$

By Hurwitz, we have that $\chi(C) = deg(h)\chi(D)$ as the covering map is unramified which allows for the elimination of $g(C)$ in the computation and reducing K_X^2 to the numbers $g(D)$, $deg(h)$, $|G|$, and r.

Next, we have that $c_2(X) = \chi(X)$. To compute then, let F be a general fiber of $f : X \to C$ so that $\chi(X) = \chi(F)\chi(C)$, so all we have to compute is $\chi(F)$. Let $P \in C$ be a point so $p_C^{-1}(P) = P \times D \subset C \times D$. Then we have the map $k : F \to P \times D$ of degree r and ramified over $(P \times D) \cap B$. This gives

$$\chi(F) = deg(k)\chi(D) - (P \times D) \cdot B = r\chi(D) - (r-1)|G| \quad (5.8)$$

hence

$$c_2(X) = \chi(X) = \chi(C)(r\chi(D) - (r-1)|G|) \quad (5.9)$$

Lastly, we have by Riemann-Roch for surfaces that

$$\chi(\mathcal{O}_{X_1}) = \frac{1}{12}(c_1^2(X_1) + c_2(X_1)) \quad (5.10)$$

which we will need for later.

5.2 LEVEL 2 K-FIBRATION

As these are built inductively, let $f = f_1 \circ p_{X_1} \circ k_2 : X_2 \to C$ be a level 2 K-Fibration over the level 1 K-Fibration $f_1 = p_C \circ k_1 : X_1 \to C$. We will use the calculations for the level 1 to get the level 2 calculations. Let B be the ramification divisor on $C \times D$ and B_1 the ramification divisor on $X_1 \times D$.

For the 3-fold X_2, we need to compute $c_3(X_2)$, $c_1c_2(X_2)$, and $c_1^3(X_2)$. We have the following facts for any 3-fold X which we will use:

1) $c_3(X) = \chi(X)$

2) $c_1c_2(X) = 24\chi(\mathcal{O}_X)$

3) $c_1^3(X) = -K_X^3$

with the second item a consequence of Hirzebruch–Riemann–Roch [9].

Again, we have that (with $\mathcal{L}_2^{\otimes r} = \mathcal{O}(B_1)$ the covering bundle)

$$K_{X_2} = k_2^*(K_{X_1\times D} \otimes \mathcal{L}_2^{\otimes(r-1)}) \tag{5.11}$$

so that

$$-c_1^3(X_2) = K_{X_2}^3 = r\left(K_{X_1\times D} + \left(\frac{r-1}{r}\right)B_1\right)^3 \tag{5.12}$$

where K_{X_1} is the canonical bundle from the level 1 K-Fibration. Expanding this out we get

$$c_1^3(X_2) = rc_1^3(X_1\times D) - 3\frac{(r-1)^2}{r}K_{X_1\times D}\cdot B_1^2 - 3(r-1)K_{X_1\times D}^2\cdot B_1 - \frac{(r-1)^3}{r^2}B_1^3. \tag{5.13}$$

For $c_1^3(X_1 \times D)$ we have that

$$\begin{aligned} c_1^3(X_1\times D) &= -K_{X_1\times D}^3 \\ &= -P_{X_1}^*K_{X_1}^3 - 3P_{X_1}^*K_{X_1}^2\cdot P_D^*K_D - 3P_{X_1}^*K_{X_1}\cdot P_D^*K_D^2 - P_D^*K_D^3 \\ &= 3c_1^2(X_1)\chi(D) \end{aligned} \tag{5.14}$$

To see $K_{X_1\times D}\cdot B_1^2 = rdeg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|)$ we have

$$K_{X_1\times D}\cdot B_1^2 = P_{X_1}^*K_{X_1}\cdot B_1^2 + P_D^*K_D\cdot B_1^2. \tag{5.15}$$

Now, let $l = h_1 \times id : X_1 \times D \to D \times D$ where $h_1 = h \circ f_1 : X_1 \to D$. Then as

$$B_1^2 = \sum_{g\in G} g^*\Gamma(h_1)^2 + \sum_{g_1\neq g_2\in G} g_1^*\Gamma(h_1)\cdot g_2^*\Gamma(h_1) = \sum_{g\in G} g^*\Gamma(h_1)^2$$

as G acts on D freely so that each component of B_1 is disjoint. Then we have

$$\Gamma(h_1)^2 = l^*(\Delta^2) = \chi(D)l^*(P\times P)$$

for some $P \in D$. Hence we can write

$$B_1^2 = |G|\chi(D)l^*(P\times P) = |G|\chi(D)h_1^*(P)\times P$$

on X_1. Moreover, as $h_1 = h \circ P_C \circ k$ we have that

$$\begin{aligned} h^*(P) &= k^* P_C^* h^*(P) \\ &= k^* P_C^{-1}(Q) deg(h) \qquad (5.16) \\ &= k^*(Q \times D) deg(h) \\ &= r deg(h) Q \times D \qquad (5.17) \end{aligned}$$

as curves on X_1 leaving us to compute $K_{X_1} \cdot Q \times D$.

$$\begin{aligned} K_{X_1} \cdot Q \times D &= (P_C^* K_C + P_D^* K_D + \frac{r-1}{r} B) \cdot (Q \times D) \\ &= -\chi(D) + \frac{r-1}{r}|G| \qquad (5.18) \end{aligned}$$

so that

$$P_{X_1}^* K_{X_1} \cdot B_1^2 = deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|). \qquad (5.19)$$

Next we have that

$$P_D^* K_D \cdot B_1^2 = |G|\chi(D)(-\chi(D))(C \times P) \cdot (h^{-1}(P) \times P) = 0 \qquad (5.20)$$

as we have that $C \times P$ is birational to $C \times Q$ for some Q not any of the potential Ps. Thus

$$K_{X_1 \times D} \cdot B_1^2 = deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|). \qquad (5.21)$$

For $K_{X_1 \times D}^2 \cdot B_1$, as

$$\begin{aligned} P_{X_1}^* K_{X_1} \cdot P_D^* K_D \cdot B_1 &= -\chi(D) P_{X_1}^* K_{X_1} \cdot (X_1 \times Q) \cdot B_1 \\ &= -\chi(D)|G| K_{X_1} \cdot h_1^*(Q) \qquad (5.22) \\ &= -\chi(D)|G| deg(h)(-r\chi(D) + (r-1)|G|) \end{aligned}$$

we have that

$$\begin{aligned} K_{X_1 \times D}^2 \cdot B_1 &= (P_{X_1}^* K_{X_1} + P_D^* K_D)^2 \cdot B_1 \\ &= P_{X_1} K_{X_1}^2 \cdot B_1 + 2 P_{X_1}^* K_{X_1} \cdot P_D^* K_D \cdot B_1 + P_D^* K_D^2 \cdot B_1 \qquad (5.23) \\ &= |G| c_1^2(X_1) - 2\chi(D)|G| deg(h)(-r\chi(D) + (r-1)|G|) \end{aligned}$$

Lastly, using adjunction ($K_{B_1} = (K_{X_1\times D} + B_1)|_{B_1}$) and the prior parts along with the fact that B_1 is isomorphic to $|G|$ copies of X_1 we have

$$\begin{aligned} B_1^3 &= K_{B_1}^2 - K_{X_1\times D}\cdot B_1^2 - 2K_{X_1\times D}\cdot B_1^2 \\ &= |G|c_1^2(X_1) - deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|) \\ &- 2(|G|c_1^2(X_1) - 2\chi(D)|G|deg(h)(-r\chi(D) + (r-1)|G|)) \end{aligned} \tag{5.24}$$

Thus we have

$$\begin{aligned} c_1^3(X_2) &= rc_1^3(X_1\times D) - 3\frac{(r-1)^2}{r}K_{X_1\times D}\cdot B_1^2 - 3(r-1)K_{X_1\times D}^2\cdot B_1 - \frac{(r-1)^3}{r^2}B_1^3 \\ &= 3rc_1^2(X_1)\chi(D) - 3\frac{(r-1)^2}{r}(deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|)) \\ &- 3(r-1)(c_1^2(X_1) - 2\chi(D)|G|deg(h)(-r\chi(D) + (r-1)|G|)) \\ &- \frac{(r-1)^3}{r^2}\Big[(|G|c_1^2(X_1) - deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|) \\ &- 2(|G|c_1^2(X_1) - 2\chi(D)|G|deg(h)(-r\chi(D) + (r-1)|G|))\Big] \end{aligned} \tag{5.25}$$

Next, we have $c_3(X_2) = \chi(X_2)$, the topological Euler number. This gives

$$\begin{aligned} c_3(X) = \chi(X_2) &= r\chi(X_1\times D) - (r-1)\chi(B_1) \\ &= r\chi(X_1)\chi(D) - (r-1)|G|\chi(X_1) \\ &= c_2(X_1)(r\chi(D) - (r-1)|G|) \end{aligned} \tag{5.26}$$

Lastly, we have by Cor. (2) and $c_1c_2(X) = 24\chi(\mathcal{O}_X)$ that

$$\begin{aligned} c_1c_2(X_2) &= 24r\chi(\mathcal{O}_{X_1\times D}) + (r-1)\chi(B_1) + (r-1)K_{B_1}^2 \\ &- \left(\frac{1-r^2}{r}\right)(K_{X_1\times D}\cdot B_1^2 + B_1^3) \end{aligned} \tag{5.27}$$

where $\chi(\mathcal{O}_{X_1\times D})$ is the arithmetic genus. Using the above computations we get that

$$\begin{aligned}
c_1c_2(X_2) &= 24r\chi(\mathcal{O}_{X_1\times D}) + (r-1)(\chi(B_1) + K_{B_1}^2 \\
&- \left(\frac{1-r^2}{r}\right)(K_{X_1\times D}\cdot B_1^2 + B_1^3) \qquad (5.28)\\
&= 12r\chi(\mathcal{O}_{X_1})\chi(D) + (r-1)|G|(c_1^2(X_1) + c_2(X_1)) \\
&- \left(\frac{1-r^2}{r}\right)(deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|)) \qquad (5.29)\\
&- \left(\frac{1-r^2}{r}\right)\Big[(|G|c_1^2(X_1) - deg(h)|G|\chi(D)(-r\chi(D) + (r-1)|G|) \\
&- 2(|G|c_1^2(X_1) - 2\chi(D)|G|deg(h)(-r\chi(D) + (r-1)|G|))\Big]
\end{aligned}$$

5.3 DISCUSSION OF COMPUTATIONS

A. Kas gives an explicit construction of an example of a level 1 K-Fibration (Example 1 in [13]) which we will discuss a generalization of it here to allow for numerical computation. As in section 4, take a curve D_0 of genus 2 and an unbranched cover $D \to D_0$ of order r with cyclic covering group G being the group of sheet interchange on the cover of order r with generator ρ. Then the genus of D is $g = r+1$. Choose a basis $\{\beta_1, \ldots, \beta_{2g}\}$ of $H_1(D,\mathbb{Z})$ such that the map $\rho_* : H_1(D,\mathbb{Z}) \to H_1(D,\mathbb{Z})$ operates on this basis as

$$\begin{aligned}
&\rho_*(\beta_i) = \beta_i \text{ for } i = 1, 2;\\
&\rho_*(\beta_i) = \beta_{i+1} \text{ for } i = 3, \ldots, r+1;\\
&\rho_*(\beta_{r+2}) = \beta_3;\\
&\rho_*(\beta_i) = \beta_{i+1} \text{ for } i = r+3, \ldots, 2r+1;\\
&\rho_*(\beta_{2r+2}) = \beta_{r+3}.
\end{aligned}$$

Then define a surjective homomorphism $f : H_1(D,\mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$ by taking

$$\beta = \sum_i m_i\beta_i \mapsto \left(\sum_{i=3}^{r+2} m_i \bmod r, \sum_{i=r+3}^{2r+2} m_i \bmod r\right) \qquad (5.30)$$

in $\mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$. This satisfies the criterion of r-divisibility we discussed in Chapter 4 and thus gives a covering map $h : C \to D$ with covering group $\mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$ and degree r^2. The main point is that determining the degree of the covering map h is dependent upon construction and challenging in most generality.

As an example, consider the situation with $r = 2$, $|G| = 2$, $\chi(D) = -4$, $deg(h) = 4$, and $\chi(C) = deg(h)\chi(D) = -16$ as in Example 1 of [13]. The Chern numbers for this are

$$c_1^2(X_1) = 368$$
$$c_2(X_1) = 160$$
$$\chi(\mathcal{O}_{X_1}) = 44$$

so that $c_1^2(X_1)/c_2(X_1) = 2 + \frac{3}{10}$.

In the book Compact Complex Surfaces [12], the authors claim that the ratio $c_1^2(X)/c_2(X)$ satisfies

$$2 < \frac{c_1^2(X)}{c_2(X)} < \frac{7}{3} \tag{5.31}$$

by taking the limit as $r \to \infty$ which is a sharper bound than $c_1^2(X) \leq 3c_2(X)$ [18]. This is not quite correct, but not far off. To see this, write $\chi(D) = |G|\chi(D/G)$ and note that $\chi(D/G)$ is at most -2 since the genus of D/G is at least 2. The quotient is

$$\frac{c_1^2(X)}{c_2(X)} = \frac{2r\chi(C)\chi(D) - 2(r-1)|G|\chi(C) - \left(\frac{r^2-1}{r}\right) deg(h)|G|\chi(D)}{\chi(C)(r\chi(D) - (r-1)|G|)} \tag{5.32}$$

$$= 2 - \frac{(r^2-1)\,|G|}{r^2\chi(D) - r(r-1)|G|} \tag{5.33}$$

$$= 2 - \frac{(r^2-1)}{r^2\chi(D/G) - r(r-1)} \tag{5.34}$$

$$\leq 2 + \frac{(r^2-1)}{3r^2 - r} \tag{5.35}$$

This rational function achieves a maximum at $r = 6$ of $2+\frac{35}{102}$. This can be constructed by taking D as a 6-fold cover over a curve of genus 2 and covering group of order 6 via the A. Kas construction discussed just prior. of Hence we have the following:

Theorem 6. $2 < \frac{c_1^2(X)}{c_2(X)} \leq 2 + \frac{35}{102}$.

The issue is that the authors in [12] took the limit at Eqn. (5.34), but the $\chi(D/G)$ has r dependence as well. Asymptotically, taking the limit as r goes to infinity in Eqn. (5.34), the quotient $\frac{c_1^2(X)}{c_2(X)}$ approaches 7/3, so for larger covering spaces, we will be away from the upper bound and close to 7/3.

Taking the A. Kas general construction, we have $\chi(D/G) = -2$ as our base D_0 is genus 2 and we can explicity write the Chern numbers explicitly in r only:

$$c_1^2(X) = 14r^5 - 4r^4 - 2r^3, \tag{5.36}$$

$$c_2(X) = 6r^5 - 2r^4. \tag{5.37}$$

Thus Eqn. (5.34) is

$$\frac{c_1^2(X)}{c_2(X)} = 2 + \frac{r^2 - 1}{3r^2 - r} \tag{5.38}$$

which is precisely the maximal upper bound. Moreover, as this is a decreasing function for $r \geq 6$, this approaches the asymptote 7/3 from above.

For 3-folds, we have the following inequalities with their Chern numbers as Chapter three on relations on Chern Numbers:

$$c_1^3 \geq \frac{8}{3} c_1 c_2, \tag{5.39}$$

$$\frac{c_3}{c_1 c_2} \geq -2 - 7\frac{c_1^3}{c_1 c_2}. \tag{5.40}$$

In general, not much can be said about c_3 for 3-fold [11], but for level 2 K-Fibrations we can say a bit in special cases. In our A. Kas example we have the explicit formulas for the Chern numbers:

$$c_1^3(X) = -12r^4(12r^3 - 7r^3 - 2r + 1), \tag{5.41}$$

$$c_1 c_2(X) = 2r^4(-33r^3 + 20r^2 + 3r - 2), \tag{5.42}$$

$$c_3(X) = -2r^5(1 - 3r)^2. \tag{5.43}$$

As an example, using $r = 2$, $|G| = 2$, $\chi(D) = -4$, and $deg(h) = 4$ we have

$$c_1^3(X_2) = -12480$$

$$c_1 c_2(X_2) = -5760$$

$$c_3(X_2) = -1600$$

which satisfy the Chern bounds above.

Notice that $c_1^3(X)$, $c_3(X)$, and $c_1c_2(X)$ are all in degree 7 in r, taking the ratio and the leading coefficient we get some asymptotic results.

Theorem 7.

$$\frac{c_3(X_2)}{c_1c_2(X_2)} \to \frac{9}{33}, \tag{5.44}$$

$$\frac{c_1^3(X_2)}{c_1c_2(X_2)} \to \frac{72}{33}, \tag{5.45}$$

$$\frac{c_1^3(X_2)}{c_3(X_2)} \to 8. \tag{5.46}$$

as $r \to \infty$.

Moreover, a quick analysis yields that for this example, $c_3(X_2)/c_1c_2(X_2)$ attains a minimum at $r = 6$ of $51/188$, $c_1^3(X_2)/c_1c_2(X_2)$ a max at $r = 6$ of $411/188$, and $c_1^3(X_2)/c_3(X_2)$ a max at $r = 6$ as well of $137/17$ so that with the initial value at $r = 2$ being the corresponding maximum or minimum we have the theorem:

Theorem 8.

$$\frac{51}{188} \leq \frac{c_3(X_2)}{c_1c_2(X_2)} \leq \frac{5}{18}, \tag{5.47}$$

$$\frac{13}{6} \leq \frac{c_1^3(X_2)}{c_1c_2(X_2)} \leq \frac{411}{188}, \tag{5.48}$$

$$\frac{31}{5} \leq \frac{c_1^3(X_2)}{c_3(X_2)} \leq \frac{137}{17}. \tag{5.49}$$

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APPENDIX

A. VECTOR BUNDLES AND SHEAVES

In the construction of the K-Fibration we relied on the construction of the total space of a sheaf. In this appendix, I will review the relationship between vector bundles and locally finite free sheaves along with the construction of the tautological section. The principle source for the correspondence is from [19] Exercise II.5.17.

Let $p : L \to X$ be a vector bundle. Define for an open set U of X a set

$$\mathcal{L}(U) = \{s : U \to L : p \circ s = id_U\}$$

called the set of sections.

Lemma 4. *$\mathcal{L}$ is a sheaf on X as a locally free $\mathcal{O}_X$-module of rank $r = rank(L)$.*

This gives the map from the category of vector bundles on X to the category of locally free $\mathcal{O}_X$-modules, up to isomorphism.

Let $\mathcal{L}$ be a locally finite free sheaf of rank r on a scheme X. The corresponding vector bundle L of $\mathcal{L}$ is $L = \mathbf{Spec}(\mathrm{Sym}(\mathcal{L}^\vee))$. This creates L is a rank r vector bundle over X via $p : L \to X$. This construction gives the sheaf corresponding to L being $\mathcal{L}^{\vee\vee} = \mathcal{L}$ (hence why we need the dual).

This construction yields a bijection between vector bundles over X and locally finite free sheafs on X, so the total space L of $\mathcal{L}$ is the corresponding vector bundle.

For L a vector bundle, we want to define a tautological section $t \in \Gamma(L, p^*\mathcal{L})$ (as named in [12]). For any morphism $f : Y \to X$, we have the bijection

$$Hom_X(Y, L) \longleftrightarrow \Gamma(Y, f^*\mathcal{L})).$$

For $Y = L$, then t is the section corresponding to the morphism induced by the natural map $f^*f_*\mathcal{O}_L \to \mathcal{O}_L$.

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